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Analytical C^2 smooth blending surfaces

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Abstract

Two factors are important in the generation of blending surfaces for interactive graphical and CAD applications, computational speed and the degree of smoothness. Most surface-blending methods blend surfaces with tangent continuity. However, curvature continuity has recently become increasingly important in various applications. In this paper, we present a method that is able to achieve curvature continuity based on the use of partial differential equations (PDE). The blending surfaces are generated as the solution to a sixth-order PDE with one vector-valued parameter. To achieve interactive performance, we propose an effective analytical method for the resolution of this sixth-order PDE.

Keywords: Surface blending; Curvature continuity; Sixth-order partial differential equation; Analytical solution

1. Introduction

Geometric objects in CAD and computer graphics are mostly modelled with regular surfaces, such as B-spline and quadratic surfaces. For aesthetic or manufacturing purposes, however, a typical real object or mechanical component is not always ‘regular’ and therefore cannot be completely represented with regular surfaces. Instead, the regular surfaces of an object or mechanical component, known as the primary surfaces, are usually connected (blended) smoothly by transition surfaces to give a more natural shape of the object. These transition surfaces are called the blending surfaces. These blending surfaces, although appear secondary, play a very important role in aesthetics and manufacturing functions. These have been an important research subject for CAD and computer graphics for many years [25].

It is well known the rolling-ball blend is a common method to blend two primary surfaces. This approach works by (imaginarily) rolling a ball between the two primary surfaces and the swept surface of the rolling ball is the generated blending surface. Depending on whether the radius of the rolling ball varies or not, rolling-ball blends can be grouped into constant-radius rolling-ball blends and varying-radius rolling-ball blends. For constant-radius rolling-ball blends, Barnhill et al. presented a method to blend two parametric surfaces following the intersection of two offset

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surfaces using only the first-order derivatives of the progenitors [4]. By sweeping rational quadratic (conic section) curves, Choi and Ju proposed a method to blend any rectangular parametric surface patches [11]. By tracing the intersection of the corresponding offset surfaces and normal projection curves on the surfaces, Farouki and Sverrisson were able to approximate the constant-radius blends with a sequence of tangent continuous rational tensor-product patches [14]. Kós et al. investigated the problem of determining the radius of constant-radius rolling-ball blends from point data for their applications in the reverse engineering of mechanical components [17]. Lukács analysed the variable-radius rolling-ball (VRRB) blending surfaces by treating these surfaces as an envelope of one-parameter families of varying radius balls [19]. Regarding a blending surface as the envelope of a rolling sphere or sweeping circle that centers on a spine curve and touches the surface to be blended along the linkage curves, Chuang et al. described some methods that use the derived spine curve and linkage curves to compute a parametric form of the variable-radius spherical and circular blends [12]. Since the radius is difficult to specify and the spine curve is hard to trace in variable-radius blending, Chuang and Hwang proposed several geometric constraints to specify the variable radius which is translated to a non-linear system to represent the spine curves exactly [13]. Rolling-ball blends are simple and intuitive, since it is defined by a straightforward physical motion. It is also attractive from the modelling point of view that the spine, the trimlines, the assignment and the profile are automatically generated. The disadvantage, however, is that the surface swept by the rolling ball is of a high algebraic degree.

Cyclides provide a method for some frequently occurring simple blends, such as a cylinder obliquely meeting a plane. They can be described by implicit quartic equations or parametric expressions using trigonometrical parameterisation or rational biquadratic Bézier equations. Allen and Dutta studied blends between natural quadrics using both Dupin ring cyclides and parabolic cyclides with a new definition of a pure cyclide blend which forces the construction of non-singular cyclide blends, deliberately excluding cyclide joins and singular surfaces [1]. They also studied plane/cone and plane/cylinder supercyclide blends [2]. By establishing a necessary and sufficient condition for two cones to have a blending Dupin cyclide, Shene presented a complete theory of blending cones with Dupin cyclides [24].

About a decade ago, Bloor and Wilson proposed another blending method which uses the solution to a partial differential equation under specified boundary conditions to define a blending surface [5]. The boundary conditions consist of the trimlines on the primary surfaces, and usually the first-order partial derivatives of the surfaces at the trimlines between the blending surface and the primary surfaces. Since then, this approach has gained momentum due to its ability to blend complex surfaces. For such blends, a key topic is how to resolve the partial differential equation accurately and quickly, which has attracted a great deal of efforts. Due to the complexity of the PDEs, the majority of the developed methods are numerical including the finite element methods [18,9], finite difference methods [10,27,30] and the collocation method [6]. These numerical methods, although effective, are very slow and therefore are not applicable to interactive applications. In order to tackle this issue, a Fourier series method [23], a pseudo-spectral method [7] and a perturbation method [8] were developed. However, these three methods do not exactly satisfy the governing PDEs or the boundary conditions and hence are not accurate enough.

Since the vector-valued shape parameter in the PDEs has a strong influence on the shape of the blending surfaces, You and Zhang proposed a more general fourth-order partial differential equation [28]. It has three shape parameters and covers all existing fourth-order PDEs used for surface generation. The effects of these vector-valued shape parameters have been studied and proven very effective in the manipulation of the generated surfaces [32]. In addition, Zhang and You also discussed the efficiency and capacity of the orders of the PDEs used for surface creation [31].

Blending surfaces with curvature continuity are often needed in many engineering applications such as high-speed cams and turbine blades. Pegna discussed a method of designing curvature continuous fairing surfaces interactively [20]. Pegna and Wolter also presented the linkage curve theorem for the design of curvature continuous blending surfaces [21]. Jones decomposed an n-sided region into n rectangles and indicated that the rectangular patches are biseptic for curvature continuity [16]. Zheng et al. investigated the issue of curvature continuity between two adjacent rational Bézier surfaces which may be either rectangular or triangular [33]. Schichtel presented a technique for filling polygonal holes using a transfinite interpolant which achieves the second-order smoothness transitions at an arbitrary linkage curve between two surfaces [22]. Aumann proposed the so-called normal ringed surfaces to form curvature continuous connections of cones and/or cylinders [3]. Ye developed the Gaussian and mean curvature criteria for curvature control [26]. Most recently, Hartmann proposed another blending method which produces parametric blending curves and surfaces able to blend the base curves and surfaces with good smoothness. The shape of the blend curves/surfaces is manipulated with two design parameters: thumb weight and balance [15].

The PDE-based approach is powerful in the generation of blending surfaces, as it is able to tackle blending problems otherwise cannot be easily dealt with using other existing methods. But two issues have to be considered before the advantage of this approach can be realised:

- Blending with curvature continuity.
- Surface generation with high computational efficiency.

In the existing references, the PDE-based methods were mainly employed to blend primary surfaces with tangential continuity on their trimlines. Little has been reported on the study of curvature continuity with such blending surfaces. Although Bloor and Wilson stated that a sixth-order partial differential equation was required when curvature continuity is to be achieved, they did not attempt a solution to this problem [5]. In addition, there exists a weak point of the current PDE resolution methods for surface blending and generation — they are either too slow or not accurate enough.

In this paper, we introduce a sixth-order partial differential equation to accommodate the requirement of curvature continuity, which is guaranteed if C^2 smoothness is achieved. Unlike other numerical resolution methods reported in the literature, we will develop an analytical method to solve the PDE under various boundary conditions. For simple boundary conditions, the PDE will be solved with a closed form analytical solution. For complex boundary conditions, a power series function will be introduced to satisfy the boundary conditions exactly and the errors of the PDE will be minimised with the least squares method. Due to its analytical nature, the proposed method is accurate and computationally very efficient, and has an ability to cover various complicated blending problems.

2. Sixth-order partial differential equation

When using partial differential equations to generate blending surfaces, the continuities on the trimlines are represented as boundary conditions. As the number of boundary conditions depends on the order of the PDE, the higher the order of a PDE, the more boundary conditions the PDE can meet. In terms of surface blending, a second-order PDE can satisfy the condition of positional continuity and a fourth-order PDE can guarantee up to tangential continuity on the trimlines. If we hope to achieve curvature continuity, a sixth-order PDE has to be employed. With the operator defined in Eq. (3), such a sixth-order PDE can be written as

$$\frac{\partial^6 \boldsymbol{x}}{\partial u^6} + 3\boldsymbol{a}^2 \frac{\partial^6 \boldsymbol{x}}{\partial u^4 \partial v^2} + 3\boldsymbol{a}^4 \frac{\partial^6 \boldsymbol{x}}{\partial u^2 \partial v^4} + \boldsymbol{a}^6 \frac{\partial^6 \boldsymbol{x}}{\partial v^6} = 0 \qquad (1)$$

where $\boldsymbol{x} = \boldsymbol{x}(u, v) = [x \quad y \quad z]^{\mathrm{T}}$ is a vector-valued positional function (the blending surface), and $\boldsymbol{a} = [a_x \quad a_y \quad a_z]^{\mathrm{T}}$ a vector-valued shape parameter.

Surface curvature depends on the partial derivatives of the surface up to the second order. It can be proven that if two surfaces have the same second-order derivatives at their trimline, both surfaces will achieve curvature continuity at the trimline. Therefore, the boundary conditions guaranteeing curvature continuity can be defined by

$$\begin{aligned} &u = 0 \quad \boldsymbol{x} = \boldsymbol{G}_1(v) \quad \frac{\partial \boldsymbol{x}}{\partial u} = \boldsymbol{G}_2(v) \quad \frac{\partial^2 \boldsymbol{x}}{\partial u^2} = G_3(v) \\ &u = 1 \quad \boldsymbol{x} = \boldsymbol{G}_4(v) \quad \frac{\partial \boldsymbol{x}}{\partial u} = \boldsymbol{G}_5(v) \quad \frac{\partial^2 \boldsymbol{x}}{\partial u^2} = G_6(v) \end{aligned} \qquad (2)$$

where $\boldsymbol{G}_i(v)$ (i = 1, 2, ..., 6) represent the vector-valued functions describing the position, tangent and curvature values on the boundaries.

Because we use the second-order partial derivatives, the continuity conditions defined by Eq. (2) are stronger than just curvature continuity. They actually define the conditions of C^2 continuity.

Subject to boundary conditions (2), PDE (1) can be solved with various numerical methods, such as the above-mentioned finite element method, finite difference method and collocation method. Considering the importance of both computational efficiency in interactive computer graphics and the satisfaction of boundary conditions, we will in this paper develop an analytical solution of PDE (1) under boundary conditions (2).

3. Closed form solution

To facilitate the description, let us firstly define a vector operator whose two operands are two column vectors, and which produces a new column vector whose each element is the product of the corresponding elements of the two column vectors, i.e.

$$\begin{aligned}\boldsymbol{st} &= [\,s_x \quad s_y \quad s_z\,]^{\mathrm{T}}[\,t_x \quad t_y \quad t_z\,]^{\mathrm{T}}\\ &= [\,s_xt_x \quad s_yt_y \quad s_zt_z\,]^{\mathrm{T}}\end{aligned} \tag{3}$$

Decomposing them into basic functions which are not in a polynomial form, boundary conditions (2) can be rewritten as follows

$$\begin{array}{llll} u=0 & \boldsymbol{x}=\sum_{i=1}^{I}\boldsymbol{a}_{1i}\boldsymbol{g}_i(v) & \dfrac{\partial \boldsymbol{x}}{\partial u}=\sum_{i=1}^{I}\boldsymbol{a}_{2i}\boldsymbol{g}_i(v) & \dfrac{\partial^2 \boldsymbol{x}}{\partial u^2}=\sum_{i=1}^{I}\boldsymbol{a}_{3i}\boldsymbol{g}_i(v)\\ u=1 & \boldsymbol{x}=\sum_{i=1}^{I}\boldsymbol{a}_{4i}\boldsymbol{g}_i(v) & \dfrac{\partial \boldsymbol{x}}{\partial u}=\sum_{i=1}^{I}\boldsymbol{a}_{5i}\boldsymbol{g}_i(v) & \dfrac{\partial^2 \boldsymbol{x}}{\partial u^2}=\sum_{i=1}^{I}\boldsymbol{a}_{6i}\boldsymbol{g}_i(v)\end{array} \tag{4}$$

where $\boldsymbol{a}_{ji}$ (j = 1, 2, ..., 6; i = 1, 2, ..., I) are known vector-valued coefficients, and $\boldsymbol{g}_i(v)$ (i = 1, 2, ..., I) are basic functions in a vector form.

Those boundary conditions under which the closed form solution of PDE (1) is obtainable if all $\boldsymbol{g}_i(v)(i=1,2,\cdots,I)$ can be expressed in the following forms

$$\begin{aligned}&\boldsymbol{g}_i^{(2)}(v)=\boldsymbol{b}_{2i}\boldsymbol{g}_i(v)\\ &\boldsymbol{g}_i^{(4)}(v)=\boldsymbol{b}_{4i}\boldsymbol{g}_i(v)\quad (i=1,2,\ldots,I)\\ &\boldsymbol{g}_i^{(6)}(v)=\boldsymbol{b}_{6i}\boldsymbol{g}_i(v)\end{aligned} \tag{5}$$

where $\boldsymbol{g}_i^{(k)}(v)=\mathrm{d}^{(k)}\boldsymbol{g}_i(v)/\mathrm{d}v$ (k = 2, 4, 6) and $\boldsymbol{b}_{ki}$ (k = 2, 4, 6) are known vector-valued coefficients.

With the method of variable separation, the following unified closed form solution of PDE (1) subject to boundary conditions (4) can be developed

$$\boldsymbol{x}(u,v)=\sum_{i=1}^{I} f_i(u)\,\boldsymbol{g}_i(v) \tag{6}$$

Function $f_i(u)$ is determined by substituting (6) into PDE (1). Depending on the values of b_{ki} (k = 2, 4, 6), $f_i(u)$ has three different forms. Here, we take the x-component as an example to present its formulation.

If $b_{x2i} = b_{x4i} = b_{x6i} = 0$,

$$\begin{aligned}f_{xi}(u) &= c_{xi0}+c_{xi1}u+c_{xi2}u^2+c_{xi3}u^3\\ &\quad +c_{xi4}u^4+c_{xi5}u^5\end{aligned} \tag{7}$$

If $b_{x2i}=b_{xi}^2, b_{x4i}=b_{xi}^4, b_{x6i}=b_{xi}^6$,

$$\begin{aligned}f_{xi}(u) &= (c_{xi0}+c_{xi1}u+c_{xi2}u^2)\cos a_xb_{xi}u\\ &\quad +(c_{xi3}+c_{xi4}u+c_{xi5}u^2)\sin a_xb_{xi}u\end{aligned} \tag{8}$$

And if $b_{x2i}=-b_{xi}^2, b_{x4i}=b_{xi}^4, b_{x6i}=-b_{xi}^6$,

$$\begin{aligned}f_{xi}(u) &= (c_{xi0}+c_{xi1}u+c_{xi2}u^2)\,\mathrm{e}^{a_xb_{xi}u}\\ &\quad +(c_{xi3}+c_{xi4}u+c_{xi5}u^2)\,\mathrm{e}^{-a_xb_{xi}u}\end{aligned} \tag{9}$$

where b_{xi} is the coefficient of parametric variable v in basic functions $\boldsymbol{g}_i(v)$.

The unknown constants in Eqs. (7), (8) and (9) are determined by substituting them into (6), then substituting (6) into boundary conditions (4).

In order to demonstrate the application of the above closed form solution in surface blending, we will give three examples of blending surfaces.

The first example is to blend an elliptic cylinder and a sphere which is used to explain the application of closed form solution (9). Suppose that the boundary conditions for this blending task have the forms of

$$
\begin{array}{llll}
u=0 & x=a\cos v & \dfrac{\partial x}{\partial u}=0 & \dfrac{\partial^2 x}{\partial u^2}=0 \\
 & y=b\sin v & \dfrac{\partial y}{\partial u}=0 & \dfrac{\partial^2 y}{\partial u^2}=0 \\
 & z=h_0-h_1 & \dfrac{\partial z}{\partial u}=-h_1 & \dfrac{\partial^2 z}{\partial u^2}=0 \\
u=1 & x=r\sin u_0\cos v & \dfrac{\partial x}{\partial u}=r\cos u_0\cos v & \dfrac{\partial^2 x}{\partial u^2}=-r\sin u_0\cos v \\
 & y=r\sin u_0\sin v & \dfrac{\partial y}{\partial u}=r\cos u_0\sin v & \dfrac{\partial^2 y}{\partial u^2}=-r\sin u_0\sin v \\
 & z=r\cos u_0 & \dfrac{\partial z}{\partial u}=-r\sin u_0 & \dfrac{\partial^2 z}{\partial u^2}=-r\cos u_0
\end{array}
\tag{10}
$$

Taking the x-component as an example and comparing (10) with (4) and (5), we have $g_{x1}(v)=\cos v$ and $b_{x21}=b_{x61}=-1$, $b_{x41}=1$. Therefore, the closed form solution of PDE (1) for the x-component is

$$
x=\left(\sum_{j=0}^{2} c_{x1j}u^j\,\mathrm{e}^{a_x u}+\sum_{j=3}^{5} c_{x1j}u^{j-3}\,\mathrm{e}^{-a_x u}\right)\cos v \tag{11}
$$

Substituting (11) into boundary conditions (10), all the unknown constants in (11) can be determined. With this closed form solution, we obtain the blending surfaces given in Fig. 1a and b. Fig. 1a is the same as Fig. 1b except that three different colours are used to distinguish the three surface patches. All position, tangent and curvature continuities are satisfied at the boundary curves.

The second example is to blend a circular cylinder and a plane at a specified straight line. Its main aim is to demonstrate the application of the closed form solution (7) in surface blending. The boundary conditions for this blending problem are

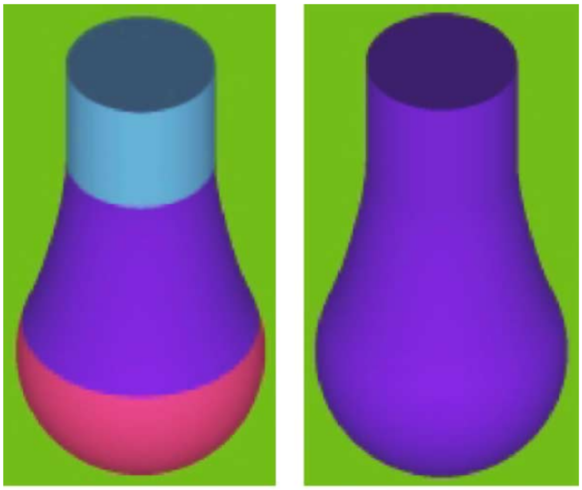
$$
\begin{array}{llll}
u=0 & x=a_0-a_1 v & \dfrac{\partial x}{\partial u}=2(a_0-a_1 v) & \dfrac{\partial^2 x}{\partial u^2}=2(a_0-a_1 v) \\
 & y=b_0+b_1 v & \dfrac{\partial y}{\partial u}=2(b_0+b_1 v) & \dfrac{\partial^2 y}{\partial u^2}=2(b_0+b_1 v) \\
 & z=0 & \dfrac{\partial z}{\partial u}=0 & \dfrac{\partial^2 z}{\partial u^2}=0 \\
u=1 & x=r\cos v & \dfrac{\partial x}{\partial u}=0 & \dfrac{\partial^2 x}{\partial u^2}=0 \\
 & y=r\sin v & \dfrac{\partial y}{\partial u}=0 & \dfrac{\partial^2 y}{\partial u^2}=0 \\
 & z=h_0+h_1 & \dfrac{\partial z}{\partial u}=h_1 & \dfrac{\partial^2 z}{\partial u^2}=0
\end{array}
\tag{12}
$$

For the x-component, we have $g_1(v)=1$, $g_2(v)=v$ and $g_3(v)=\cos v$. Therefore, the closed form solution of PDE (1) for the x-component takes the following form

$$
\begin{aligned}
x(u,v) &= \sum_{j=0}^{5} c_{x1j}u^j+\sum_{j=0}^{5} c_{x2j}u^j v \\
&\quad +\left(\sum_{j=0}^{2} c_{x3j}\ u^j\,\mathrm{e}^{a_x u}+\sum_{j=3}^{5} c_{x3j}u^{j-3}\,\mathrm{e}^{-a_x u}\right)\\
&\quad \cos v
\end{aligned}
\tag{13}
$$

In the same way as earlier, all the unknown constants in closed form solution (13) can be determined by substituting it into boundary conditions (12). The blending surface generated with (13) is depicted in Fig. 2a and b, which are from different viewing angles of the same blending surface.

The final example, which demonstrates the application of the closed form solution (8), is to blend an elliptic cylinder and a plane at a specified curve. The boundary conditions for this blending task can be



(a) (b)

Fig. 1. Blending between an elliptic cylinder and a sphere.

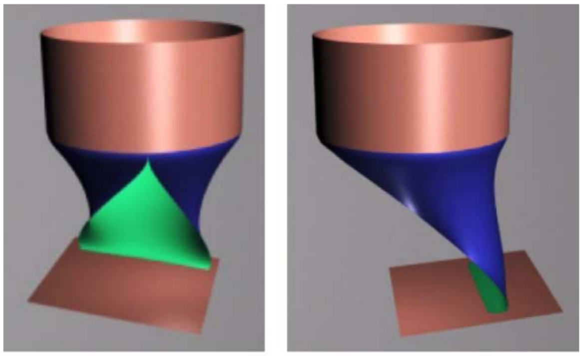
written as follows

$$\begin{array}{llll} u=0 & x=c\cosh dv & \dfrac{\partial x}{\partial u}=0 & \dfrac{\partial^2 x}{\partial u^2}=2c\cosh dv \\ & y=c\sinh dv & \dfrac{\partial y}{\partial u}=0 & \dfrac{\partial^2 y}{\partial u^2}=2c\sinh dv \\ & z=0 & \dfrac{\partial z}{\partial u}=0 & \dfrac{\partial^2 z}{\partial u^2}=0 \\ u=1 & x=a\cos v & \dfrac{\partial x}{\partial u}=0 & \dfrac{\partial^2 x}{\partial u^2}=0 \\ & y=b\sin v & \dfrac{\partial y}{\partial u}=0 & \dfrac{\partial^2 y}{\partial u^2}=0 \\ & z=h_0 & \dfrac{\partial z}{\partial u}=2h_1 & \dfrac{\partial^2 z}{\partial u^2}=2h_1 \end{array} \tag{14}$$

The basic functions for the x-component given by the above boundary conditions are $g_{x1}(v)=\cosh dv$ and $g_{x2}(v)=\cos v$. For $g_{x1}(v)$, $b_{x21}=d^2$, $b_{x41}=d^4$, $b_{x61}=d^6$.

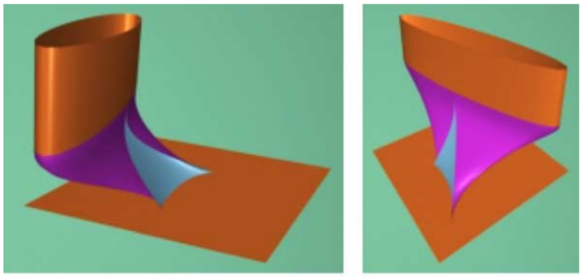
Therefore, the closed form solution of PDE (1) for the x-component is

$$x(u,v)=\left(\sum_{j=0}^{2} c_{x1j}u^j\cos a_x\,du+\sum_{j=3}^{5} c_{x1j}u^{j-3}\sin a_x\,du\right)\cosh dv+\left(\sum_{j=0}^{2} c_{x2j}u^j\,e^{a_x u}+\sum_{j=3}^{5} c_{x2j}u^{j-3}\,e^{-a_x u}\right)\cos v \tag{15}$$



(a) (b)

Fig. 2. Blending between a circular cylinder and a plane at a specified straight line.



(a) (b)

Fig. 3. Blending between an elliptic cylinder and a plane at a specified curve.

Substituting (15) into boundary conditions (14) and determining all its unknown constants, we obtain the blending surface shown in Fig. 3a and b. These two images are the same blending surface viewed from different viewing angles.

The last two examples are more complex and would be very difficult to model using other blending approaches. The blending surface connects a closed curve of a circular/elliptic cylinder to an open curve (line/curve segment) where the blending surface smoothly merges from a closed shape at one end to an open shape at the other. We have not found similar blending surfaces in existing literature. They suggest that the closed form solution of a partial differential equation has an ability to generate some complex blending surfaces.

Because the solution is analytical, the developed closed form method is computationally very fast. We have timed the process of determining the unknown constants in the closed form solutions and found the process took less than 10^{-6} of a second on a 800 MHz PC for all three blending tasks, no problem for interactive computer graphics applications.

4. Least squares solution

A closed form solution of PDE (1) subject to the boundary conditions (2) is obtainable only when the basic functions of Eq. (4) satisfy conditions (5). However, in practical blending problems, the boundary curves can be of any continuous shape. Thus, it is unlikely that a closed form solution of PDE (1) is obtainable. In this section, we introduce a least squares method, which is also efficient and accurate, for the resolution of this problem.

For the sake of description, we assume that the closed form solution of PDE (1) is unobtainable for all basic functions. However, this does not affect the treatment of the cases where both closed form and non-closed form solutions coexist. What we need to do is to separate the boundary conditions into two groups: one for the closed form solution and the other for the non-closed form solution. Then, we can use the closed form solution method and the least squares method to solve these two groups, respectively.

Here, we propose an approximate solution of PDE (1) subject to boundary conditions (4) using a power series of u

$$\boldsymbol{x}(u,v) == \sum_{i=1}^{I}\sum_{m=0}^{M} \boldsymbol{p}_{im} u^m \boldsymbol{g}_i(v) \tag{16}$$

All $\boldsymbol{g}_i(v)$ are the basic functions of (4). In order to meet the boundary conditions exactly, some unknown constants in Eq. (16) can be determined by substituting (16) into (4). Then substituting these unknown constants back to (16), the approximate solution is written as

$$\begin{aligned}\boldsymbol{x}(u,v) &= \sum_{i=1}^{I}\left\{\sum_{j=1}^{6} F_j(u)\boldsymbol{a}_{ji}\right.\\ &\quad \left. + \left[\sum_{m=6}^{M} H(m,u) + u^m\right]\boldsymbol{p}_{im}\right\}\boldsymbol{g}_i(v)\end{aligned} \tag{17}$$

The concrete forms of functions $F_j(u)$ and $H(m, u)$ are not given here. The interested reader should be able to derive them fairly easily following the above described deduction steps.

With this treatment, the boundary conditions are satisfied exactly, which is crucial to surface blending. However, solution (17) only approximate the accurate solution of PDE (1), there exist errors. In what is following, we will present a method to minimise these errors. An error (residual) function is obtained by substituting (17) into PDE (1), which represents the divergence of the approximate solution and the accurate solution. By uniformly distributing some points within the resolution region, we can calculate the residual (error) values at these collocation points. The collection of these residual values leads to the following equation containing the remaining unknown constants p_{im} (i = 6, 7, ..., M). This can be arranged as a set of linear equations expressed in the following matrix form

$$\boldsymbol{R} = \boldsymbol{K}\boldsymbol{P} - \boldsymbol{A} \tag{18}$$

where R is an array consisting of the residual values, P consists of p_{im} (i = 6, 7, ..., M), which are the unknown constants to be determined, K a matrix consisting of the coefficients of p_{im} (i = 6, 7, ..., M), and A an array of known constants.

To obtain unknown constants p_{im} (i = 6, 7, ..., M) using the least squares method [29], we minimise the square sum of the residual values, i.e.

$$\min\ \boldsymbol{R}^{\mathrm{T}}\boldsymbol{R} \tag{19}$$

Using the least squares method, one can obtain the following linear equations in a matrix form

$$\boldsymbol{K}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{P} = \boldsymbol{K}^{\mathrm{T}}\boldsymbol{A} \tag{20}$$

By solving Eq. (20), the unknown constants in the approximate solution (17) are determined, so is the function of the blending surface. In the following, we present a more complex example to demonstrate the applications of this approximate solution in surface blending.

The blending of two intersecting surfaces is not solvable using the closed form solution method presented above. With an existing PDE-based blending approach, such a problem can only be solved using numerical methods such as the finite element method and finite difference method. As discuss earlier, the downside of these numerical methods is their excessive computational cost. Here, we solve it using the above-developed least squares method.

The boundary conditions at the trimlines can be summarised as

With the above-developed least squares method, we can determine all the unknown constants of Eq. (22). The generated blending surface is depicted in Fig. 4.

This example indicates that the least squares solution developed above is applicable to various boundary conditions and an arbitrary combination of the shape parameters of Eq. (1). It is applicable to any free-form surfaces whose trimlines and partial derivatives at the trimlines may or may not be analytically obtainable. Therefore, it is more powerful and able to cover a much wider range of blending problems than many existing methods.

$$
\begin{aligned}
u=0\quad & x=s\cos v && \frac{\partial x}{\partial u}=0 && \frac{\partial^2 x}{\partial u^2}=0\\
& y=s\sin v && \frac{\partial y}{\partial u}=0 && \frac{\partial^2 y}{\partial u^2}=0\\
& z=\sqrt{(r+k_1)^2-s^2\cos^2 v} && \frac{\partial z}{\partial u}=\frac{r+k_1}{\sqrt{(r+k_1)^2-s^2\cos^2 v}} && \frac{\partial^2 z}{\partial u^2}=\frac{1}{\sqrt{(r+k_1)^2-s^2\cos^2 v}}-\frac{(r+k_1)^2}{\sqrt[3]{(r+k_1)^2-s^2\cos^2 v}}\\
u=1\quad & x=(s+l_1)\cos v && \frac{\partial x}{\partial u}=\cos v && \frac{\partial^2 x}{\partial u^2}=0\\
& y=(s+l_1)\sin v && \frac{\partial y}{\partial u}=\sin v && \frac{\partial^2 y}{\partial u^2}=0\\
& z=\sqrt{r^2-(s+l_1)^2\cos^2 v} && \frac{\partial z}{\partial u}=\frac{(s+l_1)\cos^2 v}{\sqrt{r^2-(s+l_1)^2\cos^2 v}} && \frac{\partial^2 z}{\partial u^2}=\frac{\cos^2 v}{\sqrt{r^2-(s+l_1)^2\cos^2 v}}-\frac{(s+l_1)^2\cos^4 v}{\sqrt[3]{r^2-(s+l_1)^2\cos^2 v}}
\end{aligned}
\tag{21}
$$

The basic function for the x-component is $\cos v$, for y-component is $\sin v$, and those for z-component are $\sqrt{(r+k_1)^2-s^2\cos^2 v}$, $1/\sqrt{(r+k_1)^2-s^2\cos^2 v}$, $1/\sqrt[3]{(r+k_1)^2-s^2\cos^2 v}$, $\sqrt{r^2-(s+l_1)^2\cos^2 v}$, $\cos^2 v/\sqrt{r^2-(s+l_1)^2\cos^2 v}$ and $\cos^4 v/\sqrt[3]{r^2-(s+l_1)^2\cos^2 v}$. From Eq. (16), the approximate analytical solution of the blending surface can be written as

$$
\begin{aligned}
x&=\sum_{m=0}^{M_{x0}} p_{0m}u^m\cos v\\
y&=\sum_{m=0}^{M_{y0}} q_{0m}u^m\sin v\\
z&=\sum_{m=0}^{M_{z0}} r_{0m}u^m\sqrt{(r+k_1)^2-s^2\cos^2 v}+\sum_{m=0}^{M_{z1}}\frac{r_{1m}u^m}{\sqrt{(r+k_1)^2-s^2\cos^2 v}}+\sum_{m=0}^{M_{z2}}\frac{r_{2m}u^m}{\sqrt[3]{(r+k_1)^2-s^2\cos^2 v}}\\
&\quad+\sum_{m=0}^{M_{z3}} r_{3m}u^m\sqrt{r^2-(s+l_1)^2\cos^2 v}+\sum_{m=0}^{M_{z4}}\frac{r_{4m}u^m\cos^2 v}{\sqrt{r^2-(s+l_1)^2\cos^2 v}}+\sum_{m=0}^{M_{z5}}\frac{r_{5m}u^m\cos^4 v}{\sqrt[3]{r^2-(s+l_1)^2\cos^2 v}}
\end{aligned}
\tag{22}
$$

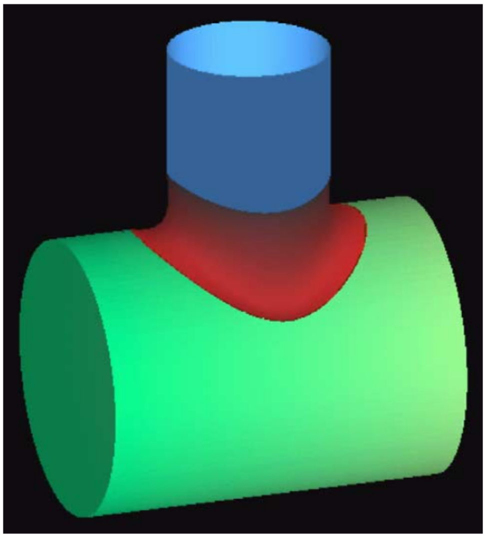


Fig. 4. The blending between two intersecting cylinders.

5. Conclusions and future work

The PDE-based approach is powerful in generating blending surfaces. However, existing methods are only able to achieve tangent continuity. In this paper, we introduce a sixth-order partial differential equation which provides enough degrees of freedom to accommodate the curvature boundary conditions.

Since computational speed is an important factor in interactive graphical and geometric modelling applications, we have developed an analytical solution. It consists of the closed form solution and the least squares solution. For the former, the second-, fourth- and sixth-order partial derivatives of the basic functions must have the same forms as the functions themselves. The closed form solution of the PDE has three different forms depending on the properties of the boundary functions. For the latter, the boundary conditions are exactly satisfied and the errors of the solution to the sixth-order PDE are minimised. Therefore, the proposed method is efficient, effective and accurate.

We have demonstrated the closed form solution method with three examples. They are the blending between an elliptic cylinder and a sphere, between a circular cylinder and a plane at a specified straight line, and between an elliptic cylinder and a plane at a given curve. The blending of two intersecting cylinders was given to demonstrate the least squares solution method. It is found that the developed blending method is able to solve complex blending problems. In addition, it is fast enough for interactive graphics and geometric applications.

PDE-based surface modelling is a general and powerful approach. It has many unique advantages over other mainstream alternatives. Although, what has been discussed in this paper is primarily for surface blending, the same idea is directly applicable to surface construction applications. In general, however, the PDE-based approach requires sophisticated mathematical manipulation, which may be a disadvantage in some sense. To facilitate the use of our method, our next step is to convert the theoretical operations into an easy to use interactive modelling tool. It will hopefully be more acceptable to a larger community of geometric and graphical designers and practitioners.

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