1. *Solution.*
   Assume every even number greater than 2 can be expresssed as a sum of two primes. Let p_a and p_b be two arbitrary primes. Then, for some natural number k, $p_a + p_b = 2k$. Let p_c be an arbitrary prime number. Then, p_c is either 2 or odd. If p_c is odd, p_c can be written as $2m + 1$ for some natural number m. Then, $p_a + p_b + p_c = 2k + 2m + 1$. Since the integers are closed under addition, $k + m$ is an integer. Therefore, $p_a + p_b + p_c = 2(k + m) + 1$ which is an odd number.

2. *Solution.*
   Suppose there exist integers n and m such that $21n + 6m = 5$. Then, since both 21 and 6 are divisible by 3, by distribution, $21n + 6m = 3(7n + 2m)$. Since the integers are closed under addition and multiplication, $7n + 2m$ is an integer. Since $7n + 2m$ is an integer, this number produces a multiple of three. However, 5 is not a multiple of 3. Therefore, there do not exist integers n, m such that $21n + 6m = 3(7n + 2m) = 5$.

3. *Solution.*
   Let $c, a \in \mathbb{Z}$, $c > 0$, a is odd, c divides a and c divides $a + 2$. Then, $a = 2k + 1$ for some $k \in \mathbb{Z}$. Since c divides a and $a + 2$, $cm = a$ and $cn = a + 2$ for some numbers $m, n \in \mathbb{Z}$.

4. *Solution.* Assume that m is not a multiple of 5. Then, $m = 5d + 1$ or $5d + 2$ or $5d + 3$ or $5d + 4$.

   (a) Case 1: If $m = 5d + 1$, then $m^2 = (5d + 1)^2 = 5(5d^2 + 2d) + 1$
   (b) Case 2: If $m = 5d + 2$, then $m^2 = (5d + 2)^2 = 5(5d^2 + 4d) + 4$
   (c) Case 3: If $m = 5d + 3$, then $m^2 = (5d + 3)^2 = 5(5d^2 + 6d) + 9$
   (d) Case 4: If $m = 5d + 4$, then $m^2 = (5d + 4)^2 = 5(5d^2 + 8d) + 16$

   Thus, if m is not a multiple of 5, then m^2 is not a multiple of 5. Therefore, if m^2 is a multiple of five, then m must also be a multiple of five.

5. *Solution.* Assume, to the contrary, that $\sqrt{5}$ was a rational number. Then, $\sqrt{5} = \frac{m}{n}$ for two integers $m, n \in \mathbb{Z}$. Assume m and n are in lowest terms. Then, $5 = \frac{m^2}{n^2}$ and therefore, $5n^2 = m^2$. Since m^2 is a multiple of 5, by the previously proven theorem, m must also be a multiple of five. Therefore, $m = 5d$ for some $d \in \mathbb{Z}$. Then, since $m = 5d$, $m^2 = 25d^2$. Thus, $5n^2 = 25d^2$. By cancellation, $n^2 = 5d^2$. Thus, n^2 is a multiple of five, and by the previous theorem, n must also be a multiple of five. Since n and m are both multiples of five, this contradicts the assumption that m and n are in lowest terms. Thus, we reach a contradiction and we conclude that $\sqrt{5}$ cannot be a rational number.