1. The statement when $n = 0$ is true since:

$$Q(0) = c_0 x^0 = c_0 \tag{1}$$

Since $Q(0) = c_0$ and c_0 is a constant, $Q(0)$ is a continuous function.

Assume $Q(n)$ for some number $n \in \mathbb{N}$. Consider the polynomial $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_N x^N + c_{N+1} x^{N+1}$. By algebra:

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_N x^N + c_{N+1} x^{N+1}$$
$$p(x) = c_0 + x(c_1 + c_2 x + \cdots + c_N x^{N-1} + c_{N+1} x^N)$$

Since $c_1 + c_2 x + \cdots + c_N x^{N-1} + c_{N+1} x^N$ is a N^{th} degree polynomial with real coefficients, $x(c_1 + c_2 x + \cdots + c_N x^{N-1} + c_{N+1} x^N)$ is a continuous function. Since c_0 is a constant function, $c_0 + x(c_1 + c_2 x + \cdots + c_N x^{N-1} + c_{N+1} x^N)$ is a continuous function.

Thus, by the Principle of Mathematical Induction, all polynomial functions with real coefficients are continuous functions.

2. Assume the Intermediate Value Theorem, and assume $f(x) = x^2 - 2$ on the interval $[a, b]$. By the theorem proved by question 1, $f(x)$ is continuous, since x^2 and -2 are both continuous functions, and the sum of two continuous functions is continuous. Then, since $f(1) = -1 < 0$ and $f(2) = 2 > 0$, by the Intermediate Value Theorem, we know $\exists c \in (a, b)$ such that $f(c) = 0$. Therefore, at some point c, $f(c) = c^2 - 2 = 0$. Solving for c, we get $c^2 = 2$ and thus, $c = \sqrt{2}$. To prove $\sqrt{2}$ is irrational, assume to the contrary, $\sqrt{2}$ is rational. Then, $\sqrt{2}$ can be written by two integers p and q such that $\sqrt{2} = \frac{p}{q}$ and $\frac{p}{q}$ are in lowest terms. Then, $2 = \frac{p^2}{q^2}$ Thus, $2q^2 = p^2$ and from a previously proven theorem, if p^2 is even, p is also even since if p was odd, p^2 would also be odd. Since p is even, $p = 2k$ for some integer k. Since $p = 2k$, $p^2 = 4k^2$. Thus, $2q^2 = 4p^2$. By cancellation, $q^2 = 2p^2$. Since q^2 is even, q must also be even. This is a contradiction, since we assumed p and q were in lowest terms. Therefore, $\sqrt{2}$ must be an irrational number.

$$S = \{z \in \mathbb{R} : \exists j, k \in \mathbb{Z} \;\; s.t. \;\; z = 43j + 18k\}$$

3. Case $S \subseteq \mathbb{Z}$

   Let x be an arbitrary element of S. Since $x \in S$, x is some real number of the form $43j + 18k$ for some integers $j, k \in \mathbb{Z}$ such that $x = 43j + 18k$. Since j is an integer, $43j$ is an integer and since k is an integer, $18k$ is an integer by closure. Since $43j$ and $18k$ are integers, their sum, $43j + 18k$, is an integer. Since x was arbitrary, $S \subseteq \mathbb{Z}$.

   Case $\mathbb{Z} \subseteq S$

   Let y be an arbitrary element in $\mathbb{Z}$. Since 43 and 18 are relatively prime, there exist two integers n and m, such that $1 = 43n + 18m$. Since $1 = 43n + 18m$, $y = 43ny + 18my$. Since y was arbitrary, it follows that $\mathbb{Z} \subseteq S$.

   Thus, since S is a subset of Z and Z is a subset of S, S = Z

(a) Case $\bigcap_{r\in\mathbb{R}^+} A_r \subseteq \varnothing$
Assume for every real number, the set A_r is the interval $[\frac{r+1}{r}, \frac{5r+3}{r})$. If $r = 2$ then, $A_2 = [1.5, 6.5)$. If $r = \frac{1}{6}$ then $A_{\frac{1}{6}} = [7, 23)$. Thus we have found two A_r which are disjoint. Thus since the intersection of these two sets is the null set, the intersection of every A_r must be the null set. Thus the intersection of all of the A_r is a subset of the null set.

Case $\varnothing \subseteq \bigcap_{r\in\mathbb{R}^+} A_r$
Since the null set is a subset of every set, we find that the null set is must be a subset of the infinite intersection of the A_r. Thus, the null set is a subset of the infinite intersection of A_r.

Thus, since the infinite intersection is a subset of the null set and the null set is a subset of the infinite intersection, $\bigcap_{r\in\mathbb{R}^+} A_r = \varnothing$

(b) Case $\bigcup_{r\in\mathbb{R}^+} A_r \subseteq (1,\infty)$

Let x be an arbitrary element of $\bigcup_{r\in\mathbb{R}^+}$. Since $x \in \bigcup_{r\in\mathbb{R}^+} A_r$, x must be in at least one of the A_r. Since $x \in A_r$ for some positive real number r, $x \in [\frac{r+1}{r}, \frac{5r+3}{r}]$, thus $\frac{r+1}{r} \leq x$. Since r is a positive real number, $r < r+1$. Since $r < r+1$, $1 < \frac{r+1}{r}$. Since $1 < \frac{r+1}{r} \leq x$, $1 < x$. Since x was arbitrary, $\bigcup_{r\in\mathbb{R}^+} A_r \subseteq (1,\infty)$

Case $(1,\infty) \subseteq \bigcup_{r\in\mathbb{R}^+} A_r$

Let y be an arbitrary element of the interval $(1,\infty)$. We want to show $y \in [\frac{r+1}{r}, \frac{5r+3}{r}]$. Let $y = \frac{r+1}{r}$. Then, by algebra, $r = \frac{1}{(y-1)}$. By closure, r is a real number. Since $y > 1$, $y - 1 > 0$. Thus, $\frac{1}{y-1}$ is positive. Thus, we have found a positive real number $r = \frac{1}{(y-1)}$ for which $y \in A_r$. Since y was arbitrary, $(1,\infty) \subseteq \bigcup_{r\in\mathbb{R}^+} A_r$.

Thus, since each subset is a subset of the other, $\bigcup_{r\in\mathbb{R}^+} A_r = (1,\infty)$