Foundations of Applied Math
HW #2 Due Thursday

1. 1.1) #7 *Solution.* $a_n(1.005) + 500, a_0 = 5000$

2. 1.2) #2 *Solution.* Plotting the data in Figure 1, we can see the data follows an exponential pattern.

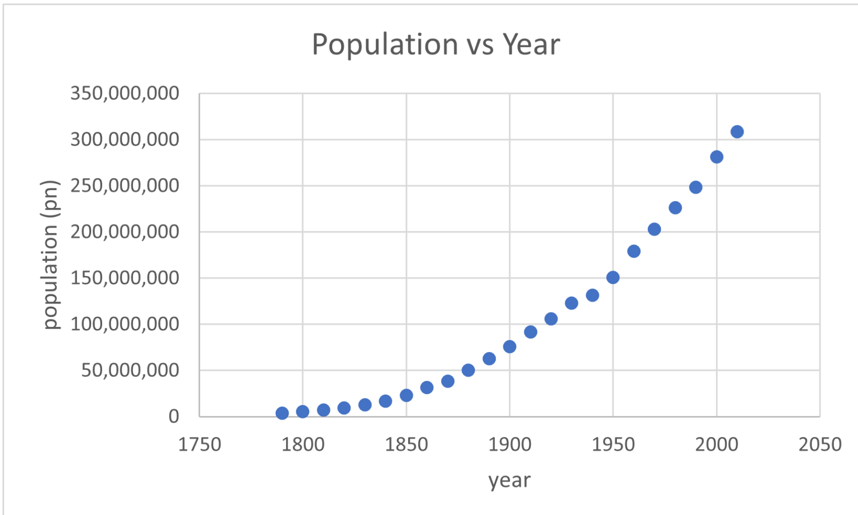

Figure 1: Plot of the data given by the problem.

Exponential models can be modelled using the equation $\Delta pn = kpn$. To find the value of k, we can plot the Δpn vs pn graph and use the slope of that graph to find k.

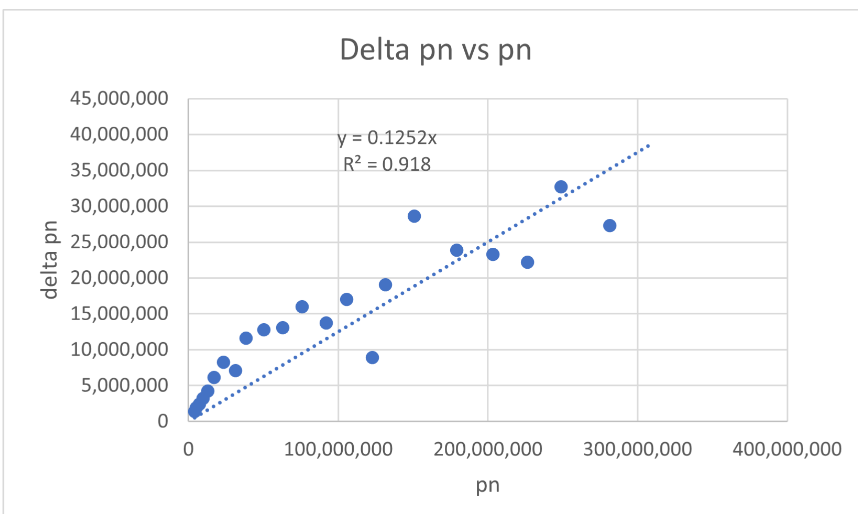

Figure 2: Plot of Δpn vs pn.

As our R^2 value is quite close to 1, we can confidently use the slope we found in Figure 2 as our k value giving us an exponential model of:

$$\Delta pn = .1252pn$$
$$p_{n+1} - p_n = .1252p_n$$
$$p_{n+1} = 1.1252p_n$$

After iterating this model over the same amount of years given in the data, we see the model plot in orange and the actual data in blue.

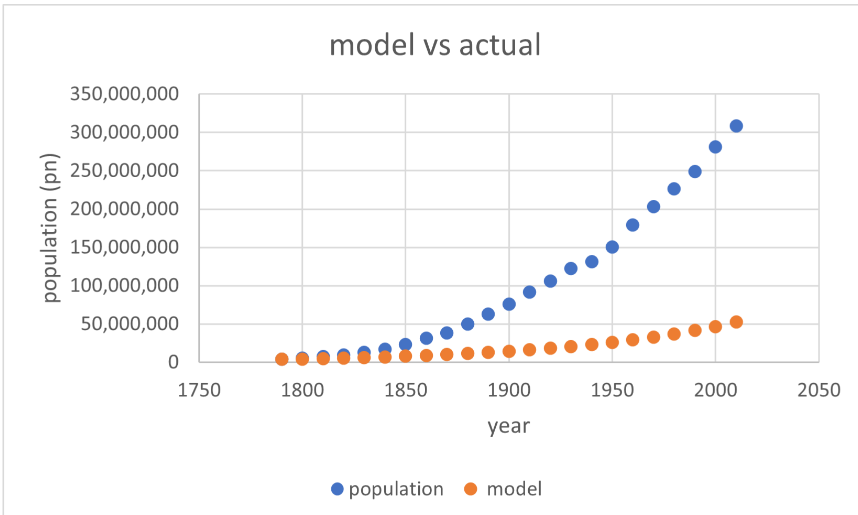

Figure 3: Plot of the model vs the actual data given.

3. 1.2) #9 *Solution.*

(a) Plotting the graph, we see that there is a clear linear relationship between Δa_n and n:

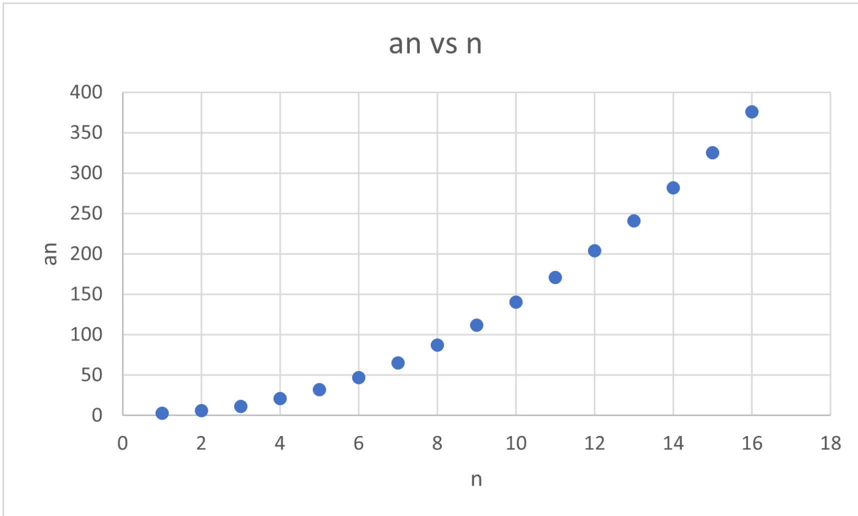

Figure 4: Plot of Δpn vs pn.

(b) Based on the graph of part (a), we know the Δa_n vs n graph is linear. Using this information, we can find a difference equation of the form . $a_{n+1} = a_n + 3.1395n$ Plotting the errors in the predicted values against n:

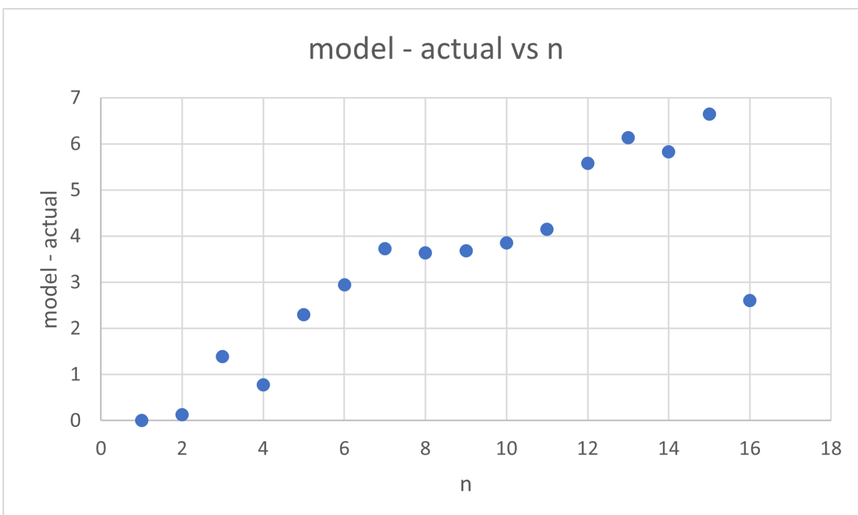

Figure 5: Plot of the difference between the model and the actual data for every n.

Since the errors in the plotted graph in Figure 5 is quite small for every single data point, we can see this model is appropriate and should accuately model the data given.

4. 1.3) #1c

   Solution. The solution for an equation in the form $a_{n+1} = ra_n$ is $a_k = r^k a_0$. So, substituting the values given to us, we get the solution: $a_k = (3/4)^k 64$

5. 1.3) #1f

   Solution. Our solution is of the form $a_k = r^k c + \frac{b}{1-r}$ By plugging in our values of r and b, we get:

   $$a_k = (.1)^k c + \tfrac{3.2}{.9} \ a_k = (.1)^k c + \tfrac{32}{9}$$

   To this, we can plug in our initial condition, $a_0 = 1.3$:

   $$\begin{gathered} 1.3 = (.1)^0 c + \tfrac{32}{9} \\ 1.3 = c + \tfrac{32}{9} \\ 1.3 - \tfrac{32}{9} = c \\ c \approx -2.256 \end{gathered}$$

   So, our solution to the equation takes the form:

   $$a_k = -2.256(.1)^k + \tfrac{32}{9}$$

6. 1.3) #3a

   Solution. From the given equation, we find that the equilibrium solution is:

   $$a_0 = \tfrac{50}{2.2}$$

   However, because our $|r|$ value is greater than 1, we can conclude that our function will be unstable, as shown in Figure 6 below.

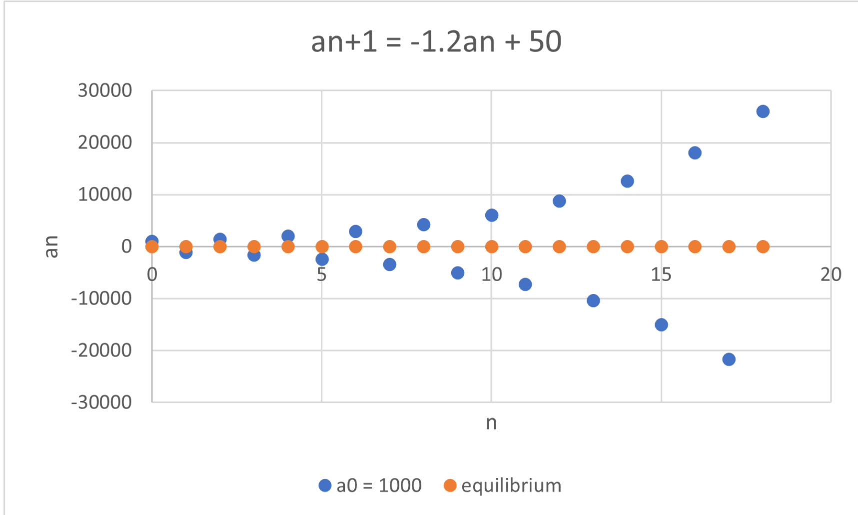

Figure 6: Plot of the model vs the actual data given.

7. 1.3) #3c

Solution. From the given equation, we find the equilibrium solution is:

$$a_0 = -500$$

which happens to be the same initial condition given to us by the problem. The $|r|$ value for our equation is less than one in this case, so we can conclude this will be a stable equilibrium as shown below.

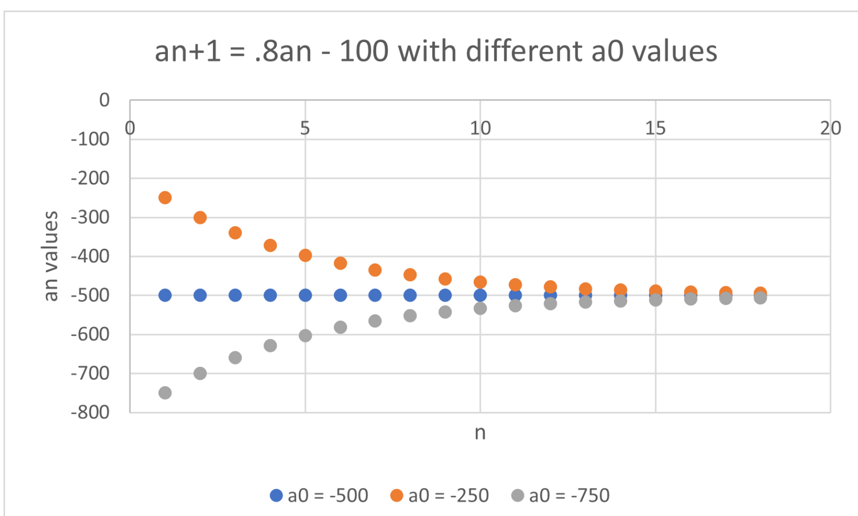

Figure 7: Plot of the model vs the actual data given.

8. Suppose for a drug, we have the following data where the values in rows 2 and 3 are in mg and the top row is in days.

n	0	1	2	3	4	5	6	7	8
a_n	.5	.345	.238	.164	.113	.078	.054	.037	.026
Δa_n	-.155	-.107	-.074	-.051	-.035	-.024	-.017	-.011	

Formulate a model representing the amount of drug left in the body after n days. Note this doesn't seem realistic you'll see!

Solution. Plotting the graph above, we see the drug decays exponentially

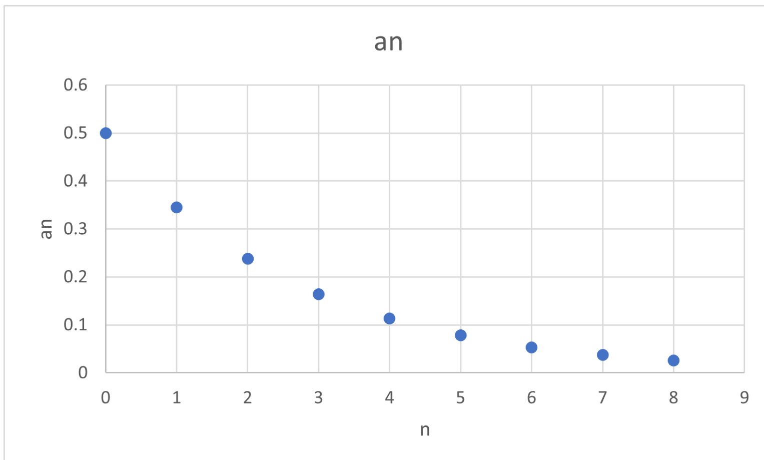

Figure 8: Plot of a_n vs n

So we can use the exponential difference equation $\Delta a_{n+1} = ka_n$. In order to find the value of k, we can plot the graph of Δa_n vs a_n of which the slope of the graph will be our value for k which we have done below:

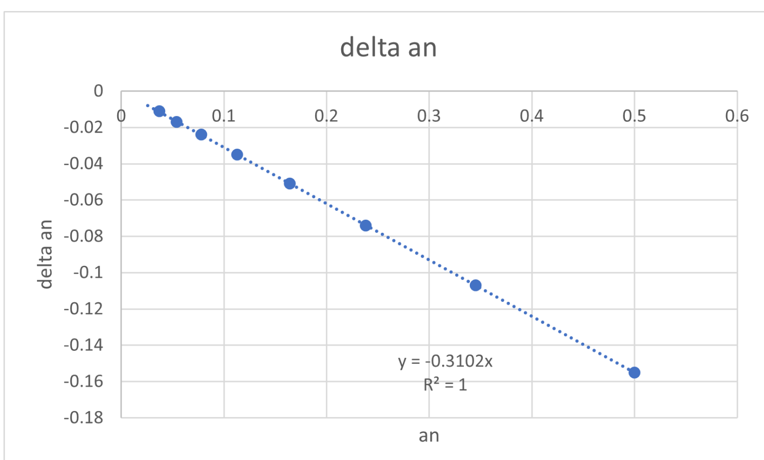

Figure 9: Plot of Δa_n vs a_n

Using the slope of Figure 9, we derive our value for k

$$k = -0.3102$$

from which we get our difference equation:

$$\begin{aligned}
\Delta a_{n+1} &= k a_n \\
\Delta a_{n+1} &= -0.3102 a_n \\
a_{n+1} - a_n &= -0.3102 a_n \\
a_{n+1} &= 0.6898 a_n
\end{aligned}$$

9. Suppose for yet another drug we prescribe a daily drug dosage of 0.1 mg and know that half the drug remains in the system at the end of each dosage period.

   Write down a model to describe this.

   Solution. $a_{n+1} = .5a_n + .1$

   Given the three starting doses, 0.1, 0.2, and 0.3 (mg). Is there a value in mg that the system will approach after a fews days for each of these three initial dosages?
   Solution. From the model above, we can find the equilibrium value a:

$$\begin{aligned}
a &= \tfrac{b}{1-r} \\
a &= \tfrac{.1}{1-0.5} \\
a &= \tfrac{.1}{.5} \\
a &= .2
\end{aligned}$$

   Therefore, since $|r| < 1$, we know this equilibrium solution will be stable and for any given initial value, the drug will converge to .2 mg.

10. Consider a model for the long-term studying behavior of the Wentworth Applied Math majors, pre-Covid.
    It is found that 90% of the students who study in the math department (3rd floor of Ira Allen) return to study there again, whereas those who study in the library have a 75% return rate. Suppose that those are the only two places that are really good for studying on campus, and
    assume that all applied math majors do study each time period and that they do study in one of these two places only.
    Let's define the following variables:

$$\begin{aligned}
m_n &= \text{the percentage of students who study in the math department in period } n. \\
\ell_n &= \text{the percentage of students who study in the library in period } n.
\end{aligned}$$

a. Using the above information and the above variables, write down system of equations that models student studying behavior.

$$f(n) := \begin{cases} m_{n+1} = .9m_n + .25\ell_n \\ \ell_{n+1} = .75\ell_n + .1m_n \end{cases}$$

b. At each time step, n, what value should $m_n + \ell_n$ equal and why?

$m_n + \ell_n$ should be the total amount of students in the Applied Math major, as m_n and ℓ_n are both percentages of the total number of students, and we assumed that these were the only two places Applied Math majors study and every Applied Math major studied at these two places.

c. What is the long term studying behavior? In other words, what percentage of students study in the the math department long term and what percentage study in the library long term? Explain how you obtained your answer.

Long term, the students seem to be much more inclined to study at the math department than the library. By using our model above, we can plot the data and see where the system of equations are stable:

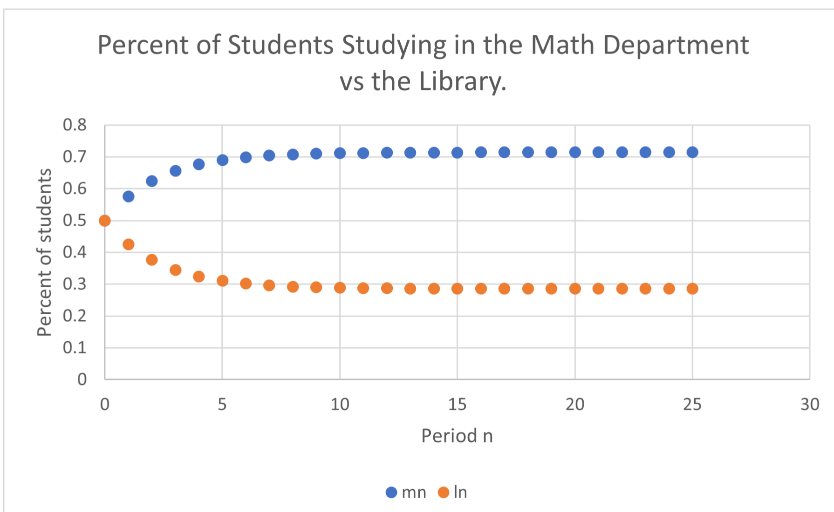

Figure 10: Plot of the percent of students in the math department vs the library

From our graph, we see the percent of students who study in the math department converges to a value around .71 or 71% and those who study in the library converge to a value around .29 or 29%. We see this behavior with varying initial values in Figure 11 and 12 as well:

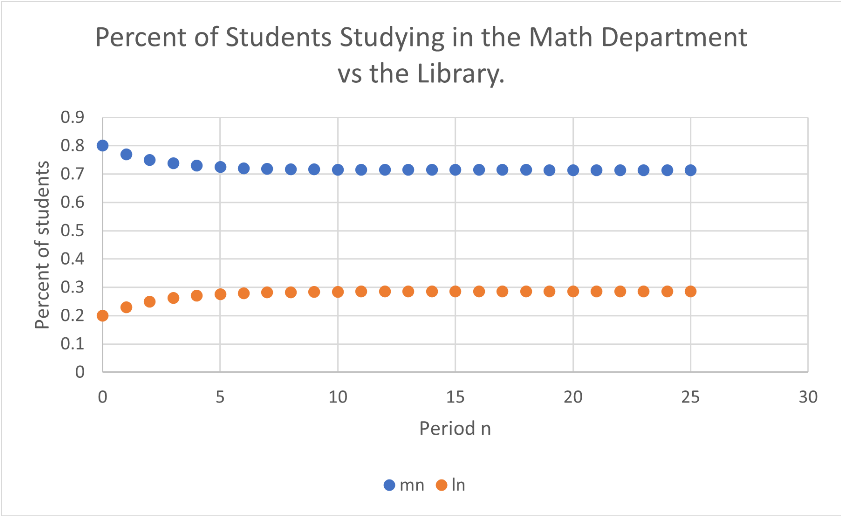

Figure 11: Plot of the percent of students in the math department vs the library with $m_0 = .8$ and $\ell_0 = .2$

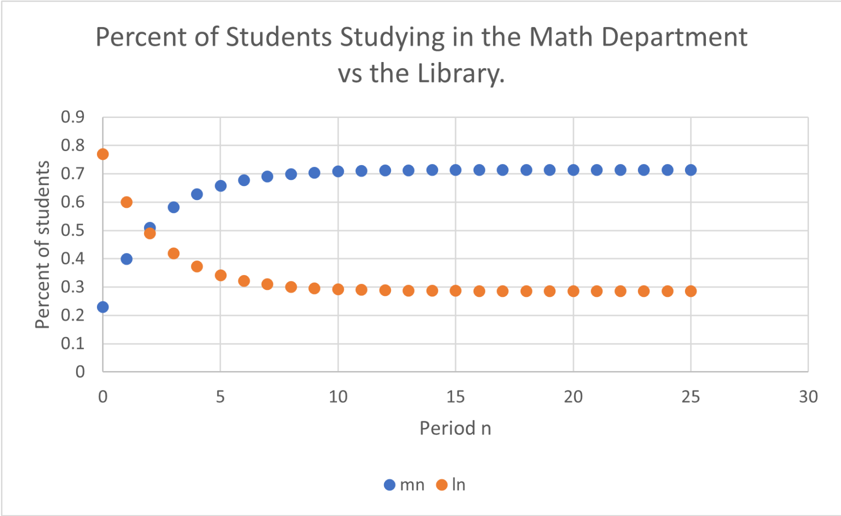

Figure 12: Plot of the percent of students in the math department vs the library with $m_0 = .8$ and $\ell_0 = .2$