1. Let (a_n) be a seqence which converges to a limit L. Then, by definition, there exists $N \in \mathbf{N}$ such that if $n \geq N$, $|a_n - L| < \epsilon$. Since $b_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$, if $n \geq N$, $b_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_{N-1}) + \frac{1}{n}(a_N + \cdots + a_{n-1} + a_n)$. Then, $b_n \leq |b_n| = |\frac{1}{n}(a_1 + a_2 + \cdots + a_{N-1}) + \frac{1}{n}(a_N + \cdots + a_{n-1} + a_n)| \leq |\frac{1}{n}(a_1 + a_2 + \cdots + a_{N-1})| + |\frac{1}{n}(a_N + \cdots + a_{n-1} + a_n))|$. Then, since $|a_n - L| < \epsilon$, the max value a_n can take is $L + \epsilon$ for all $n \geq N$. Thus, $|\frac{1}{n}(a_N + \cdots + a_{n-1} + a_n))| < |\frac{(n-N)(L+\epsilon)}{n}| < L + \epsilon$. Thus, $b_n < |\frac{1}{n}(a_1 + a_2 + \cdots + a_{N-1})| + L + \epsilon$ which is a finite number. Therefore, at some point b_n must converge.

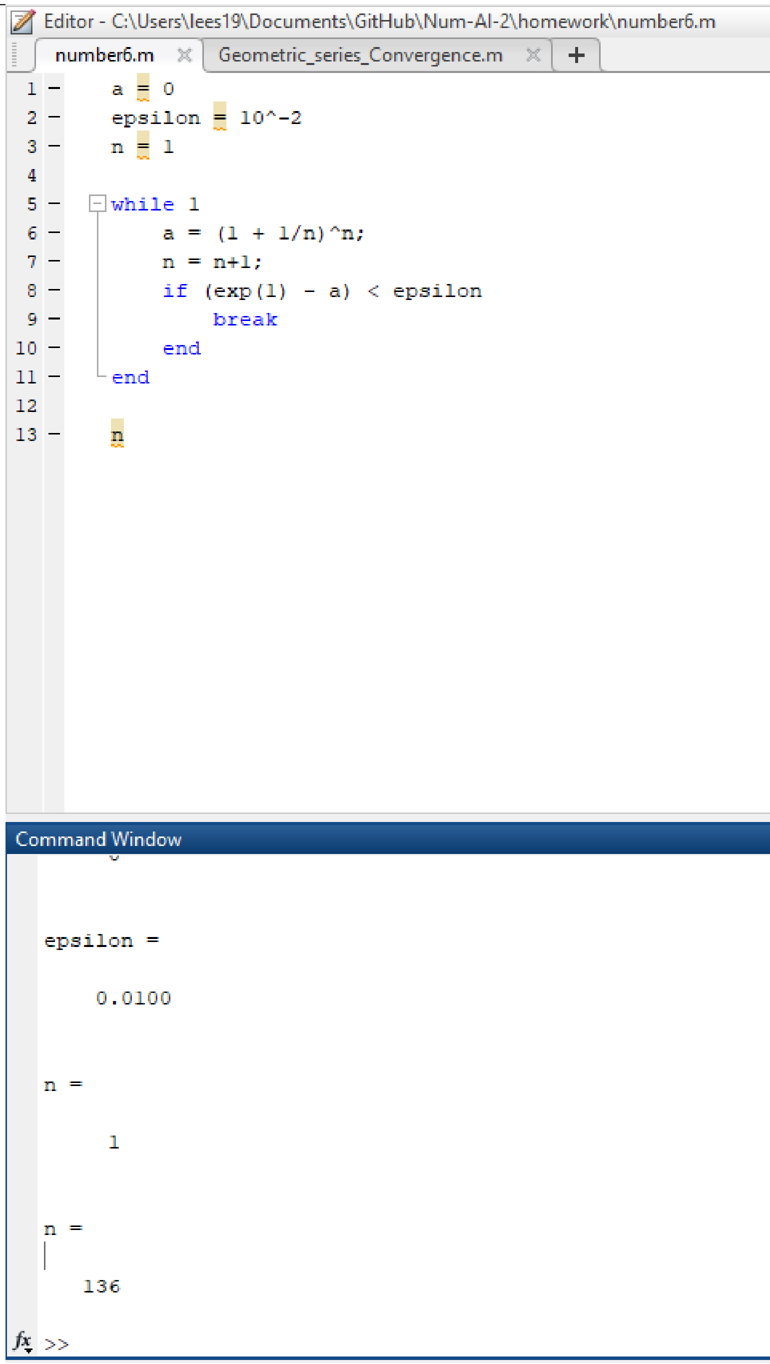
2. Without loss of generality, assume $a > b$. Then since $a > b$, $(a^n + b^n)^{1/n} < (2a^n)^{1/n}$ and $(a^n)^{1/n} < (a^n + b^n)^{1/n}$. Then, $a < (a^n + b^n)^{1/n} < (2a^n)^{1/n} = 2^{1/n}a$ and as $n \to \infty$, $a < (a^n + b^n)^{1/n} < a$ and thus, by the squeeze theorem $(a^n + b^n)^{1/n} \to a$.

3. Let $\frac{1}{2} < \epsilon$. Claim: $|\sqrt{n+1} - \sqrt{n} - 0| < \epsilon$. Since, $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}}$. Since $\sqrt{n+1} + \sqrt{n} > 2\sqrt{n}$, $\frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}}$. Since $n \geq 1$, and $\frac{1}{2\sqrt{n}} > \frac{1}{2\sqrt{n+1}}$, $\frac{1}{2\sqrt{n}} < \frac{1}{2} < \epsilon$. Thus, $|\sqrt{n+1} - \sqrt{n} - 0| < \epsilon$, thus $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$.

4. Assume $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ Then, $|(x_n + y_n) - (x + y)| = |x_n - x + y_n - y| \leq |x_n - x| + |y_n - y|$. Since $(x_n) \to x$ and $(y_n) \to y$, we can find N_1, N_2 such that $|x_n - x| < \frac{\epsilon}{2}, \forall n \geq N_1$. and $|y_n - y| < \frac{\epsilon}{2}, \forall n \geq N_2$. Thus, if we take $N = max\{N_1, N_2\}$, we get that $|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| = \epsilon$.

5. Let $\{a_n\}$ be a sequence of real numbers such that $|a_n| \leq 2$ and $|a_{n+2} - a_{n+1}| \leq \frac{1}{8}|a_{n+1}^2 - a_n^2| \forall n \geq 1$. Then, $\frac{1}{8}|a_{n+1}^2 - a_n^2| = \frac{1}{8}|(a_{n+1} - a_n)(a_{n+1} + a_n)| \leq \frac{1}{8^2}|a_n^2 - a_{n-1}^2||a_{n+1} + a_n| \leq \cdots \leq \frac{1}{8^{n-2}}|a_2^2 - a_1^2||a_{n+1} + a_n| \cdots |a_3 + a_2|$. Since $|a_n| \leq 2$, $\frac{1}{8^{n-2}}|a_2^2 - a_1^2||a_{n+1} + a_n| \cdots |a_3 + a_2| \leq \frac{1}{8^{n-2}}|a_2^2 - a_1^2||2 + 2| \cdots |2 + 2| = \frac{1}{8^{n-2}}|a_2^2 - a_1^2|4^{n-2} = \frac{1}{2^{n-2}}|a_2^2 - a_1^2|$. Since $\frac{1}{2^{n-2}} \to 0$ as $n \to \infty$, $\{a_n\}$ is Cauchy. Thus $\{a_n\}$ converges.

6. Using matlab:



Editor - C:\Users\lees19\Documents\GitHub\Num-AI-2\homework\number6.m

number6.m | Geometric_series_Convergence.m

```
a = 0
epsilon = 10^-2
n = 1

while 1
    a = (1 + 1/n)^n;
    n = n+1;
    if (exp(1) - a) < epsilon
        break
    end
end

n
```

Command Window

```
epsilon =

    0.0100


n =

     1


n =

   136

>>
```

We find that this sequence does indeed converge to e:

```
Command Window
M =

  Columns 1 through 16

    2.0000    2.2500    2.3704    2.4414    2.4883    2.5216    2.5465    2.5658    2.5812    2.5937    2.6042    2.6130    2.6206    2.6272    2.6329    2.6379

  Columns 17 through 32

    2.6424    2.6464    2.6500    2.6533    2.6563    2.6590    2.6615    2.6637    2.6658    2.6678    2.6696    2.6713    2.6728    2.6743    2.6757    2.6770

  Columns 33 through 48

    2.6782    2.6794    2.6804    2.6815    2.6824    2.6834    2.6842    2.6851    2.6859    2.6866    2.6873    2.6880    2.6887    2.6893    2.6899    2.6905

  Columns 49 through 64

    2.6911    2.6916    2.6921    2.6926    2.6931    2.6935    2.6940    2.6944    2.6948    2.6952    2.6956    2.6960    2.6963    2.6967    2.6970    2.6973

  Columns 65 through 80

    2.6977    2.6980    2.6983    2.6986    2.6988    2.6991    2.6994    2.6996    2.6999    2.7001    2.7004    2.7006    2.7008    2.7011    2.7013    2.7015

  Columns 81 through 96

    2.7017    2.7019    2.7021    2.7023    2.7025    2.7026    2.7028    2.7030    2.7032    2.7033    2.7035    2.7037    2.7038    2.7040    2.7041    2.7043

  Columns 97 through 112

    2.7044    2.7045    2.7047    2.7048    2.7049    2.7051    2.7052    2.7053    2.7054    2.7056    2.7057    2.7058    2.7059    2.7060    2.7061    2.7062

  Columns 113 through 128

    2.7064    2.7065    2.7066    2.7067    2.7068    2.7069    2.7069    2.7070    2.7071    2.7072    2.7073    2.7074    2.7075    2.7076    2.7077    2.7077

  Columns 129 through 135

    2.7078    2.7079    2.7080    2.7081    2.7081    2.7082    2.7083


n =

   136

>>
```

And one the last line, we find it took 136 iterations to be within $\epsilon = 10^{-2}$. Thus, our $N = 136$