이동로봇 Tutorial 1차시

Model Predictive Path Integral

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01 Stochastic Optimal Control

Stochastic Optimal Control

Why Stochastic Optimal Control?

- Real-world systems contain uncertainties:
  - Model mismatch
  - External disturbances
  - Sensor and actuator noise
- Deterministic control assumes perfect models -> not realistic
- Stochastic Optimal Control aims to minimize the expected cost under uncertainty
- Foundation for algorithms such as Path Integral Control and MPPI

Stochastic Optimal Control

System Dynamics in SOC

- State: $\mathbf{x}_t \in \mathbb{R}^n$, Control input: $\mathbf{u}_t \in \mathbb{R}^m$, Disturbance: $\mathbf{w}_t \in \mathbb{R}^n$
- Stochastic dynamics:

$$\mathbf{x}_{t+1} = f(\mathbf{x}_t, \mathbf{u}_t) + \mathbf{w}_t$$

Cost Functional

- Running cost: $q(\mathbf{x}_t, \mathbf{u}_t)$
- Terminal cost: $\phi(\mathbf{x}_T)$
- Control sequence: $\mathbf{U} = \{\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{T-1}\}$
- Expected total cost of a control sequence:

$$J(\mathbf{U}) = \mathbb{E}\left[\phi(\mathbf{x}_T) + \sum_{t=0}^{T-1} \ell(\mathbf{x}_t, \mathbf{u}_t)\right]$$

- SOC objective:

$$\mathbf{U}^* = \min_{\mathbf{U}} J(\mathbf{U})$$

Stochastic Optimal Control

Value function & Bellman Optimality

- Value function:

$$V(\mathbf{x}_0) = \min_{\mathbf{U}} \mathbb{E}\left[\phi(\mathbf{x}_T) + \sum_{t=0}^{T-1} \ell(\mathbf{x}_t, \mathbf{u}_t)\right]$$

- Bellman optimality:

$$V(\mathbf{x}_t) = \min_{\mathbf{u}_t} \left[\ell(\mathbf{x}_t, \mathbf{u}_t) + \mathbb{E}\left[V(\mathbf{x}_{t+1})\right]\right]$$

- Hamilton−Jacobi−Bellman (HJB) PDE

$$-\partial_t V(\mathbf{x}, t) = \min_{\mathbf{u}} \left\{\ell(\mathbf{x}, \mathbf{u}, t) + \nabla_{\mathbf{x}} V(\mathbf{x}, t)^\top f(\mathbf{x}, \mathbf{u}, t) + \frac{1}{2} \operatorname{tr}\big(\sigma(\mathbf{x}, \mathbf{u}, t)\sigma(\mathbf{x}, \mathbf{u}, t)^\top \nabla_{\mathbf{x}}^2 V(\mathbf{x}, t)\big)\right\}$$

$$V(\mathbf{x}, T) = \phi(\mathbf{x})$$

02

Stochastic Sampling & Monte Carlo Basics

Stochastic Sampling & Monte Carlo Basics

Why Sampling in Control?

- In stochastic systems, future outcomes are random
- Expectations cannot be computed analytically in most cases
- Monte Carlo sampling provides a simple numerical approximation
- MPPI relies entirely on sampling to evaluate and improve control sequences

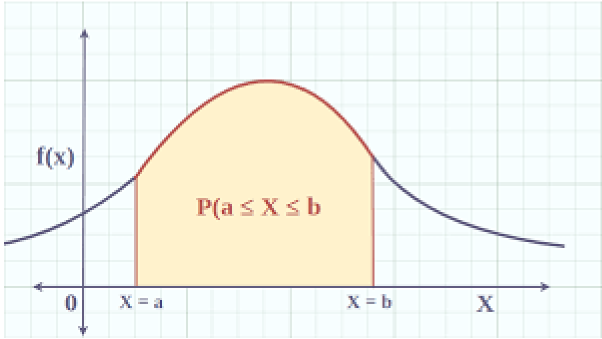
Stochastic Sampling & Monte Carlo Basics

Understanding Probability Density Function (PDF)

- Role of $p(x)$ in Expectation
- In the expectation integral $\mathbb{E}[f(X)] = \int f(x)p(x)dx$
  the term $p(x)$ represents the Probability Density Function
- It acts as a "weighting function":
  regions with high $p(x)$ are sampled more frequently

- Key Property: Normalization
  - The total area under the curve must equal 1

$$\int_{-\infty}^{\infty} p(x)dx = 1$$



Ref: https://www.geeksforgeeks.org/engineering-mathematics/probability-density-function

Stochastic Sampling & Monte Carlo Basics

Monte Carlo Approximation of Expectations

- For a random variable X and function f,
  the expectation:

$$\mathbb{E}[f(X)] = \int f(x)p(x)dx$$

  can be approximated via N samples

- Monte Carlo estimator:

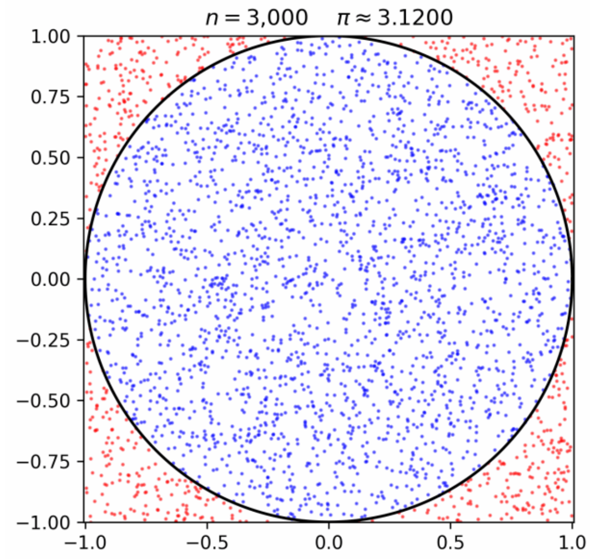
$$\mathbb{E}[f(X)] \approx \frac{1}{N}\sum_{i=1}^{N} f(x^{(i)})$$

- The approximation becomes mor accurate as N increases

Stochastic Sampling & Monte Carlo Basics

Example

$$
\begin{aligned}
\mathbb{E}[f(X)] &= \iint f(x,y)p(x,y)\,dx\,dy \\
&= \int_{-1}^{1}\int_{-1}^{1} f(x,y)\cdot\tfrac{1}{4}\,dx\,dy \\
&= \frac{1}{4}\iint_{x^2+y^2\le 1} 1\,dx\,dy \\
&= \frac{1}{4}\times(\text{Area of Circle}) \\
&= \frac{\pi}{4}
\end{aligned}
$$



$$\mathbb{E}[f(X)] \approx \frac{1}{N}\sum_{i=1}^{N} f(x^{(i)})$$

Ref: https://www.geeksforgeeks.org/artificial-intelligence/what-is-monte-carlo-simulation

Stochastic Sampling & Monte Carlo Basics

Properties of Monte Carlo Estimation

- Unbiased estimator
- Variance decreases as 1/N
- Works in high dimensions
- MPPI uses Monte Carlo not just for expectation, but also for control update weighting

03

Importance Sampling

Importance Sampling

Why Importance Sampling?

- Monte Carlo uses uniform sampling from a distribution
- Works well when the integrand is smooth
- But fails when:
  - Rare events dominate the expectation
  - Some samples contribute much more than others
- We need a method to focus samples on important regions

Importance Sampling

Changing the Sampling Distribution

- Suppose the target expectation is:

$$\mathbb{E}_p[f(x)] = \int f(x)p(x)\,dx$$

- But we sample from another distribution $q(x)$
- Rewrite:

$$f(x)p(x) = f(x)\frac{p(x)}{q(x)}q(x)$$

- Then:

$$\mathbb{E}_p[f(x)] = \mathbb{E}_q\left[f(x)\frac{p(x)}{q(x)}\right]$$

Importance Sampling

Importance Sampling Estimator

- Using samples $x^{(i)} \sim q(x)$,
  the estimator becomes:

$$\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} f(x^{(i)}) w^{(i)}$$

- Importance weights:

$$w^{(i)} = \frac{p(x^{(i)})}{q(x^{(i)})}$$

Importance Sampling

Normalized Importance Weights

- Often we normalize the weights:

$$\tilde{w}^{(i)} = \frac{w^{(i)}}{\sum_{j=1}^{N} w^{(j)}}$$

- Then the estimator becomes:

$$\hat{\mu} = \sum_{i=1}^{N} \tilde{w}^{(i)} f(x^{(i)})$$

- Normalization improves numerical stability

04

Free Energy

Free Energy

Why Is Free Energy?

- Free Energy is a fundamental concept from statistical physics.
- It measures the trade-off between:
    - Low energy (preferred states)
    - High entropy (diverse distribution)
- Used to describe how probabilistic systems select states

Free Energy

Why Do We Need Free Energy

- Probabilistic systems face two opposing forces:
  - 1. Minimize Energy – prefer good (low-cost) states
  - 2. Maximize Entropy – avoid collapsing to a single state
- Free Energy combines both in a single objective

Free Energy

Mathematical Definition

- Free Energy of a distribution $q(x)$:

$$F(q) = \mathbb{E}_q[E(x)] - T\,\mathcal{H}(q)$$

$$\mathcal{H}(q) = -\int q(x) \log q(x)\, dx$$

- $E(x)$: energy (or cost) of state
- $\mathcal{H}(q)$: entropy of distribution
- T : temperature controlling the balance

Free Energy

What is Entropy?

- Entropy measures the uncertainty of a distribution

$$\mathcal{H}(q) = -\int q(x) \log q(x)\, dx$$

- High entropy -> spread-out, diverse distribution
- Low entropy -> sharply peaked, concentrated distribution

Free Energy

How to Interpret Free Energy

- The system seeks a distribution $q(x)$ that:
  - Assigns high probability to low-energy states
  - Maintains adequate entropy
- Free Energy quantifies this balance
- Minimizing $F(q)$ yields the most natural distribution

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Boltzmann Weights

Boltzmann Weights

Free Energy Minimization Yields the Boltzmann Distribution

- Free Energy functional:

$$F(q) = \mathbb{E}_q[E(x)] - T\,\mathcal{H}(q)$$

- Optimization problem:

$$\begin{aligned} q^* &= \arg\min_q \, F(q) \\ &= \arg\min_q \left[\int q(x)\, E(x)\, dx + T \int q(x) \log q(x)\, dx \right] \end{aligned}$$

- Result:

$$q^*(x) \propto e^{-E(x)/T}$$

Boltzmann Weights

Energy-Based View of Probability

- Free Energy minimization yields an exponential-form distribution:

$$q^*(x) \propto e^{-E(x)/T}$$

- $E(x)$: energy or cost
- T : Temperature
- Lower energy -> higher probability
- Exponential form ensures positivity and normalization

Boltzmann Weights

Boltzmann Distribution

- To be a valid probability distribution,
  the exponential for must be normalized:

$$q^*(x) = \frac{e^{-E(x)/T}}{Z}$$

- Normalization constant:

$$Z = \int e^{-E(x)/T} dx$$

- Ensures:

$$\int q^*(x)\, dx = 1$$

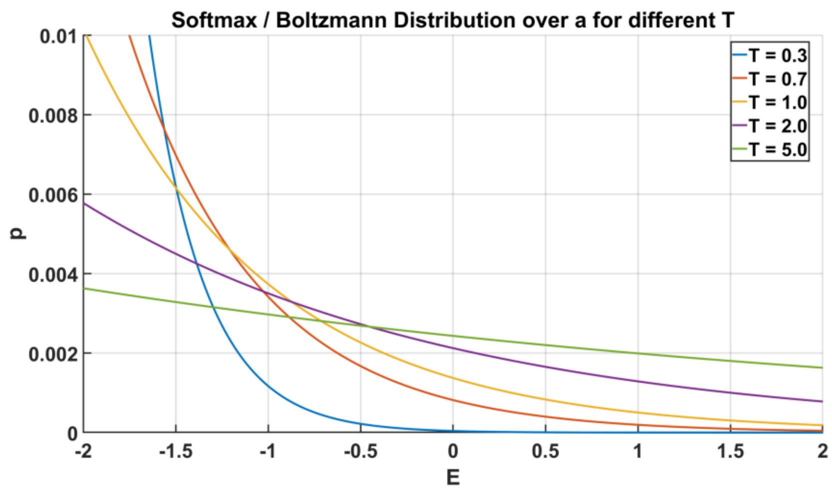
Boltzmann Weights

Softmax as a Discrete Boltzmann Distribution

- For discrete samples $x^{(i)}$, Boltzmann normalization becomes:

$$p^{(i)} = \frac{e^{-E^{(i)}/T}}{\sum_j e^{-E^{(j)}/T}}$$

- This is exactly the softmax operation
- Equivalent to a Boltzmann distribution over discrete set
- Low-energy samples -> exponentially higher weight



06

KL Divergence

KL Divergence

What is KL Divergence?

- KL divergence measures the difference between two probability distributions
- Defined as:

$$D_{\mathrm{KL}}(p\|q) = \int p(x)\ \log \frac{p(x)}{q(x)}\, dx$$

- Always non-negative
- Zero only when $p(x) = q(x)$ almost everywhere
- Asymmetric

$$D_{\mathrm{KL}}(p\|q) \neq D_{\mathrm{KL}}(q\|p)$$

KL Divergence

Interpretation of KL Divergence

- KL measures how inefficient it is to approximate p with q
- Equivalent to the expected log-difference:

$$D_{\mathrm{KL}}(p||q) = \mathbb{E}_p\left[\log p(x) - \log q(x)\right]$$
$$= \int log \frac{p(x)}{q(x)} p(x) dx$$

- Large KL -〉 q places low probability on regions where p is high
- KL is a natural objective for distribution matching

KL Divergence

What is Cross–Entropy

- Cross–entropy measures how well q represents p:

$$\mathcal{H}(p,q) = -\int p(x)\log q(x)\,dx$$

- Uses p for sampling
- Uses q for evaluating the log–probability
- Represents the coding cost of using q to encode samples form p
- If $q = p$, cross–entropy becomes the entropy

KL Divergence

KL Divergence in Terms of Entropy and Cross-Entropy

- Entropy

$$\mathcal{H}(p) = -\int p(x)\log p(x)\,dx$$

- Cross-entropy

$$\mathcal{H}(p,q) = -\int p(x)\log q(x)\,dx$$

- KL divergence:

$$\begin{aligned} D_{\mathrm{KL}}(p\|q) &= \int p(x)\ \log \frac{p(x)}{q(x)}\,dx \\ &= -\mathcal{H}(p) + \mathcal{H}(p,q) \end{aligned}$$

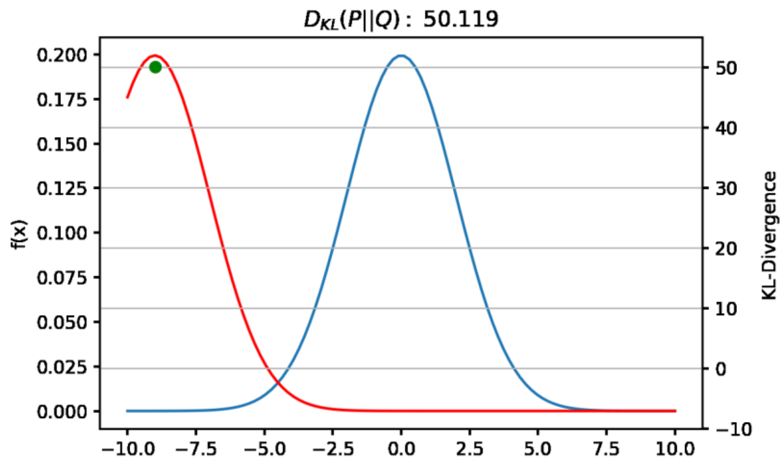
KL Divergence

Meaning of KL Divergence

- KL measures how inefficient it is to use q to represent
- Entropy: minimum required coding cost of p
- Cross-entropy: actual cost when encoding p using q
- KL is the extra cost:

$$D_{\mathrm{KL}}(p||q) = \mathcal{H}(p,q) - \mathcal{H}(p)$$

- Larger KL –〉 q poorly matches p's important regions (high inefficiency)



Ref: https://bekaykang.github.io/posts/KL_divergence

07

Jensen’s Inequality

Jensen's Inequality

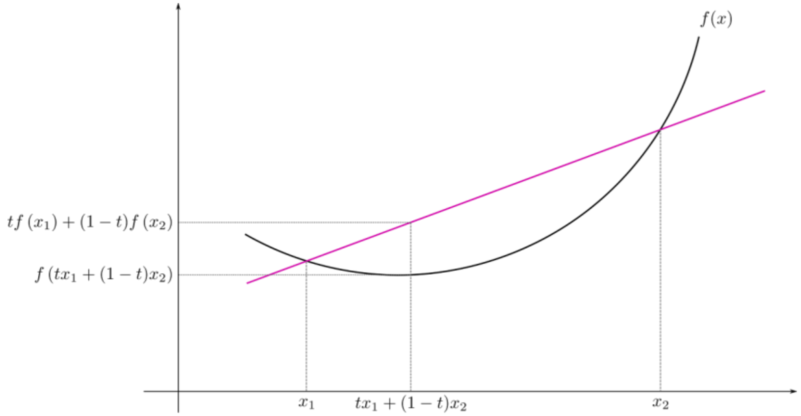
Jensen's Inequality

- For any convex function f

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$$

- For concave function (e.g., log)

$$\log \mathbb{E}[X] \geq \mathbb{E}[\log X]$$



- Proof
  - Let f be convex. By definition

$$f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2)$$

  - Extend to a finite mixture

$$f\left(\sum_i w_i x_i\right) \leq \sum_i w_i f(x_i), \qquad \sum_i w_i = 1$$

  - Extend to a continuous distribution

$$f\left(\int x\, p(x)\, dx\right) \leq \int f(x)\, p(x)\, dx$$

Ref: https://187cm.tistory.com/96

Thank you!