

If $\sigma \geq 0$, W_α^* is an exponentially tilted stable random variable so that there exist exact samplers.

[Jim Pitman, 5.3](#) said

if Levy density has the form $\rho_\alpha(x) = \frac{\alpha x^{-\alpha-1}}{\Gamma(1-\alpha)}$ ($x > 0$), then total length $T =: S_\alpha$ follows stable(α) with pdf $f(t)$

Furthermore, from Levy khintchine formula, we can get a Laplace transform as

$$\mathbb{E} \{e^{-tT}\} = \exp \left\{ \int_{(0,\infty]} (1 - e^{-tx}) \rho_\alpha(dx) \right\} = \exp(-t^\alpha)$$

[Jim Pitman, 4.2](#) said

If we set $\rho_{\alpha,b}(x) = \rho_\alpha(x)e^{-bx}$ as a Levy density, then $T =: S_{\alpha,b}$'s pdf $f^{(b)}(t) = f(t)e^{\psi(b)-bt}$ where $\psi(b) = \int_0^\infty (1 - e^{-bx}) \rho_\alpha(x)dx = \exp(-b^\alpha)$

Therefore, using a simple trick, we can get a Laplace transform as

$$\mathbb{E} \{e^{-tS_{\alpha,\lambda}}\} = \mathbb{E}_\alpha \left\{ e^{\lambda^\alpha - (t+\lambda)S_\alpha} \right\} = e^{\lambda^\alpha - (t+\lambda)^\alpha}$$

[Caron, 5.3](#) considers Levy measure

$$\begin{aligned} \rho(dw)\lambda([0, \alpha]) &= \frac{\alpha}{\Gamma(1-\sigma)} w^{-1-\sigma} \exp(-\tau w) dw \\ &= \frac{\alpha}{\sigma} \times \frac{\sigma}{\Gamma(1-\sigma)} w^{-1-\sigma} \exp(-\tau w) dw \\ &= \frac{\alpha}{\sigma} \rho_{\sigma,\tau}(dw) \end{aligned}$$

Then, from Levy khintchine formula,

$$\begin{aligned} \mathbb{E} \{e^{-tW_\alpha^*}\} &= \exp \left\{ \int_{(0,\infty]} (1 - e^{-tx}) \rho(dx)\lambda([0, \alpha]) \right\} \\ &= \exp \left\{ \frac{\alpha}{\sigma} \times \int_{(0,\infty]} (1 - e^{-tx}) \rho_{\sigma,\tau}(dx) \right\} \\ &= \exp \left\{ \frac{\alpha}{\sigma} \times (\tau^\sigma - (t+\tau)^\sigma) \right\} \\ &=: \exp\{M \times (\tau^\sigma - (t+\tau)^\sigma)\} \\ &= \exp \left\{ \left(\left(M^{\frac{1}{\sigma}} \tau \right)^\sigma - \left(M^{\frac{1}{\sigma}} t + M^{\frac{1}{\sigma}} \tau \right)^\sigma \right) \right\} \\ &= \mathbb{E} \left[\exp \left(-t M^{\frac{1}{\sigma}} S_{\sigma, M^{1/\sigma} \tau} \right) \right] \end{aligned}$$

Hence, we can sample W_α^* as

1. Sample $T = S_{\sigma, M^{1/\sigma} \tau}$
2. Multiply $T = T \times M^{\frac{1}{\sigma}}$

