1. Let $(V, ||\cdot||)$ be a normed vector space and $x, y \in V$. Then, $||x|| = ||x-y+y|| \leq ||x-y|| + ||y||$. Since $||x|| \leq ||x-y|| + ||y||$, $||x|| - ||y|| \leq ||x-y||$. Then, $||x-y|| = ||x+(-y)|| \leq ||x|| + ||-y|| = ||x|| + |-1|||y|| \leq ||x|| - ||y||$.

2. Without loss of generality, let $b > a$.
For Positive Definiteness:
$< f, f >_w = \int_a^b f^2(x)w(x)$. Since f^2 is always positive and w is also always positive $f^2(x)w(x)$ is always positive. Since $f^2(x)w(x)$ is always positive and $b > a$, we are going to have a positive value for our integral.
For Symmetry:
$< f, g >_w = \int_a^b f(x)g(x)w(x) = \int_a^b g(x)f(x)w(x) = < g, f >_w$
For Bilinearity:
$< \alpha f + \beta g, h >_w = \int_a^b (\alpha f(x) + \beta g(x))h(x)w(x) = \int_a^b \alpha f(x)h(x)w(x) + \beta g(x)h(x)w(x) = \int_a^b \alpha f(x)h(x)w(x) + \int_a^b \beta g(x)h(x)w(x) = < f, h >_w + < g, h >_w$
And thus, since this inner product definition satisfies the properties of an inner product, $< f, g >_w = \int_a^b f(x)g(x)w(x)$ is an inner product.

3. Let $||x|| = ||y||$. Then, $< x-y, x+y > = (x-y)^T(x+y) = \Sigma(x_i - y_i)(x_i + y_i) = \Sigma x_i^2 - y_i^2 = \Sigma x_i^2 - \Sigma y_i^2$. Since $||x|| = ||y||$, $\Sigma x_i^2 - \Sigma y_i^2 = 0$. Thus, $x - y$ is orthogonal to $x + y$ if and only if $||x|| = ||y||$.

4. Let $v_0 = 1, v_1 = x, v_2 = x^2, v_3 = x^3$ be the basis vectors given. Then,

$$y_0 = 1$$
$$y_1 = v_1 - < v_1, y_0 > \frac{y_0}{||y_0||^2} = x - 0 = x$$
$$y_2 = v_2 - < v_2, y_0 > \frac{y_0}{||y_0||^2} - < v_2, y_1 > \frac{y_1}{||y_1||^2} = x^2 - \frac{1}{3}$$
$$y_3 = v_3 - < v_3, y_0 > \frac{y_0}{||y_0||^2} - < v_3, y_1 > \frac{y_1}{||y_1||^2} - < v_3, y_2 > \frac{y_2}{||y_2||^2} = x^3 - \frac{3x}{5}$$

Since none of the y_i are zero, we will use all of our y_i in our orthogonal basis. To turn this into an orthonormal basis, we divide each y_i by its inner norm.

$$f_0 = \frac{y_0}{||y_0||} = \frac{1}{\sqrt{2}}$$
$$f_1 = \frac{y_1}{||y_1||} = \frac{x}{\sqrt{\frac{2}{3}}}$$
$$f_2 = \frac{y_2}{||y_2||} = \frac{x^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}}$$
$$f_3 = \frac{y_3}{||y_3||} = \frac{x^3 - \frac{3x}{5}}{\sqrt{\frac{8}{175}}}$$

Since all of the basis vectors are orthogonal, our matrix A will be the 4 by 4 identity matrix.

5. Using Matlab, we can calculate the first 4 terms in our fourier series:

```
g(x) =

0.63661977 - 0.084882636*cos(4.0*x) - 0.036378273*cos(6.0*x) - 0.42441318*cos(2.0*x)
```

And thus, our fourier series estimate for $|sin(x)|$ is $0.63661977 - 0.084882636cos(4.0x) - 0.036378273cos(6.0x) - 0.42441318cos(2.0x)$.

6. Since we are no longer in the interval $[-\pi, \pi]$, we must take back into consideration L, our period. We also now have a disctoninuous piecewise function and so when we are integrating, we must split our integral into two parts. Taking this into account, we can write a script in Matlab to get our first four fourier series terms:

```
g(x) =

0.63661977*sin(4.712389*x) + 1.9098593*sin(1.5707963*x) + 0.38197186*sin(7.8539816*x) + 0.5
```

Plotting these four terms: