A model of rational choice

- *Microeconomic Theory*, Mas-Collel
- *An Introduction to Game Theory*, Osborne
- *Evolutionary Game Theory*, Weibull
- *Game Theory and Economic Modelling*, Kreps

A model of rational choice

게임이론은 rational choice로 설명할 수 있나? — 아주 중요한 문제

A model of rational choice

- A set A of actions from which the decision maker makes a choice.
- A set C of possible consequences of these actions.

Osbone's definition

one to one function

- A consequence function (or a belief) $g : A \to C$ that associates a consequence with each action.

=belief function

action set의 모든 요소가 consequence set의 모든 요소와 하나씩 mapping 되는 함수
일대일대응 one to one correspondence 일 필요는 없음

임의의 두 요소 간 선호관계를 언제나 판단 가능

언제나 자기자신은 자기자신보다 약선호됨 $x \succsim x$

- A preference relation (a complete, transitive, reflexive binary relation) $\succsim$ on the set C.

꼬임이 없음 $x \succsim y$ & $y \succsim z \Rightarrow x \succsim z$

- Or sometimes the decision maker's preferences are specified by giving a utility function $u : C \to R$, which defines a preference relation $\succsim$ by the condition $x \succsim y$ iff $u(x) \geq u(y)$.

preference relation 숫자로 표현

utility function

feasible set에 있는 하나의 action을 찾는 것 : choice

- Given any set $B \subseteq A$ of actions that are feasible in some particular case, a rational decision maker chooses an action a^* that is feasible ($a^* \in B$) and optimal in the sense that $g(a^*) \succsim g(a)$ for all $a \in B$, alternatively he solves the problem $\max_{a \in B} u(g(a))$.

수치로 표현하면 max 찾는 것으로 대체 가능

이 consequence가 어떠한 a를 취했을 때의 consequence 보다 약선호되면
→ a*를 선택하는 것이 rational choice!

Binary Relation

X 집합에서 하나 Y 집합에서 하나 뽑아서 순서쌍 만듦
→ 어떤 특성을 만족하는 부분 집합 R

$X = \{1,2,3\}$ $X \otimes Y = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$
$Y = \{1,2,3\}$

binary relation 예시

$x \succsim y$를 만족하는 순서쌍 뽑음
↓
$R = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3)\}$

R에 없는 조합은 순서를 뒤집으면 R에 있음 …… complete
xRy or yRx or both (모두 성립)

(2,1), (3,2)가 있으면 (3,1)도 있음 …… transitive

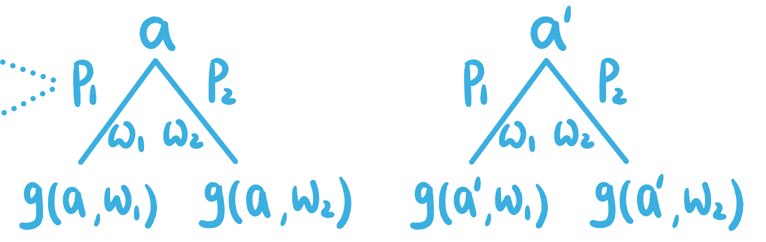
(1,1), (2,2), (3,3)도 있음 …… reflexive

A model of rational choice under risk

상황이 어떻게 될지 모름
= risk 존재

- A set A of actions from which the decision maker makes a choice.
- A set C of possible consequences of these actions.
- 가능한 모든 미래를 만듦
  A state space Ω and a probability measure over Ω.
- A라는 행동을 할 때 이 state에서는 결과 C가 나옴 g : A → C 였는데 Ω 고려함
  A consequence function $g : A \times \Omega \rightarrow C$ that associates a consequence with each action (a) and each state (ω).
- e.g., $A = \{a_1, a_2\}$, $\Omega = \{\omega_1, \omega_2\}$. Then by $g : A \times \Omega \rightarrow C$ we have $C = \{g(a_1, \underset{}{\overset{\omega \equiv (\omega_1, \omega_2)}{\overline{\omega}}}), g(a_2, \omega)\}$, where $g(a_1, \omega) = \underbrace{((a_1, \omega_1), (a_1, \omega_2))}_{(g(a_1,\omega_1),\ g(a_2,\omega_2))\text{로 쓰면 좀 더 정확함}}$ and and $g(a_2, \omega) = ((a_2, \omega_1), (a_2, \omega_2))$.
- Suppose Ω is the support of p and denoted by $supp(p)$, and each $\omega \in \Omega = supp(p)$, $p(\omega) \geq 0$ and $\Sigma_{\omega \in supp(p)} p(w) = 1$. The set of simple probability distributions on X will be denoted by P.
- 각각의 State가 시행될 확률 모두 계산 → Lottery 만듦
  With $p = (p(\omega_1), p(\omega_2)) \in P$, we can define each consequence as a lottery, $L(g(a, \omega) \mid \underline{p(\omega)})$, and the set of Lottery, L as a modified the set of consequences.
  복권을 삼 / 당첨금 ₩100, ₩50, ₩0 / ₩100, ₩50, ₩0 당첨될 확률
- Can we define $u : L \rightarrow R$ such that $u(L_1) \geq u(L_2)$ iff $L_1 \succsim L_2$? See Expected Utility Theorem.
  Von Neumann – Morgenstern utility function

a와 a'의 P_1, P_2는 같아야함 (p는 상수)
bayes' rule ← assign하는 p가 적절한가? 적절하게 업데이트 하고 있나? 어떤 P를 assign해도 괜찮음 (주관적이어도)
Newcomb's paradox ← assign과 관련된 유명한 paradox



Choice의 문제 → 수치로 표현하고 싶다면
↑
둘 중 어떤 복권을 살까?

A model of rational choice under strategic interaction

- A set N of individuals who participate in the interaction.
- A set A_i of actions from which decision maker i makes a choice.
- A set $A = \Pi_{j \in N} A_j$ is the set of the action profiles.

  $A_1 = \{a_1, a_2, a_3\}$
  $A_2 = \{b_1, b_2, b_3\} \rightarrow (a_1, b_1, c_1), (a_1, b_1, c_2), \cdots$
  $A_3 = \{c_1, c_2, c_3\}$
  하나하나가 action profile

- A set C of possible consequences of these action profiles.

  action profile set 이지만 action profile을 선택하는 것이 아님 action을 선택함 → rational choice 모형이 잘 맞는지 벌써 문제됨

- A consequence function $g : A \rightarrow C$ that associates a consequence with each action profile.

  중요한 문제이지만 게임이론 하는 사람들은 크게 신경안씀

- A preference relation (a complete, transitive, reflexive binary relation) of individual i, $\succsim_i$, on the set C. Or sometimes the decision maker's preferences are specified by giving a utility function $u_i : C \rightarrow R$, which defines a preference relation $\succsim_i$ by the condition $x \succsim_i y$ iff $u_i(x) \geq u_i(y)$.

A model of rational choice under empathetic preferences

- A set N of individuals who participate in the interaction.
- A set A_i of actions from which decision maker i makes a choice.
- A set $A = \Pi_{j \in N} A_j$ is the set of the action profiles.
- A set C of possible consequences of these action profiles.
- A consequence function $g : A \rightarrow C$ that associates a consequence with each action profile.
- A preference relation (a complete, transitive, reflexive binary relation) of individual i, $\succsim_i$, on the set $C \times N$. For example, for $x, y \in C$ and $m, n \in N$, individual i's preference can be either $(x, m) \succsim_i (y, n)$ or $(y, n) \succsim_i (x, m)$ or both. (or between (x, m) and (x, n)).

preference relation

지금 이 백신을 사람1,2
누구에게 주는 게 나은지
비교할 수 있게 됨

모든 결과에 사람 꼬리표가 붙음

- Or sometimes individual i's empathetic preferences are specified by giving a utility function $v_i : C \times N \rightarrow R$, which defines a preference relation $\succsim_i$ by the condition $(x, m) \succsim_i (y, n)$ iff $v_i(x, m) \geq v_i(y, n)$.

Basic Elements of Noncooperative Games

What is a game?

CS에서 simulation할 때 4가지 모두 정의해두면 그 다음부터는 편함

- To describe a situation of strategic interaction, we need to know four things:
  - i) The players: Who is involved?

    순차적으로 움직이지만 상대방의 선택을 모름 ...... simultaneous game

    행동을 하는 그 시점에 무엇을 알고있는지 ...... 상대 결과의 일부라도 내가 알고 움직임 ...... sequential game

  - ii) The rules: Who moves when? What do they know when they move? What can they do?

    action profiles

  - iii) The outcomes: For each possible set of ~~actions~~ by the players, what is the outcome of the game?
  - iv) The payoffs: What are the players' preferences (i.e., utility functions) over the possible outcomes?

What is a game?

- Example. Matching Pennies
  - *Players*: There are two players, 1 and 2.
  - *Rules*: Each player simultaneously puts a penny down, either heads up or tails up.
  - *Outcomes*: Two pennies match or do not match.
  - *Payoffs*: If the two pennies match (either both heads up or both tails up), player 1 pays 1 dollar to player 2: otherwise, player 2 pays 1 dollar to player 1.

player 2

		H	T
player 1	H	-1, 1	1, -1
	T	1, -1	-1, 1

(normal form)

What is a game?

- Meeting in New York
  - *Players*: Two players, Mr. Thomas and Mr. Schelling.
  - *Rules*: The two players are separated and cannot communicate. They are supposed to meet in New York City at noon for lunch but have forgotten to specify where. Each must decide where to go (each can make only one choice).
  - *Outcomes*: If they meet each other, they get to enjoy each other's company at lunch. Otherwise, they must eat alone.
  - *Payoffs*: They each attach a monetary value of 100 dollars to the other’s company (their payoffs are each 100 dollars if they meet, 0 dollars if they do not).

		Schelling	
		A	B
Tomas	A	100, 100	0, 0
	B	0, 0	100, 100

(normal form)

What is a game?

- Payoff 경제학에서 payoff는 모호해서 정하는 것이 문제임

  숫자가 의미하는 바가 무엇인가?

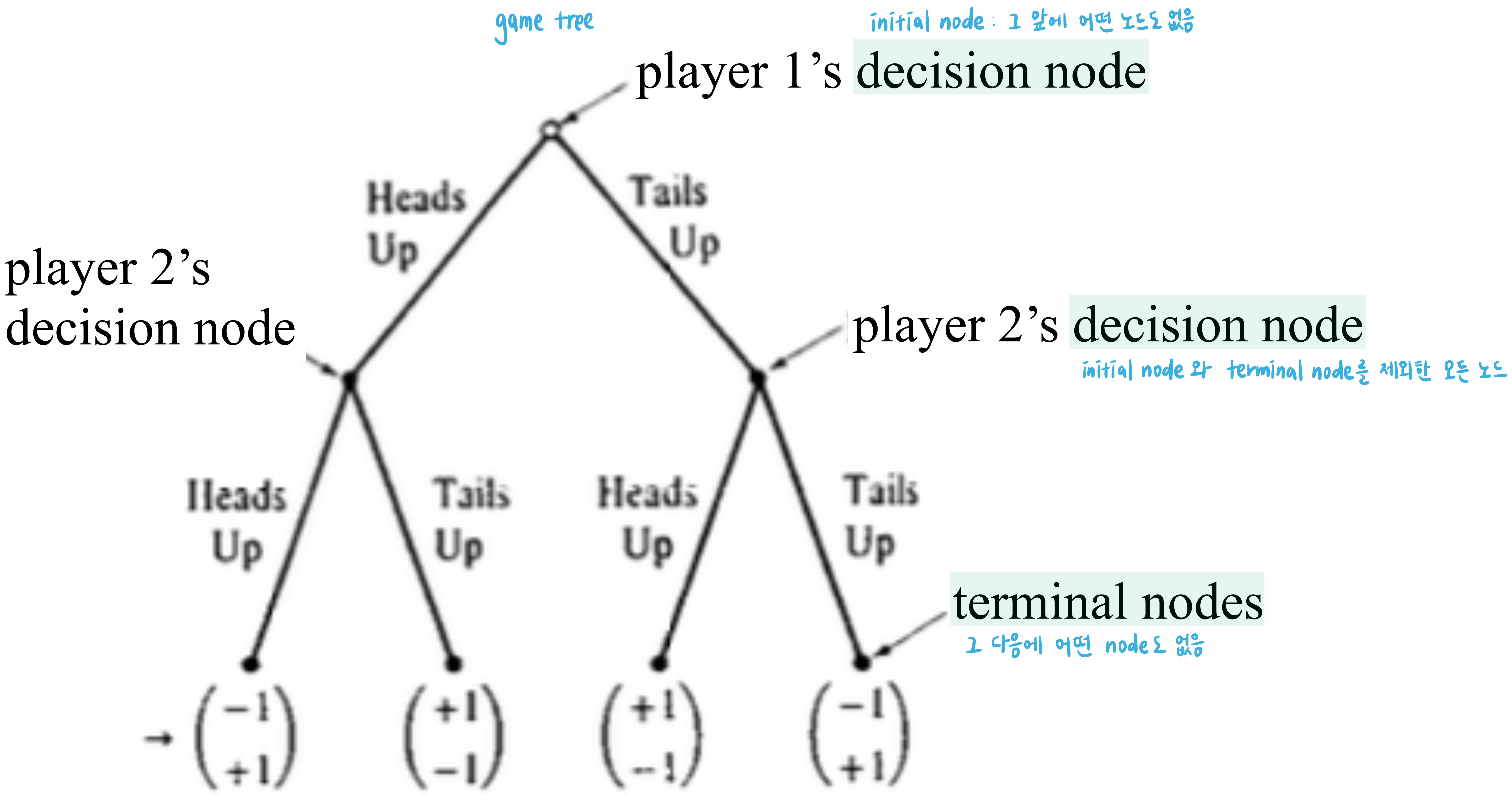
  숫자를 어떻게 부여해야 하나?

  preference를 utility function으로 나타낼 때 utility를 서수적으로만 고려하면 되나? .... 그래서 expected utility function 쓰는 게 이론적으로 문제가 있긴 함

  - As a general matter, we describe a plaver's preferences by a utility function that assigns a utility level for each possible outcome. It is common to refer to the player's utility function as her *payoff* function and the utility level as her payoff. Throughout, we assume that these utility functions take an expected utility form so that when we consider situations in which outcomes are random, we can evaluate the random prospect by means of the player's expected utility.

The extensive form representation of a game.

- Matching Pennies v.B and its extensive form. Version B is identical to Matching Pennies (see Example 7.B.I) except that the two players move sequentially, rather than simultaneously.



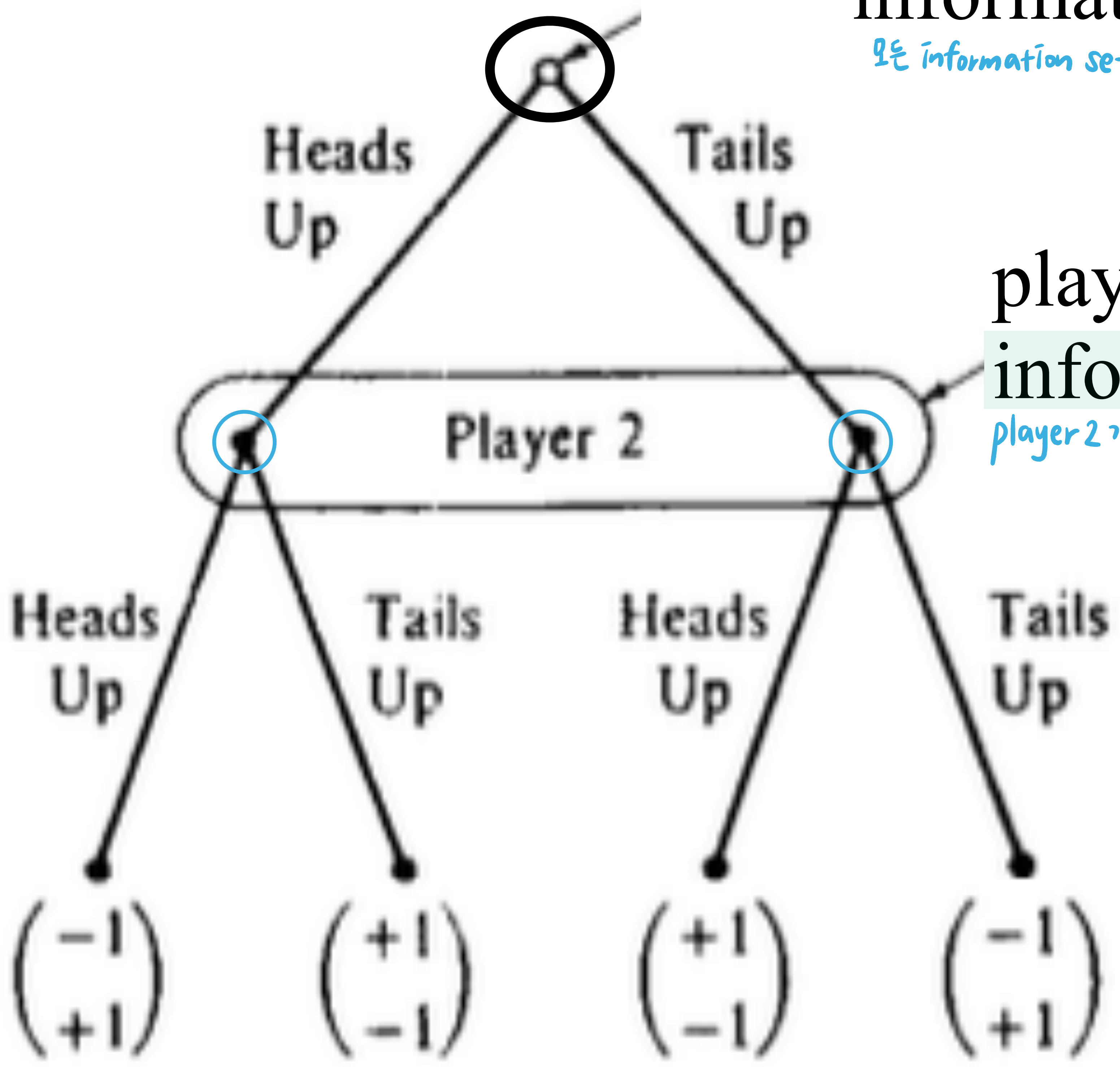
The extensive form representation of a game.

- Matching Pennies v.C and its extensive form. This version is just like Matching Pennies Version B except that when player I puts her penny down, she keeps it covered with her hand. Hence, player 2 cannot see player l's choice until after player 2 has moved.

순차적으로 움직였지만 상대의 선택을 모름 → simultaneous game

player 1's decision node and information set (singleton)

모든 information set이 노드가 하나이면 player가 모르는 것은 없음

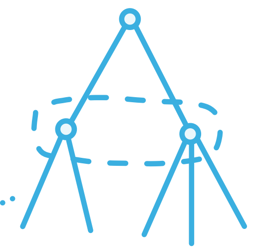


player 2's information set

player2가 두 노드를 구분할 수 없음

Player 2's two decision nodes to indicate that these two nodes are in a single information set, meaning that this information set is that when it is player 2's turn to move, she cannot tell which of these two nodes she is.

Two natural **restrictions on information sets**:

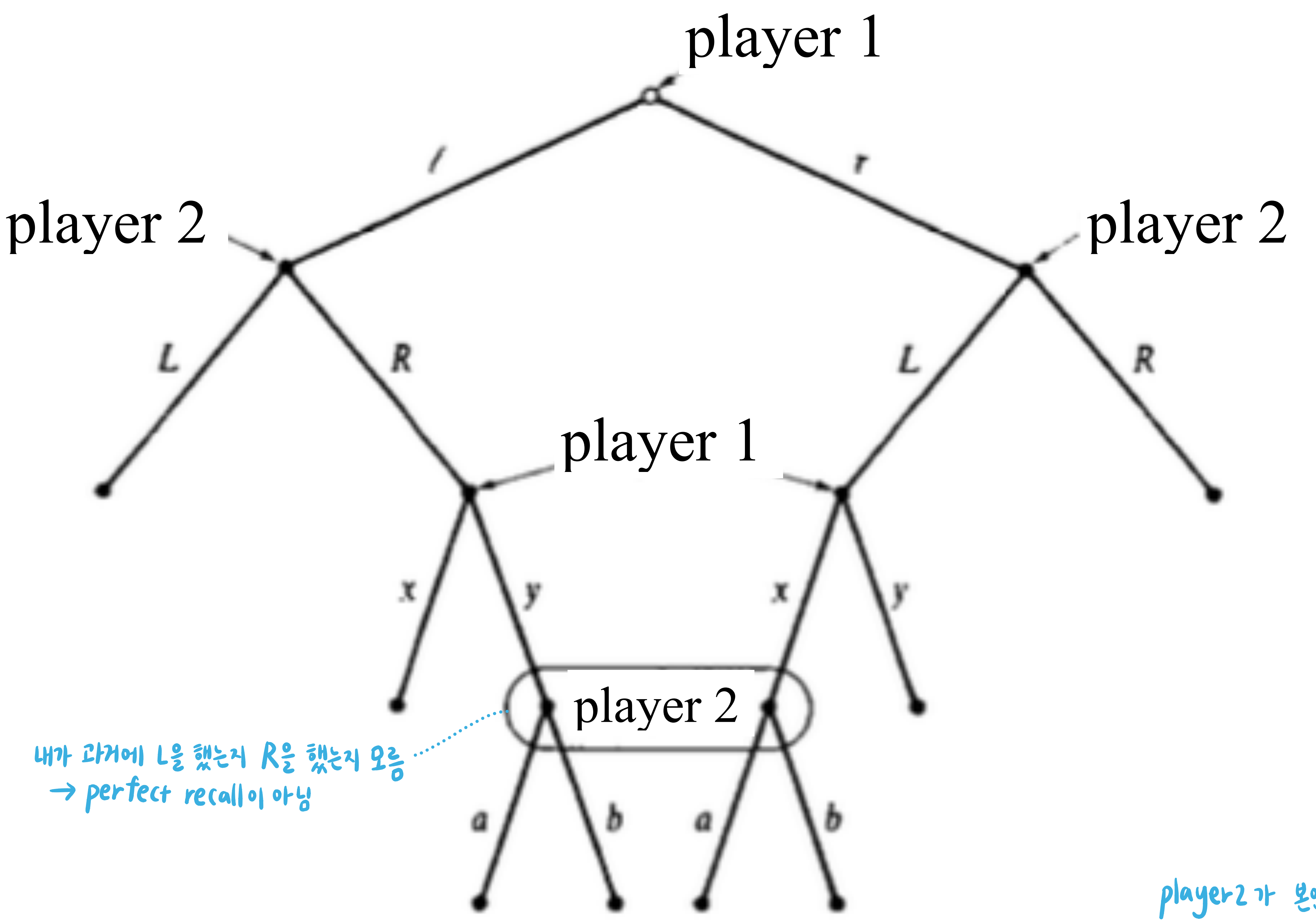


이런 그림은 그려질 수 없음

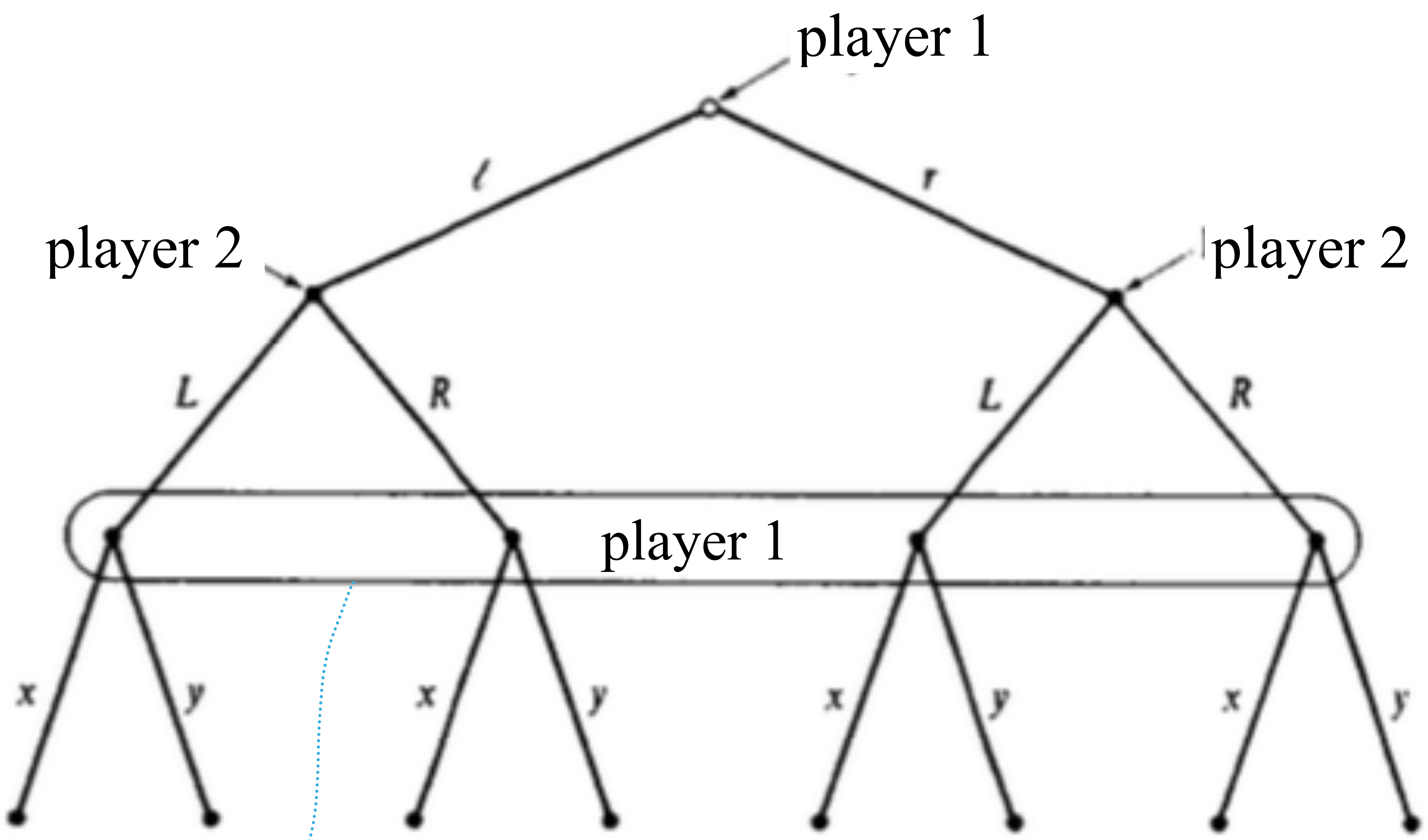
(1) At every node within a given information set, a player must have the same set of possible actions.

(2) Players possess what is known as *perfect recall*, meaning that a player does not forget what she once knew, including her own actions.

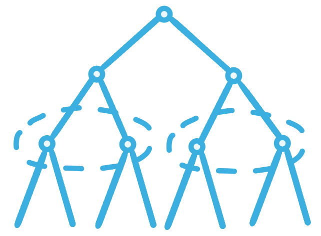
The extensive form representation of a game.



내가 과거에 L을 했는지 R을 했는지 모름
→ perfect recall이 아님



player2가 본인이 L을 했는지 R을 했는지 모름
→ perfect recall이 아님



이건 perfect recall

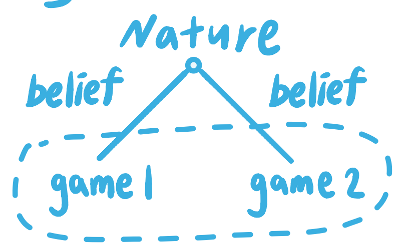
과거에 알았던 건 현재에도 알아야함 보통 perfect recall은 성립함

- The games not satisfying perfect recall, meaning that a player does not forget what she once knew, including her own actions.

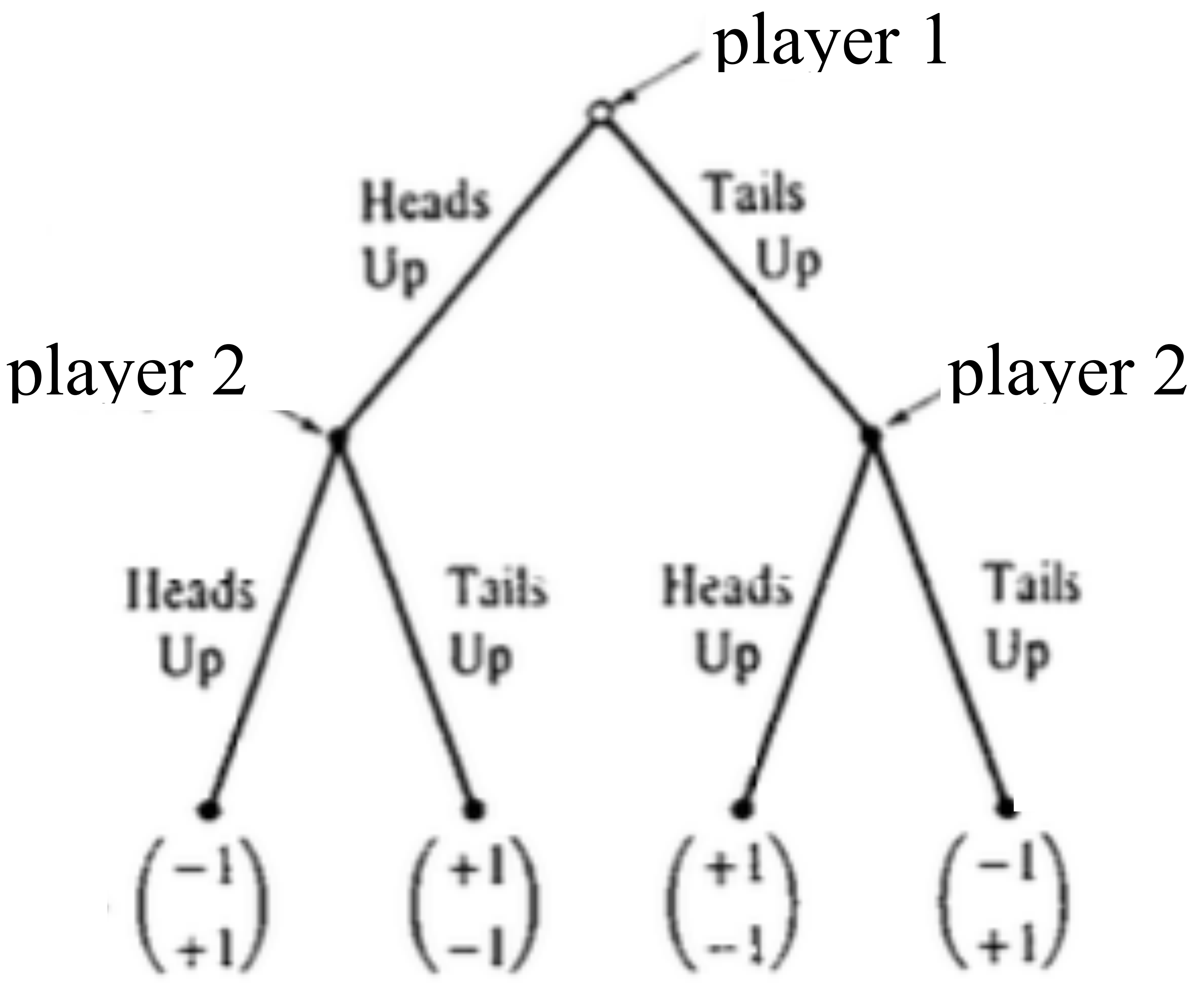
The extensive form representation of a game.

player1은 상대의 정보를 모두 앎 → complete information
player2는 상대의 정보를 모름 → incomplete information
(player2는 belief를 가짐)

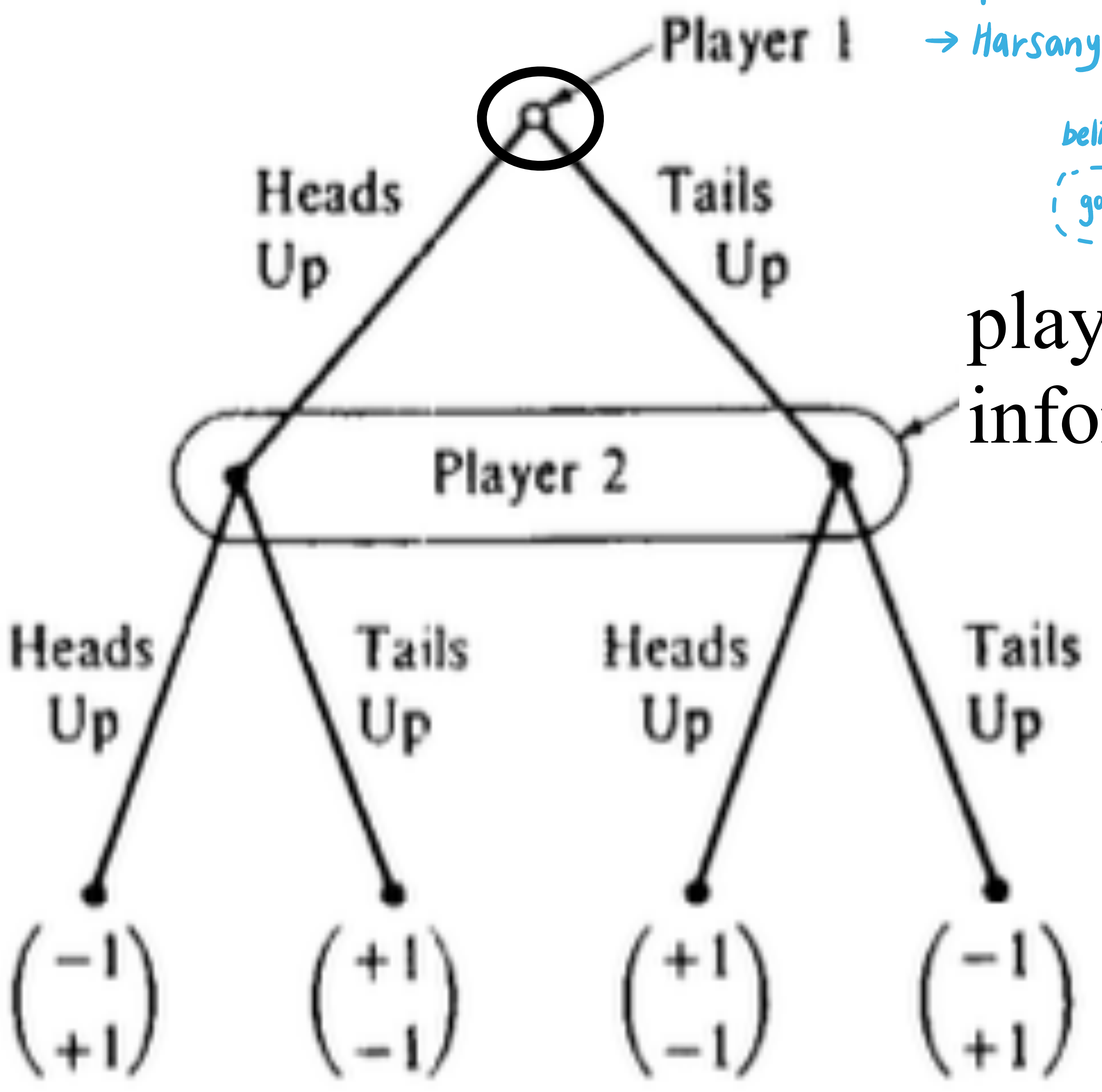
incomplete information game을 "moves by nature"를 도입해서
imperfect information game으로 전환해서 풀 수 있음
→ Harsanyi transformation



perfect information을 가진 game
=information set에는 single decision node만 있음



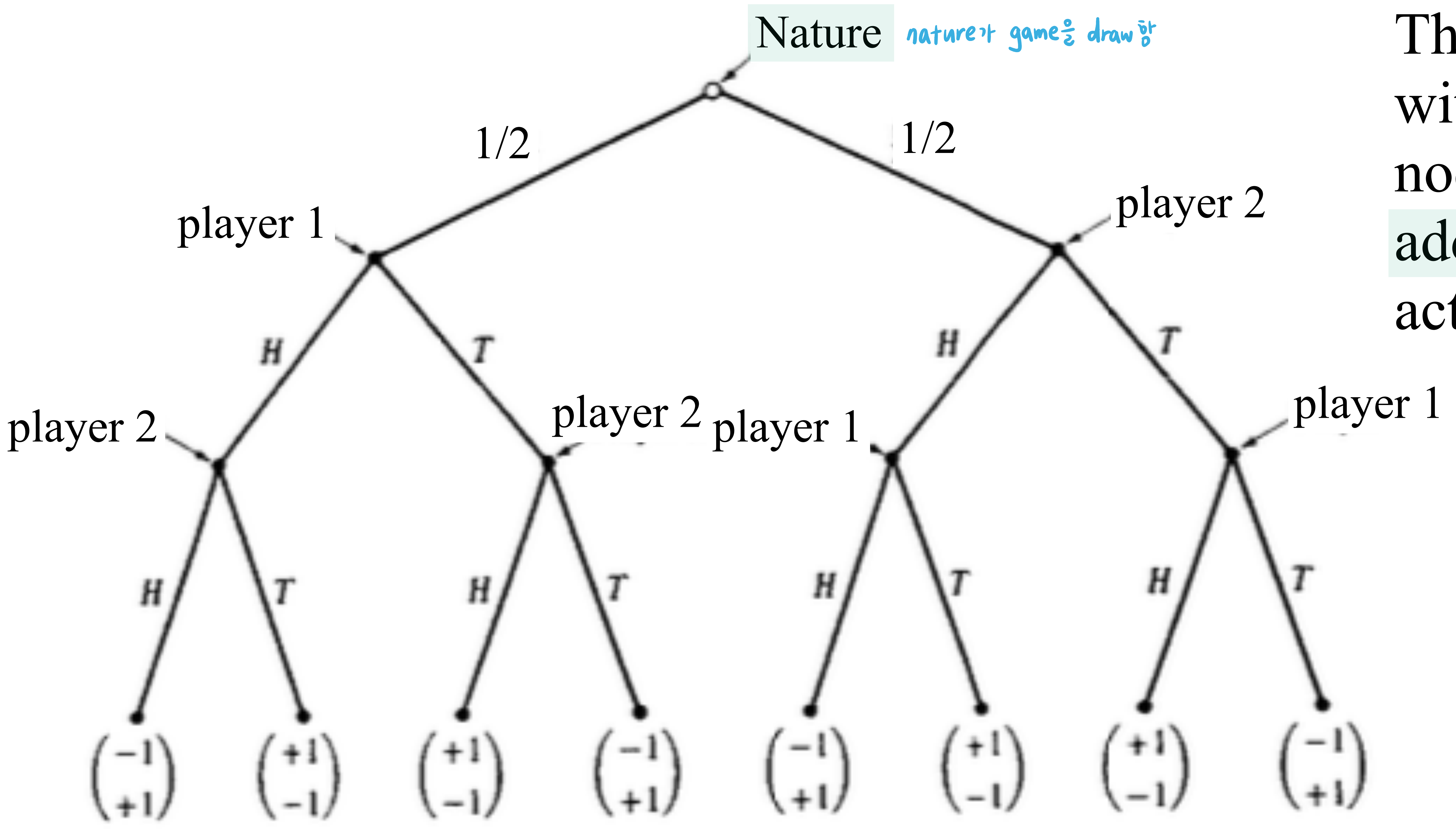
imperfect information을 가진 game



- The game on the left is of perfect information. When it is a player's turn to move, she is able to observe all her rival's previous moves.
- **Definition**. A game is one of *perfect information* if each information set contains a single decision node. Otherwise, it is a game of imperfect information.

The extensive form representation of a game.

- Matching Pennies v.D and its extensive form. Suppose that prior to playing Matching Pennies Version B, the two players flip a coin to see who will move first. Thus, with equal probability either player 1 will put her penny down first, or player 2 will.



This game is depicted as beginning with a move of nature at the initial node. *Nature* in this game is an additional player who must play its two actions with fixed probabilities.

The extensive form representation of a game.

Formally, a game represented in extensive form consists of the following items:

(i) A finite set of nodes $\mathscr{X}$, a finite set of possible actions $\mathscr{A}$, and a finite set of players $\{1\dots, I\}$.

node와 node를 mapping하는 function

initial node는 선행노드가 없음

(ii) A function $p : \mathscr{X} \rightarrow \{\mathscr{X} \cup \varnothing\}$ specifying a single immediate predecessor of each node x; $p(x)$ is nonempty for all $x \in \mathscr{X}$ but one, designated as the initial node x_0. The immediate successor nodes of x are then $s(x) = p^{-1}(x)$, and the set of all predecessors and all successors of node x can be found by iterating $p(x)$ and $s(x)$. To have a tree structure, we require that these sets be disjoint (a predecessor of node x

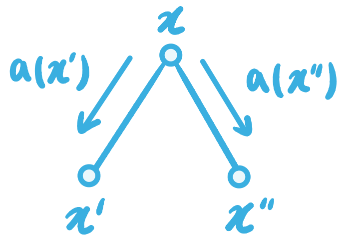
엄밀히 말하면 함수가 아님 후행노드는 여러개일 수 있음

cannot also be a successor to it). The set of terminal nodes is $T = \{x \in \mathscr{X} : s(x) = \varnothing\}$. All other nodes $\mathscr{X} \backslash T$ are known as decision nodes.

모든 노드집합에서 terminal node를 뺌

(iii) A function $\alpha : \mathscr{X} \backslash \{x_0\} \rightarrow \mathscr{A}$ giving the action that leads to any noninitial node x from its immediate predecessor $p(x)$ and satisfying the property that if $x', x'' \in s(x)$ and $x' \neq x''$, then $\alpha(x') \neq \alpha(x'')$. The set of choices available at decision node x is $c(x) = \{a \in \mathscr{A} : a = \alpha(x') \text{ for some } x' \in s(x)\}$.

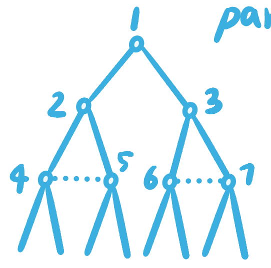
어떤 결과가 있으면 (여기서 결과는 node) 그 결과를 가능하게 하는 action을 찾는 함수



당연히 action 다름

node x에서 가능한 action set (choice)

The extensive form representation of a game.



partition: (1)(2)(3)(4 5)(6 7)

x와 x'이 같은 information set에 속해있다면
x, x'에서 가능한 choices 같아야함
→ Strategy는 information set에서 나오는 것

node를 information set으로 연결하는 함수

(iv) A collection of information sets $\mathscr{H}$, and a function $H : \mathscr{X} \to \mathscr{H}$ assigning each decision node x to an information set $H(x) \in \mathscr{H}$. Thus, the information sets in $\mathscr{H}$ form a partition of $\mathscr{X}$. We require that all decision nodes assigned to a single information set have the same choices available; formally, $c(x) = c(x')$ if $H(x) = H(x')$. We can therefore write the choices available at information set H as $C(H) = \{a \in \mathscr{A} : a \in c(x) \text{ for } x \in H\}$. information set을 기준으로 choice 정의

information set을 각 player에게 할당하는 함수

(v) A function $\iota : \mathscr{H} \to \{0,\ldots,I\}$ assigning each information set in $\mathscr{H}$ to the player (or to nature: formally, player 0) who moves at the decision nodes in that set. We can denote the collection of player i's information sets by $\mathscr{H}_i = \{H \in \mathscr{H} : i = \iota(H)\}$. i가 포함된 information set을 찾음 (역함수)

nature가 play하게 되는 information set의 collection

(vi) A function $\rho : \mathscr{H}_0 \times \mathscr{A} \to [0,1]$ assigning probabilities to actions at information sets where nature moves and satisfying $\rho(H, a) = 0$ if $a \notin C(H)$ and $\Sigma_{a \in C(H)} \rho(H, a) = 1$ for all $H \in \mathscr{H}_0$.

nature가 draw하는 probability 합은 1

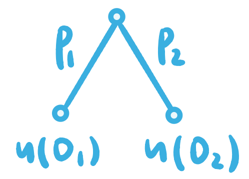
(vii) A collection of payoff functions $u = \{u_1(\cdot), \ldots, u_I(\cdot)\}$ assigning utilities to the players for each terminal node that can be reached, $u_i : T \to R$. Because we want to allow for a random realization of outcomes we take each $u_i(\cdot)$ to be a Bernoulli utility function.

The extensive form representation of a game.

- Thus, formally, a game in extensive form is specified by the collection
$$\Gamma_E = \{\mathscr{X}, \mathscr{A}, p(\cdot), \alpha(\cdot), \mathscr{H}, H(\cdot), \iota(\cdot), \rho(\cdot), u\}.$$

node set | action set | predecessor | choices | collection of information set | information set | assigning information set to player | nature draw | utility function

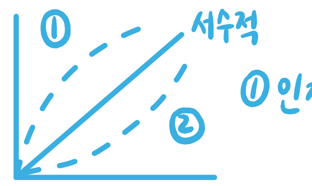
Von Neumann utility function



$EU = p_1 \cdot u(O_1) + p_2 \cdot u(O_2)$

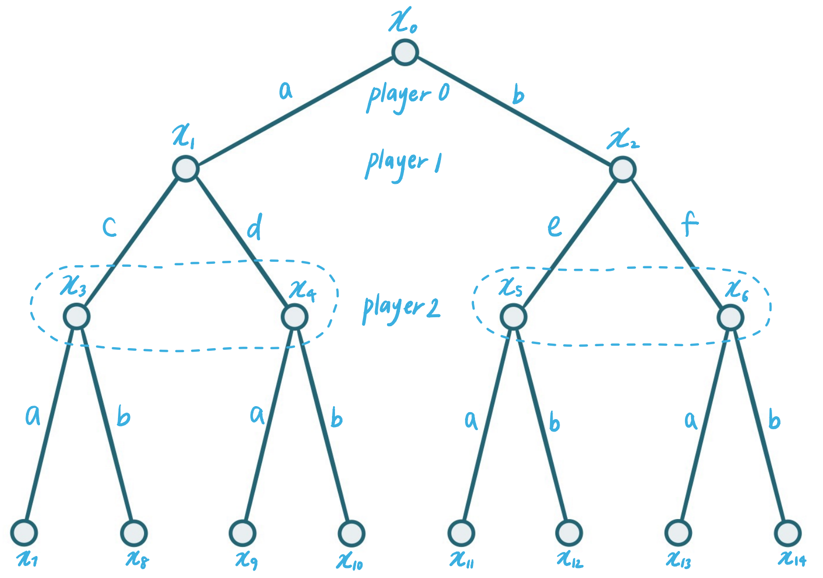
0.0	0.0
0.0	0.0

payoff 각각은 Bernoulli utility function
반은 기수적이고 반은 서수적임
= 곡률이 중요함



①인지 ②인지 중요함

(i) A finite set of nodes $\mathscr{X}$, a finite set of possible actions $\mathscr{A}$, and a finite set of players $\{1,\ldots, I\}$.



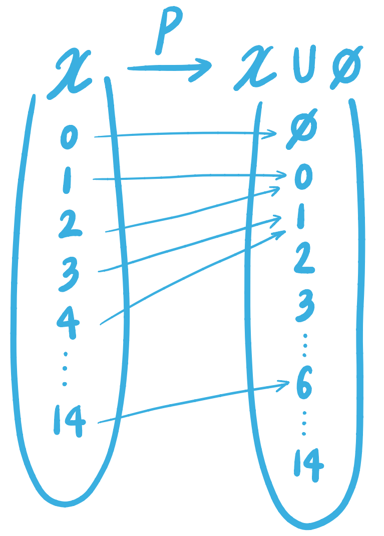
node set $\mathcal{X} = \{x_0, x_1, x_2, \cdots, x_{14}\}$

action set $\mathcal{A} = \{a, b, c, d, e, f\}$

player set $I = \{0, 1, 2\}$

(ii) A function $p : \mathscr{X} \to \{\mathscr{X} \cup \varnothing\}$ specifying a single immediate predecessor of each node x; $p(x)$ is nonempty for all $x \in \mathscr{X}$ but one, designated as the initial node x_0. The immediate successor nodes of x are then $s(x) = p^{-1}(x)$, and the set of all predecessors and all successors of node x can be found by iterating $p(x)$ and $s(x)$. To have a tree structure, we require that these sets be disjoint (a predecessor of node x wlr

cannot also be a successor to it). The set of terminal nodes is $T = \{x \in \mathscr{X} : s(x) = \varnothing\}$. All other nodes $\mathscr{X} \backslash T$ are known as decision nodes.



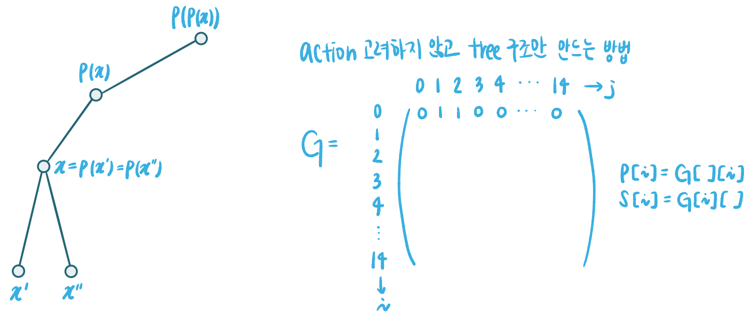
$P = \{P(0), P(1), P(2), \cdots, P(14)\}$

$= \{\varnothing, 0, 0, 1, 1, 2, 2, \cdots, 6, 6\}$

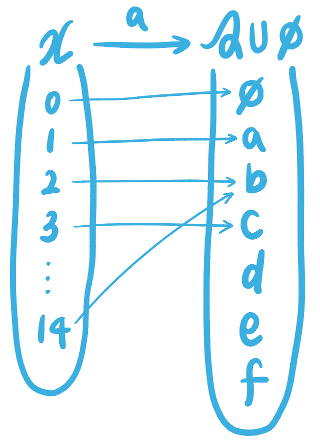
$S = \{S(0), S(1), S(2), \cdots, S(14)\}$

$= \{P^{-1}(0), P^{-1}(1), P^{-1}(2), \cdots, P^{-1}(14)\}$

$= \{(1,2), (3,4), (5,6), \cdots, (13,14), \varnothing, \varnothing, \cdots, \varnothing\}$



(iii) A function $\alpha : \mathscr{X}\backslash\{x_0\} \to \mathscr{A}$ giving the action that leads to any noninitial node x from its immediate predecessor $p(x)$ and satisfying the property that if $x', x'' \in s(x)$ and $x' \neq x''$, then $\alpha(x') \neq \alpha(x'')$. The set of choices available at decision node x is $c(x) = \{a \in \mathscr{A} : a = \alpha(x')$ for some $x' \in s(x)\}$.



$a = \{\varnothing, a, b, c, d, e, f, a, b, \cdots, a, b\}$

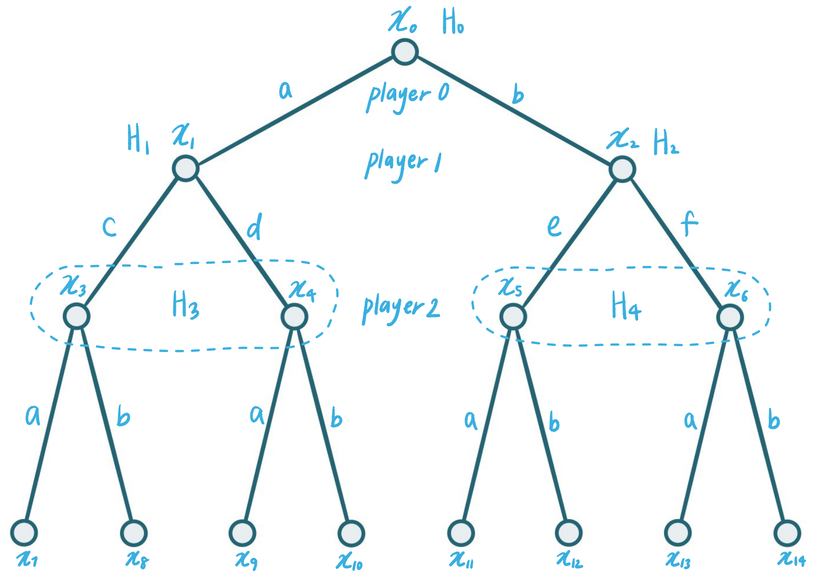
$x \in D$ decision node

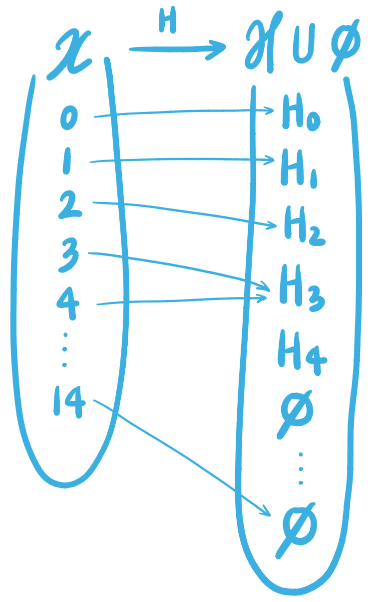
$C(x) = \{C(0), C(1), C(2), \cdots, C(6)\}$

$= \{(a(1), a(2)), (a(3), a(4)), \cdots, (a(13), a(14))\}$

$= \{(a,b), (c,d), (e,f), (a,b), \cdots, (a,b)\}$

(iv) A collection of information sets $\mathscr{H}$, and a function $H : \mathscr{X} \to \mathscr{H}$ assigning each decision node x to an information set $H(x) \in \mathscr{H}$. Thus, the information sets in $\mathscr{H}$ form a partition of $\mathscr{X}$. We require that all decision nodes assigned to a single information set have the same choices available; formally, $c(x) = c(x')$ if $H(x) = H(x')$. We can therefore write the choices available at information set H as $C(H) = \{a \in \mathscr{A} : a \in c(x)$ for $x \in H\}$.





$H = \{H(0), H(1), H(2), \cdots, H(14)\}$

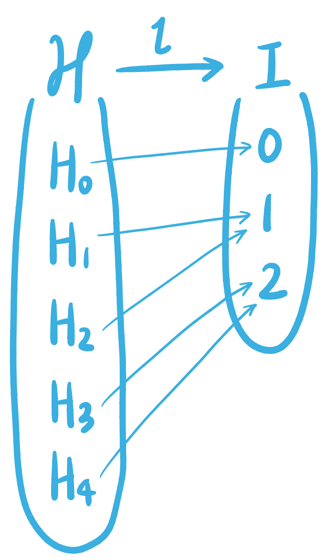
$= \{H_0, H_1, H_2, H_3, H_3, H_4, H_4, \varnothing, \varnothing, \cdots, \varnothing\}$

$C(H) = \{C(H_0), C(H_1), C(H_2), C(H_3), C(H_4)\}$

$C(x) = \{C(0), C(1), C(2), C(3), C(4), C(5), C(6)\}$

$= \{(a,b), (c,d), (e,f), (a,b), (a,b), (a,b), (a,b)\}$

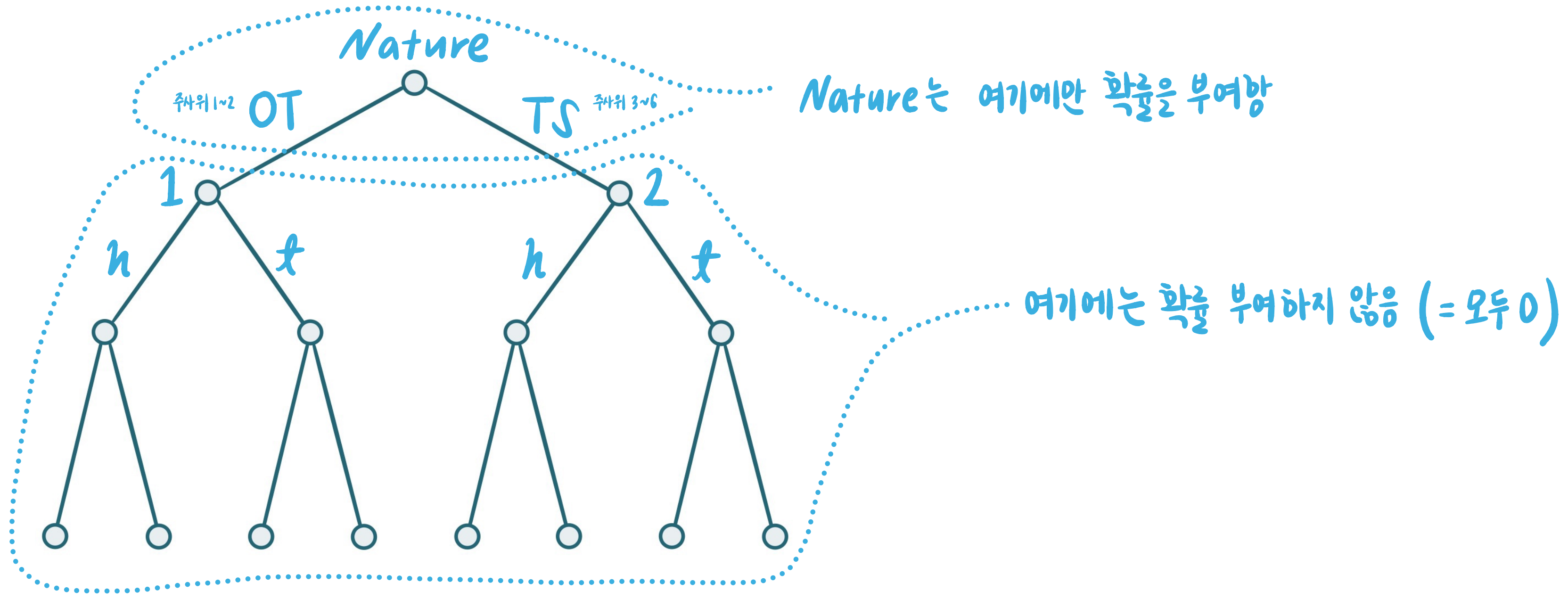
(v) A function $\iota : \mathscr{H} \to \{0,\ldots, I\}$ assigning each information set in $\mathscr{H}$ to the player (or to nature: formally, player 0) who moves at the decision nodes in that set. We can denote the collection of player i's information sets by $\mathscr{H}_i = \{H \in \mathscr{H} : i = \iota(H)\}$.



$\iota = \{\iota(H_0), \iota(H_1), \iota(H_2), \iota(H_3), \iota(H_4)\}$

$= \{0, 1, 1, 2, 2\}$

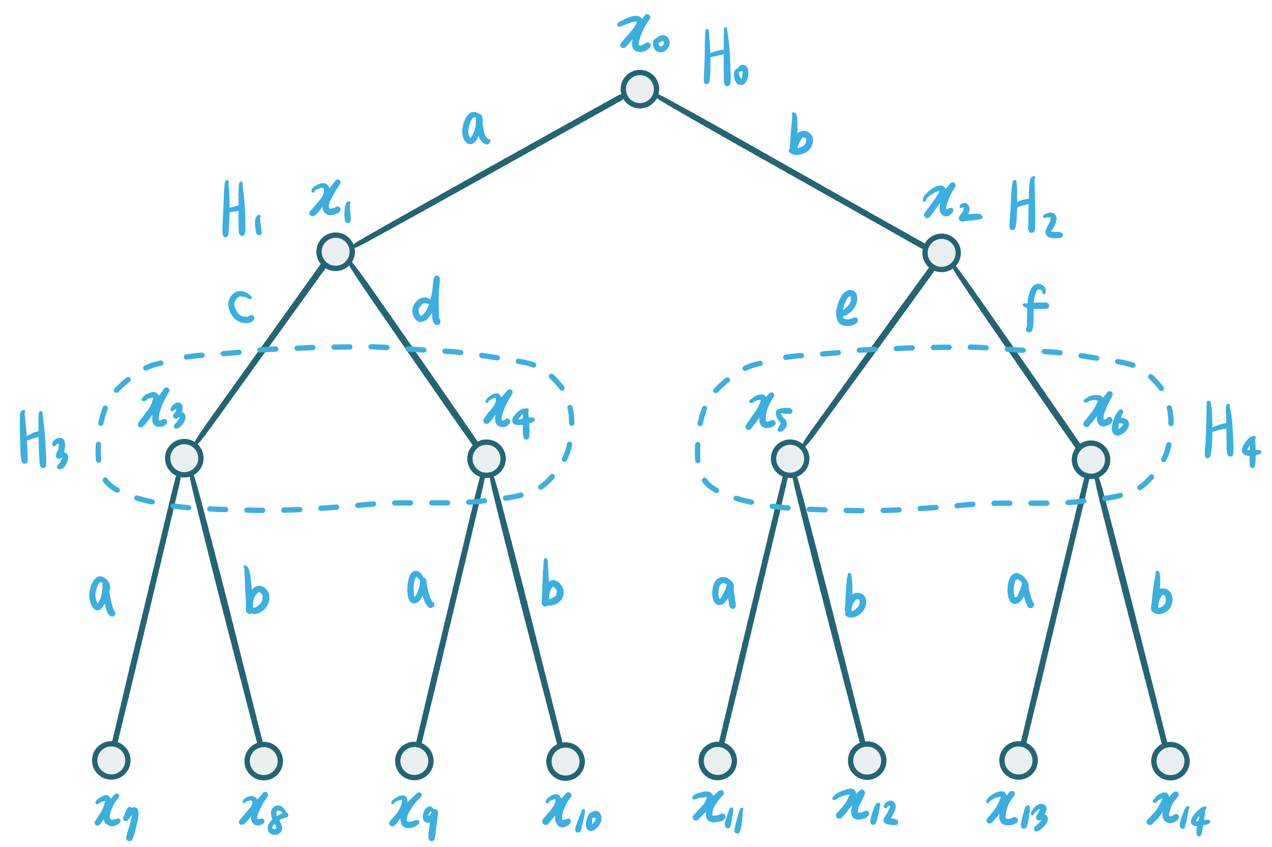
(vi) A function $\rho : \mathscr{H}_0 \times \mathscr{A} \to [0,1]$ assigning probabilities to actions at information sets where nature moves and satisfying $\rho(H, a) = 0$ if $a \notin C(H)$ and $\Sigma_{a\in C(H)}\rho(H, a) = 1$ for all $H \in \mathscr{H}_0$.



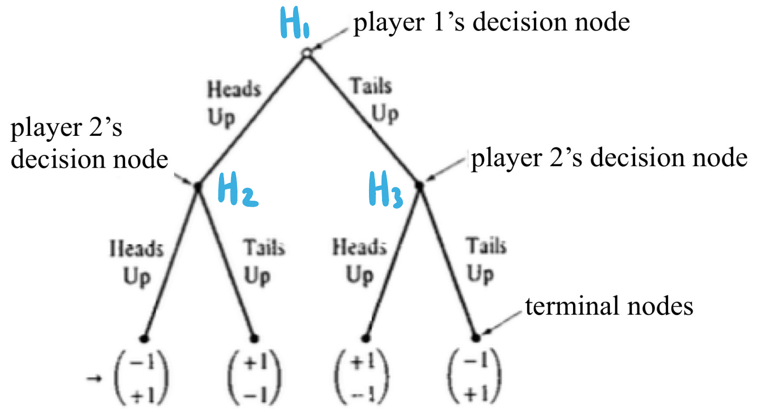
Strategies and the normal form representation of a game

- **Definition.** Let $\mathscr{H}_i$ denote the collection of player i's information sets, $\mathscr{A}$ the set of possible actions in the game, and $C(H) \subset \mathscr{A}$ the set of actions possible at information set H. A (*pure*) *strategy* for player i is a function $s_i : \mathscr{H}_i \to \mathscr{A}$ such that $s_i(H) \in C(H)$ for all $H \in \mathscr{H}_i$.

C(H)에 속한 action 중에서 선택



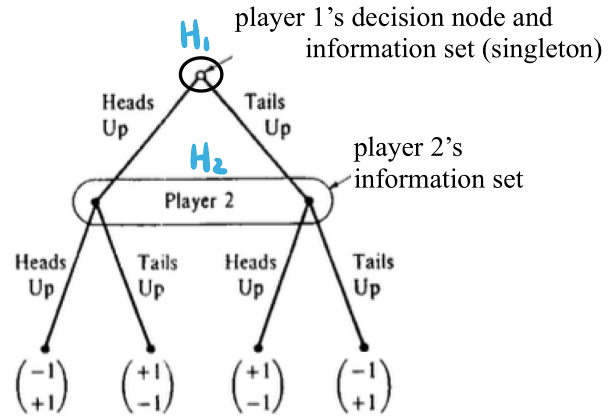
- Example. Strategies in Matching Pennies Version B



$\mathscr{H}_1 = \{H_1\}$ $\quad S_1 = (h, t)$

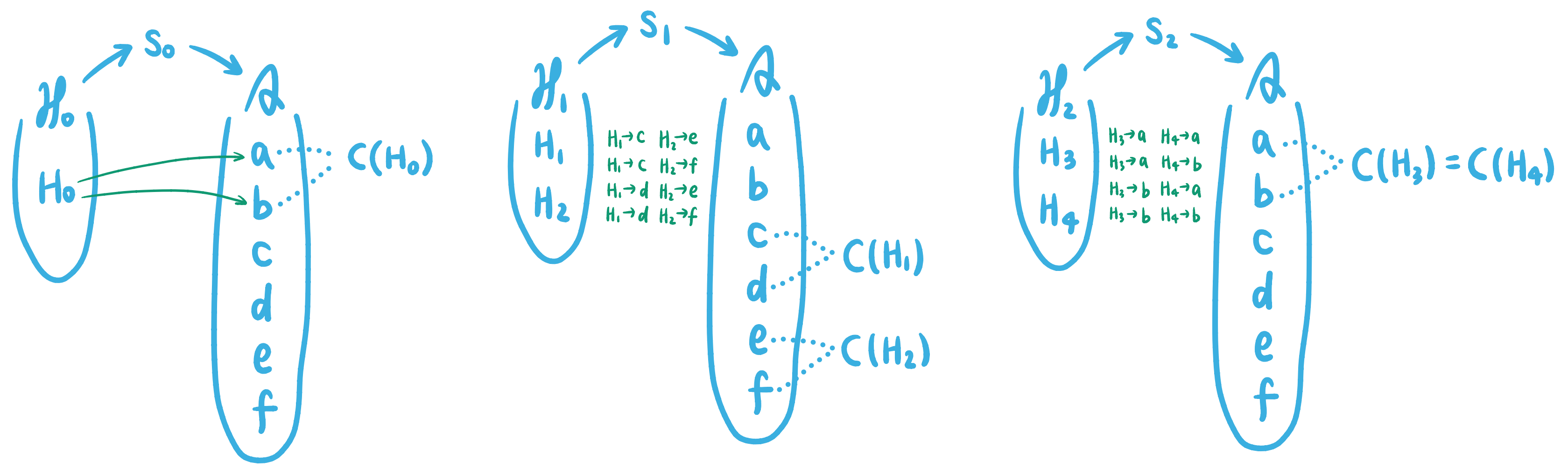
$\mathscr{H}_2 = \{H_2, H_3\}$ $\quad S_2 = (hh, ht, th, tt)$ $\quad$ H_2 H_3

- Example. Strategies in Matching Pennies Version C

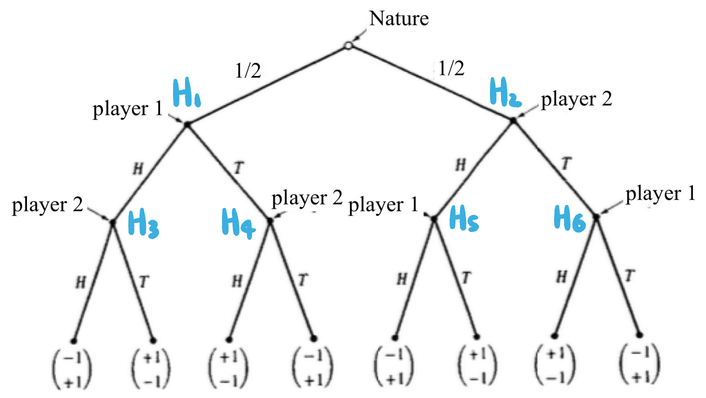


$\mathscr{H}_1 = \{H_1\}$ $\quad S_1 = (h, t)$

$\mathscr{H}_2 = \{H_2\}$ $\quad S_2 = (h, t)$



- Example. Strategies in Matching Pennies Version D



$\mathscr{H}_1 = \{H_1, H_5, H_6\}$ $\quad S_1 = (hhh, hth, \cdots, ttt)$

$\mathscr{H}_2 = \{H_2, H_3, H_4\}$ $\quad S_2 = (hhh, hth, \cdots, ttt)$

2^3 pure strategies

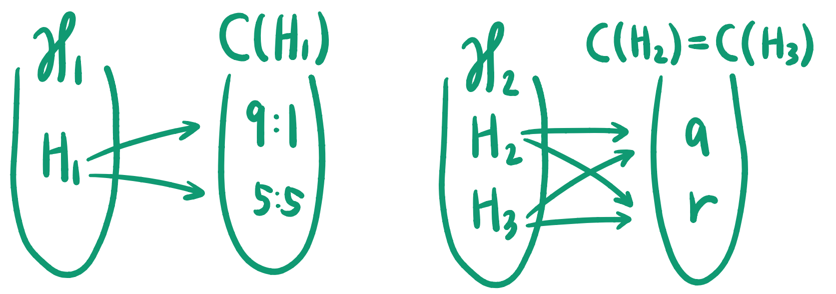
pure strategy가 정의되면 normal form으로 표현가능

Strategies and the normal form representation of a game

- **Definition.** For a game with I players, the normal form representation Γ_N specifies for each player i a set of strategies S_i (with $s_i \in S_i$) and a payoff function $u_i(s_1 \ldots, s_I)$ giving the von Neumann-Mrgenstern utiliy levels associated with the (possibly random) outcome arising from a strategy profile $(s_1, \ldots, s_I)$. Formally we write $\Gamma_N = \{I, \{S_i\}, \{u_i(\cdot)\}\}$.

이렇게 3요소로만 정의해도 extreme form (tree) 정의하는 것과 다르지 않음

-



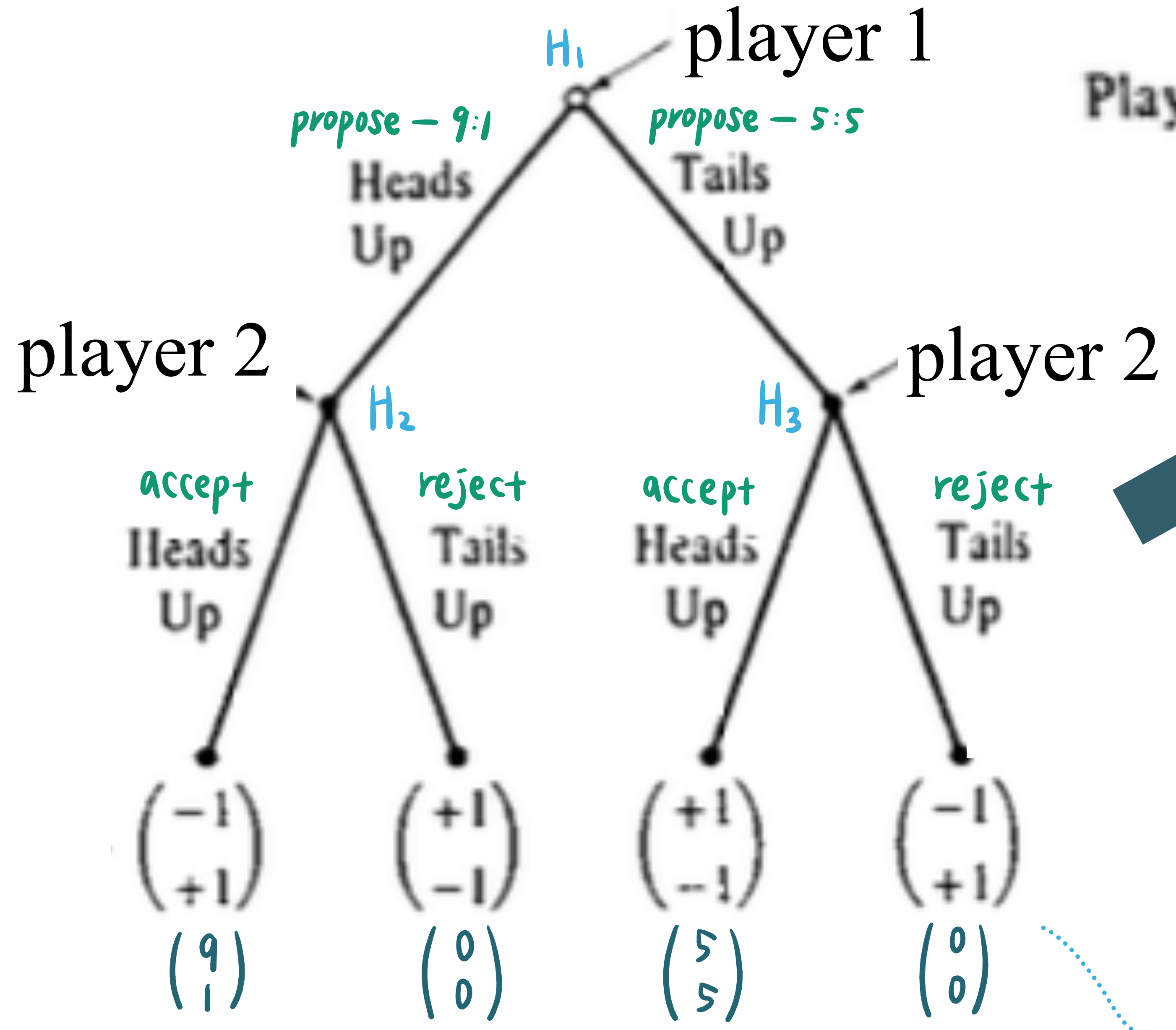
	aa	ar	ra	rr
9:1	9,1	9,1	0,0	0,0
5:5	5,5	0,0	5,5	0,0

$S_1 = (9:1,\ 5:5)$
$S_2 = (aa, ar, ra, rr)$

Player 2

	HH s_1	HT s_2	TH s_3	TT s_4
Player 1 H	−1, +1	−1, +1	+1, −1	+1, −1
Player 1 T	+1, −1	−1, +1	+1, −1	−1, +1

payoff matrix



$S_1 = (H, T)$
$S_2 = (HH, HT, TH, TT)$

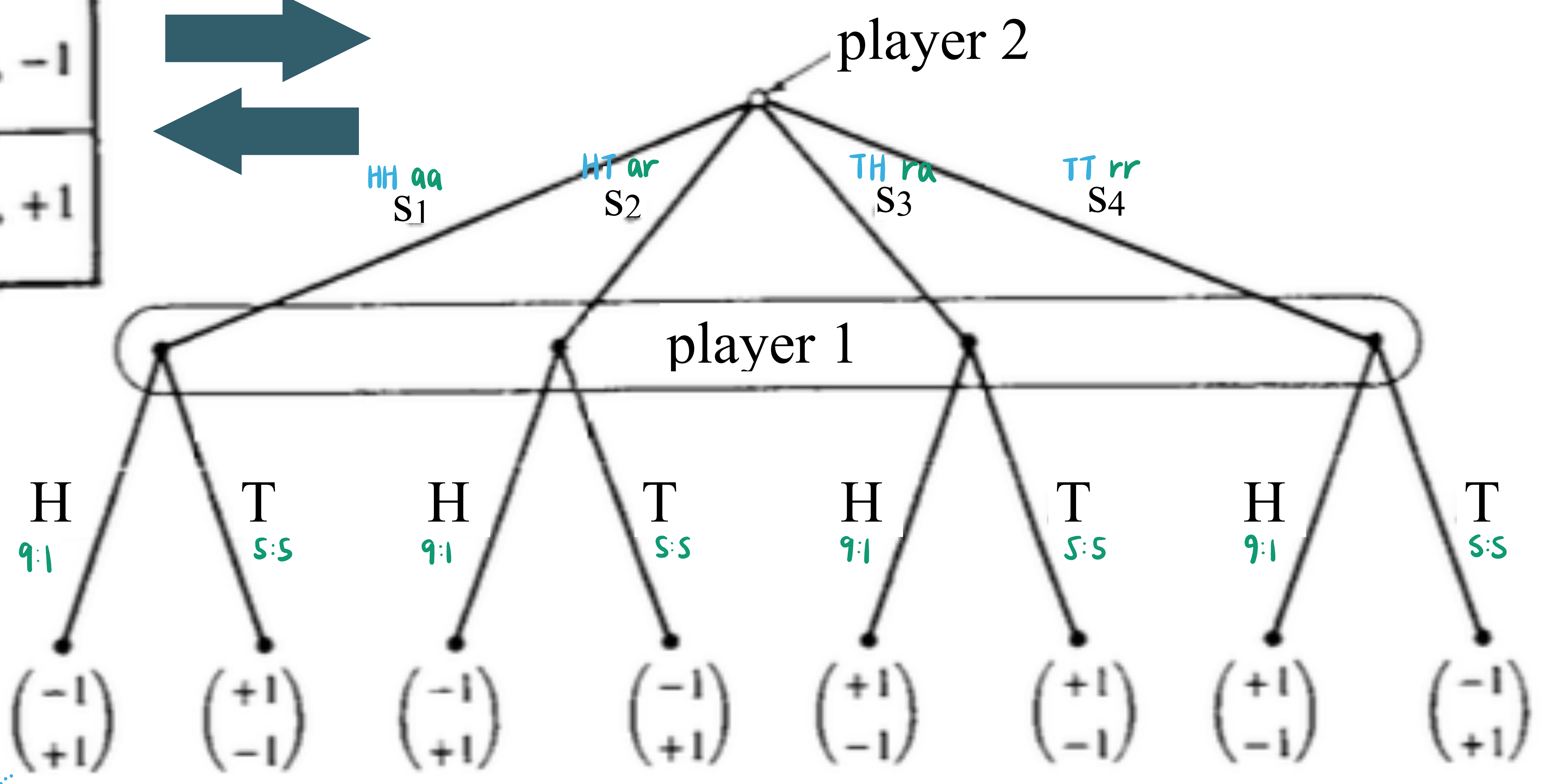
Strategy profile $s \in S_1 \times S_2 \equiv S$

$S = \{(H,HH), (H,HT), (H,TH), (H,TT), (T,HH), (T,HT), (T,TH), (T,TT)\}$

$\pi_1 : S \to \mathbb{R}$: $(H,HH) \to -1$, $(H,HT) \to -1$, ⋮, $(T,TT) \to -1$

$\pi_2 : S \to \mathbb{R}$: $(H,HH) \to 1$, $(H,HT) \to 1$, ⋮, $(T,TT) \to 1$

payoff matrix 그릴수 있음



전혀 다른 게임처럼 보이지만 결과적으로나 구조적으로나 동일함

Randomized choices

각각의 pure strategy를 얼마의 확률로 할지 probability distribution을 가지고 있으면 그 확률분포가 mixed strategy

ex) $S_1 = \{HH, HT, TH, TT\}$

$\sigma_1(HH), \sigma_2(HT), \sigma_3(TH), \sigma_4(TT) \xrightarrow{\text{SUM}} 1$

1/4 1/4 1/4 1/4 가능한 하나의 mixed strategy

연속된 공간이니깐 무수히 많은 mixed strategy 만들 수 있음

- **Definition.** Given player i's (finite) pure strategy set S_i, a mixed strategy for player i, $\sigma_i : S_i \to [0,1]$ assignes to each pure strategy $s_i \in S_i$, a probability $\sigma_i(s_i) \geq 0$ that it will be played, where $\Sigma_{s_i \in S_i} \sigma_i(s_i) = 1$.

- Suppose that player i has M pure strategies in set $S_i = \{s_{1i}, \ldots, s_{Mi}\}$. Player i's set of possible mixed strategies can therefore be associated with the points of the following simplex:

$$\underbrace{\Delta(S_i)}_{\text{simplex}} = \{(\sigma_{1i}, \ldots, \sigma_{Mi}) \in R^M : \sigma_{mi} \geq 0 \text{ for all } m = 1, \ldots, M \text{ and } \Sigma_{m=1}^{M} \sigma_{mi} = 1\}$$

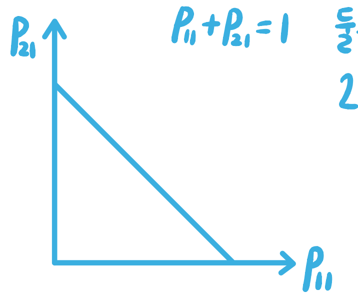
- This simplex is called the *mixed extension* of S.

- Note that a pure strategy can be viewd as a special case of a mixed strategy in which the probability distribution over the element of S_i is degenerate.

$S_1 = \{A, B\}$

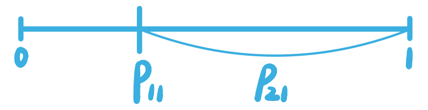
$\sigma_1 = (P_{11}, P_{21})$ 단 $P_{11} + P_{21} = 1$

mixed: A를 P_{11} 확률로 B를 P_{21} 확률로 함



$P_{11} + P_{21} = 1$ 둘은 독립적이지 않음 $\sigma_1 = (P_{11}, 1 - P_{11})$으로 적어도 정확히 같음

2차원 공간에 정의 되는 줄 알았는데 사실상 1차원에서 정의됨



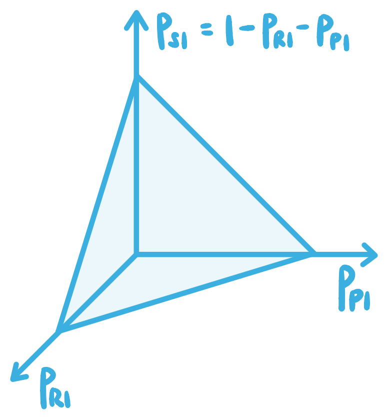
simplex 공간

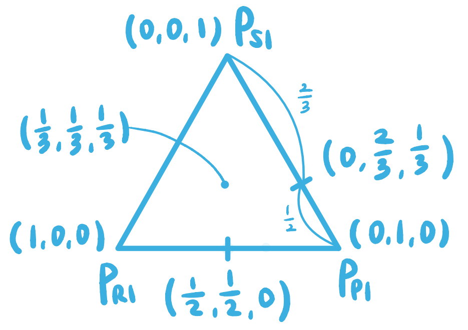
$S_1 = \{R, P, S\}$

$\sigma_1 = (P_{R1}, P_{P1}, P_{S1})$ 단 $P_{R1} + P_{P1} + P_{S1} = 1$

만약 (1, 0, 0) 이면 R을 pure strategy로 사용한 것

degenerate game





삼각형 평면 안에 모든 가능한 mixed strategy 표현가능

Randomized choices

	s_{21}	s_{22}
s_{11}	$u_1(s_{11},s_{21}), u_2(s_{11},s_{21})$	$u_1(s_{11},s_{22}), u_2(s_{11},s_{22})$
s_{12}	$u_1(s_{12},s_{21}), u_2(s_{12},s_{21})$	$u_1(s_{12},s_{22}), u_2(s_{12},s_{22})$

Bernoulli utility

pure strategy $S_1=\{s_{11}, s_{12}\}$, $S_2=\{s_{21}, s_{22}\}$

mixed strategy $\sigma_1=\{\sigma_1(s_{11}), \sigma_1(s_{12})\}$, $\sigma_2=\{\sigma_2(s_{21}), \sigma_1(s_{22})\}$

$$
\begin{aligned}
E_\sigma[u_1(s)] = {} & \sigma_1(s_{11})\sigma_2(s_{21})u_1(s_{11},s_{21}) \\
& + \sigma_1(s_{11})\sigma_2(s_{22})u_1(s_{11},s_{22}) \\
& + \sigma_1(s_{12})\sigma_2(s_{21})u_1(s_{12},s_{21}) \\
& + \sigma_1(s_{12})\sigma_2(s_{22})u_1(s_{12},s_{22})
\end{aligned}
$$

- When players randomize over their pure strategies, the induced outcome is itself random, leading to a probability distribution over the terminal nodes of the game. Since each player i's normal form payoff function $u_i(s)$ is of the von Neumann-Morgenstern type, player i's payoff given a profile of mixed strategies $\sigma = (\sigma_1, \ldots, \sigma_I)$ for I players is her expected utility $E_\sigma[u_i(s)]$, the expectation being taken with respect to the probabilities induced by σ on pure strategy profiles $s = (s_1, \ldots, s_I)$. That is, letting $S = S_1 \times \cdots \times S_I$, player i's von Neumann-Morgenstern utility from mixed strategy profile σ is

$$E_\sigma[u_i(s)] = \Sigma_{s \in S}[\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_I(s_I)]u_i(s)$$

  which, with a slight abuse of notation, we denote by $u_i(\sigma)$. Note that because we assume that $u_i(s)$ 한번 썼는데 $u_i(\sigma)$로 또 씀 each player randomizes on her own, we take the realizations of players' randomizations to be independent of one another.

- The basic definition of the normal form representation is $\Gamma_N = [I, \{\Delta(S_i\}, \{u_i(\cdot)\}]$.

Randomized choices

- **Another way to randomize**.
- Rather than randomizing over the pure strategies in S_i, she could randomize separately over the possible actions at each of her information sets $H \in \mathscr{H}_i$. This way of randomizing is called a *behavior strategy*.
- **Definition**. Given an extensive form game Γ_E, a *behavior strategy* for player *i* specifies, for every information set $H \in \mathscr{H}_i$ and action $a \in C(H)$, a probability $\lambda_i(a, H) \geq 0$, with $\Sigma_{a \in C(H)} \lambda_i(a, H) = 1$ for all $H \in \mathscr{H}_i$.

Simultaneous-Move Games

Dominant and dominated strategies

- A strategy that is strictly dominated is never a best response.
- A strategy might never be a best response even though it is not strictly dominated.
- Eliminating strategies that are never a best response must eliminate at least as many strategies as eliminating just dominated strategies and may eliminate more.
- A pure strategy that is a best reply to some mixed-strategy profile cannot be strictly dominated. Pearce (1984) showed that the converse holds in any two-player game: A pure strategy that is not strictly dominated in such a game is necessarily a best reply to some mixed-strategy profile.
- A pure strategy that is a best reply to some completely mixed-strategy profile is undominated. Pearce (1984) showed that in any two-player game, the converse also to this holds: An undominated pure strategy is then a besy reply to some completely mixed strategy profile. Neither of these two converses are generally valid in games with more than two players.
- Proposition 1 in Weibull (1995). Consider any two-player game, $s_i \in S_i$ is not strictly dominated if and only if $s_i \in \beta(y)$ for some $y \in \Theta$, $s_i \in S_i$ is undominated if and only if $s_i \in \beta(y)$ for some $y \in int(\Theta)$.

Dominant and dominated strategies

- Consider games, $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ whose strategy sets allow for only pure strategies.
- **Definition**. A strategy $s_i \in S_i$ is a *strictly dominant strategy* for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if for all $s_i' \neq s_i$, we have

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

  for all $s_{-i} \in S_{-i}$

Don't confess = strictly dominant strategy
상대의 rationality, 상대의 선택 몰라도 됨
나의 payoff matrix만 알면됨
즉, 전략적 상호작용이 없음

		Prisoner 2	
		Don't Confess	Confess
Prisoner 1	Don't Confess	$-2, -2$	$-10, -1$
	Confess	$-1, -10$	$-5, -5$

Playing "confess" is each player's best strategy regardless of what the other player does. This type of strategy is known as a strictly dominant strategy.

- **Definition**. A strategy $s_i \in S_i$ is *strictly dominated* for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if there exists another strategy $s_i' \in S_i$ and $s_i' \neq s_i$ such that for all $s_{-i} \in S_{-i}$,

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$$

- In this case, we say that strategy s_i' *strictly dominates* strategy s_i
- Strategy s_i is a *strictly dominant strategy* for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if it strictly dominates every other strategy in S_i.

Dominant and dominated strategies

- **Definition**. A strategy $s_i \in S_i$ is *weakly dominated* in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if there exists another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$,

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$$

  with strict inequality for some s_{-i}

- A strategy is a *weakly dominant strategy* for player i in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if it weakly dominates every other strategy in S_i

Player 2

Player 1		L	R
	U	1, −1	−1, 1
	M	−1, 1	1, −1
	D	−2, 5	−3, 2

D is strictly dominated

하지만 strictly dominant strategy 존재하지 않음

Player 2

Player 1		L	R
	U	5, 1	4, 0
	M	6, 0	3, 1
	D	6, 4	4, 4

U and M are weakly dominated

Dominant and dominated strategies

- Note for weakly dominated strategies. Unlike a strictly dominated strategy, a strategy that is only weakly dominated cannot be ruled out based solely on principles of rationality. Caution might therefore rule out weakly dominated strategies. More generally, weakly dominated strategies could be dismissed if players always believed that there was at least some positive probability that any strategies of their rivals could be chosen.

Player 2

Player 1		L	R
	U	1, −1	−1, 1
	M	−1, 1	1, −1
	D	−2, 5	−3, 2

D is strictly dominated

상대가 L을 선택하든 R을 선택하든 D보다 M이 선호된다고 말할 수 있음
→ belief 필요없음

Player 2

Player 1		L	R
	U	5, 1	4, 0
	M	6, 0	3, 1
	D	6, 4	4, 4

U and M are weakly dominated

① U가 D에 의해 weakly dominated 되므로 U를 선택하는 것은 상대가 L을 할 확률이 positive 하다는 belief를 기반으로 함
하지만 내가 U를 지우면 상대가 L 할 이유 없어보임 → 모순발생
② M을 지워도 마찬가지

Dominant and dominated strategies

강열등전략의 반복소거

- **Iterated deletion of strictly dominated strategies**
- Note the way players' common knowledge of each other's payoffs and rationality is used to solve the game on the right. Elimination of strictly dominated strategies requires only that each player be rational. What we have just done, however, requires not only that prisoner 2 be rational but also that prisoner I know that prisoner 2 is rational. Put somewhat differently, a player need not know anything about his opponents' payoffs or be sure of their rationality to eliminate a strictly dominated strategy from consideration as his own strategy choice; but for the player to elimination of his strategies from consideration because it is dominated if his opponents never play their dominated strategies does require this knowledge.

		Prisoner 2: Don't Confess	Prisoner 2: Confess
Prisoner 1	Don't Confess	0, −2	−10, −1
	Confess	−1, −10	−5, −5

① 경기자1 dominance 존재 X
경기자2 dominance 존재 O
→ 경기자2 DC 삭제

② 경기자2가 C를 하는 상황에서는
경기자1 입장에서 C가 DC를 dominant
→ 경기자1 DC 삭제

상대의 payoff를 아는 상태에서는 상대가 DC 선택하지 않을 것이라고 생각할 수 있음
→ 각자 rational 할 뿐만 아니라 서로가 rational 하다는 생각을 가지고 있어야 함

Dominant and dominated strategies

- **Iterated deletion of *weakly* dominated strategies**
- The iterated deletion of weakly dominated strategies is harder to justify. As we have already indicated, the argument for deletion of a weakly dominated strategy for player *i* is that he contemplates the possibility that every strategy combination of his rivals occurs with positive probability. However, this hypothesis clashes with the logic of iterated deletion, which assumes, precisely, that eliminated strategies are not expected to occur.This inconsistency leads the iterative elimination of weakly dominated strategies to have the undesirable feature that it can depend on the order of deletion. The game on the right provides an example. If we first eliminate strategy U, we next eliminate strategy L, and we can then eliminate strategy M: (D. R) is therefore our prediction. If, instead, we eliminate strategy M first, we next eliminate strategy R, and we can then eliminate strategy U: now (D, L) is our prediction.

		Player 2	
		L	*R*
Player 1	*U*	5, 1	4, 0
	M	6, 0	3, 1
	D	6, 4	4, 4

① 경기자1 M, D 중에서 M 지움
② 경기자2 M이 없는 상황에서는 R 지움
③ 경기자1 위 상황을 모두 알고 U, D 중에서 D 선택
→ (D, L) 선택

지우는 순서가 달라지면 결과 달라질 수 있음!
strictly에서는 이런 문제 없는데 weakly에서는 그럴 수 있음

Dominant and dominated strategies

- **Dominant and dominated strategies *with mixed strategies.***
- **Definition**. A strateay $\sigma_i \in \Delta(S_i)$ is *strictly dominated* for player i in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if there exists another strategy $\sigma_i' \in \Delta(S_i)$ such that for all $\sigma_{-i} \in \Pi_{j \neq i} \Delta(S_j)$,

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$$

- We say that strategy σ_i' *strictly dominates* strategy σ_i. A strategy σ_i is a strictly dominant strategy for player i in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if it strictly dominates every other strategy in $\Delta(S_i)$.

pure strategy만 보면 Strictly dominant Strategy 없음

$\sigma_1 = (\sigma_1(U), \sigma_1(M), \sigma_1(D))$

example. $\sigma_1 = (\frac{1}{2}, 0, \frac{1}{2})$

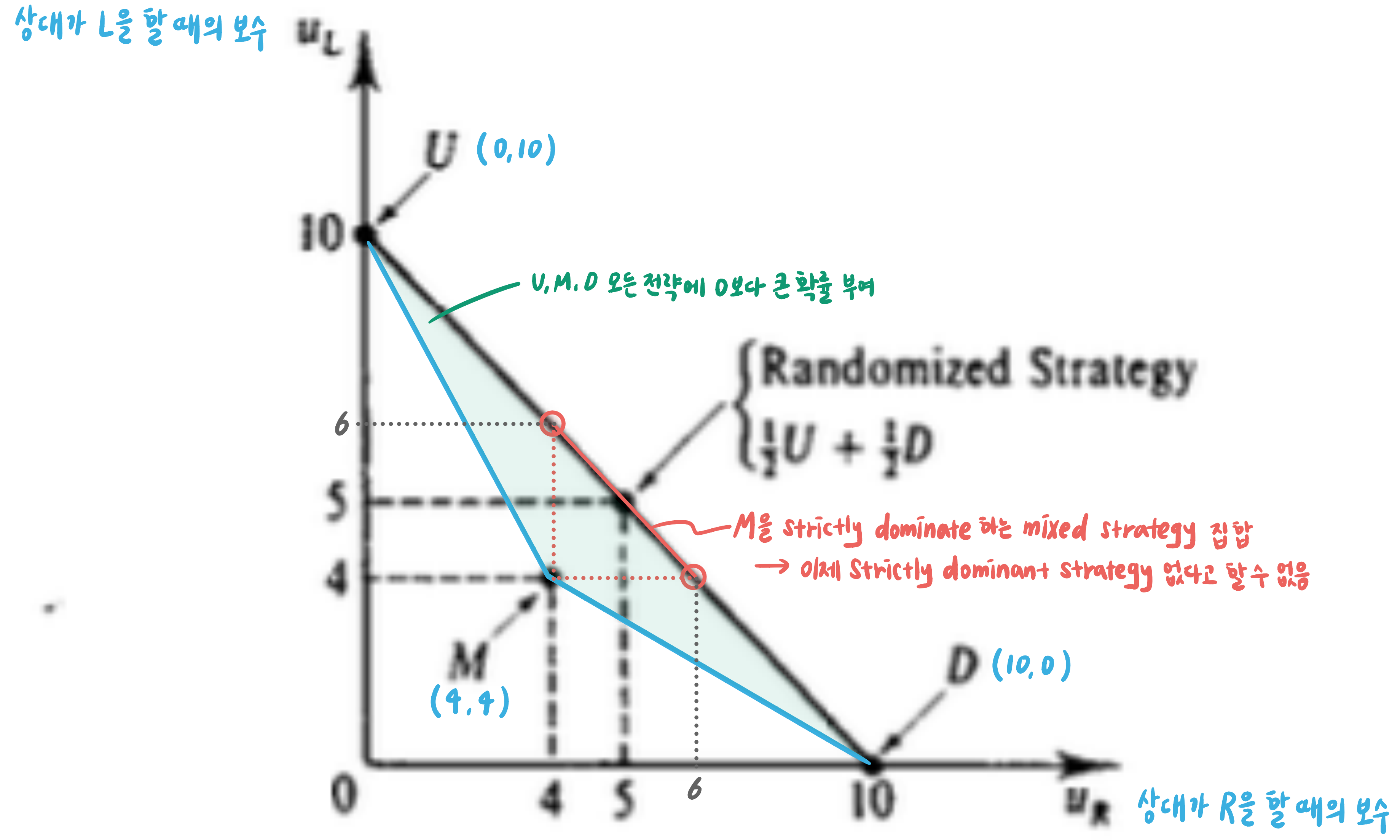
상대가 어떤 선택을 하든

$\frac{1}{2} \times 10 + \frac{1}{2} \times 0 = 5$, $\frac{1}{2} \times 0 + \frac{1}{2} \times 10 = 5$

→ M을 strictly dominant 하는 mixed strategy

※ (0.4, 0, 0.6), (0.6, 0, 0.4) 사이에만 있으면 됨

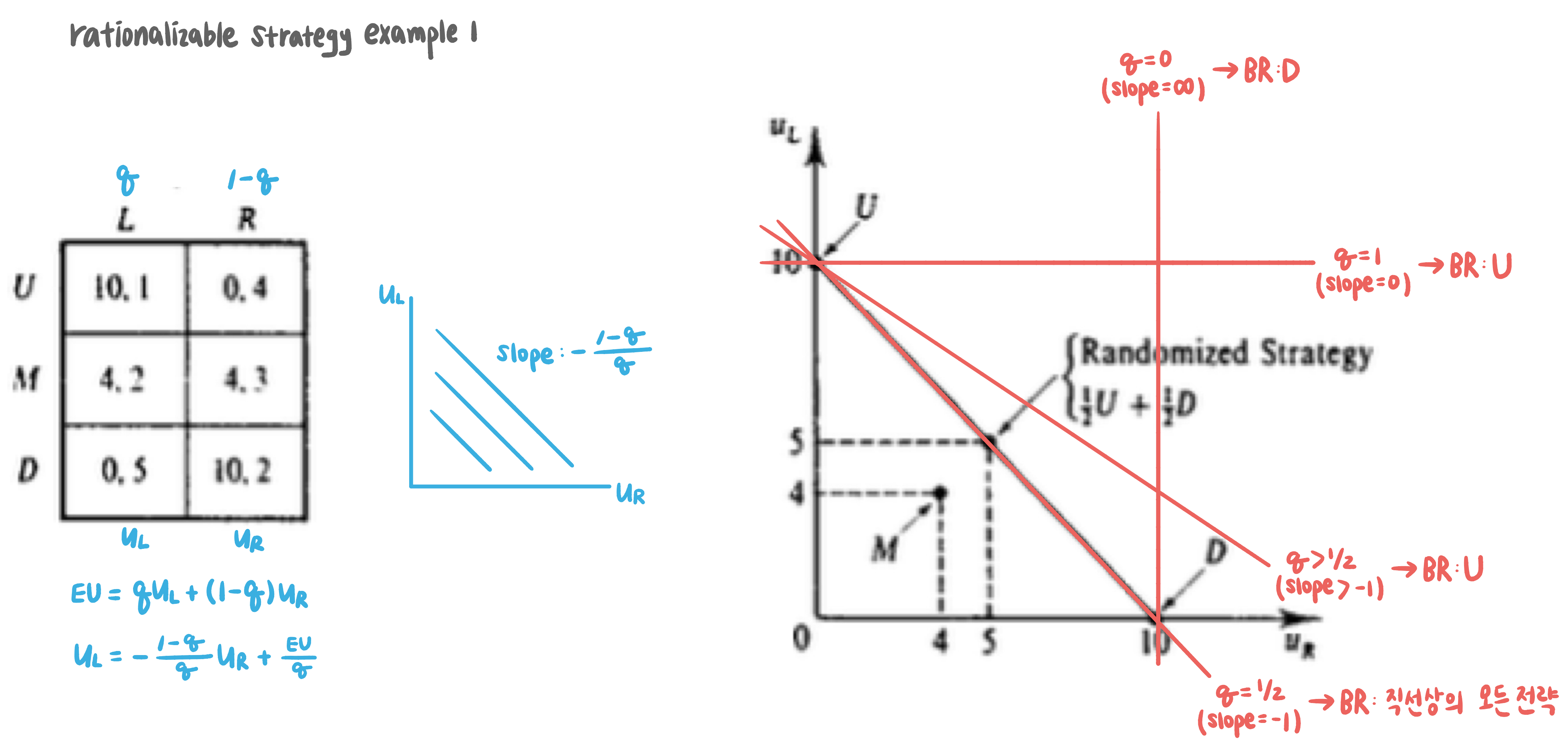
		Player 2	
		L	R
Player 1	U	10, 1	0, 4
	M	4, 2	4, 3
	D	0, 5	10, 2

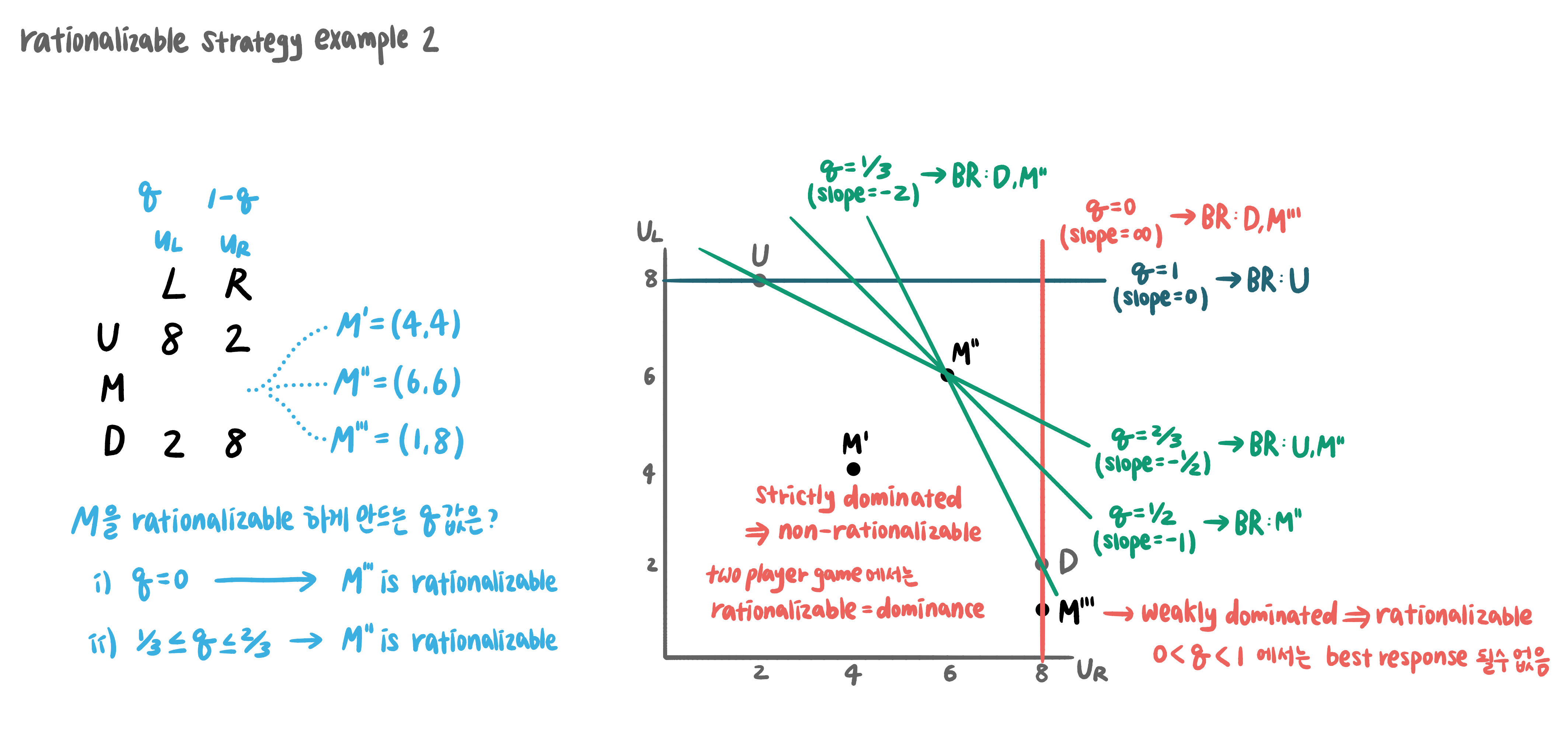


Dominant and dominated strategies

- **Dominant and dominated strategies *with mixed strategies.***
- **Proposition**. Player i's pure strategy $s_i \in S_i$ is strictly dominated in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if and only if there exists another strategy $\sigma_i' \in \Delta(S_i)$ such that for all $s_{-i} \in S_{-i}$,

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i})$$





Rationalizable strategies

각각의 전략은 나름대로 필요한 전략
하지만 어떤 전략에 대해서도 best response 가 되지 않는 전략은 rationalizable 하지 않음

- **Definition**. In game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, strategy σ_i is a *best response* for player i to his rivals' strategies σ_{-i} if for all $\sigma_i' \in \Delta(S_i)$

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

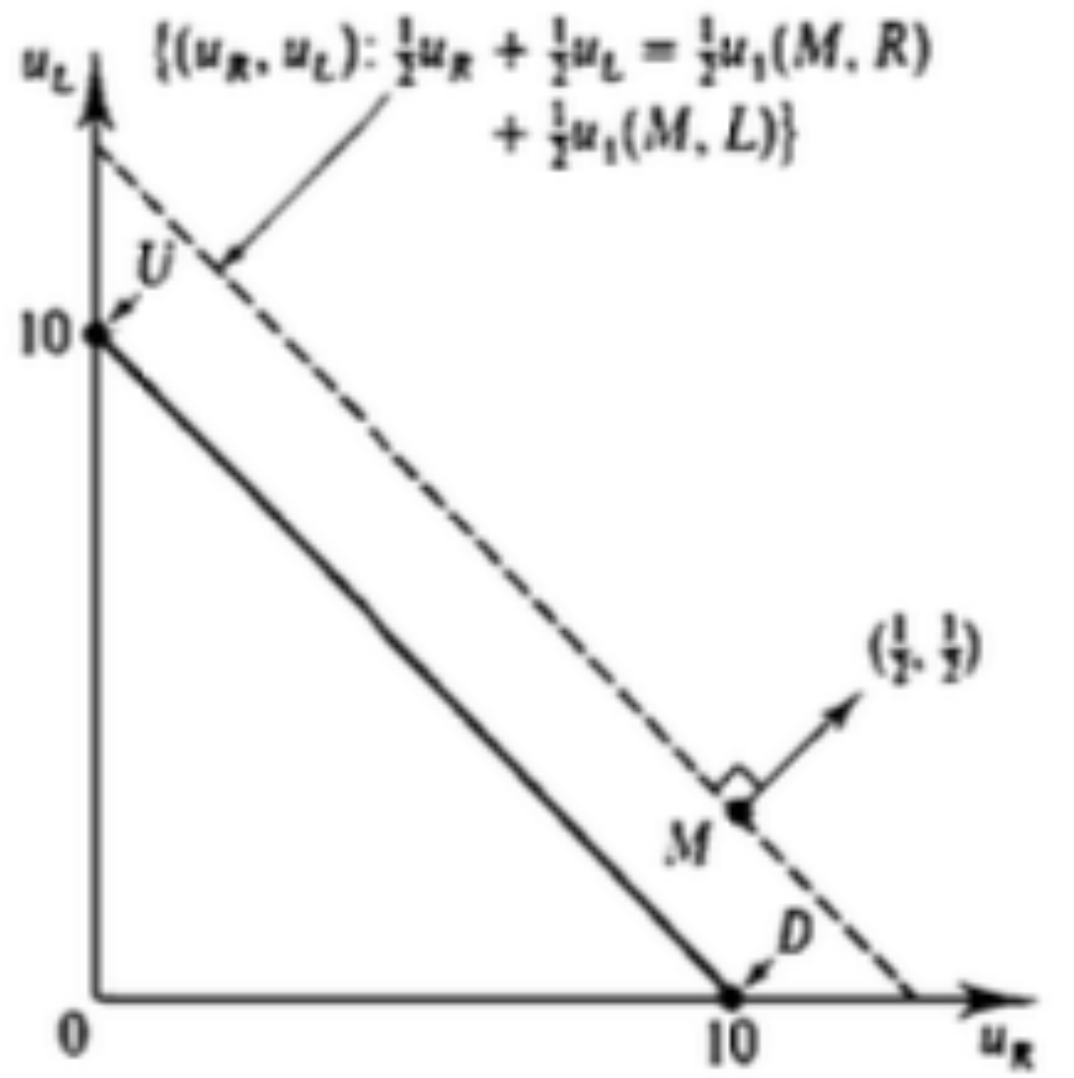
  Strategy σ_i is *never a best response* if there is no σ_{-i} for which σ_i is a best response.

- **Definition**. In game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, the strategies in $\Delta(S_i)$ that survive the iterated removal of strategies that are never a best response are known as player i's rationalizable strategies.

Player 2

		BR(a₁) BR(a₄) b_1	BR(a₂) b_2	BR(a₃) BR(a₄) b_3	non-rationalizable 존재이유 없음 b_4
Player 1	BR(b₃) a_1	0, 7	2, 5	7, 0	0, 1
	BR(b₂) a_2	5, 2	3, 3	5, 2	0, 1
	BR(b₁) a_3	7, 0	2, 5	0, 7	0, 1
	BR(b₄) a_4 상대가 b4 지우면 a4 지움	0, 0	0, −2	0, 0	10, −1

4x4 처럼 보이지만 3x3
Strictly/weakly domation 없음



Nash equilibrium

서로의 best response (최적대응)로만 묶여있는 전략

- **Definition**. A strategy profile $s = (s_1, \ldots, s_I)$ constitutes a Nash equilibrium of game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$, if for every $i = 1, \ldots, I$,

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

for all $s_i' \in S_i$.

rationality + belief correction

(Paper, Rock) ······· player 2 입장에서

P에 대한 최적대응으로 R선택 ································ rationality O

상대가 1의 확률로 P를 할 것이라는 믿음에 기반 ·········· belief correction X

∴ (Paper, Rock)는 NE가 아님

Rationalizable Strategy

Nash equilibrium

SPNE

$S_1 = \{U, M, D\}$ $S_2 = \{l, m, r\}$

$s \in S_1 \times S_2 = \{(U,l), (U,m), \cdots (M,m), \cdots (D,r)\}$

Player 2

Player 1	l	m	r
U	5, 3	0, 4	3, 5
M	4, 0	5, 5	4, 0
D	3, 5	0, 4	5, 3

NE: (M, m)

Mr. Schelling

Mr. Thomas	Empire State	Grand Central
Empire State	100, 100	0, 0
Grand Central	0, 0	100, 100

NE: (E,E), (C,C)

Nash equilibrium

나는 belief 하에서 rationalizable strategies 중 하나를 고름
그리고 그 belief가 convention 하면 Nash Equilibrium

과학적으로 제대로 된 belief란 우리의 경험에 비추어 볼 때 타당한 것
→ bayesian을 통한 업데이트

- **Discussion of the concept of Nash equilibrium**

(1) Nash equilibrium as a consequence of rational inference.

(2) Nash equilibrium as a necessary condition if there is a unique predicted outcome to a game.

(3) Focal points.

(4) Nash equilibrium as a self-enforcing agreement.

(5) Nash equilibrium as a stable social convention.

Nash equilibrium

	s_j'	s_j''	s_j'''
s_i'	2,2	0,0	0,1
s_i''	0,0	2,2	0,1
s_i'''	1,0	1,0	3,3

NE : $(s_i', s_j'), (s_i'', s_j''), (s_i''', s_j''')$

$S_i = \{s_i', s_i'', s_i'''\}$

$(\sigma_i, \sigma_j) = ((1,0,0), (1,0,0))$ pure : s_i', s_j'

$((0,1,0), (0,1,0))$ pure : s_i'', s_j''

$((0,0,1), (0,0,1))$ pure : s_i''', s_j'''

$((\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 0)) \rightarrow S_i^+ = \{s_i', s_i''\}$, $S_i / S_i^+ = \{s_i'''\}$

s_i'''을 선택하는 순간 s_j'''를 1의 확률로 사용할 것이므로 (1/2, 1/2, 0)은 BR이 아님

- **Mixed strategy Nash equilibrium**

- **Definition.** A mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ constitutes a *Nash equilibrium* of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if for every $i = 1, \cdots, I$,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

for all $\sigma_i' \in \Delta(S_i)$.

확률이 0인 전략은 제외

- **Proposition**. Let $S_i^+ \subset S_i$ denote the set of pure strategies that player i plays with positive probability in mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$. Strategy profile σ is a Nash equilibrium in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if and only if for all $i = 1, \ldots, I$,

한쪽이 조금이라도 더 크면 확률 1.0으로 부여할테니 같아야함

(i) $u_i(s_i, \sigma_{-i}) = u_i(s_i', \sigma_{-i})$ for all $s_i, s_i' \in S_i^+$; mixed strategy NE는 확률이 0보다 큰 전략들을 확률에 따라 섞어서 쓰는 것 효용이 더 높은 전략이 있다면 그 전략을 pure strategy로 사용함

(ii) $u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i})$ for all $s_i \in S_i^+$ and all $s_i' \notin S_i^+$.

BR이 아닌 전략

- **Corollary**. Pure strategy profile $s = (s_1, \ldots, s_I)$ is a Nash equilibrium of game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if and only if it is a (degenerate) mixed strategy Nash equilibrium of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$.

Nash equilibrium

- **Existence of Nash equilibrium** mixed strategy 이면 반드시 Nash Equilibrium 하나 이상 존재
- **Proposition**. Every game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ in which the sets $S_1, \ldots, S_I$ have a finite number of elements has a mixed strategy Nash equilibrium.
- **Proposition**. A Nash equilibrium exists in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if for all $i = 1, \ldots, I$,

  (i) S_i is a nonempty, convex, and the compact subset of some Euclidean space $\mathbb{R}^M$.

  (ii) $u_i(s_1, \ldots, s_I)$ is continuous in $(s_1, \ldots, s_I)$ and quasiconcave in s_i.

perfect information : 내가 어디있는지를 앎 (imperfect - 동시게임)
complete information : 상대의 payoff까지 앎. 내가 아는걸 상대가 알고 그걸 내가 알고... (incomplete - 어떤 게임을 하는지 모름)

Games of incomplete information: Bayesian Nash equilibrium

- Games of complete information. 상대의 정보를 몰라도 되는 경우는 Strictly dominance가 있는 경우가 유일
  players know all relevant information about each other, including the payoffs that each receives from the various outcomes of the game.
- Games of incomplete information. players do not know all relevant information.

$S_1 = \{C, D\}$
$S_2 = \{CC, CD, DC, DD\}$: θ_2^1, : θ_2^2

Player 1's type $\Theta_1 = \{\theta_1^1\}$
Player 2's type $\Theta_2 = \{\theta_2^1, \theta_2^2\}$ ※ $\theta_2 \in \Theta_2$, θ_2 is RV

$\Theta = \Theta_1 \times \Theta_2 = \{(\theta_1^1, \theta_2^1), (\theta_1^1, \theta_2^2)\}$

$F(\theta_1, \theta_2) \begin{cases} F(\theta_1^1, \theta_2^1) = \mu \\ F(\theta_1^1, \theta_2^1) = 1-\mu \end{cases}$

$S_i = \Theta_i \to A$: decision rule

$S_1 = \Theta_1 \to A \begin{cases} S_1(\theta_1^1) = C \\ S_1(\theta_1^1) = D \end{cases}$

$S_2 = \Theta_2 \to A \begin{cases} S_2(\theta_2^1) = C \text{ and } S_2(\theta_2^2) = C : CC \\ S_2(\theta_2^1) = C \text{ and } S_2(\theta_2^2) = D : CD \\ S_2(\theta_2^1) = D \text{ and } S_2(\theta_2^2) = C : DC \\ S_2(\theta_2^1) = D \text{ and } S_2(\theta_2^2) = D : DD \end{cases}$

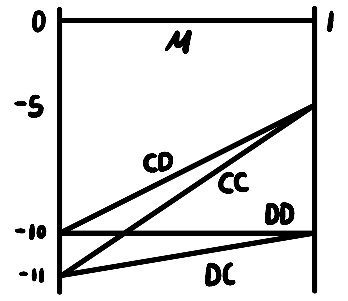
$u_1(S_1(\theta_1), S_2(\theta_2), \theta_1^1) = \begin{pmatrix} -5 & -1 \\ -10 & 0 \end{pmatrix}$

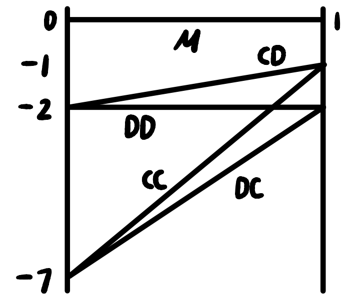
$u_2(S_1(\theta_1), S_2(\theta_2), \theta_2^1) = \begin{pmatrix} -5 & -10 \\ -1 & -2 \end{pmatrix}$

$u_2(S_1(\theta_1), S_2(\theta_2), \theta_2^2) = \begin{pmatrix} -1 & -10 \\ -7 & -2 \end{pmatrix}$

	$S_2(\theta_2^1)=C$ $S_2(\theta_2^2)=C$	$S_2(\theta_2^1)=C$ $S_2(\theta_2^2)=D$	$S_2(\theta_2^1)=D$ $S_2(\theta_2^2)=C$	$S_2(\theta_2^1)=D$ $S_2(\theta_2^2)=D$
C	-5 BR	$-1-4\mu$ $\mu>1/6 \to$ BR	$-5+4\mu$ $\mu<5/6 \to$ BR	-1
D	-10	-10μ $\mu<1/6 \to$ BR	$-10+10\mu$ $\mu>5/6 \to$ BR	0 BR

	$S_2(\theta_2^1)=C$ $S_2(\theta_2^2)=C$	$S_2(\theta_2^1)=C$ $S_2(\theta_2^2)=D$	$S_2(\theta_2^1)=D$ $S_2(\theta_2^2)=C$	$S_2(\theta_2^1)=D$ $S_2(\theta_2^2)=D$
C	$-11+6\mu$	$-10+5\mu$	$-11+\mu$	-10
D	$-7+6\mu$	$-2+\mu$	$-7+5\mu$	-2





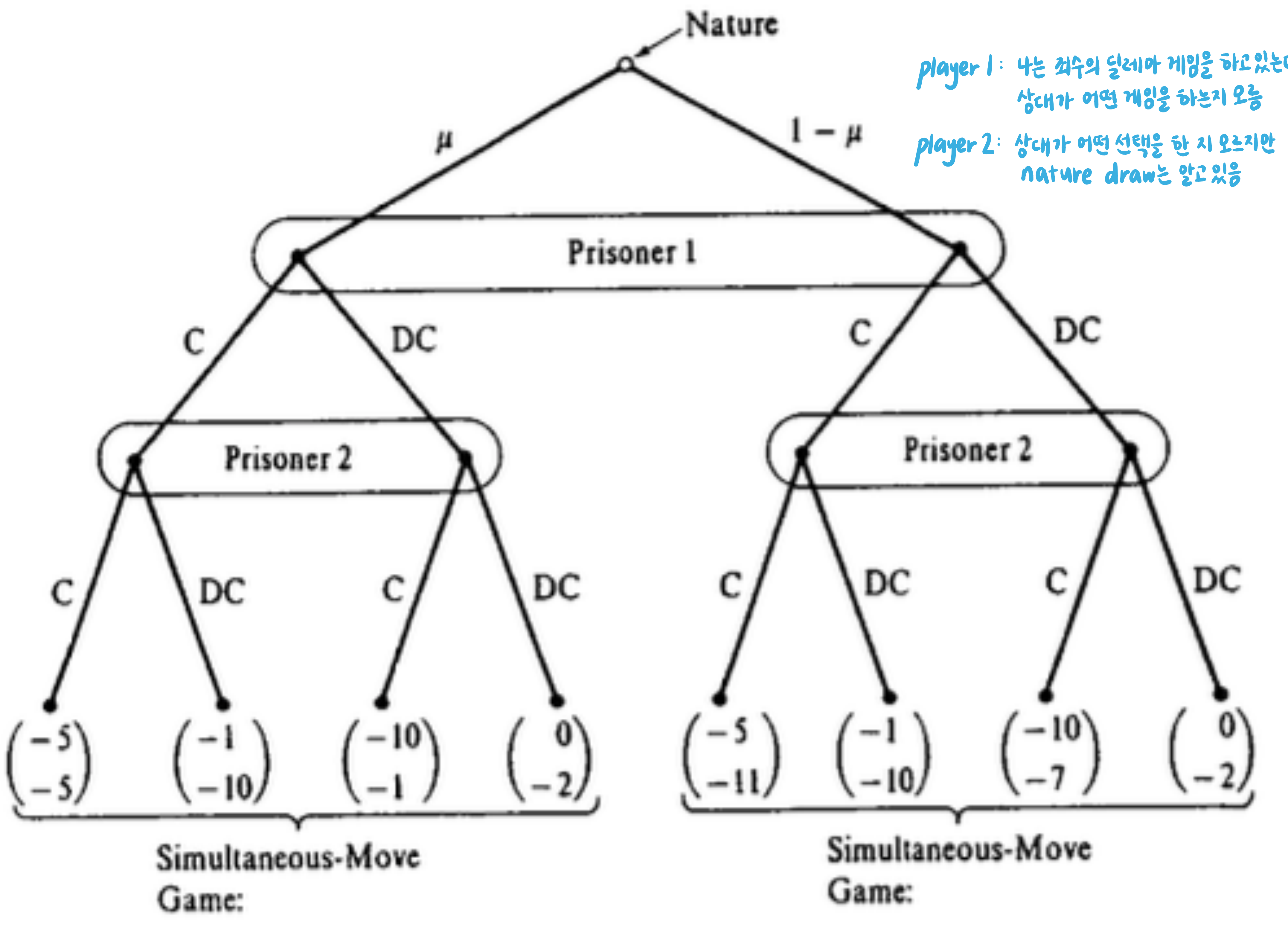
$B_2(C) = \begin{cases} CD & \text{if } 0<\mu<1 \\ CD, DD & \text{if } \mu=0 \\ CD, CC & \text{if } \mu=1 \end{cases}$

$B_2(D) = \begin{cases} CD & \text{if } 0<\mu<1 \\ CD, DD & \text{if } \mu=0 \\ CD, CC & \text{if } \mu=1 \end{cases}$

Nash equilibrium
if $\mu=0$ (D,CD), (D,DD)
if $\mu=1$ (C,CD), (C,CC)
if $1/6<\mu<1$ (C,CD)
if $0<\mu<1/6$ (D,CD)
θ_2^1 또는 θ_2^2로 실행됐을 때를 가정하고 풀면 둘만 남음

Harsanyi transformation
: incomplete game —"moves by nature"→ imperfect game

player 1 : 나는 죄수의 딜레마 게임을 하고있는데 상대가 어떤 게임을 하는지 모름
player 2 : 상대가 어떤 선택을 한 지 모르지만 nature draw는 알고 있음



		Prisoner 2	
		DC	C
Prisoner 1	DC	0, −2	−10, −1
	C	−1, −10	−5, −5

Strictly dominated

		Prisoner 2	
		DC	C
Prisoner 1	DC	0, −2	−10, −7
	C	−1, −10	−5, −11

Strictly dominated

Games of incomplete information: Bayesian Nash equilibrium

- In a Bayesian game
- Each player i has a payoff function $u_i(s_i, s_{-i}, \theta_i)$ where $\theta_i \in \Theta_i$ is a random variable chosen by nature that is observed only by player i.
- The joint probability distribution of the θ_i's is given by $F(\theta_1, \ldots, \theta_I)$, which is assumed to be common knowledge among the players.
- Letting $\Theta = \Theta_1 \times \ldots \times \Theta_I$, a Bayesian game is summarized by the data $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$.
- A pure strategy for player i in Bayesian game is a function $s_i(\theta_i)$, or *decision rule*, that gives the player's strategy choice for each realization of his type θ_i. Player i's pure strategy set $\mathcal{S}_i$ is the set of all such functions.
- Player i's expected payoff given a profile of pure strategies for the I players $(s_1(\cdot), \ldots, s_I(\cdot))$ is given by

$$\bar{u}_i(s_1(\cdot), \ldots, s_I(\cdot)) = E_\theta[u_i(s_1(\theta_1), \ldots, s_I(\theta_I), \theta_i)]$$

Games of incomplete information: Bayesian Nash equilibrium

- **Definition**. A (pure strategy) Baysian Nash equilibrium for the Bayesian game $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$ is a profile of decision rules $(s_1(\cdot), \ldots, s_I(\cdot))$ that constitutes a Nash equilibrium of game $\Gamma_N = [I, \{\mathcal{S}_i\}, \{\bar{u}_i(\cdot)\}]$. That is, for every $i = 1, \ldots, I$

$$\bar{u}_i(s_1(\cdot), s_{-i}(\cdot)) \geq \bar{u}_i(s_1'(\cdot), s_{-i}(\cdot))$$

for all $s_i'(\cdot) \in \mathcal{S}_i$.

- **Proposition**. A profile of decision rules $(s_1(\cdot), \ldots, s_I(\cdot))$ is a Bayesian Nash equilibrium in Bayesian game $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$ if and only if, for all i and all $\bar{\theta}_i \in \Theta_i$ occuring with positive probability

$$E_{\theta_{-i}}[u_i(s_i(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i) \,|\, \bar{\theta}_i)] \geq E_{\theta_{-i}}[u_i(s_i'(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i) \,|\, \bar{\theta}_i)]$$

for all $s_i' \in S_i$, where the expectation is taken over realizations of the other players' random variables conditional on player i's realization of his signal $\bar{\theta}_i$.

Possibility of mistakes: Trembling hand perfection

교란된 게임

- For any normal form game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, a *perturbed game* is defined by choosing for each player i and strategy $s_i \in S_i$ a number $\varepsilon_i(s_i) \in (0,1)$, with $\Sigma_{s_i \in S_i} \varepsilon_i(s_i) < 1$ and player i's perturbed strategy set is defined as

  실수할 확률의 합이 1일 필요는 없음

  $$\Delta_\varepsilon(S_i) = \{\sigma_i : \sigma_i(s_i) \geq \varepsilon_i(s_i) \text{ for all } s_i \in S_i \text{ and } \Sigma_{s_i \in S_i} \sigma_i(s_i) = 1\}$$

- **Definition**. A Nash equilibrium σ of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ is *(normal form) trembling-hand perfect* if there is some sequence of perturbed games $\{\Gamma_{\varepsilon^k}\}_{k=1}^{\infty}$ that converges to Γ_N [in a sense that $\lim_{k\to\infty} \varepsilon_i^k(s_i) = 0$ for all i and $s_i \in S_i$], for which there is some associated sequence of Nash equilibria $\{\sigma^k\}_{k=1}^{\infty}$ that converges to σ [i.e., such that $\lim_{k\to\infty} \sigma^k = \sigma$].

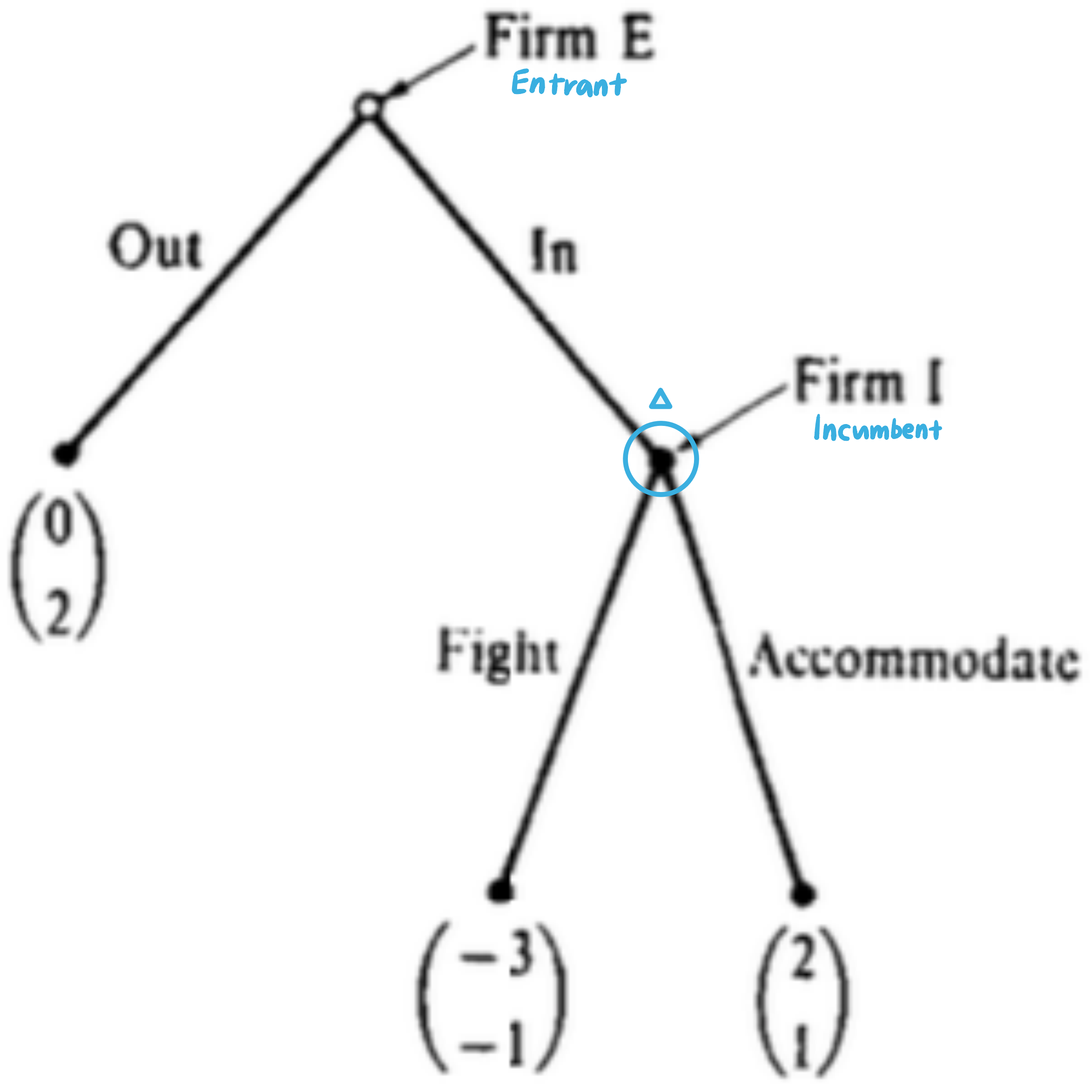
	L	R (weakly dominated)
U	1, 1	0, −3
(weakly dominated) D	−3, 0	0, 0

Possibility of mistakes: Trembling hand perfection

- **Proposition.** A Nash equilibrium σ of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ is (normal form) trembling-hand perfect if and only if there is some sequence of totally mixed strategies $\{\sigma^k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty}\sigma^k = \sigma$ and σ_i is a best response to every element of sequence $\{\sigma_{-i}^k\}_{k=1}^{\infty}$ for all $i = 1,\ldots,I$.
- **Proposition**. If $\sigma = (\sigma_1, \ldots, \sigma_I)$ is a (normal form) trembling-hand perfect Nash equilibrium, then σ_i is not a weakly dominated strategy for any $i = 1,\ldots,I$. Hence, in any (normal form) trembling-hand perfect Nash equilibrium, no weakly dominated pure strategy can be played with positive probability.

Dynamic Games

Sequential rationality, backward induction, and subgame perfection



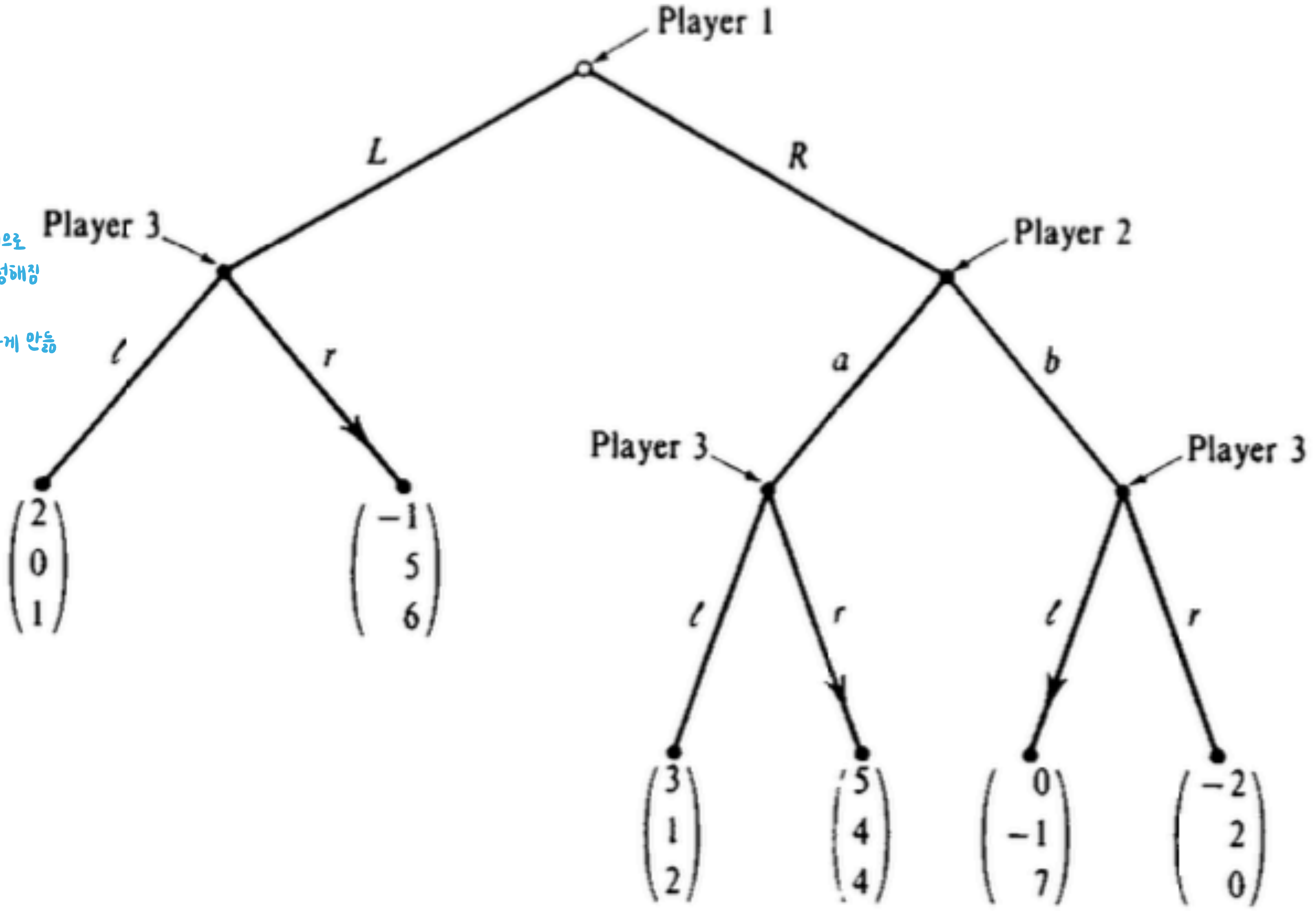
		Firm I: Fight if Firm E Plays "In"	Firm I: Accommodate if Firm E Plays "In"
Firm E	Out	0, 2	0, 2
	In	−3, −1	2, 1

NE = {(out, fight), (in, accommodate)}

이상한 전략인데 왜 균형이 되나? ······· 게임이 △에서 일어날 확률이 0이으로 out 하는순간 뭘 하든 보수 2로 정해짐 = rationality 문제없음

trembling hand ······ △에서 일어날 확률 아주 조금이라도 존재하게 만듦 ······ fight은 더 이상 BR이 아님

backward induction



Sequential rationality, backward induction, and subgame perfection

- **Proposition**. (Zermelo's Theorem) Every finite game of perfect information Γ_E has a pure strategy Nash equilibrium that can be derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a uniqe Nash equilibrium that can be derived in this manner.

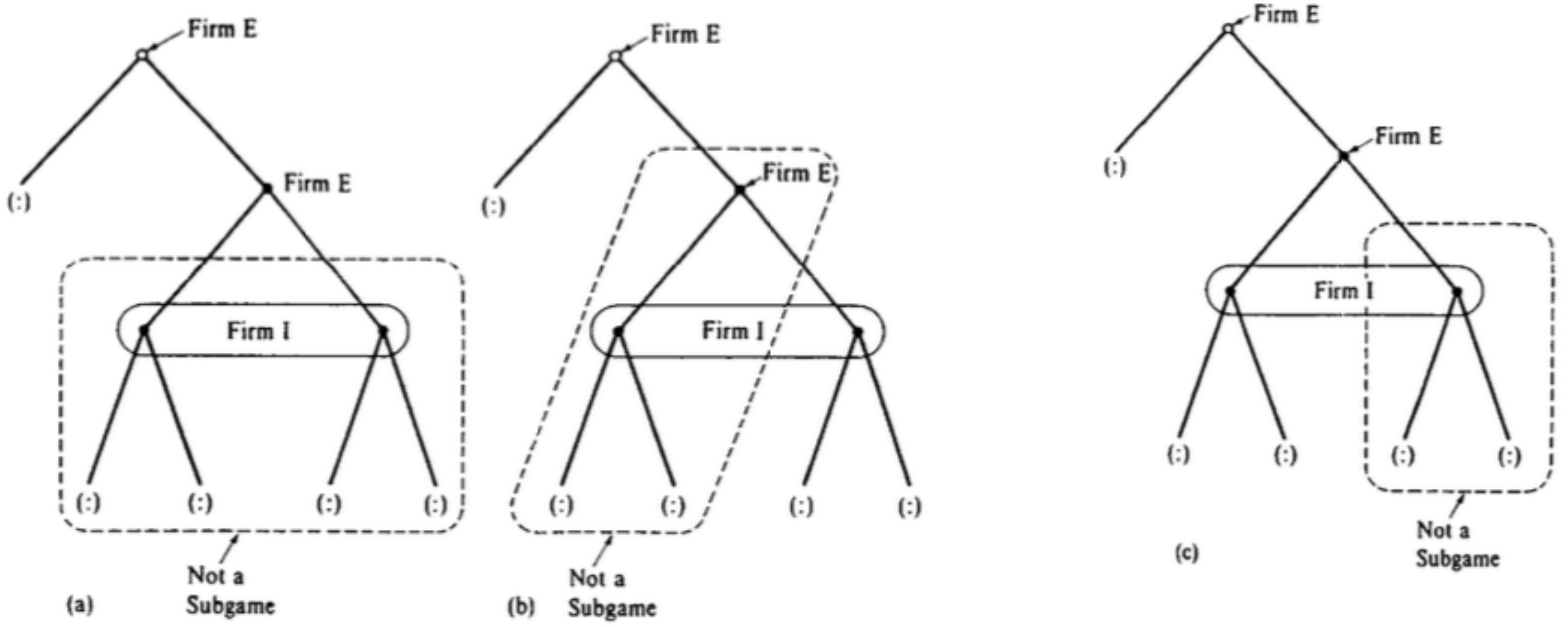
모든 노드에서 play되는걸 전제로 한다면
= backward induction으로 게임을 푼다면
= 모든 subgame에서 본다면

Sequential rationality, backward induction, and subgame perfection

- **Definition**. A *subgame* of an extensive form game Γ_E is a subset of the game having the following properties:

  (1) It begins with an information set containing a single decision node, contains all the decision nodes that are successors (both immediate and later) of this node, and contains *only* these nodes.

  (2) If decision node x is in the subgame, then every $x' \in H(x)$ is also, where $H(x)$ is the information set that contains decision node x. (That is, there are no "broken" information sets.)



Sequential rationality, backward induction, and subgame perfection

- **Definition**. A profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_I)$ in an I-player extensive form game Γ_E is a *subgame perfect Nash equilibrium* (SPNE), if it induces a Nash equilibrium in every subgame of Γ_E.
- **Proposition.** Every finite game of perfect information Γ_E has a pure strategy subgame perfect Nash equilibrium. Moreover, if no player h a s the same payoffs at any two terminal nodes, then there is a unique subgame perfect Nash equilibrium.
- **Proposition**. Consider an extensive form game I, and some subgame S of Γ_E. Suppose that strategy profile σ^S is an SPNE in subgame S, and let $\hat{\Gamma}_E$ be the reduced game formed by replacing subgame S by a terminal node with payoffs equal to those arising from play of σ^S. Then

  (i) In any SPNE σ of Γ_E in which σ^S is the play in subgame S, players' moves at information sets outside subgame S must constitute an SPNE of reduced game $\hat{\Gamma}_E$.

  (ii) If $\hat{\sigma}$ is an SPNE of $\hat{\Gamma}_E$, then the strategy profile σ that specifies the moves in σ^S at information sets in subgame S and that specifies the moves in $\hat{\sigma}$ at information sets no in S in an SPNE of Γ_E.

Sequential rationality, backward induction, and subgame perfection

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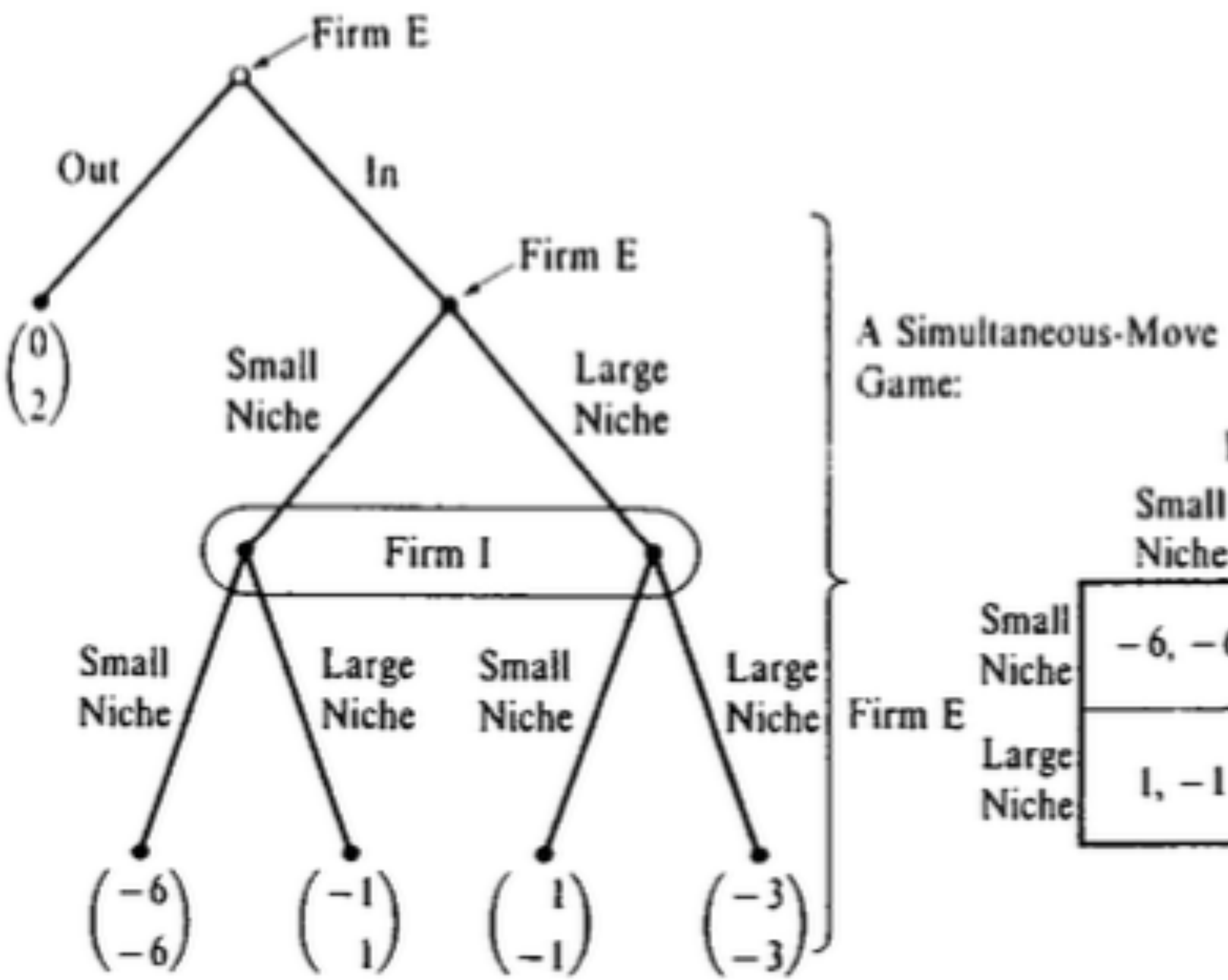
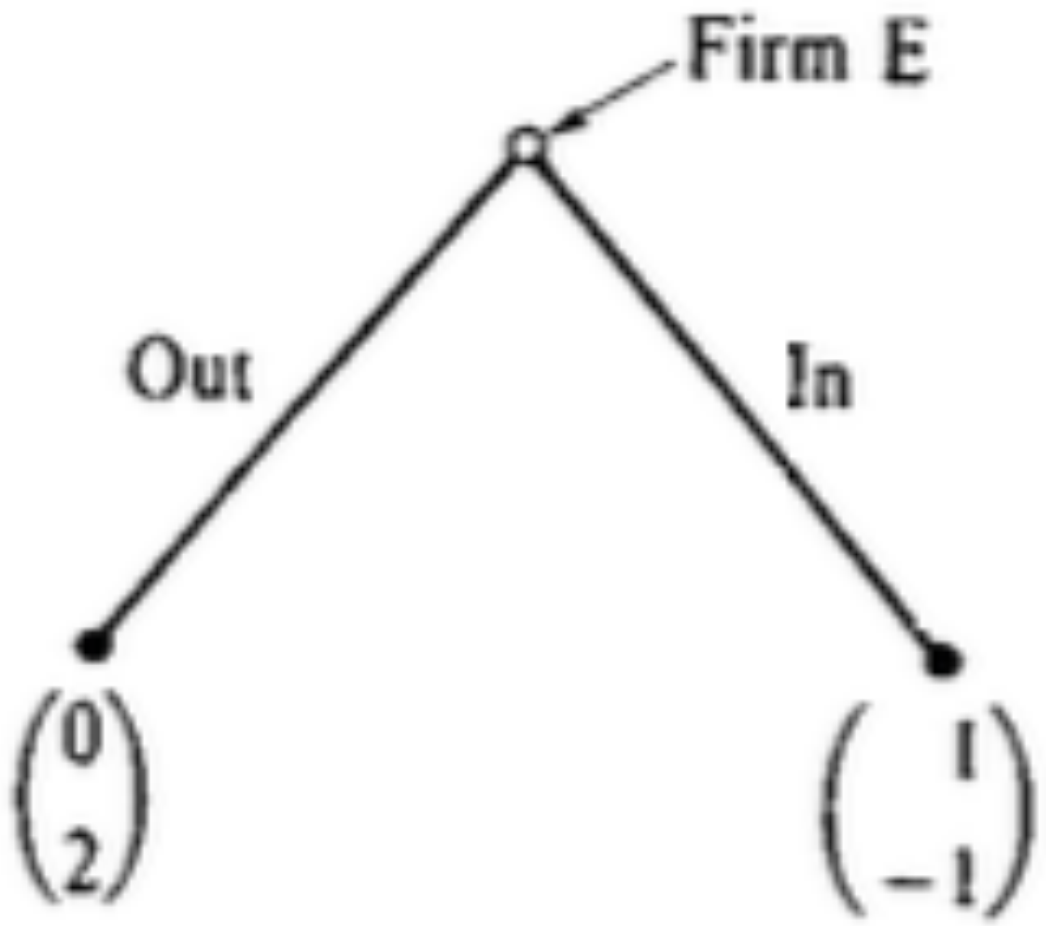
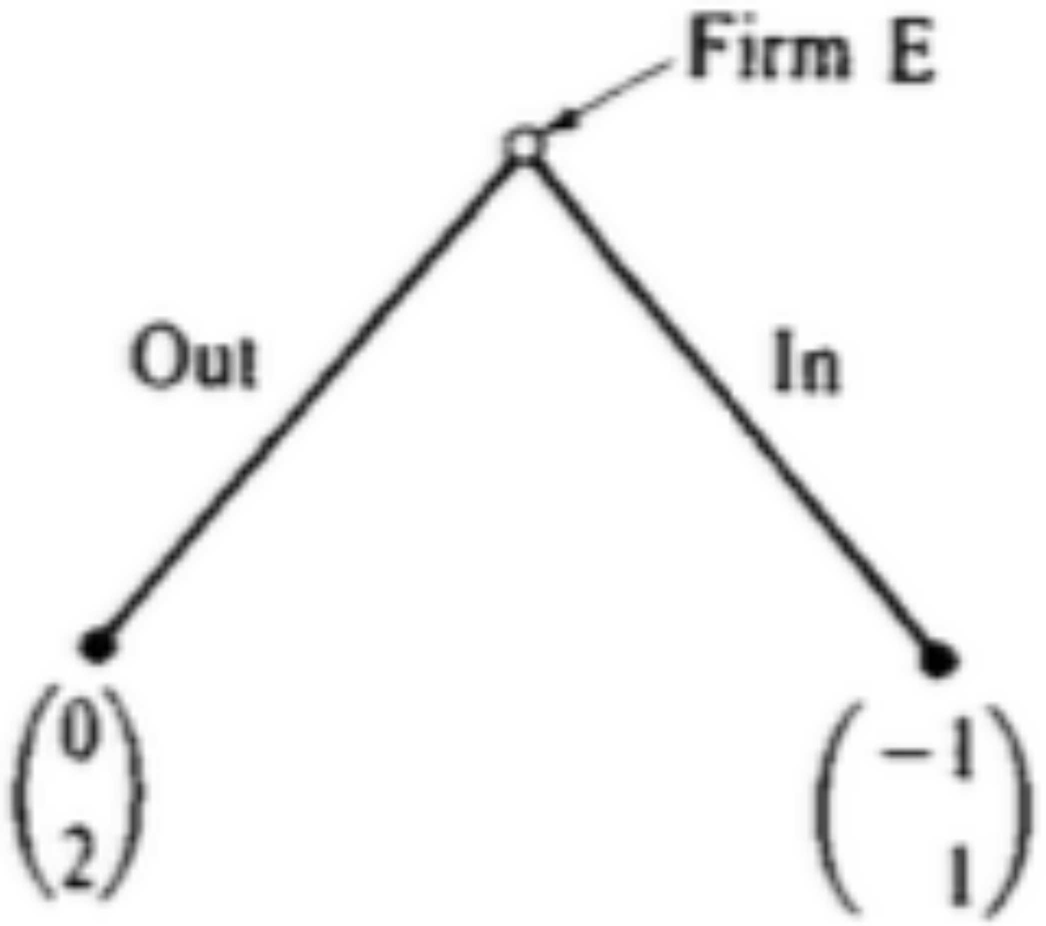


Firm E \ Firm I	Small Niche	Large Niche
Small Niche	−6, −6	−1, 1
Large Niche	1, −1	−3, −3





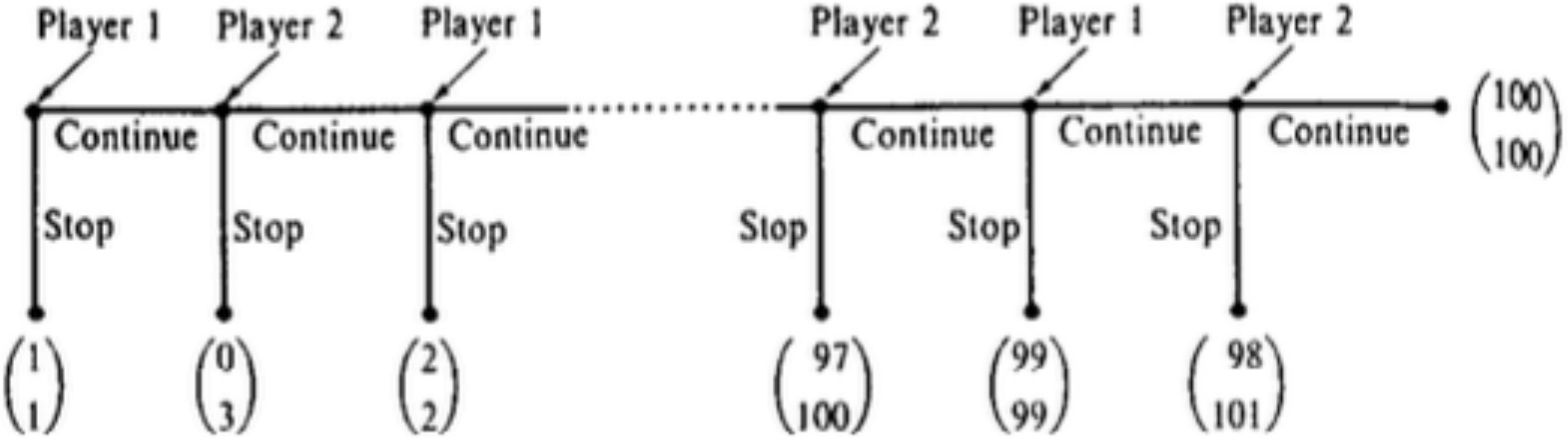
Sequential rationality, backward induction, and subgame perfection

- **Proposition**. Consider an I-player extensive form game Γ_E involvina successive play of T simultaneous-move games, $\Gamma_N^t = \{I, \{\Delta(S_i^t\}, \{u_i^t(\cdot)\}]$ for $t = 1,\ldots, T$ with the players observing the pure strategies played in each game immediately after its play is concluded. Assume that each player's payoff is equal to the sum of her payoffs in the plays in the T games. If there is a unique Nash equilibrium in each game say $\sigma^t = (\sigma_1^t, \ldots, \sigma_I^t)$, then there is a unique SPNE of Γ_E and it consists of each player i playing strategy σ_i^t in each game Γ_N^t regardless of what has appened previously.

Sequential rationality, backward induction, and subgame perfection

- An interesting tension present in the SPNE concept that is related to the bounded rationality issue that does not arise in the context of simultaneous move games.

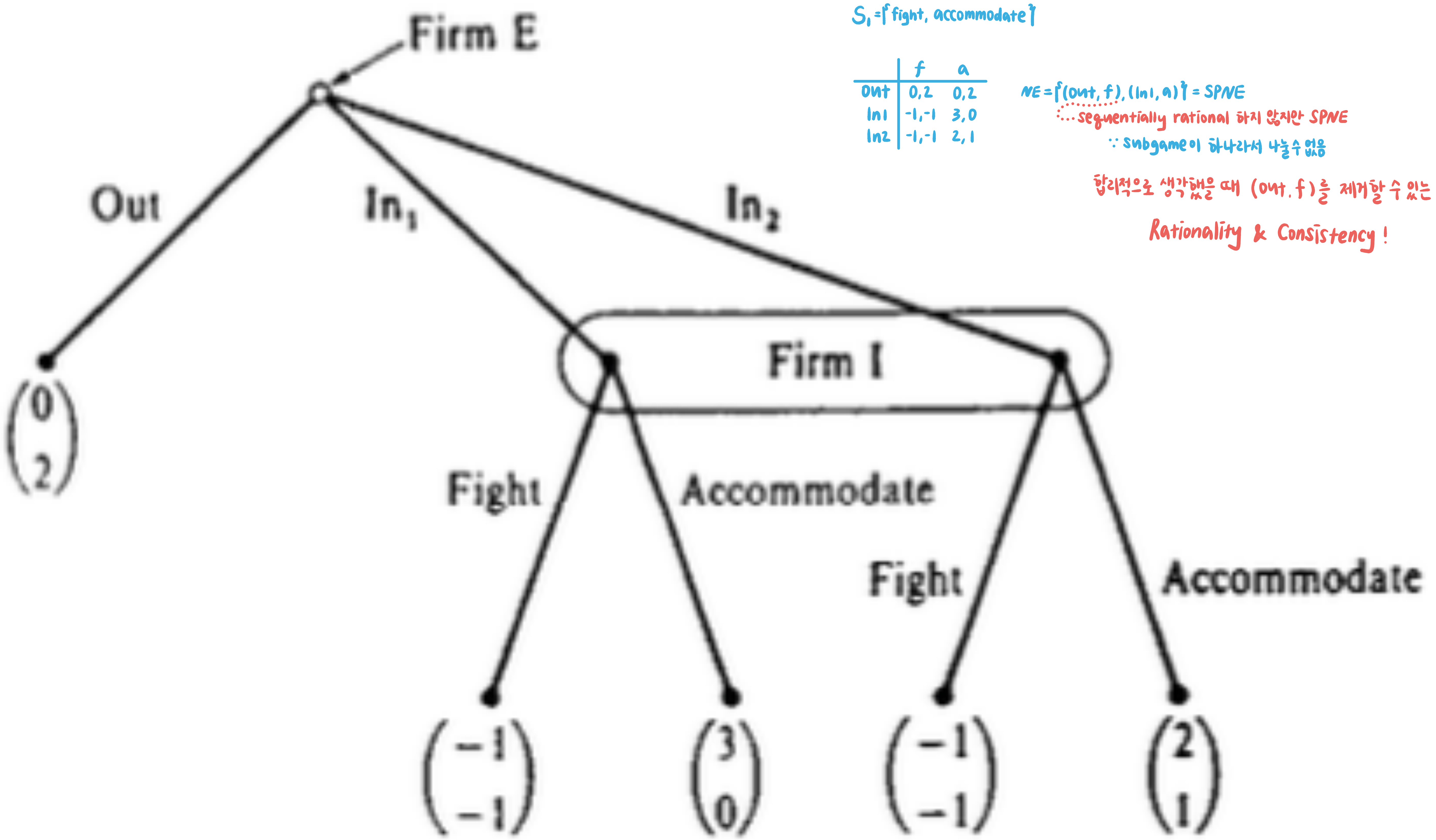
centpede game



균형이지만 실제로 첫 경기에서
Stop하는 일은 없을 것임

Beliefs and Sequential Rationality

- When SPNE concept fails to insure sequential rationality



$S_E = \{Out, In1, In2\}$
$S_I = \{fight, accommodate\}$

	f	a
Out	0,2	0,2
In1	-1,-1	3,0
In2	-1,-1	2,1

$NE = \{(Out, f), (In1, a)\} = SPNE$
sequentially rational 하지 않지만 SPNE
∵ subgame이 하나라서 나눌수 없음

합리적으로 생각했을 때 (Out. f)를 제거할 수 있는 조건
Rationality & Consistency !

① system of belief
subgame을 더 못나누니깐 information set 기준으로 생각
fight 하는 게 accommodate 하는 것보다 나은 belief system (μ) 가 존재하는가?
→ 존재하지 않음 → sequentially rational 하지 않음
(belief system의 rationality 따지기도 전에 이 전략조합은 제외됨)

② weak perfect Bayesian equilibrium
belief system (μ) 가 적절한 Bayes' rule 에 의해 정해졌나?
= consistent 한가?

Beliefs and Sequential Rationality

- **Definition**. *A system of belief* μ in extensive form game Γ_E is a specification of a probability $\mu(x) \in [0,1]$ for each decision node x in Γ_E such that $\Sigma_{x \in H}\mu(x) = 1$ for all information sets H. $E[u_i \mid H, \mu, \sigma_i, \sigma_{-i}]$: player i's expected utility starting at her information set H
  if her beliefs regarding the conditional probabilities of being at the various nodes in H are given by μ

- **Definition**. A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ in extensive form game Γ_E is *sequentially rational at information set H given a system of beliefs* μ if, denoting by $\iota(H)$ the player who moves at information set H, we have

information set 단위에서 BR이 되어야 함

$$E[u_{\iota(H)} \mid H, \mu, \sigma_{\iota(H)}, \sigma_{-\iota(H)}] \geq E[u_{\iota(H)} \mid H, \mu, \bar{\sigma}_{\iota(H)}, \sigma_{-\iota(H)}]$$

for all $\bar{\sigma}_{\iota(H)} \in \Delta(S_{\iota(H)})$. If strategy profile σ satisfies this condition for *al*l information sets H, then we say that σ is *sequentially rational given belief system* μ.

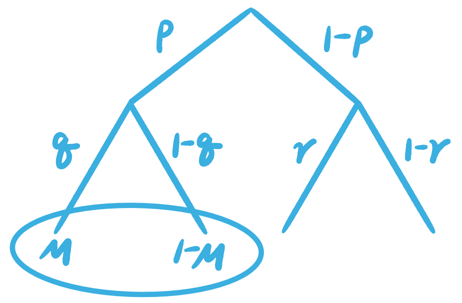
Beliefs and Sequential Rationality

First, Strategies must be sequentially rational given belief
Second, whenever possible beliefs must be consistent with the strategies
0보다 크다면

- **Definition**. A profile of strategies and system of beliefs (σ, μ) is a weak perfect Bayesian equilibrium (weak PBE) in extensive form game Γ_E if it has the following properties:

  (i) The strategy profile σ is sequentially rational given belief system μ.

  (ii) The system of beliefs μ is derived from strategy profile σ through Bayes' rule whenever poossible. That is for any information set H such that $\text{Prob}(H \mid \sigma) > 0$ (read as "the probability of reaching information set H is positive under strategies σ"), we must have



$$\frac{pq}{pq + p(1-q)} = q = \mu$$

$$\mu(x) = \frac{\text{Prob}(x \mid \sigma)}{\text{Prob}(H \mid \sigma)} \text{ for all } x \in H.$$

0 이면 consistency 판단 불가

Beliefs and Sequential Rationality

- **Proposition**. A strategy profile σ is a Nash equilibrium of extensive form game Γ_E if and only if there exists a system of beliefs μ such that

  (i) The strateay profile σ is sequentially rational given belief system μ at all information sets H such that $\text{Prob}(H \mid \sigma) > 0$.

  모든 H가 아닌 도달가능한 H에서만

  (ii) The system of beliefs μ is derived from strategy profile a through Bayes' rule whenever possible.

Beliefs and Sequential Rationality

$S_{E1} = \{$Out/In, Out/Out, Enter/In, Enter/Out, Propose/Out, Propose/In$\}$

$S_{E2} = \{$accept, decline$\}$

$S_1 = \{$fight, accommodate$\}$

SPNE = NE = $\{$(Propose/In, accept, accommodate), WPBE

(Propose/Out, accept, accommodate), ... $\}$

NOT sequentially rational

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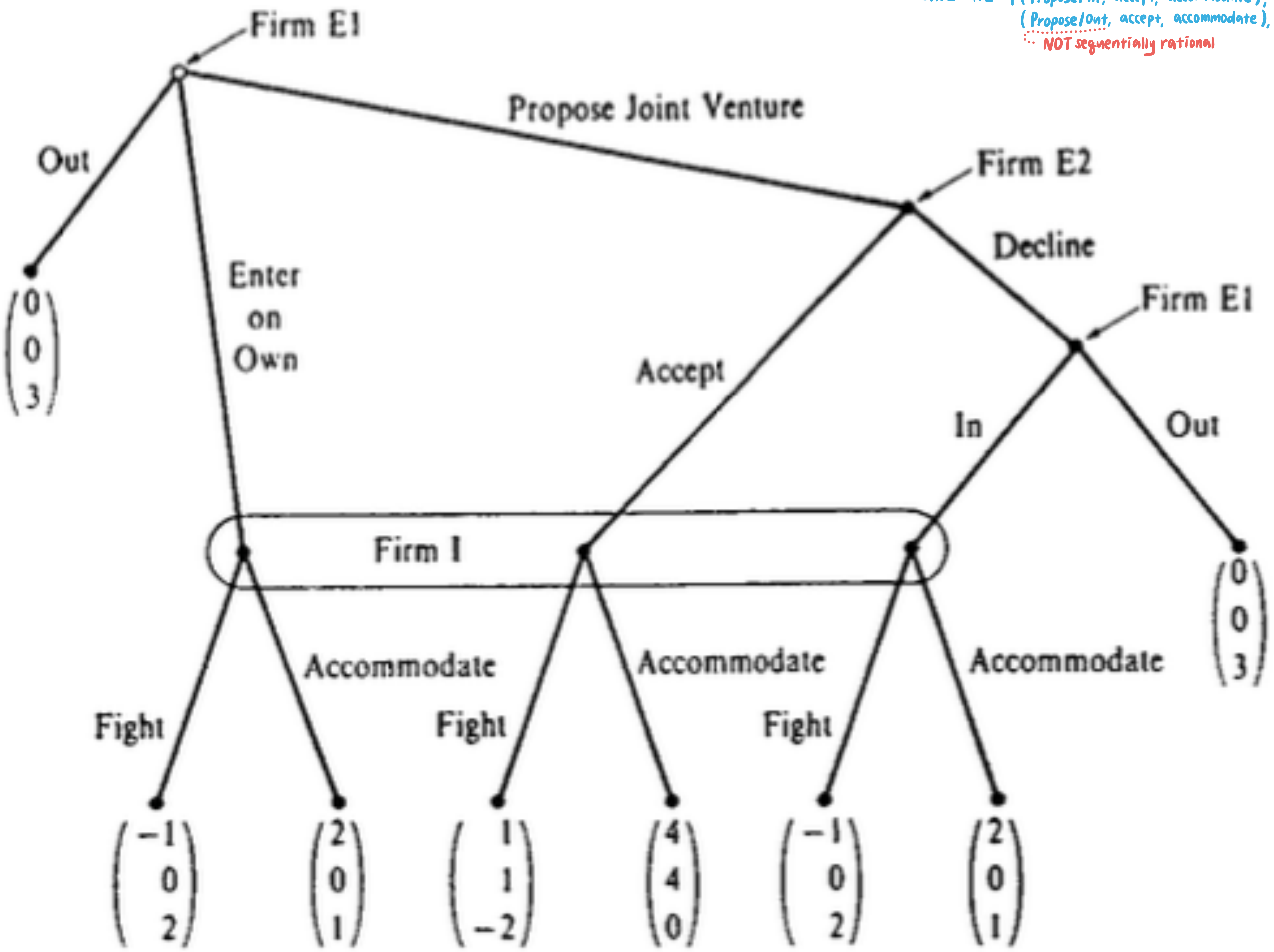

Beliefs and Sequential Rationality

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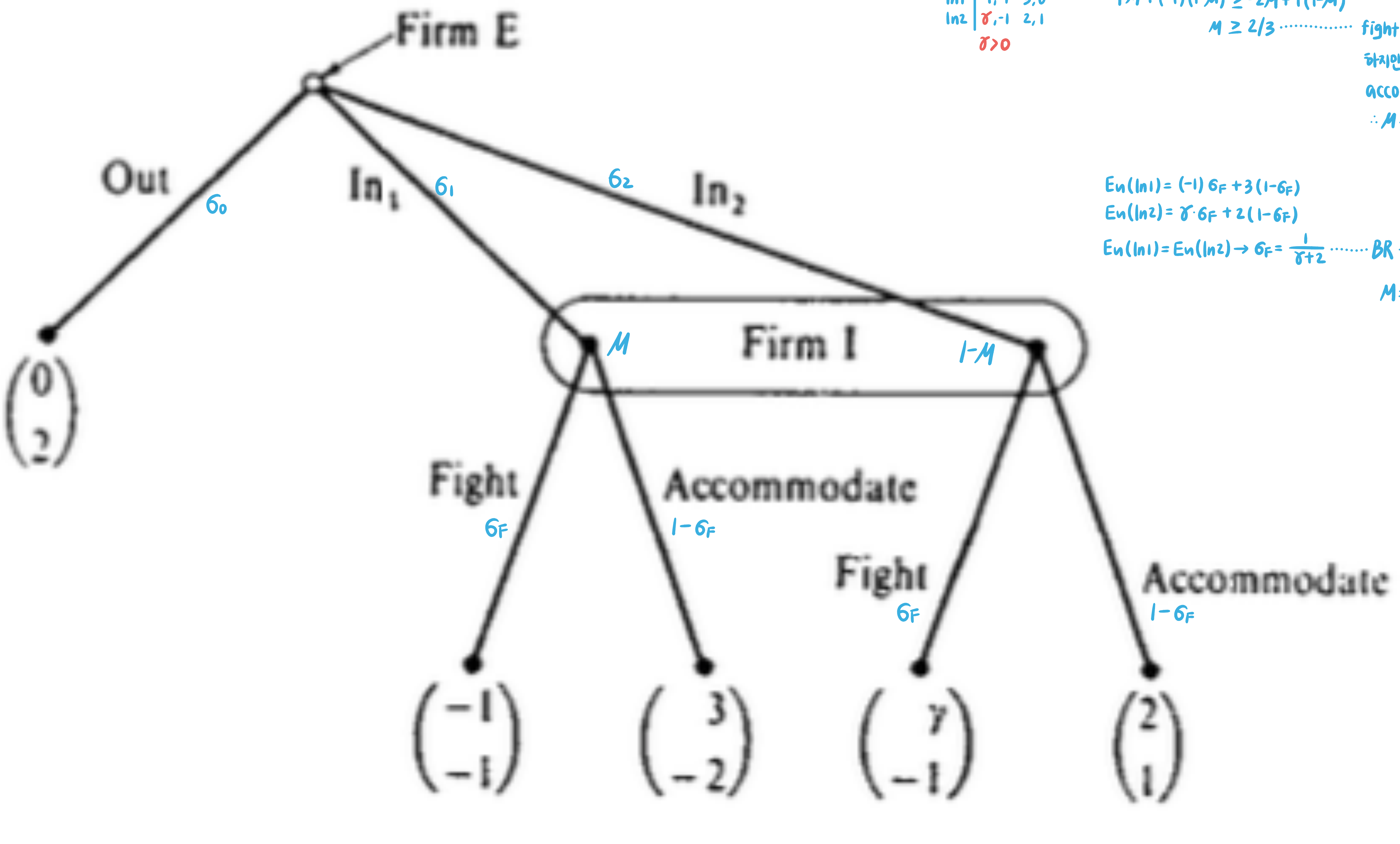


$\gamma > -1$

$S_E = \{Out, In1, In2\}$
$S_1 = \{fight, accommodate\}$

	f	a
Out	0,2	0,2
In1	-1,-1	3,0
In2	γ,-1	2,1

$\gamma > 0$

Pure strategy NE 존재하지 않음

$-1\cdot\mu + (-1)(1-\mu) \geq -2\mu + 1(1-\mu)$

$\mu \geq 2/3$ ········ fight을 sequentially rational 하게 만드는 μ 존재
하지만 (In2, fight) 가 consistent 하지 않음
accommodate 에 대해서도 부등호방향만 반대고 같음
$\therefore \mu = \frac{2}{3}$ 일때 randomize 할 것임

$Eu(In1) = (-1)\sigma_F + 3(1-\sigma_F)$
$Eu(In2) = \gamma\cdot\sigma_F + 2(1-\sigma_F)$
$Eu(In1) = Eu(In2) \rightarrow \sigma_F = \frac{1}{\gamma+2}$ ········ BR to $\sigma_F = \frac{1}{2+\gamma}$: $\sigma_1 \in [0,1]$

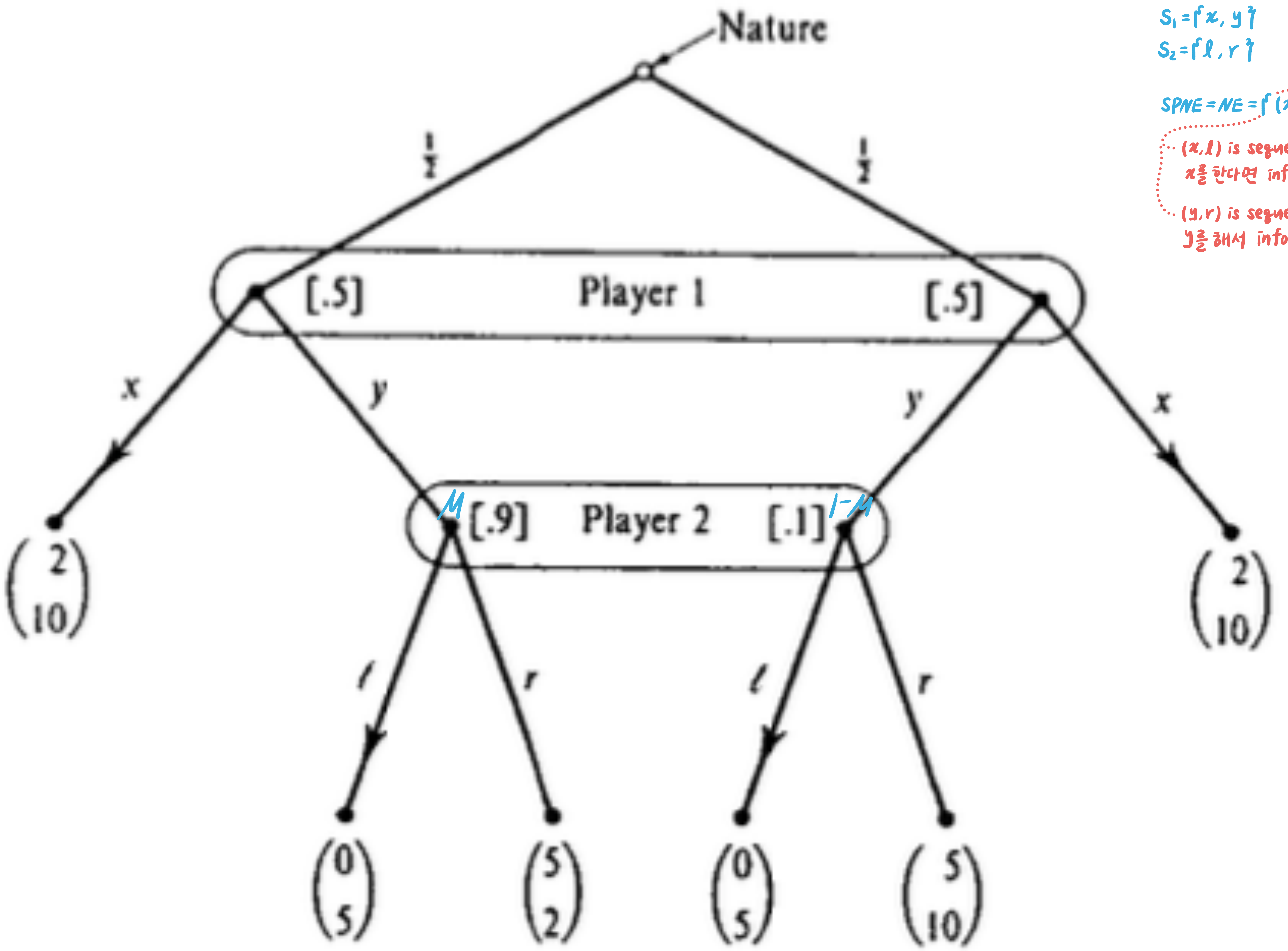
$\mu = \frac{2}{3}$ 이기 때문에 $\sigma_0 = 0$, $\sigma_1 = \frac{2}{3}$, $\sigma_2 = \frac{1}{3}$ 되어야 consistent
$\sigma_0 = \frac{1}{2}$, $\sigma_1 = \frac{2}{6}$, $\sigma_2 = \frac{1}{6}$ 도 가능

Beliefs and Sequential Rationality

- Strengthenings of the weak perfect Bayesian equilibrium concept

Belief in a weak PBE may not be structually consistent

Weak PBE may not even be subgame perfect



- Belief in a weak PBE may not be structually consistent

$S_1 = \{x, y\}$

$S_2 = \{l, r\}$

$SPNE = NE = \{(x, l), (y, r) \cdots\}$ weak PBE

$\mu = \lim_{k \to \infty} \mu^k$ 인데 $0.9 \neq 1/2$ 이므로

NOT sequential equilibrium

(x, l) is sequentially rational if $\mu > 5/8$

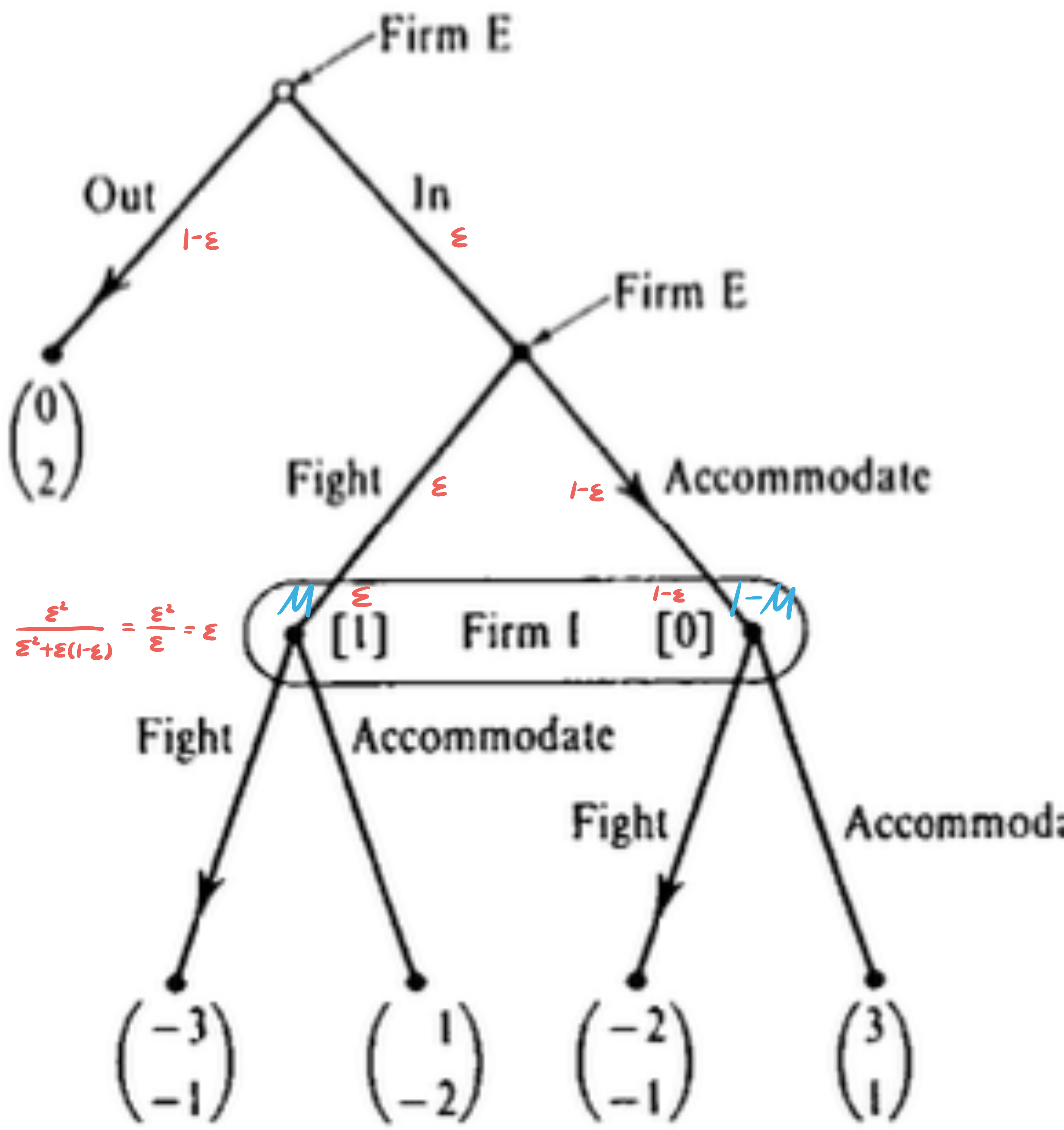
x를 한다면 information set에 들어올 확률이 0 이므로 consistency 판단 불가

(y, r) is sequentially rational if $\mu < 5/8$

y를 해서 information set 에 들어오려면 Nature가 $1/2$, $1/2$ 확률을 assign 하므로 $\mu = \frac{1}{2}$이면 consistent

Beliefs and Sequential Rationality

- Perfect Bayesian equilibrium
- Definition. A strategy profile and systme of beliefs (σ, μ) is a sequential equilibrium of extensive form game Γ_E if it has the following properties:
- (i) Strategy profile σ is sequentially rational given belief system μ.
- (ii) There exists a sequence of completely mixed strategies $\{\sigma^K\}_{k=1}^{\infty}$, with $\lim_{k\to\infty}\sigma^k = \sigma$, such that $\mu = \lim_{k\to\infty}\mu^k$, where μ^k denotes the beliefs derived from strategy profile σ^k using Bayes' rule.



- Belief in a weak PBE may not be structually consistent
- Weak PBE may not even be subgame perfect

$S_E = \{Out/f,\ Out/a,\ In/f,\ In/a\}$
$S_I = \{f, a\}$

$NE = \{(Out/f, f), (Out/a, f), (In/a, a)\}$
$SPNE = \{(In/a, a)\}$
$WPBE = \{(In/a, a), (Out/a, f)\}$

fight accommodate
$-1\cdot\mu + (-1)(1-\mu) > -2\cdot\mu + 1(1-\mu)$

sequentially rational if $\mu > 2/3$
Out하면 도달할 확률 0이므로 consistency 판단불가 .... $\mu = \lim_{k\to\infty}\mu^k$ 인데 $1 \neq 0$ 이므로
NOT sequential equilibrium

모든 information set에 도달할 확률을 0보다 크게 부여해서 해결
(trembling hand)

Beliefs and Sequential Rationality

- Perfect Bayesian equilibrium
- Proposition. In every sequential equilibrium (σ, μ) of an extensive form game Γ_E, the equilibrium strategy profile σ constitutes a subgame perfect Nash equilibrium of Γ_E.

Reasonable Beliefs and Forward Induction

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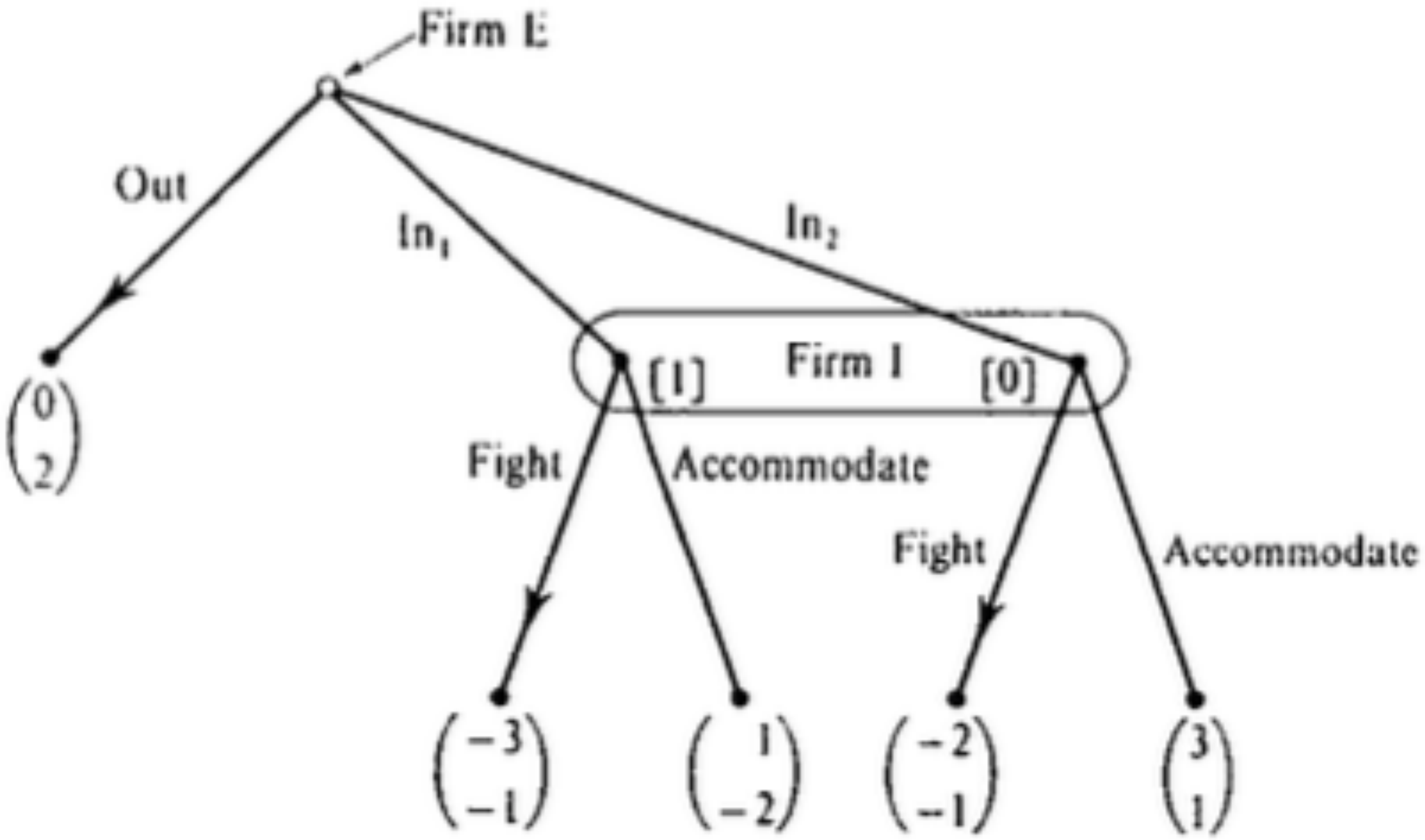

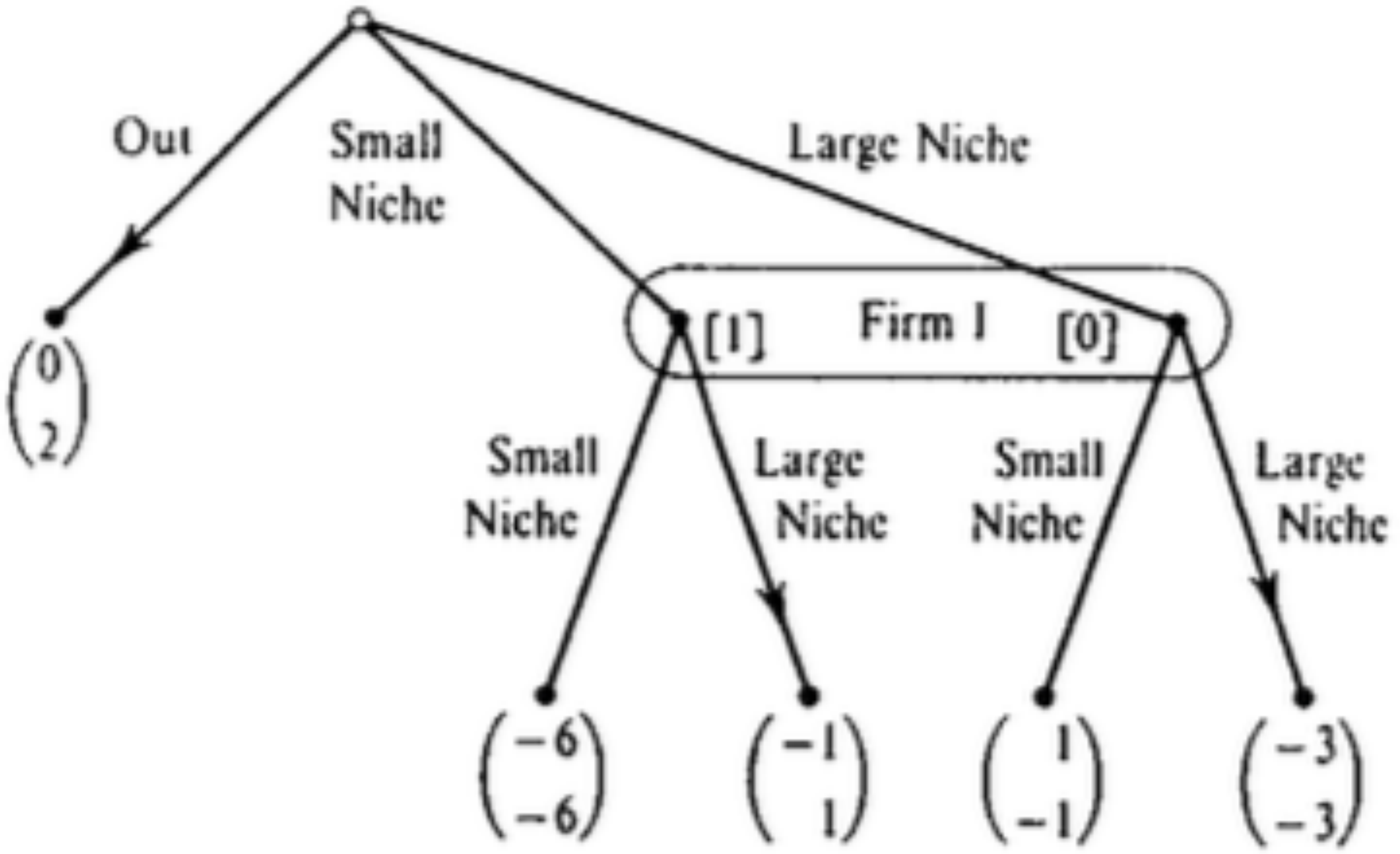

Reasonable Beliefs and Forward Induction

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