1. Let $\{a_n\}$ be a sequence of real numbers such that $|a_n| \leq 2$ and $|a_{n+2} - a_{n+1}| \leq \frac{1}{8}|a_{n+1}^2 - a_n^2| \forall n \geq 1$. Then, $\frac{1}{8}|a_{n+1}^2 - a_n^2| = \frac{1}{8}|(a_{n+1} - a_n)(a_{n+1} + a_n)| \leq \frac{1}{8^2}|a_n^2 - a_{n-1}^2||a_{n+1} + a_n| \leq \cdots \leq \frac{1}{8^{n-2}}|a_2^2 - a_1^2||a_{n+1} + a_n| \cdots |a_3 + a_2|$. Since $|a_n| \leq 2$, $\frac{1}{8^{n-2}}|a_2^2 - a_1^2||a_{n+1} + a_n| \cdots |a_3 + a_2| \leq \frac{1}{8^{n-2}}|a_2^2 - a_1^2||2 + 2| \cdots |2 + 2| = \frac{1}{8^{n-2}}|a_2^2 - a_1^2|4^{n-2} = \frac{1}{2^{n-2}}|a_2^2 - a_1^2|$. Since $\frac{1}{2^{n-2}} \to 0$ as $n \to \infty$, $\{a_n\}$ is Cauchy. Thus $\{a_n\}$ converges.

2. For all $x, y \in \mathbb{R}$, $|cos(x) - cos(y)| = 2|sin(\frac{x+y}{2})sin(\frac{x-y}{2})| \leq 2|sin(\frac{x-y}{2})| < 2|\frac{x-y}{2}| = |x - y|$. Thus, if we let $\delta = \epsilon$, $|f(x) - f(y)| < \epsilon$ if $|x - y| < \delta$. Thus, $f(x) = cos(x)$ is uniformly continuous on all of $\mathbb{R}$.

3. Let $f(x) = cos(x)$ where $x \in \mathbb{R}$. Then, $f'(x) = -sin(x)$ Since $sin(x)$ is bounded by $[-1, 1]$, the derivative is bounded below by -1 and above by 1. Thus, $|f'| \leq 1$. Since f is a function with a bounded derivative, $cos(x)$ is continuous on all of $\mathbb{R}$.

4. Since we are no longer in the interval $[-\pi, \pi]$, we must take back into consideration L, our period. We also now have a disctoninuous piecewise function and so when we are integrating, we must split our integral into two parts. Taking this into account, we can write a script in Matlab to get our first four fourier series terms:

```
g(x) =

0.63661977*sin(4.712389*x) + 1.9098593*sin(1.5707963*x) + 0.38197186*sin(7.8539816*x) + 0.5
```

Plotting these four terms: