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Foundations of Applied Math
HW #6 11.4 Graphical Solutions of Autonomous Differential Equations and review Due Friday, Oct. 23 by 7 am

Reminder You need to turn in a .zipped folder that contains your .tex file, your image files, your python files, your Excel file(s), and the tex file must compile. Rename the .tex file: HW6_YourLastName.tex and call the folder which you will compress: HW6_YourLastName

1. Verify that the following functions

$$x = -\frac{1}{2} + \frac{e^{2t}}{2}, y = -\frac{3}{4} + \frac{3e^{2t}}{8} + \frac{3e^{-2t}}{8}$$

   solve the following system of differential equations: $\frac{dx}{dt} = 2x + 1$, $\frac{dy}{dt} = 3x - 2y$
   Solution

   (a) To verify the function x is a solution to $\frac{dx}{dt}$:

$$\begin{aligned}
\frac{dx}{dt} &= 2x + 1 \\
\frac{d}{dt}(-\frac{1}{2} + \frac{e^{2t}}{2}) &= 2x + 1 \\
0 + e^{2t} &= 2x + 1 \\
e^{2t} &= 2(-\frac{1}{2} + \frac{e^{2t}}{2}) + 1 \\
e^{2t} &= -1 + e^{2t} + 1 \\
e^{2t} &= e^{2t}
\end{aligned}$$

   Therefore, we have shown that $x = -\frac{1}{2} + \frac{e^{2t}}{2}$ is a solution to $\frac{dx}{dt} = 2x + 1$

   (b) To verify the function y is a solution to $\frac{dy}{dt}$:

$$\begin{aligned}
\frac{dy}{dt} &= 3x - 2y \\
\frac{d}{dt}(-\frac{3}{4} + \frac{3e^{2t}}{8} + \frac{3e^{-2t}}{8}) &= 3(-\frac{1}{2} + \frac{e^{2t}}{2}) - 2(-\frac{3}{4} + \frac{3e^{2t}}{8} + \frac{3e^{-2t}}{8}) \\
\frac{3e^{2t}}{4} - \frac{3e^{-2t}}{4} &= -\frac{3}{2} + \frac{3e^{2t}}{2} + \frac{3}{2} - \frac{3e^{2t}}{4} - \frac{3e^{-2t}}{4} \\
\frac{3e^{2t}}{4} - \frac{3e^{-2t}}{4} &= \frac{3e^{2t}}{2} - \frac{3e^{2t}}{4} - \frac{3e^{-2t}}{4} \\
\frac{3e^{2t}}{4} - \frac{3e^{-2t}}{4} &= \frac{3e^{2t}}{4} - \frac{3e^{-2t}}{4}
\end{aligned}$$

Therefore, we have shown that $y = -\frac{3}{4} + \frac{3e^{2t}}{8} + \frac{3e^{-2t}}{8}$ is a solution to $\frac{dy}{dt} = 3x - 2y$

2. Given the following differential equation system (again):

$$\begin{aligned} \frac{dR}{dt} &= 0.65I(t) \\ \frac{dI}{dt} &= -0.65I(t) + .0015I(t)S(t) \\ \frac{dS}{dt} &= -.0015I(t)S(t) \end{aligned}$$

How would you write this as a difference equation system?
Solution.
Since we are given derivatives for our susceptible, infected and removed, we can apply Euler's method assuming some step size h:

$$\begin{aligned} R_{n+1} &\approx R_n + h\frac{dR}{dt} \\ I_{n+1} &\approx I_n + h\frac{dI}{dt} \\ S_{n+1} &\approx S_n + h\frac{dS}{dt} \end{aligned}$$

Plugging in our derivatives for each approximation:

$$\begin{aligned} R_{n+1} &\approx R_n + h(0.65I_n) \\ I_{n+1} &\approx I_n + h(-0.65I_n + .0015I_nS_n) \\ S_{n+1} &\approx S_n + h(-.0015I_nS_n) \end{aligned}$$

3. Suppose $\frac{dy}{dt} - \lambda y = 0$, $y(0) = y_0$ where λ is any real number. Apply Euler's Method to this differential equation to write a difference equation for y_{n+1} in terms of h (the step size), λ , and y_n.
Solution.
From our equation above:

$$\frac{dy}{dt} - \lambda y = 0$$

$$\frac{dy}{dt} = \lambda y$$

Therefore, we can approximate using Euler's method:

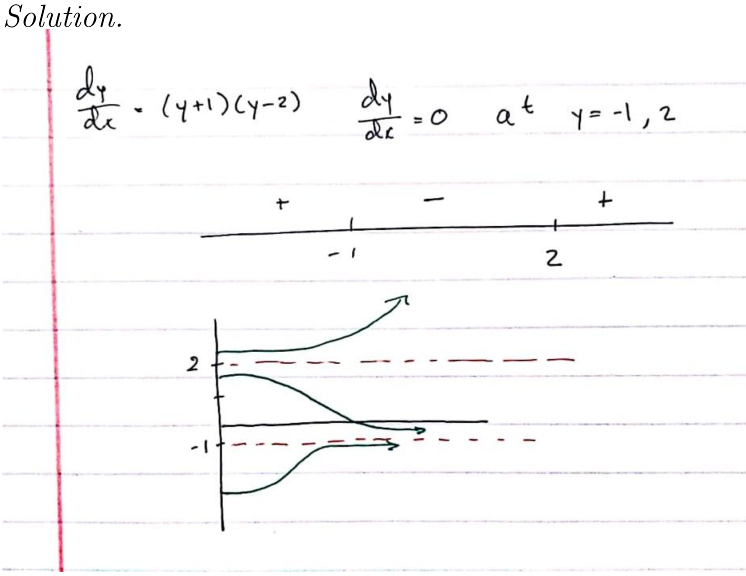
$$y_{n+1} \approx y_n + h\frac{dy}{dt}$$
$$y_{n+1} \approx y_n + h\lambda y$$

Thus, we have found a difference equation using Euler's method in terms of h, λ and y_n

4. Suppose we have the following differential equation

$$\frac{dy}{dx} = (y+1)(y-2)$$

Use phase line analysis to sketch approximate solutions to this differential equation. You can include a hand drawn drawing. Just use the `includegraphics` command to paste your sketch in to go with your typed analysis.
Solution.

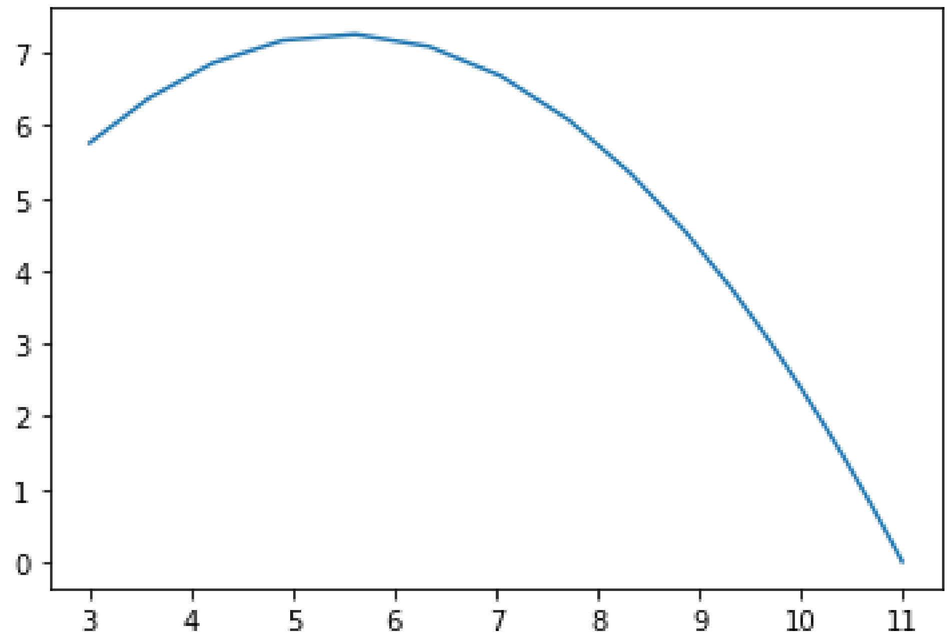


From our equation $\frac{dy}{dx} = (y+1)(y-2)$ we find that the derivative of y is equal to zero when $y = -1, 2$. From there, we can use the snake method to see if the derivative is positive or negative when $y < -1, -1 < y < 2$ and $y > 2$. After finding whether the derivative is positive or negative within these respective intervals, we can draw approximate solutions.

5. Consider the following initial value problem again:
$\frac{dP}{dt} = 0.24P(11 - P), P(0) = 3$ where P is measured in 100's.

a) Why is this a differential equation?
Solution.
This equation has a derivative, which is what makes this an ordinary differential equation.

b) Use Python to graph $\frac{dP}{dt}$ vs P. Include your .py file and paste your code and your graph below.
Solution.

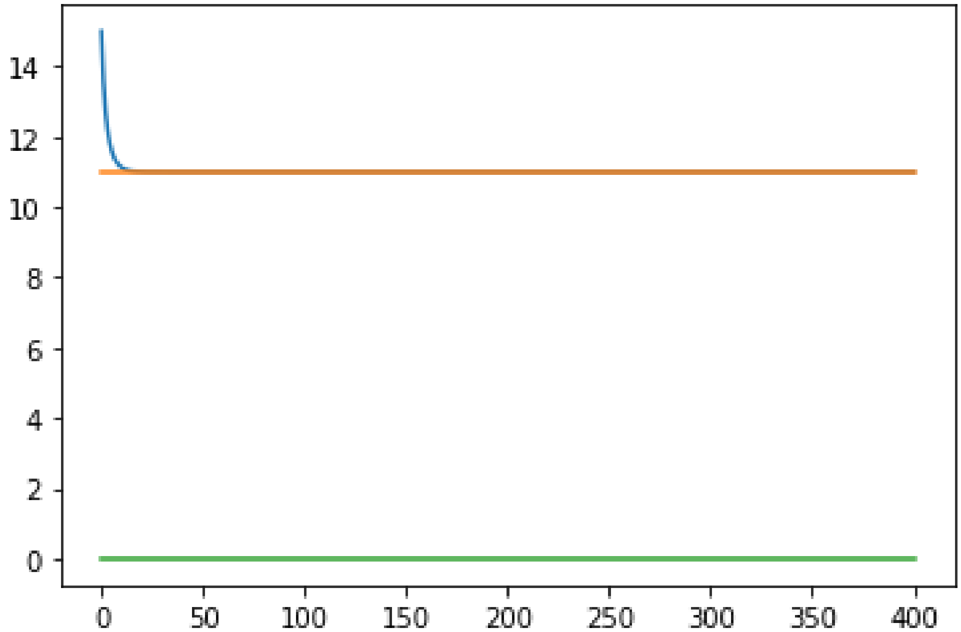


```python
# -*- coding: utf-8 -*-
"""
Created on Thu Oct 22 09:10:39 2020

@author: lees19
"""

import numpy as np
import matplotlib.pyplot as plt

def pPrime(p):
    return .24*p*(11-p)

h = .1
#x = np.arange(-10-h, 21+h, h)
x = np.arange(0, 40+h, h)
P = np.zeros_like(x)
P[0] = 3
dP = np.zeros_like(x)

for i in range(1, x.size):
    dP[i-1] = pPrime(P[i-1])
    P[i] = P[i-1] + h*(pPrime(P[i-1]))

print(dP)
print(P)

plt.plot(P, dP)

```

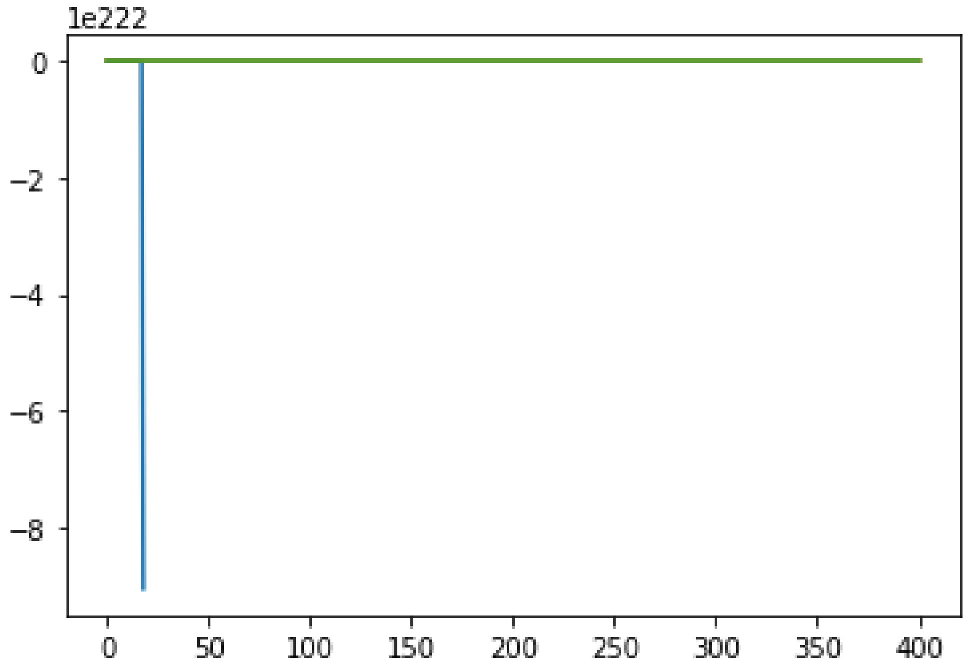
c) Using your answer to the graph above identify which value(s) of P yield $\frac{dP}{dt} = 0$. Verify that your value(s) of P that you obtained from the graph area also the value(s) of P which algebraically yield $\frac{dP}{dt} = 0$. These are called equilibrium solutions or values or sometimes they are called rest points.
Solution.
Using the graph above, we see that there is one equlibrium value at $P = 11$. This makes sense, as if we plug in 11 into our $\frac{dP}{dt}$, we find that our derivative equals zero, meaning when $P = 11$, there is no longer a change in P. There is also another equlibrium point when $P = 0$ which we can find algebraically, however, we never reach this point on the graph.

d) How do equilibrium solutions for a differential equation compare to equilibrium solutions of difference equations? Explain each. With equilibrium solutions for a differential equation, we are trying to find where the derivative is zero. This makes sense, since if the derivative is zero, it means the change in the dependent variable is zero, meaning there is no change. With difference equations, however, we are finding where the next iteration of the difference equation is the same as the previous.

e) Describe the stability of each rest point. Justify your work with graphs and clear mathematical explanations. Your graphs for this part can be hand drawn and pasted in as an image using the `includegraphics` command.



From this graph where the initial value of P is greater than 11, we find when the initial value is greater than 11, P will still converge back down to 11. When the initial value is between 11 and 0, we find that P still converges to 11. Therefore, we can conclude 11 is a stable equilibrium point.



Here we see if the initial value of P is below 0, it will blow up to negative infinity. Therefore, 0 is not a stable equilibrium value.

f) For which P value is $\frac{dP}{dt}$ the largest? What does this mean in terms of the model if this is a model about a disease or rumor spread?
Solution.
$\frac{dP}{dt}$ seems to be the largest when P is further away from a stable equilibrium point and then slows down as it gets closer to an equilibrium point. This seems to be a model about disease spread.

g) In terms of this being a rumor spread or disease spread model, why do your explanations for stability of the equilibrium point(s) make sense? Zero as an equilibrium point makes sense as if nobody is infected, there is no one to spread the disease so the amount of infected will not change. As for 11 as an equilibrium point, if there is a small amount of infected people, that means there will be a large amount of susceptible people, which will result in a fast increase in infected people until it hits an equilibrium value. Starting above the equilibrium value, many people are going to be infected, meaning more people are going to be removed, until, again, we reach an equilibrium value.

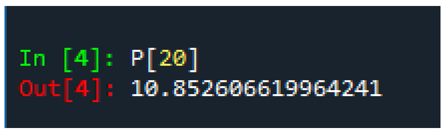
6. Another method (besides Euler's Method) that is used to solve $y' = g(t, y), y(0) = y_0$ is the following:

   Let

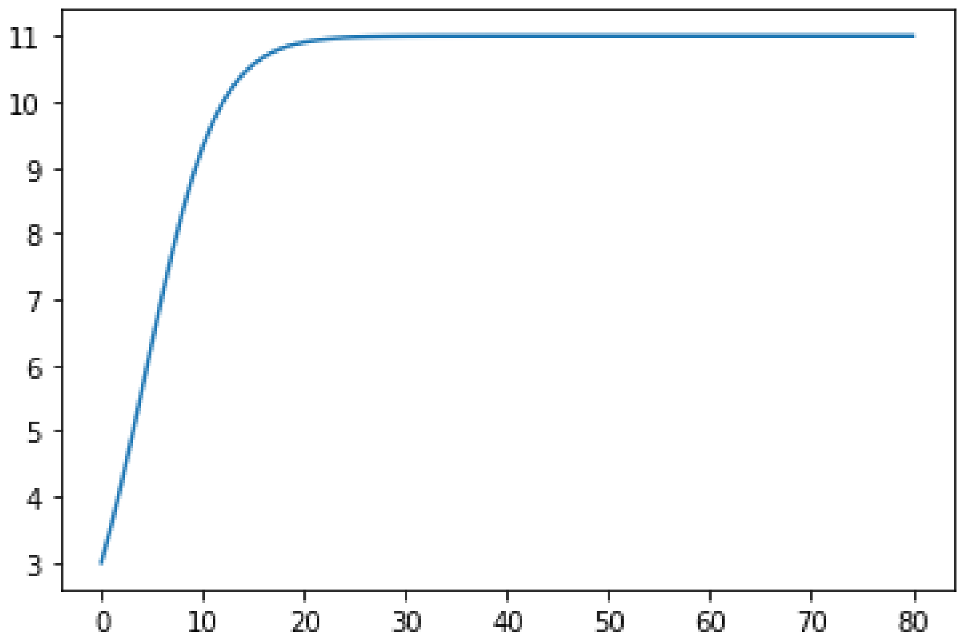
$$
\begin{aligned}
y_{n+1} &= y_n + \frac{K_1 + 2K_2 + 2K_3 + K_4}{6}, \text{ where} \\
K_1 &= g(t_n, y_n)h \\
K_2 &= g(t_n + h/2, y_n + K_1/2)h \\
K_3 &= g(t_n + h/2, y_n + K_2/2)h \\
K_4 &= g(t_n + h, y_n + K_3)h
\end{aligned}
$$

Use this method with a step size of $h = 0.1$ to approximate the solution for the initial value problem in #5. Also say the value of $P(2)$ Also obtain a plot of the solution over $t \in [0, 8]$. Include your code here and in your compressed folder.

Solution.

The value for $P(2)$: 

The graph using our new approximation algorithm:



Code:

```python
# -*- coding: utf-8 -*-
"""
Created on Thu Oct 22 15:22:28 2020

@author: lees19
"""
import numpy as np
import matplotlib.pyplot as plt

#this is also dP/dt
def g(p):
    return .24*p*(11-p)

def k1(p, h):
    return g(p) * h

def k2(p, h):
    return g(p + (k1(p, h)/2))* h

def k3(p, h):
    return g(p + (k2(p, h)/2))* h

def k4(p, h):
    return g(p+k3(p, h))* h

h = .1
x = np.arange(0, 8+h, h)
P = np.zeros_like(x)
dP = np.zeros_like(x)

#initial value = 3
P[0] = 3

for i in range(1, x.size):
    #dP[i-1] = g(P[i-1])
    P[i] = P[i-1] + ((k1(P[i-1], h) + 2*k2(P[i-1], h) + 2*k3(P[i-1], h) + k4(P[i-1], h)

#print P(2)
print(P[20])

plt.plot(P)
```