Due: Monday, October 19

1. (a) Provide two predicates $P(x)$ and $Q(x)$ (and a domain D for the variable x) for which

$$[\forall x \in D\ (P(x) \vee Q(x))] \Rightarrow [(\forall x \in D,\ P(x)) \vee (\forall x \in D,\ Q(x))]$$

is false.
Solution.
$P(x) = x$ is even, $Q(x) = x$ is odd, $D = \mathbb{N}$

(b) Provide a predicate $P(x, y)$ (and domains D_x and D_y for the variables x and y respectively) for which

$$[\forall y \in D_y,\ \exists x \in D_x \text{ s.t. } P(x, y)] \Rightarrow [\exists x \in D_x \text{ s.t. } \forall y \in D_y\ P(x, y)]$$

is false.
Solution.
$D_y = \mathbb{R}, D_x = \mathbb{R}, P(x, y) : y/x = 1$

2. Suppose that A, B, C, and D are sets for which

   - $A \cup B \subseteq C \cup D$,
   - $A \cap B = \varnothing$, and
   - $C \subseteq A$.

   Prove that $B \subseteq D$.
   Solution.
   Assume $A \cup B \subseteq C \cup D$, $A \cap B = \varnothing$, and $C \subseteq A$. Let $x \in A \cup B$. Then, since $x \in A \cup B$ and $A \cup B \subseteq C \cup D$, $x \in C \cup D$. Since $A \cap B = \varnothing$, x must be in A or B. Let $x \in B$. Then, since $B \subseteq C \cup D$, $B \subseteq C$ or $B \subseteq D$. Let $B \subseteq C$. Then, $x \in B \subseteq C$. Since $C \subseteq A$, $x \in B \subseteq C \subseteq A$ therefore, $x \in A$. Thus we have found that if $x \in B$ and $B \subseteq C$, x must also be in A. Since $x \in A \cap B$, $A \cap B \neq \varnothing$, which results in a contradiction since we assumed that $A \cap B = \varnothing$. Thus, it must be the case that $B \subseteq D$.

3. (a) Prove or disprove: for any sets A and B, that

$$\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B).$$

Case $\mathscr{P}(A \cap B) \subseteq \mathscr{P}(A) \cap \mathscr{P}(B)$:
Let $X \in \mathscr{P}(A \cap B)$. Then, since $X \in \mathscr{P}(A \cap B), X \subseteq A \cap B$. Since $X \subseteq A \cap B, X \subseteq A$ and $X \subseteq B$. Since $X \subseteq A$, $X \in \mathscr{P}(A)$. Since $X \subseteq B$, $X \in \mathscr{P}(B)$. Then, since X is in both $\mathscr{P}(A)$ and $\mathscr{P}(B)$, X is in $\mathscr{P}(A \cap B)$. Since X was arbitrary, $\mathscr{P}(A \cap B) \subseteq \mathscr{P}(A) \cap \mathscr{P}(B)$.

Case $\mathscr{P}(A) \cap \mathscr{P}(B) \subseteq \mathscr{P}(A \cap B)$:
Let $X \in \mathscr{P}(A) \cap \mathscr{P}(B)$. Then, $X \in \mathscr{P}(A)$ and $X \in \mathscr{P}(B)$. Since $X \in \mathscr{P}(A)$, $X \subseteq A$. Since $X \in \mathscr{P}(B)$, $X \subseteq B$ Since $X \subseteq A$ and $X \subseteq B$, $X \subseteq A \cap B$. Since $X \subseteq A \cap B$, $X \in \mathscr{P}(A \cap B)$.

Thus, we have shown that both are subsets of the other, therefore are equal.

(b) Prove or disprove: for any sets A and B, that

$$\mathscr{P}(A \cup B) = \mathscr{P}(A) \cup \mathscr{P}(B).$$

Counter example: Let $A = \{a, b\}$ and $B = \{b, c\}$. Then, $A \cup B = \{a, b, c\}$ and $\mathscr{P}(A \cup B) = \{\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. However, $\mathscr{P}(A) \cup \mathscr{P}(B) = \{\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}$ Therefore, while $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$, $\mathscr{P}(A \cup B) \subsetneq \mathscr{P}(A) \cup \mathscr{P}(B)$, and therefore, $\mathscr{P}(A \cup B) \neq \mathscr{P}(A) \cup \mathscr{P}(B)$.

(c) Prove or refute the existence of suitable conditions on sets A and B that guarantee

$$\mathscr{P}(A) \cap \mathscr{P}(B) = \varnothing.$$

Assume A and B are disjoint. Then, $A \cap B = \varnothing$. Then, $\mathscr{P}(A \cap B) = \mathscr{P}(\varnothing)$. By a previously proven theorem, $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B)$. Therefore, $\mathscr{P}(A \cap B) = \mathscr{P}(A) \cap \mathscr{P}(B) = \mathscr{P}(\varnothing) = \varnothing$. Thus if A and B are disjoint sets, the intersection of their power sets is the null set.

4. (a) Find the intersection and union of the following family:

$$\mathscr{F} = \left\{ \left[\frac{1-n}{n}, \frac{n-1}{n} \right] \; : \; n \in \mathbb{N} \right\}$$

$$\bigcup_{F \in \mathscr{F}} = \{(-1,1)\}$$

$$\bigcap_{F \in \mathscr{F}} = \{0\}$$

(b) Find the intersection and union of the following family:

$$\mathscr{M} = \{n\mathbb{Z} : n \in \mathbb{N}\},$$

where $n\mathbb{Z} = \{\ldots, -3n, -2n, -n, 0, n, 2n, 3n, \ldots\}$ for each $n \in \mathbb{N}$.

$$\bigcup_{M \in \mathscr{M}} = \mathbb{Z}$$

$$\bigcap_{M \in \mathscr{M}} = 0$$

(c) Find the intersection and union of the following family:

$$\mathscr{V} = \{V_n \ : \ n \in \mathbb{N}\}$$

where for each natural number n,

$$V_n = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leq x \leq 1,\ x^n \leq y \leq \sqrt[n]{x}\}.$$

$$\bigcup_{V \in \mathscr{V}} = \{(x, y) \in \mathbb{R}^2 \ : \ 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$$

$$\bigcap_{V \in \mathscr{V}} = \{(x, y) \in \mathbb{R}^2 \ : \ x = y, 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$$