THE DOUBLE PENDULUM

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1. History

The Pendulum has been studied from a physical approach since the first century in the Han Dynasty of ancient China. Although the interest of the Chinese in the pendulum so early in world history was primarily for entertainment, they did pioneer the magnificent influence that the pendulum has had on the human knowledge of physics.

After the chinese, the pendulum was not fully explored until the time of Galileo Galilei in 1602 when he proposed that the isochronic nature of the pendulum could be used to measure time more accurately than ever before. Following Galileo's research, the proverbial floodgates opened, and the pendulum has since been used to measure quantities as familiar as a second to as foreign as a parsec to accuracies which we have come to understand them today.

The double pendulum has a smaller, but no less rich, history. It was not until the development of Lagrangian and Hamiltonian mechanics that the double pendulum was truly understood. Today, the motion of the double pendulum is essentially completely understood, and its physical nature makes it an interesting application for studying many other problems of physics. It is for this reason, that an analysis of the double pendulum from the perspective of dynamical systems is so enlightening, and it is this reason that motivated the creation of this paper.

2. Description

The generalized double pendulum (Figure 2.1) consists of a suspended mass from which another mass is suspended. There also can be added features such as a driving force, and extendable connections between the masses, but these features will be ignored for the purposes of this paper. We shall also assume that the pendulums are confined to move in a two dimensional plane, and is constructed such that the lengths cannot interfere with one another (specifically, the second pendulum is free to spin around over the first pendulum without interference). We will also assume that our pendulum will have massless connections. The coordinate system we will use is the x-y coordinate system which has its origin at the fixing point of the top pendulum, with the negative y-axis extending towards the vertical position of the second pendulum in its rest positon. The physical description we shall derive relies on the angles θ_1 and θ_2, which are measured with respect to the negative y-axis. This description will also be simplified further by assuming that the lengths and masses are of unit value, i.e. ($m_1 = 1[kg] = m_2$ and $L_1 = 1[m] = L_2$).

3. Physical Considerations

We begin our analysis of the double pendulum by deriving the potential energy (V) and kinetic energy (T) at any given time. We note that the positions of the masses are given by the equations.

$$x_1 = \sin\theta_1 \tag{1}$$

$$y_1 = -\cos\theta_1 \tag{2}$$

$$x_2 = \sin\theta_1 + \sin\theta_2 \tag{3}$$

$$y_2 = -\cos\theta_1 - \cos\theta_2 \tag{4}$$

From these equations we may derive the equations for T and V:

$$V = m_1 g y_1 + m_2 g y_2 = -g\cos\theta_1 - g\cos\theta_1 - g\cos\theta_2 = -2g * \cos\theta_1 - g\cos\theta_2 \tag{5}$$

$$T = \frac{1}{2}m_1(\dot{x_1}^2 + \dot{y_1}^2) + \frac{1}{2}m_2(\dot{x_2}^2 + \dot{y_2}^2) \tag{6}$$

$$\dot{x_1} = \dot{\theta}_1 \cos\theta_1 \tag{7}$$

$$\dot{y_1} = \dot{\theta}_1 \sin\theta_1 \tag{8}$$

$$\frac{1}{2}m_1(\dot{x_1}^2 + \dot{y_1}^2) = \frac{1}{2}\dot{\theta_1}^2 \tag{9}$$

$$\dot{x_2} = \dot{\theta}_1 \sin\theta_1 + \dot{\theta}_2 \sin\theta_2 \tag{10}$$

$$\dot{y_2} = \dot{\theta}_1 \cos\theta_1 + \dot{\theta}_2 \cos\theta_2 \tag{11}$$

(12)
$$\dot{x_2}^2 + \dot{y_2}^2 = \dot{\theta_1}^2 + \dot{\theta_2}^2 + 2\dot{\theta}_1\dot{\theta}_2 * (\cos\theta_1\cos\theta_1 + \sin\theta_1\sin\theta_2) = \dot{\theta_1}^2 + \dot{\theta_2}^2 + 2\dot{\theta}_1\dot{\theta}_2\cos\theta_1 - \theta_2$$

$$\frac{1}{2}m_2(\dot{x_2}^2 + \dot{y_2}^2) = \frac{1}{2}\dot{\theta_1}^2 + \frac{1}{2}\dot{\theta_2}^2 + \dot{\theta_1}\dot{\theta_2}\cos\theta_1 - \theta_2 \tag{13}$$

(14)
$$T = \frac{1}{2}m_1(\dot{x_1}^2+\dot{y_1}^2)+\frac{1}{2}m_2(\dot{x_2}^2+\dot{y_2}^2) = \frac{1}{2}\dot{\theta_1}^2+\frac{1}{2}\dot{\theta_1}^2+\frac{1}{2}\dot{\theta_2}^2+\dot{\theta_1}\dot{\theta_2}\cos\theta_1 - \theta_2 = \dot{\theta_1}^2+\frac{1}{2}\dot{\theta_2}^2+\dot{\theta_1}\dot{\theta_2}\cos\theta_1 - \theta_2$$

Now that we have found the energies, we can construct the Lagrangian (L) and Hamiltonian (H) of the system:

$$L = T - V = \dot{\theta_1}^2 + \frac{1}{2}\dot{\theta_2}^2 + \dot{\theta_1}\dot{\theta_2}\cos\theta_1 - \theta_2 + 2g * \cos\theta_1 + g\cos\theta_2 \tag{15}$$

For the Hamiltonian, we first calculate the generalized momenta (P_1 and P_2).

$$P_1 = \frac{\partial L}{\partial\dot{\theta_1}} = 2\dot{\theta_1} + \dot{\theta_2}\cos(\theta_1 - \theta_2) \tag{16}$$

$$P_2 = \frac{\partial L}{\partial\dot{\theta_2}} = \dot{\theta_2} + \dot{\theta_1}\cos(\theta_1 - \theta_2) \tag{17}$$

We now use the system of equation (16 and 17) to solve for $\dot{\theta_1}$ and $\dot{\theta_2}$ in terms of P_1 and P_2.

$$\dot{\theta_2} = P_2 - \dot{\theta_1}\cos(\theta_1 - \theta_2) \tag{18}$$

$$P_1 - P_2\cos(\theta_1 - \theta_2) = \dot{\theta_1}(2 - \cos(\theta_1 - \theta_2)^2) \tag{19}$$

$$\dot{\theta_1} = \frac{P_1 - P_2\cos(\theta_1 - \theta_2)}{2 - \cos(\theta_1 - \theta_2)^2} \tag{20}$$

$$\dot{\theta_2} = P_2 - \frac{P_1 - P_2\cos(\theta_1 - \theta_2)}{2 - \cos(\theta_1 - \theta_2)^2}\cos(\theta_1 - \theta_2) \tag{21}$$

We can therefore writhe the hamiltonian as:

$$H = \dot{\theta_1}P_1 + \dot{\theta_2}P_2 - L \tag{22}$$

(23)
$$H = \left(\frac{P_1 - P_2\cos(\theta_1 - \theta_2)}{2 - \cos(\theta_1 - \theta_2)^2}\right)P_1 + \left(P_2 - \frac{P_1 - P_2\cos(\theta_1 - \theta_2)}{2 - \cos(\theta_1 - \theta_2)^2}\cos(\theta_1 - \theta_2)\right)P_2 - L$$

L needs to be re-written in terms of P_1 and P_2:

$$L = \left(\frac{P_1 - P_2\cos(\theta_1 - \theta_2)}{2 - \cos(\theta_1 - \theta_2)^2}\right)^2 + \frac{1}{2} * \left(P_2 - \frac{P_1 - P_2\cos(\theta_1 - \theta_2)}{2 - \cos(\theta_1 - \theta_2)^2}\cos(\theta_1 - \theta_2)\right)^2 + 2g * \cos\theta_1 + g\cos\theta_2$$

We have now found all the required information for finding the hamiltonian. Unfortunately, the equation itself is so messy, that doing anything with it would be nearly impossible, and certainly tedious. However, after some extremely tedious algebra(not included here) we find that the hamiltonian has a reasonable simplified form:

$$H = \frac{P_1^2 + 2P_2^2 - 2P_1P_2\cos(\theta_1 - \theta_2)}{2 * (1 + \sin(\theta_1 - \theta_2)^2)} \tag{24}$$

Now that we have the Hamiltonian and Lagrangian, we may derive the equations of motion for the double pendulum in two methods. The first method uses the Euler-Lagrange equation to produce two coupled non-linear second-order ordinary differential equations.

$$0 = \frac{\partial L}{\partial \theta_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) \tag{25}$$

$$0 = \frac{\partial L}{\partial \theta_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) \tag{26}$$

$$\frac{\partial L}{\partial \theta_1} = -\dot{\theta}_1\dot{\theta}_2 - 2g\sin\theta_1 \tag{27}$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = 2\dot{\theta}_1 + \dot{\theta}_2\cos(\theta_1 - \theta_2) \tag{28}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = 2\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1 - \theta_2) - \dot{\theta}_2(\dot{\theta}_1 - \dot{\theta}_2)\sin(\theta_1 - \theta_2) \tag{29}$$

so.

$$0 = -\dot{\theta}_1\dot{\theta}_2 - 2g\sin\theta_1 - (2\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1 - \theta_2) - \dot{\theta}_2(\dot{\theta}_1 - \dot{\theta}_2)\sin(\theta_1 - \theta_2)) \tag{30}$$

which, after some algebra reduces to:

$$0 = 2\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1 - \theta_2) + \dot{\theta}_2^{\,2}\sin(\theta_1 - \theta_2) + 2g\sin\theta_1 \tag{31}$$

a similar procedure for θ_2 gives us the expression:

$$0 = 2\ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_1 - \theta_2) - \dot{\theta}_1^{\,2}\sin(\theta_1 - \theta_2) + 2g\sin\theta_2 \tag{32}$$

Equations (31) and (32) completely govern the system of the double pendulum. The Hamiltonian derivation uses Hamilton's cononical equations of motion to produce a system of four, coupled, first-order, non-linear ODEs:

$$\dot{\theta}_1 = \frac{\partial H}{\partial P_1} \tag{33}$$

$$\dot{\theta}_2 = \frac{\partial H}{\partial P_2} \tag{34}$$

$$\dot{P}_1 = -\frac{\partial H}{\partial \theta_1} \tag{35}$$

$$\dot{P}_2 = -\frac{\partial H}{\partial \theta_2} \tag{36}$$

$$\dot{\theta}_1 = \frac{\partial H}{\partial P_1} = \frac{P_1 - P_2 \cos(\theta_1 - \theta_2)}{1 + \sin(\theta_1 - \theta_2)^2} \tag{37}$$

$$\dot{\theta}_2 = \frac{\partial H}{\partial P_2} = \frac{4P_2 - 2P_1 \cos(\theta_1 - \theta_2)}{1 + \sin(\theta_1 - \theta_2)^2} \tag{38}$$

$$-\dot{P}_1 = \frac{2(1+\sin(\theta_1-\theta_2)^2)\sin(\theta_1-\theta_2)+(P_1^2+2P_2^2-2P_1P_2\cos(\theta_1-\theta_2))(2\sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2))}{4(1+\sin(\theta_1-\theta_2)^2)^2}$$

$$-\dot{P}_2 = \frac{-2(1+\sin(\theta_1-\theta_2)^2)\sin(\theta_1-\theta_2)-(P_1^2+2P_2^2-2P_1P_2\cos(\theta_1-\theta_2))(2\sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2))}{4(1+\sin(\theta_1-\theta_2)^2)^2}$$

If we introduce a substitution, we can simplify the expressions for $\dot{P}_1$ and $\dot{P}_2$. Let:

$$C_1 = \frac{P_1 P_2 \sin(\theta_1 - \theta_2)}{1 + \sin(\theta_1 - \theta_2)^2} \tag{39}$$

$$C_2 = \frac{P_1^2 + 2P_2^2 - P_1 P_2 \cos(\theta_1 - \theta_2)}{2(1 + \sin(\theta_1 - \theta_2)^2)} \sin 2(\theta_1 - \theta_2) \tag{40}$$

This substitution allows for Hamilton's equations to be written more succinctly as:

$$\dot{\theta}_1 = \frac{P_1 - P_2 \cos(\theta_1 - \theta_2)}{1 + \sin(\theta_1 - \theta_2)^2} \tag{41}$$

$$\dot{\theta}_2 = \frac{2P_2 - P_1 \cos(\theta_1 - \theta_2)}{1 + \sin(\theta_1 - \theta_2)^2} \tag{42}$$

$$\dot{P}_1 = -2g \sin\theta_1 - C_1 + C_2 \tag{43}$$

$$\dot{P}_2 = -g\sin\theta_2 + C_1 - C_2 \tag{44}$$

4. Small Oscillations

In studying the pendulum, introductory courses often make the assumption that the angles through which the pendulum swings are small in an attempt to learn more about the situation. While this situation is, in general, not a realistic one, it is nonetheless an important part of our analysis because it confirms the legitimacy of the equations above. We will begin by assuming that θ_1 and θ_2 are small, while allowing for any velocities ($\dot{\theta}_1$ and $\dot{\theta}_2$). If we restrict the pendulum to small angles, we should change the energies of the system to reflect this change. Equation (5) tells us that:

$$V = -2g * \cos\theta_1 - g\cos\theta_2 \tag{45}$$

But if we are assuming small oscillations then:

$$\cos\theta_1 \approx 1 \tag{46}$$

$$\cos\theta_2 \approx 1 \tag{47}$$

so:

$$V \approx -3g \tag{48}$$

Equation (14) tells us that:

$$T = \dot{\theta_1}^2 + \frac{1}{2}\dot{\theta_2}^2 + \dot{\theta}_1\dot{\theta}_2\cos\theta_1 - \theta_2 \tag{49}$$

If we expand formula (49) in a taylor series about (θ_1,θ_2)=(0,0), we see that:

$$T \approx \dot{\theta_1}^2 + \frac{1}{2}\dot{\theta_2}^2 \tag{50}$$

We can now construct the modified Lagrangian:

$$L \approx \dot{\theta_1}^2 + \frac{1}{2}\dot{\theta_2}^2 + 3g \tag{51}$$

Again we use the Euler-Lagrange equations (25) and (26) to construct the equations of motion.

$$0 = \frac{\partial L}{\partial \theta_1} - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot{\theta}_1}\Big) = -\frac{d}{dt}\Big(2\dot{\theta}_1)\Big) \tag{52}$$

or:

$$0 = \ddot{\theta}_1 \tag{53}$$

which tells us that:

$$\theta_1 = At + B \tag{54}$$

for some A,B$\in \mathbb{R}$. Similarly for θ_2, we see that:

$$0 = \ddot{\theta}_2 \tag{55}$$

so:

$$\theta_2 = Ct + D \tag{56}$$

likewise for some C,D$\in \mathbb{R}$. This tells us that the position of lower pendulum is given by:

$$(x(t), y(t)) = (\sin(At + B) + \sin(Ct + B), -\cos(At + B) - \cos(Ct + B)) \tag{57}$$

This is to be expected, as a double pendulum behaves a lot like a normal pendulum when it only oscillates through small angles.

5. Double to Single Pendulum Reduction

Another important test of the accuracy of our physical model is to assume that $\theta_1 = \theta_2$ in an attempt to reduce the problem to a single pendulum. This can be achieved easily by starting with the Lagrangian model we derived in the third section. If we add equations (31) and (32) and replace the values of θ_i, $\dot{\theta}_i$, and $\ddot{\theta}_i$ for $i = 1, 2$ with the value θ, $\dot{\theta}$, and $\ddot{\theta}$, we see that:

$$0 = 6\ddot{\theta} + 4g\sin\theta \tag{58}$$

Equation (58) is a version of the normal pendulum equation ($0 = \ddot{\theta} + g\sin\theta$), which has been modified to account for the fact that the "single pendulum" dose not have a massless extension. The results from section 4 and 5 should be convincing enough to believe that the model derived in section 3 is accurate.

6. Numerical Integration

6.1. Euler's method.

The first type of numerical integration which we shall considered is known as Euler's method. The method is theoretically derived directly from the definition of the derivative, and it has its advantages and disadvantages. For example, Euler's method is theoretically beautiful, and can be programmed fairly easily for problems that are not too complex; however, Euler's method grows exponentially in complexness as more and more steps are taken, and if the problem isn't exactly smooth or continuous then Euler's method can have accuracies which are not acceptable in many circumstances. For the purposes of numerical integration, we will use the system of equations given by Hamilton's equations of motion in section 3 (equations (41)-(44)).

We begin our analysis of the double pendulum in the context of Euler's method by stating the formulas which will be used for numerical integration:

$$t_{k+1} = t_k + \Delta h \tag{59}$$

$$\theta_{1,k+1} = \theta_{1,k} + \dot{\theta}_{1,k}\Delta h \tag{60}$$

$$\theta_{2,k+1} = \theta_{2,k} + \dot{\theta}_{2,k}\Delta h \tag{61}$$

$$P_{1,k+1} = P_{1,k} + \dot{P}_{1,k}\Delta h \tag{62}$$

$$P_{2,k+1} = P_{2,k} + \dot{P}_{2,k}\Delta h \tag{63}$$

where we have taken k to be an integer greater than or equal to zero, and where the meaning of $F_{i,j}$ is the value of F_i (P_i or θ_i) $i = 1, 2$ at the time $t = j$.

Although more information would be extremely enlightening, limitations on time and ability, have only allowed for a small taste of what is the power of numerical integration. Provided are examples of Euler's method for four initial values:

$$(\theta_1, \theta_2, P_1, P_2) = (\frac{\pi}{4}, \frac{\pi}{4}, 0, 0) \tag{64}$$

$$(\theta_1, \theta_2, P_1, P_2) = (\frac{\pi}{4}, \frac{3\pi}{4}, 0, 0) \tag{65}$$

$$(\theta_1, \theta_2, P_1, P_2) = (\frac{\pi}{2}, \frac{\pi}{2}, 0, 0) \tag{66}$$

$$(\theta_1, \theta_2, P_1, P_2) = (\frac{\pi}{2}, \frac{3\pi}{2}, 0, 0) \tag{67}$$

Equation (64) corresponds to figure 6.1.1, equation (65) corresponds to figure 6.1.2, and so on. Although there might be many patterns in the given graphs, the most important may be that the "craziness" of the graph is proportional to the amount of energy initially given to the system. This seems to imply that the more energy that the double pendulum is started with, the more chaotic the motion seems to be. This should not come as too big a surprise seeing as how most mechanical systems behave in such a way, but it is interesting to think about the point (or edge) of initial potential energy that separates normal (well-behaved) behavior from chaotic behavior.

Another interesting observations of the graphs is that the behavior of the system is more chaotic when the second pendulum is started at the same or a higher level of potential energy as the first pendulum. This observation is interesting, but it also makes sense if it is put in the right light. We can conclude without too big a jump in logic that the motion of the second pendulum is more complex than the motion of the first pendulum, after all, the first pendulum is confined to move on a circle, the second one is not. That being said, if the second pendulum is given more energy than the first pendulum, it would make sense that the motion of the system would be affected more by the motion of the second pendulum than the first pendulum. Since this motion is more chaotic, the system should, in theory, also behave more chaotically.

6.2. Runge-Kutta Method.

The next method of numerical integration we will consider is the Runge-Kutta method. This method is considered accurate and efficient, which explains its wide usage for many mechanical systems and indeed any given physical system. We begin again with the statement of the formula:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \frac{1}{6}\Big(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4\Big) \tag{68}$$

$$\mathbf{k}_1 = \mathbf{f}(\mathbf{x}_n)\Delta t \tag{69}$$

$$\mathbf{k}_2 = \mathbf{f}\Big(\mathbf{x}_n + \frac{1}{2}\mathbf{k}_1\Big)\Delta t \tag{70}$$

$$\mathbf{k}_3 = \mathbf{f}\Big(\mathbf{x}_n + \frac{1}{2}\mathbf{k}_2\Big)\Delta t \tag{71}$$

$$\mathbf{k}_4 = \mathbf{f}\Big(\mathbf{x}_n + \mathbf{k}_3\Big)\Delta t \tag{72}$$

Where we will define:

$$\mathbf{x}_n = \Big(\theta_1, \theta_2, P_1, P_2\Big) \tag{73}$$

and.

$$\mathbf{f} = \left(\frac{P_1 - P_2\cos(\theta_1 - \theta_2)}{1 + \sin(\theta_1 - \theta_2)^2}, \frac{2P_2 - P_1\cos(\theta_1 - \theta_2)}{1 + \sin(\theta_1 - \theta_2)^2}, -2g\sin\theta_1, -g\sin\theta_2\right) \tag{74}$$

The same initial conditions (equations (64)-(67)) are graphed using the Runge-Kutta method in figures 6.2.1 through 6.2.1.

At first glance, the graphs seem to be of completely different trajectories, or completely different physical systems for that matter. However, the Runge-Kutta method has merely managed to create more accurate approximations for the same scenario. The method does take a bit more calculation, but as you can see from comparing the graphs, the time and processing power are worth it.

Unfortunately the small taste of numerical integration shown here is not extremely enlightening (except perhaps on method choice). Although we were able to conclude several characteristics about initial condition relations, we were not able to prove that any chaotic motion (according to the definition, not the adjective) had actually occurred. Consider the initial condition(s) which we noticed were "chaotic", the small glimpse we saw of the motion said nothing about a possible stable limit cycle, it just indicated that the initial movement was crazy. This predicament prompts a more formal discussion of chaotic motion and limit cycles. But first we should identify some unique characteristics of the double pendulum.

7. Symmetries, Quantities and Preliminary Conclusions

7.1. **Conservation of Energy.**

One of the most fundamental axioms of modern physics, and classical physics for that matter, it that of conservation of energy. The topic can be both a method of solution and a method of refutation, but one thing is for certain, without conservation of energy the physical processes we have come to know very well would behave differently than we would expect. This should not be taken as some sort of lame proof, but rather as an argument for conservation of energy, whatever we may define energy as.

The double pendulum system we have been studying thus far is a physical system which does, and must undergo conservation of energy. In this case, the only two types of energy we are concerned with are kinetic energy and potential energy, seeing as how we assumed that that there is no driving force, and no friction. These two assumptions lead to the conclusion that the system has no energy leaving and no energy coming in, so the system is closed, and the only energy involved is that which is contained in the initial conditions provided. We therefore have a conserved quantity given by formula (24)-the hamiltonian.

This one conclusion provides us with many interesting corollaries. The system we are studying cannot have any fixed attracting points. This is an interesting concept, particularly when one thinks of a double pendulum with an arbitrary initial condition. Such a picture has the impression of having a stable limit cycle at the condition $(\theta_1, \theta_2, P_1, P_2) = (0, 0, 0, 0)$, however, this conclusion requires the assumption of a fricative force, which we did not assume. Another interesting conclusion we can derive from having a fixed quantity is that any isolated fixed point of the system that is also a local minimum of H constitutes a non-linear center, in that all trajectories in a sufficiently small neighborhood of the fixed point are necessarily closed.

7.2. **Time-Reversal Symmetry.**

Another interesting characteristic of physical systems is time-reversal symmetry. Such a system is invariant under the substitution $t \to -t$. To demonstrate the double pendulum's time reversal symmetry, consider any function:

$$\Theta = \Theta(t) \tag{75}$$

if we let $-\tau = t$, then:

$$\Theta(t) = \Theta(-\tau) \tag{76}$$

and therefore.

$$\dot{\Theta(t)} = \frac{d\Theta}{d\tau} * \frac{d\tau}{dt} = (-1)\Theta'(\tau) \tag{77}$$

taking one more derivative we see that.

$$\ddot{\Theta}(t) = \Theta''(\tau) \tag{78}$$

This says that under time-reversal substitution, velocities reverse sign, but accelerations preserve their sign. We can apply this information to equations (31) and (32) to show the time reversal symmetry of the double pendulum.

$$0 = 2\ddot{\theta_1} + \ddot{\theta_2}\cos(\theta_1 - \theta_2) + \dot{\theta_2}^2 \sin(\theta_1 - \theta_2) + 2g\sin\theta_1 \tag{79}$$

$t \to -t$

$$0 = 2\ddot{\theta_1} + \ddot{\theta_2}\cos(\theta_1 - \theta_2) + (-\dot{\theta_2})^2 \sin(\theta_1 - \theta_2) + 2g\sin\theta_1 \tag{80}$$

$$= 2\ddot{\theta_1} + \ddot{\theta_2}\cos(\theta_1 - \theta_2) + \dot{\theta_2}^2 \sin(\theta_1 - \theta_2) + 2g\sin\theta_1 \tag{81}$$

similarly for equation (32).

$$0 = 2\ddot{\theta_2} + \ddot{\theta_1}\cos(\theta_1 - \theta_2) - \dot{\theta_1}^2 \sin(\theta_1 - \theta_2) + 2g\sin\theta_2 \tag{82}$$

$t \to -t$

$$0 = 2\ddot{\theta_2} + \ddot{\theta_1}\cos(\theta_1 - \theta_2) - (-\dot{\theta_1})^2 \sin(\theta_1 - \theta_2) + 2g\sin\theta_2 \tag{83}$$

$$= 2\ddot{\theta_2} + \ddot{\theta_1}\cos(\theta_1 - \theta_2) - \dot{\theta_1}^2 \sin(\theta_1 - \theta_2) + 2g\sin\theta_2 \tag{84}$$

Therefore the governing equations of the double pendulum are invariant under the substitution $t \to -t$, and therefore it has time-reversal symmetry.

Apart from just being interesting, time reversal symmetry also has a few interesting significances. For example, if the origin is a linear center for a continuously differentiable system that is time reversal invariant, then sufficiently close to the origin all trajectories are closed.

8. Bifurcation Analysis

8.1. **Elementary Fixed Point Analysis.**

The double pendulum, like the regular pendulum has a few conceptually stable points, and therefore we can pick out some fixed points off the top of our head. Consider for example the point $(\theta_1, \theta_2, P_1, P_2) = (0, \pi, 0, 0)$.
If we linearize about this point the Jacobian Matrix has eigenvalues:

$$\lambda_{1,2,3,4} = \pm\sqrt{-2 * g \pm \sqrt{2}g} \tag{85}$$

Equation (85) has some astounding results. If we take g to be the normal earth-value of -9.8 meters per second, we can see that the eigenvalues of the Jacobian Matrix will always be non-zero and distinct. Furthermore, two will always occur in opposite signed real values, and the other two will always occur in complex conjugate pairs, and will always be purely imaginary. This tells us that the four-dimensional space given by the coordinates $(\theta_1, \theta_2, P_1, P2)$ behaves very predictably at the point $(0, \pi, 0, 0)$. The nature of the eigenvalues tells us that there are always two dimensions in which the double pendulum has a saddle point at this point, and that in the other two dimensions the point is a nonlinear center.

In order to conceptualize this fact, imagine the double pendulum at the point $(0, \pi, 0, 0)$. Without any disturbance (i.e. no imparted momentum) on either of the masses, the pendulum will remain at rest. This is the conceptual picture of the stable manifold of the saddle point. However, if the point is disturbed even slightly (pushed off the stable manifold) then the masses will have a non-zero momentum. Using the facts that the system is closed, and that total energy is conserved, it must follow that the total momenta associated with the masses is transferred between the masses in a way which is purely periodic. This dose not say anything about periodic motion in the pendulums, just that the quantity correspond to the total momentum of the system is periodic. What it does say something about, is that the energy of the system will oscillate periodically between kinetic and potential energies, just as in the regular pendulum.

Interestingly enough, the eigenvalues of equation (85) also correspond to the fixed point (0,0,0,0). But conceptually, this too makes sense (note: there are no attracting fixed fixed points), if the pendulum is not imparted with any momentum, it will remain in state (0,0,0,0) indefinitely; however, a minor disturbance from this point implies that the pendulum will never again return to rest, and will continue to oscillate in potential and kinetic energy values.

The initial condition $(\theta_1, \theta_2, P_1, P_2) = (\pi, \pi, 0, 0)$ is a completely different animal. The linearization matrix has eigenvalues:

$$\lambda_{1,2,3,4} = \pm\sqrt{2g \pm \sqrt{2}g} \tag{86}$$

Inspection of these eigenvalues tells us that they are all real, and occur in positive/negative pairs. Once again, our four-dimensional space can be separated into two, co-dependent, two dimensional subspaces. This time however, each two-dimensional subspace has a saddle point. Like the other saddle points in the previous initial conditions, if the masses are left at rest, the system will undergo no motion, but a small disturbance will always cause motion and momenta changes.

Also, since the spatial compliment to one of the saddle point spaces is not a center, the transfer of energy between energies will not be periodic. Accounting for this is the persistent nature of the second pendulum to completely circle around the first. During this time, the second pendulum remains in constant motion with high kinetic energy while the first remains relatively fixed with a small kinetic energy. This motion cannot be periodic, since there is no center, so the pendulum will eventually return to a state where each mass has relatively comparative values of kinetic and potential energies. The problem, is that the point at which the system will return to a state where the second pendulum is circling about the first is largely unpredictable. This fact is an interesting consideration to take into account in the search for chaos in the double pendulum.

The last fixed point we shall consider is $(\theta_1, \theta_2, P_1, P_2) = (\pi, 0, 0, 0)$. The linearization for this fixed point has eigenvalues:

$$\lambda_{1,2} = \pm 2^{\frac{1}{4}}\sqrt{-g} \tag{87}$$

and.

$$\lambda_{3,4} = \pm i 2^{\frac{1}{4}}\sqrt{g} \tag{88}$$

Once again we get two eigenvalues which are real and occur in positive/negative pairs, and two eigenvalues which are purely imaginary and occur in complex conjugates. Thus the behavior of the system given this initial condition is the same as for the first two initial conditions discussed.

8.2. **Qualitative Chaos Analysis.**

The previous section demonstrated that even old initial condition for the double pendulum will cause it to behave in a qualitatively different manner. Although it was demonstrated for noticeably different initial conditions, this dependence also holds for small changes in certain initial conditions. The initial conditions for which the total energy of the system is greater than any possible potential energy for any one (and only one) of the masses are generally considered highly dependent on initial conditions. This requirement allows for the system to obtain kinetic energies which can take the second pendulum into regions where any normal pendulum could not go. At this energy leve, small differences in initial conditions can contribute to large differences in trajectories, i.e. an even higher dependence on initial conditions than expected. Such a dependence is characteristic of chaotic motion, and indeed, can be conceptually considered a "threshold" for chaotic motion.

Along with certain energy levels, other parameter dependencies can contribute to a more chaotic motion. Consider for example, if we were to allow for different masses in each of the pendulums, but fix the values of the arm lengths to be equal. If the second pendulum's mass (m_2) is large in comparison to first pendulum's mass, the motion of the system would be dominated by the motion of the second pendulum, and therefore the system as a whole would tend to behave more like a regular pendulum, than a double pendulum. If we go the opposite way, i.e. make m_1 equal to, larger than, or much larger than m_2, the motion of the system would be dominated by the first pendulum. From the analysis of the system, and from numerical pictures obtained here, it should be obvious that the motion of the second pendulum is greatly dependent on the motion of the first pendulum. It should therefore come as no surprise that the chaotic behavior of the second pendulum will increase with the increase of the ratio $\frac{m_1}{m_2}$. Therefore, we can qualitatively concluded that the greater the mass of the first pendulum in comparison to the second pendulum, the more chaotic the motion of the system will be.

If we now consider the situation in which we fix the masses to be the same, and allow the arm lengths to vary, the parameter dependence changes significantly. The magnitude of the motion of a particle that is forced to pivot about a single point goes up as the distance from of the particle to the pivoting point goes up. If we were to allow one length of the double pendulum to become large in comparison to the other, the system as a whole could be viewed as a single pendulum by ignoring the smaller length. It is for this reason, that the most chaotic motion of the double pendulum will tend to occur when the lengths of the arms are comparable.

If we allow all four parameters to vary at the same time, qualitative reasoning points us to the situation in which $m_1 \gg m_2$ and $L_1 \approx L_2$ in order to achieve the most chaotic motion possible in the double pendulum.

9. References

The following resources were used in the creation of this project:

9.1. websites.

http://www.myphysicslab.com/dblpendulum.html.
http://en.wikipedia.org/wiki/Doublependulum.
http://scienceworld.wolfram.com/physics/DoublePendulum.html.
http://www.youtube.com/watch?v=anwl6OZ1UuQ (Chaos Demonstration highly reccomended).

9.2. Literature.

Classical Dynamics, by: Stephen T Thorton, Jerry B Marion, ISBN: 0-534-40896-6.
The Pendulum-A Case Study in Physics, by: Gregory L Baker, James A Blackburn, ISBN: 0-19-856754-5.
Non-Linear Oscillations, by: Peter Hagerdorn, ISBN: 0-19-856142-3.
Nonlinear Dynamics and Chaos, by: Steven H Strogatz.

9.3. Publications.

On The Solutions of the Differential Equations of Motion of a Double Pendulum, by: William E Cederberg, Augustana Library Publications.
Solving Classical Mechanics Problems by Numerical Integration of Hamilton's Equations, by: Daniel R Stump, American Journal of Physics.
Chaos in a Double Pendulum, by: Troy Shinbrot, American Journal of Physics.