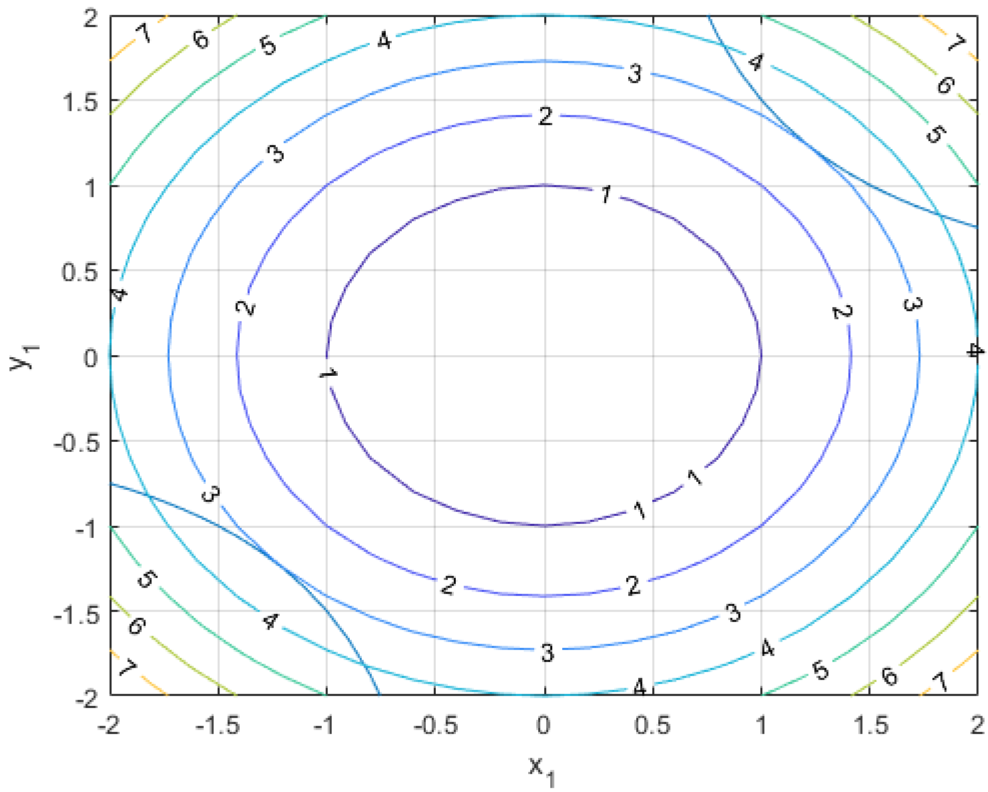
1.

$$\text{minimize } x_1^2 + x_2^2$$
$$\text{subject to } 2x_1x_2 = 3$$

The contour map of the objective function as well as the constraint:



(a) The Lagrangian function in our case is:

$$\mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) + \lambda h(x_1, x_2) = x_1^2 + x_2^2 + 2\lambda x_1x_2 - 3\lambda$$

(b)

$$\nabla\mathcal{L} = \begin{bmatrix} 2x_1 + 2\lambda x_2 \\ 2x_2 + 2\lambda x_1 \\ 2x_1x_2 - 3 \end{bmatrix}$$

(c) Setting the gradient equal to zero, we are left with the system of equations:

$$\begin{cases} 2x_1 + 2\lambda x_2 = 0 \\ 2x_2 + 2\lambda x_1 = 0 \\ 2x_1x_2 - 3 = 0 \end{cases}$$

Using the third equation, we obtain $x_1 = \frac{3}{2x_2}$. Then,

$$2(\frac{3}{2x_2}) + 2\lambda x_2 = 0$$
$$\frac{3}{x_2} = -2\lambda x_2$$
$$\lambda = \frac{-3}{2x_2^2}$$

then,

$$2x_2 + 2(\frac{-3}{2x_2^2})(\frac{3}{2x_2}) = 0$$
$$2x_2 = \frac{9}{2x_2^3}$$
$$x_2^4 = \frac{9}{4}$$
$$x_2 = \sqrt{\frac{3}{2}}$$

Thus, $\lambda = \frac{-3}{2\frac{3}{2}} = -1$ and $x_1 = \frac{3}{2\sqrt{\frac{3}{2}}}$ and we have a solution at the point $(\frac{3}{2\sqrt{\frac{3}{2}}}, \sqrt{\frac{3}{2}})$ which gives us a value of 3.

(d) We can check our answer by taking the gradient of the objective function and comparing it to the gradient of our restriction at the point:

$$\nabla(x_1^2 + x_2^2) = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}, \nabla(2x_1x_2 - 3) = \begin{bmatrix} 2x_2 \\ 2x_1 \end{bmatrix}$$

Since

$$2x_1 = 2\left(\frac{3}{2\sqrt{(\frac{3}{2})}}\right) = \frac{3}{\sqrt{\frac{3}{2}}} = 3\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{2}\sqrt{3}}{\sqrt{2}\sqrt{3}} = 3\frac{2\sqrt{3}}{3\sqrt{2}} = 2\frac{\sqrt{3}}{\sqrt{2}} = 2\sqrt{\frac{3}{2}} = 2x_2$$

we obtain $2x_1 = 2x_2$ and thus the gradients of the two functions are parallel. Thus, this is an optimal solution to our problem.

2. (a)

$$f(x_1, x_2) = 8x_1 + 12x_2 + x_1^2 - 2x_2^2$$

$$\nabla f(\mathbf{x}^*) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 8 + 2x_1 \\ 12 - 4x_2 \end{bmatrix}$$

Since both partial derivatives are linear, we will only have one stationary point at $x_1 = -4$ and $x_2 = -6$

(b) Calculating the Hessian:

$$H = \nabla^2 f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$$

Taking the determinant of the Hessian: $det(H) = 2(-4) + 0 \cdot 0 = -8$. Since the determinant is negative, we have that our stationary point is a sadddle point.

(c) Since in the objective function we have the two terms $x_1^2 - 2x_2^2$, we would expect our contour lines to be hyperbolas.