# A model of rational choice

# A model of rational choice

- A set $A$ of actions from which the decision maker makes a choice.
- A set $C$ of possible consequences of these actions.
- A consequence function (or a belief) $g : A \to C$ that associates a consequence with each action.
- A preference relation (a complete, transitive, reflexive binary relation) $\succsim$ on the set $C$.
- Or sometimes the decision maker's preferences are specified by giving a utility function $u : C \to R$, which defines a preference relation $\succsim$ by the condition $x \succsim y$ iff $u(x) \geq u(y)$.
- Given any set $B \subseteq A$ of actions that are feasible in some particular case, a rational decision maker chooses an action $a^*$ that is feasible ($a^* \in B$) and optimal in the sense that $g(a^*) \succsim g(a)$ for all $a \in B$, alternatively he solves the problem $\max_{a \in B} u(g(a))$.

# A model of rational choice under risk

- A set $A$ of actions from which the decision maker makes a choice.
- A set $C$ of possible consequences of these actions.
- A state space $\Omega$ and a probability measure over $\Omega$.
- A consequence function $g : A \times \Omega \to C$ that associates a consequence with each action ($a$) and each state ($\omega$).
- e.g., $A = \{a_1, a_2\}$, $\Omega = \{\omega_1, \omega_2\}$. Then by $g : A \times \Omega \to C$ we have $C = \{g(a_1, \omega), g(a_2, \omega)\}$, where $g(a_1, \omega) = ((a_1, \omega_1), (a_1, \omega_2))$ and and $g(a_2, \omega) = ((a_2, \omega_1), (a_2, \omega_2))$.
- Suppose $\Omega$ is the support of $p$ and denoted by $supp(p)$, and each $\omega \in \Omega = supp(p)$, $p(\omega) \geq 0$ and $\Sigma_{\omega \in supp(p)} p(w) = 1$. The set of simple probability distributions on X will be denoted by $P$.
- With $p = (p(\omega_1), p(\omega_2)) \in P$, we can define each consequence as a lottery, $L(g(a, \omega)\,|\,p(\omega))$, and the set of Lottery, $L$ as a modified the set of consequences.
- Can we define $u : L \to R$ such that $u(L_1) \geq u(L_2)$ iff $L_1 \succsim L_2$? See Expected Utility Theorem.

# A model of rational choice under strategic interaction

- A set $N$ of individuals who participate in the interaction.
- A set $A_i$ of actions from which decision maker $i$ makes a choice.
- A set $A = \Pi_{j \in N} A_j$ is the set of the action profiles.
- A set $C$ of possible consequences of these action profiles.
- A consequence function $g : A \to C$ that associates a consequence with each action profile.
- A preference relation (a complete, transitive, reflexive binary relation) of individual $i$, $\succsim_i$, on the set $C$. Or sometimes the decision maker's preferences are specified by giving a utility function $u_i : C \to R$, which defines a preference relation $\succsim_i$ by the condition $x \succsim_i y$ iff $u_i(x) \geq u_i(y)$.

# A model of rational choice under empathetic preferences

- A set $N$ of individuals who participate in the interaction.
- A set $A_i$ of actions from which decision maker $i$ makes a choice.
- A set $A = \Pi_{j \in N} A_j$ is the set of the action profiles.
- A set $C$ of possible consequences of these action profiles.
- A consequence function $g : A \to C$ that associates a consequence with each action profile.
- A preference relation (a complete, transitive, reflexive binary relation) of individual $i$, $\succsim_i$, on the set $C \times N$. For example, for $x, y \in C$ and $m, n \in N$, individual $i$'s preference can be either $(x, m) \succsim_i (y, n)$ or $(y, n) \succsim_i (x, m)$ or both. (or between $(x, m)$ and $(x, n)$).
- Or sometimes individual $i$'s empathetic preferences are specified by giving a utility function $v_i : C \times N \to R$, which defines a preference relation $\succsim_i$ by the condition $(x, m) \succsim_i (y, n)$ iff $v_i(x, m) \geq v_i(y, n)$.

# Basic Elements of Noncooperative Games

# What is a game?

- To describe a situation of strategic interaction, we need to know four things:
    i) The players: Who is involved?
    ii) The rules: Who moves when? What do they know when they move? What can they do?
    iii) The outcomes: For each possible set of actions by the players, what is the outcome of the game?
    iv) The payoffs: What are the players' preferences (i.e., utility functions) over the possible outcomes?

# What is a game?

- Example. Matching Pennies
  - *Players*: There are two players, 1 and 2.
  - *Rules*: Each player simultaneously puts a penny down, either heads up or tails up.
  - *Outcomes*: Two pennies match or do not match.
  - *Payoffs*: If the two pennies match (either both heads up or both tails up), player 1 pays 1 dollar to player 2: otherwise, player 2 pays 1 dollar to player 1.

## What is a game?

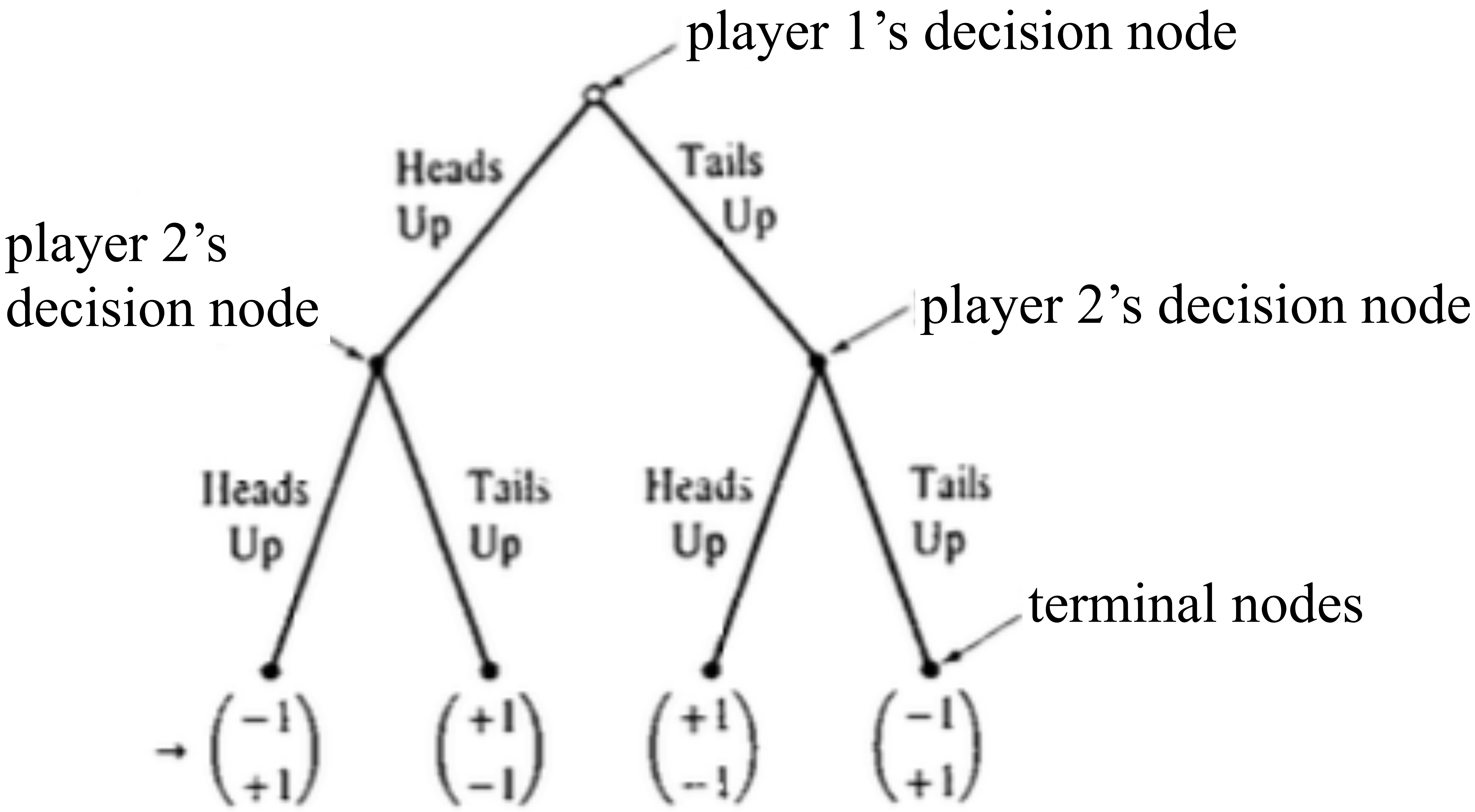
- Meeting in New York
  - *Players*: Two players, Mr. Thomas and Mr. Schelling.
  - *Rules*: The two players are separated and cannot communicate. They are supposed to meet in New York City at noon for lunch but have forgotten to specify where. Each must decide where to go (each can make only one choice).
  - *Outcomes*: If they meet each other, they get to enjoy each other's company at lunch. Otherwise, they must eat alone.
  - *Payoffs*: They each attach a monetary value of 100 dollars to the other’s company (their payoffs are each 100 dollars if they meet, 0 dollars if they do not).

# What is a game?

- Payoff
  - As a general matter, we describe a plaver's preferences by a utility function that assigns a utility level for each possible outcome. It is common to refer to the player's utility function as her *payoff* function and the utility level as her payoff. Throughout, we assume that these utility functions take an expected utility form so that when we consider situations in which outcomes are random, we can evaluate the random prospect by means of the player's expected utility.

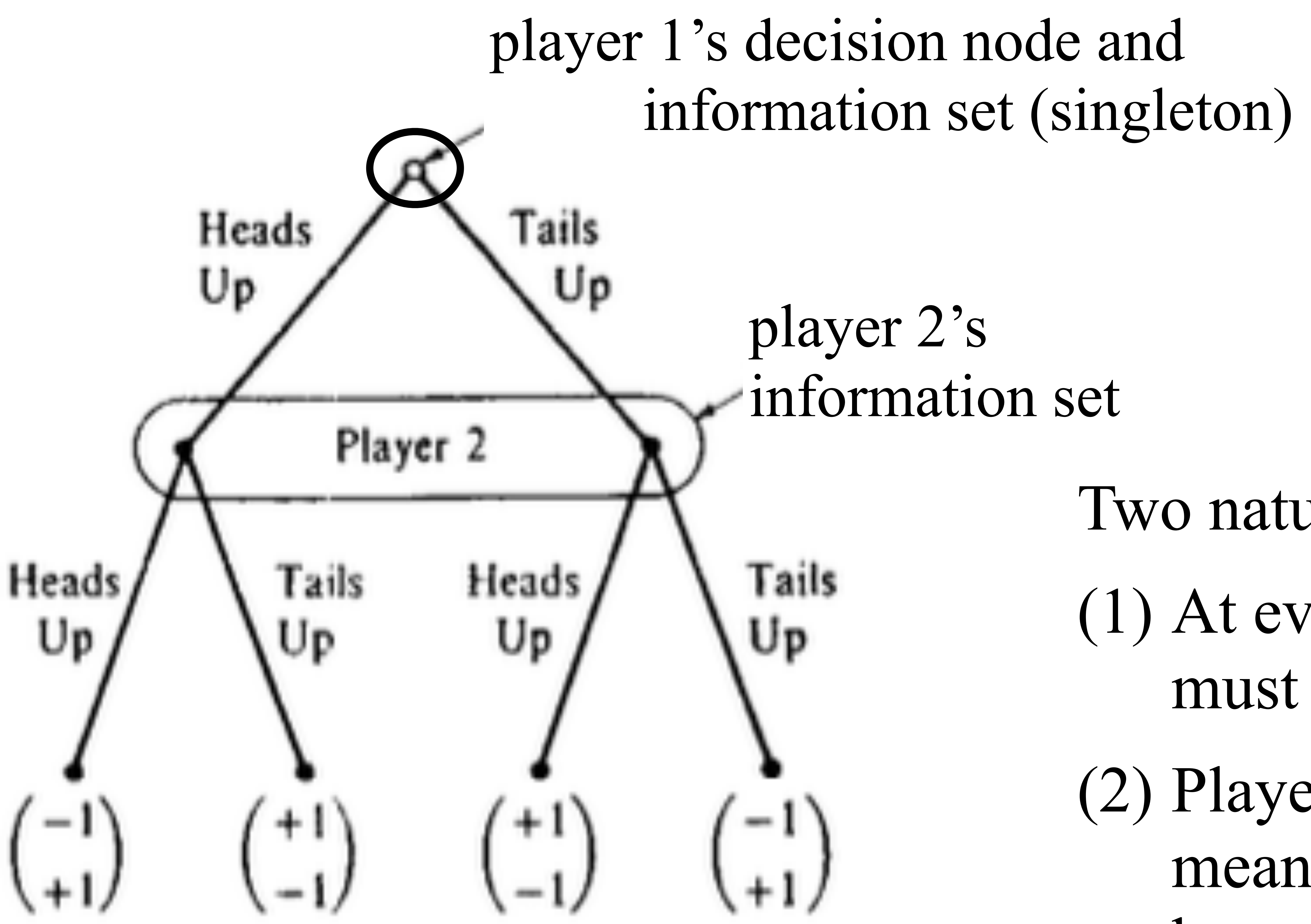
# The extensive form representation of a game.

- Matching Pennies v.B and its extensive form. Version B is identical to Matching Pennies (see Example 7.B.I) except that the two players move sequentially, rather than simultaneously.



# The extensive form representation of a game.

- Matching Pennies v.C and its extensive form. This version is just like Matching Pennies Version B except that when player I puts her penny down, she keeps it covered with her hand. Hence, player 2 cannot see player l's choice until after player 2 has moved.



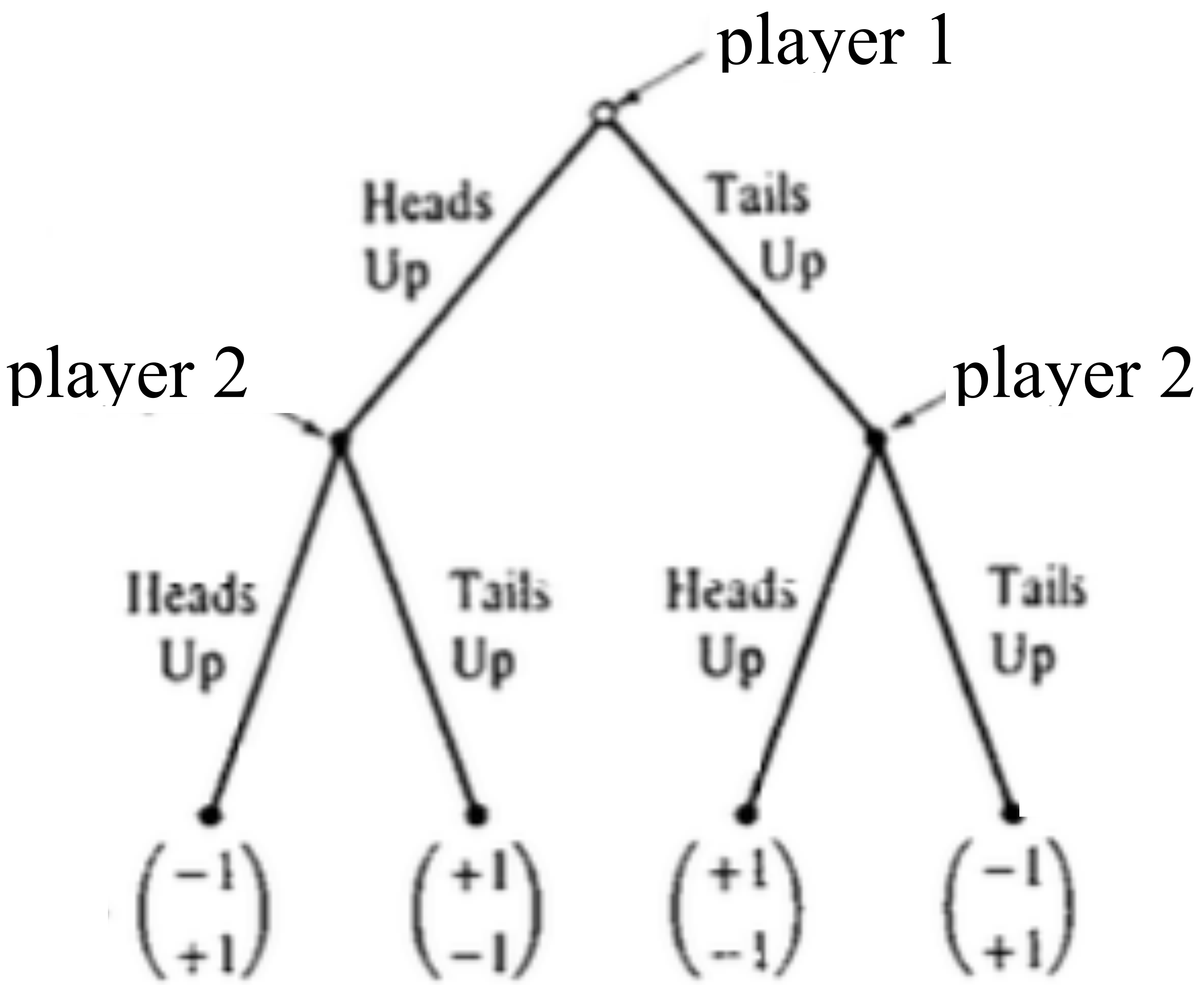

Player 2's two decision nodes to indicate that these two nodes are in a single information set, meaning that this information set is that when it is player 2's turn to move, she cannot tell which of these two nodes she is.

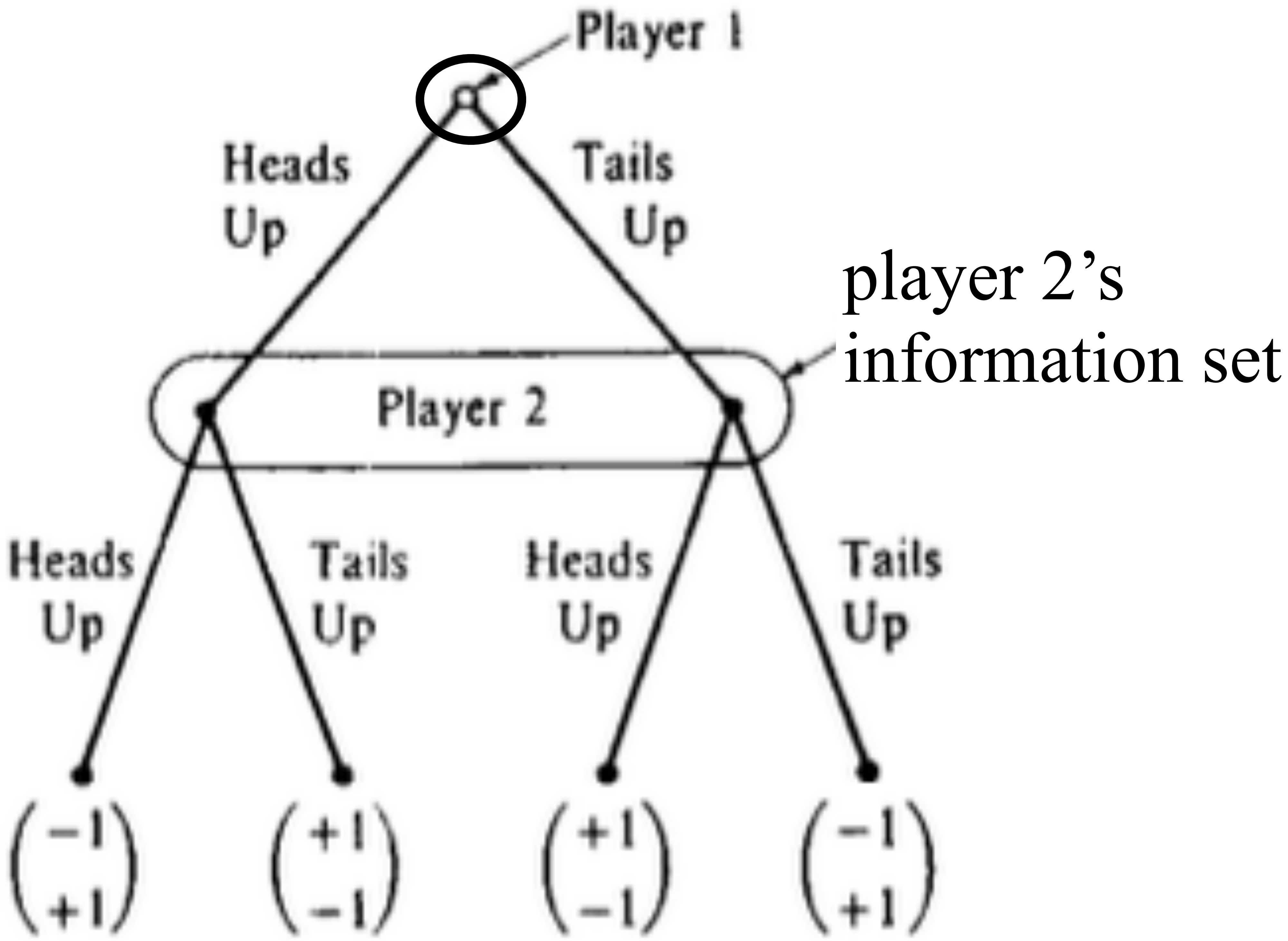
Two natural **restrictions on information sets**:

(1) At every node within a given information set, a player must have the same set of possible actions.

(2) Players possess what is known as *perfect recall*, meaning that a player does not forget what she once knew, including her own actions.

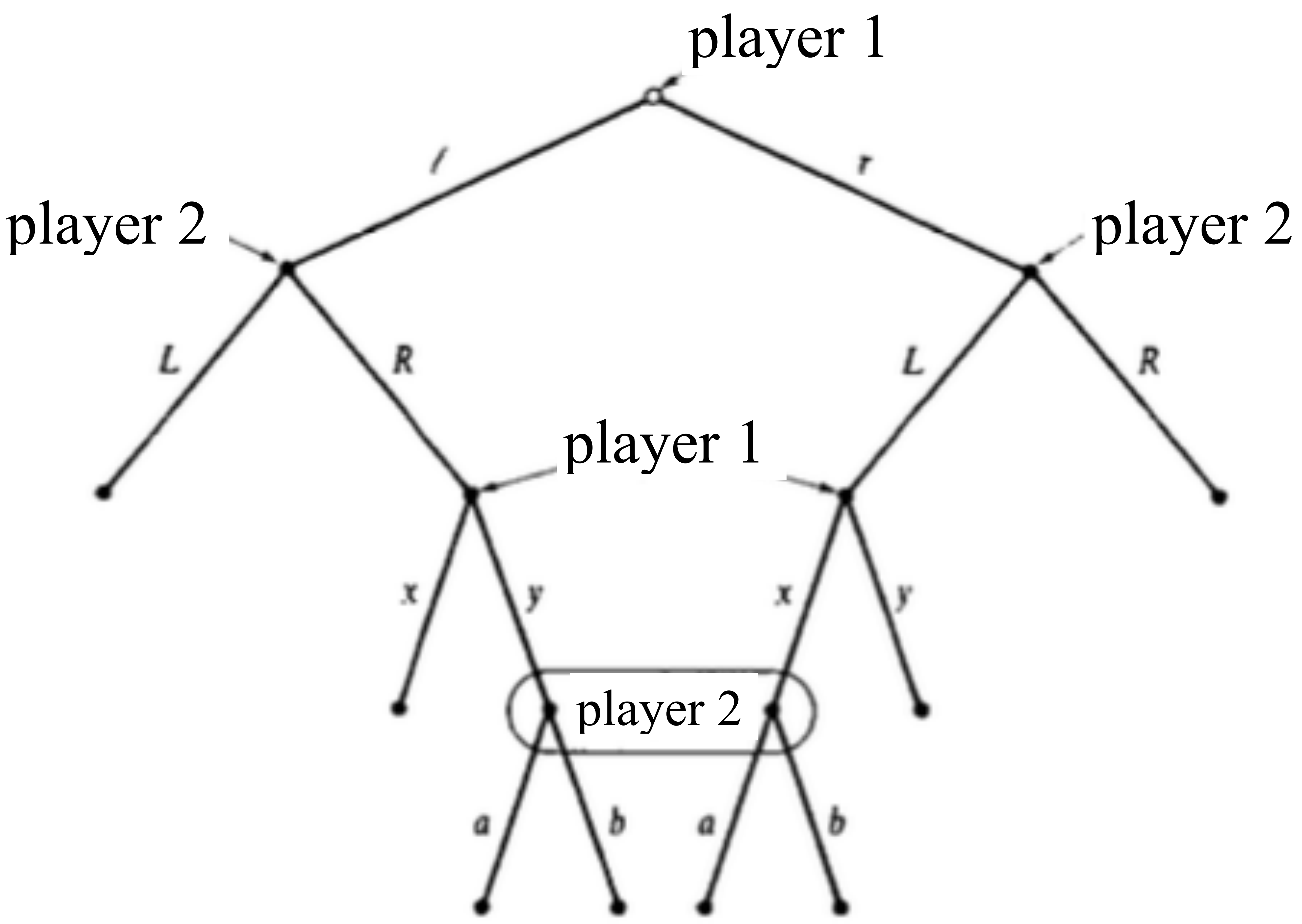
# The extensive form representation of a game.

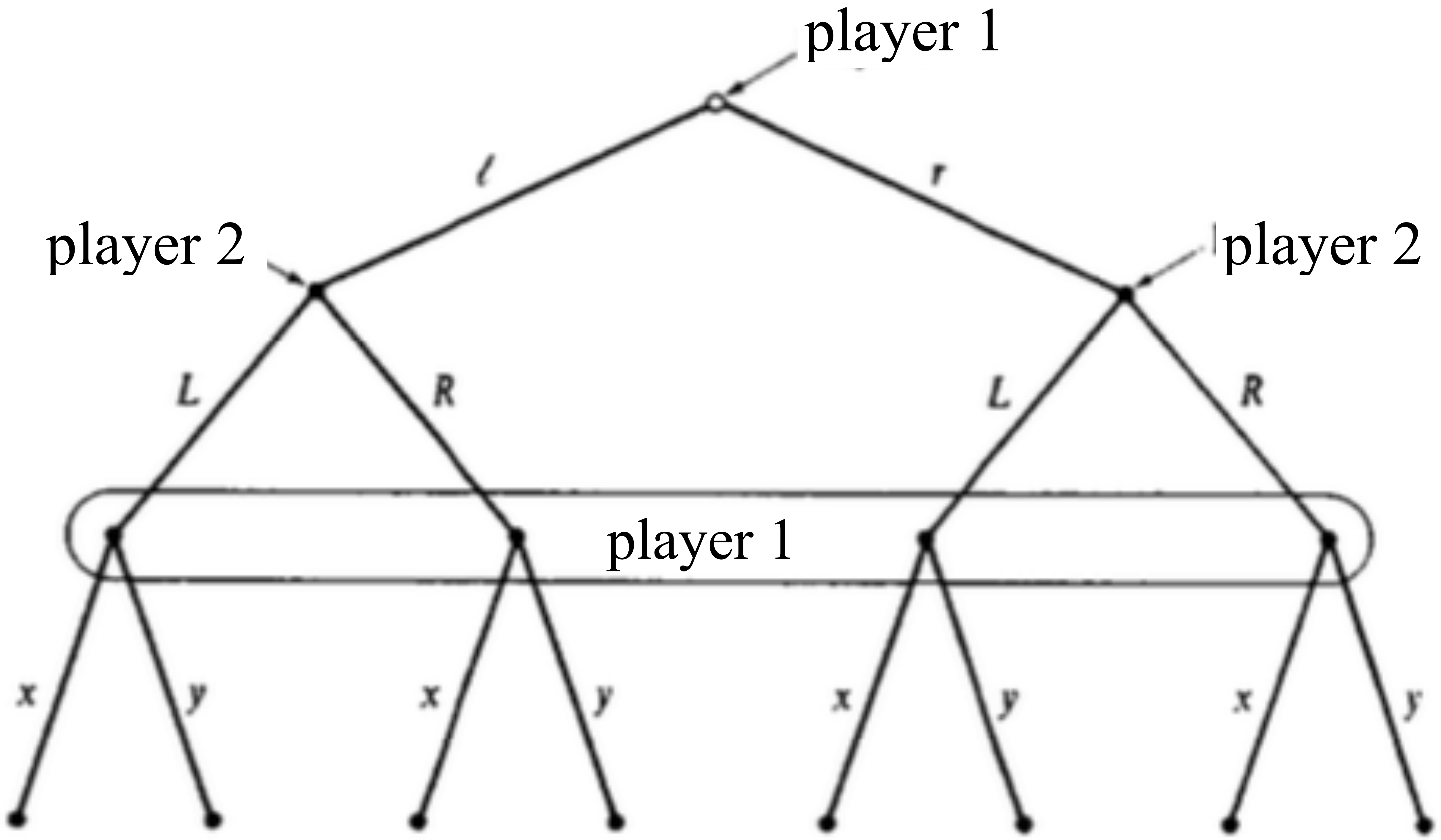



- The game on the left is of perfect information. When it is a player's turn to move, she is able to observe all her rival's previous moves.
- **Definition**. A game is one of *perfect information* if each information set contains a single decision node. Otherwise, it is a game of imperfect information.

# The extensive form representation of a game.

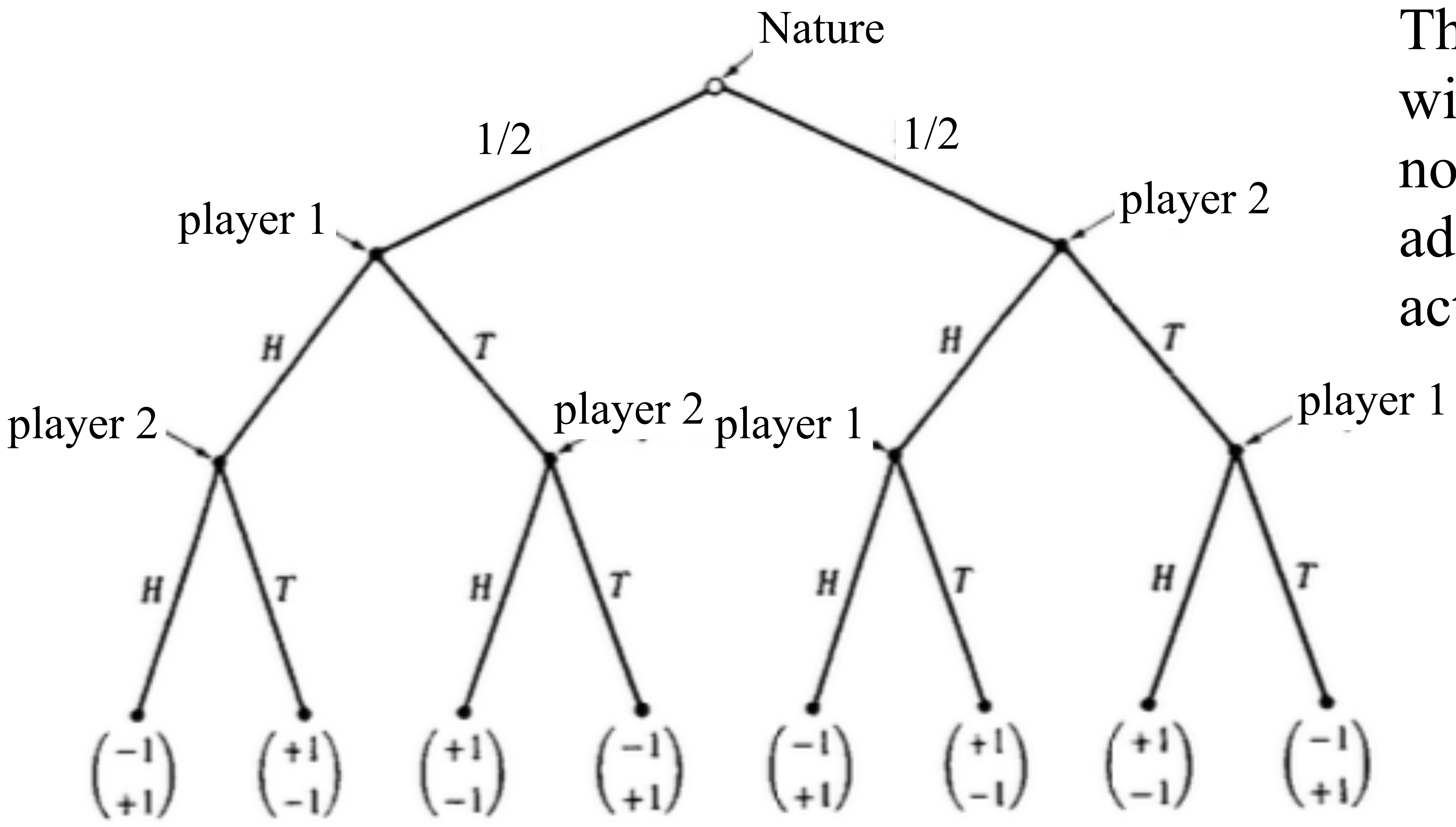



- The games not satisfying perfect recall, meaning that a player does not forget what she once knew, including her own actions.

# The extensive form representation of a game.

- Matching Pennies v.D and its extensive form. Suppose that prior to playing Matching Pennies Version B, the two players flip a coin to see who will move first. Thus, with equal probability either player 1 will put her penny down first, or player 2 will.




This game is depicted as beginning with a move of nature at the initial node. *Nature* in this game is an additional player who must play its two actions with fixed probabilities.

## The extensive form representation of a game.

Formally, a game represented in extensive form consists of the following items:

(i) A finite set of nodes $\mathscr{X}$, a finite set of possible actions $\mathscr{A}$, and a finite set of players $\{1\ldots., I\}$.

(ii) A function $p : \mathscr{X} \to \{\mathscr{X} \cup \varnothing\}$ specifying a single immediate predecessor of each node $x$; $p(x)$ is nonempty for all $x \in \mathscr{X}$ but one, designated as the initial node $x_0$. The immediate successor nodes of $x$ are then $s(x) = p^{-1}(x)$, and the set of all predecessors and all successors of node $x$ can be found by iterating $p(x)$ and $s(x)$. To have a tree structure, we require that these sets be disjoint (a predecessor of node $x$ wlr

cannot also be a successor to it). The set of terminal nodes is $T = \{x \in \mathscr{X} : s(x) = \varnothing\}$. All other nodes $\mathscr{X}\backslash T$ are known as decision nodes.

(iii) A function $\alpha : \mathscr{X}\backslash\{x_0\} \to \mathscr{A}$ giving the action that leads to any noninitial node $x$ from its immediate predecessor $p(x)$ and satisfying the property that if $x', x'' \in s(x)$ and $x' \neq x''$, then $\alpha(x') \neq \alpha(x'')$. The set of choices available at decision node $x$ is $c(x) = \{a \in \mathscr{A} : a = \alpha(x') \text{ for some } x' \in s(x)\}$.

# The extensive form representation of a game.

(iv) A collection of information sets $\mathscr{H}$, and a function $H : \mathscr{X} \to \mathscr{H}$ assigning each decision node $x$ to an information set $H(x) \in \mathscr{H}$. Thus, the information sets in $\mathscr{H}$ form a partition of $\mathscr{X}$. We require that all decision nodes assigned to a single information set have the same choices available; formally, $c(x) = c(x')$ if $H(x) = H(x')$. We can therefore write the choices available at information set $H$ as $C(H) = \{a \in \mathscr{A} : a \in c(x) \text{ for } x \in H\}$.

(v) A function $\iota : \mathscr{H} \to \{0,\ldots,I\}$ assigning each information set in $\mathscr{H}$ to the player (or to nature: formally, player 0) who moves at the decision nodes in that set. We can denote the collection of player $i$'s information sets by $\mathscr{H}_i = \{H \in \mathscr{H} : i = \iota(H)\}$.

(vi) A function $\rho : \mathscr{H}_0 \times \mathscr{A} \to [0,1]$ assigning probabilities to actions at information sets where nature moves and satisfying $\rho(H, a) = 0$ if $a \notin C(H)$ and $\Sigma_{a \in C(H)} \rho(H, a) = 1$ for all $H \in \mathscr{H}_0$.

(vii) A collection of payoff functions $u = \{u_1(\cdot), \ldots, u_I(\cdot)\}$ assigning utilities to the players for each terminal node that can be reached, $u_i : T \to R$. Because we want to allow for a random realization of outcomes we take each $u_i(\cdot)$ to be a Bernoulli utility function.

# The extensive form representation of a game.

- Thus, formally, a game in extensive form is specified by the collection $\Gamma_E = \{\mathscr{X}, \mathscr{A}, p(\cdot), \alpha(\cdot), \mathscr{H}, H(\cdot), \iota(\cdot), \rho(\cdot), u\}$.

## Strategies and the normal form representation of a game

- **Definition.** Let $\mathscr{H}_i$ denote the collection of player $i$'s information sets, $\mathscr{A}$ the set of possible actions in the game, and $C(H) \subset \mathscr{A}$ the set of actions possible at information set $H$. A (*pure*) *strategy* for player $i$ is a function $s_i : \mathscr{H}_i \to \mathscr{A}$ such that $s_i(H) \in C(H)$ for all $H \in \mathscr{H}_i$.

- Example. Strategies in Matching Pennies Version B

- Example. Strategies in Matching Pennies Version C

- Example. Strategies in Matching Pennies Version D

# Strategies and the normal form representation of a game

- **Definition.** For a game with $I$ players, the normal form representation $\Gamma_N$ specifies for each player $i$ a set of strategies $S_i$ (with $s_i \in S_i$) and a payoff function $u_i(s_1\ldots, s_I)$ giving the von Neumann-Mrgenstern utiliy levels associated with the (possibly random) outcome arising from a strategy profile $(s_1, \ldots, s_I)$. Formally we write $\Gamma_N = \{I, \{S_i\}, \{u_i(\cdot)\}\}$.
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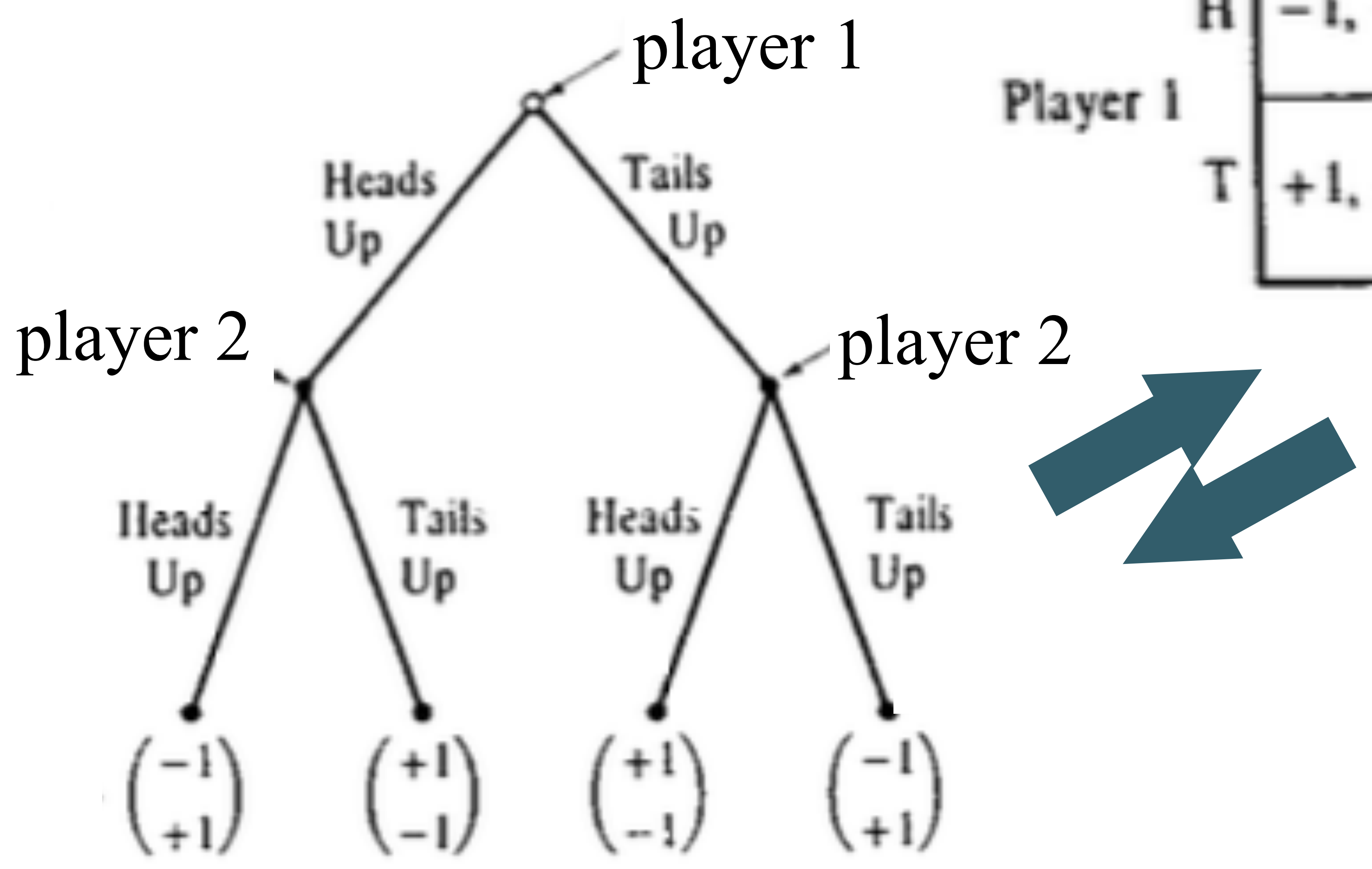
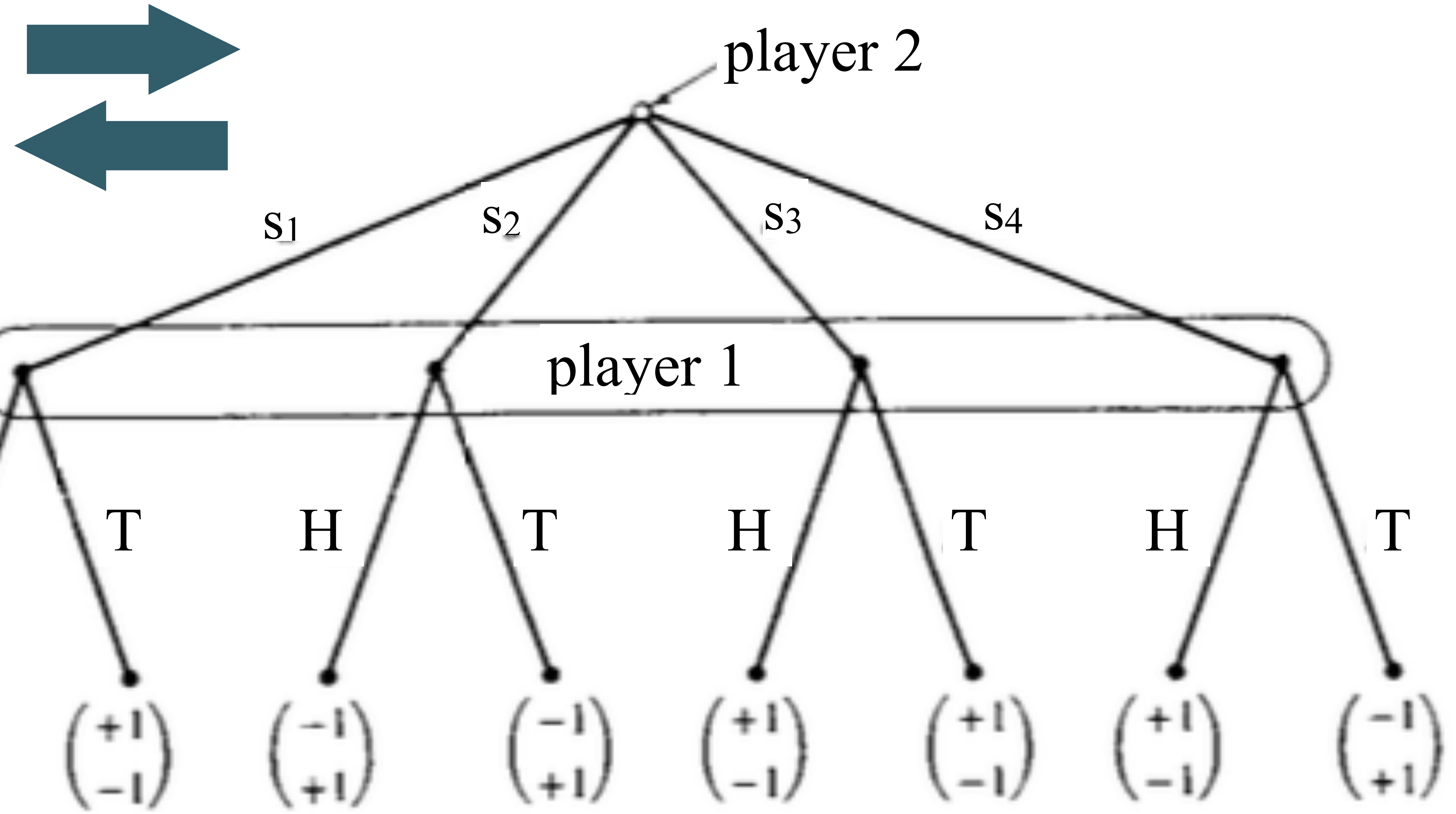



Player 2

| Player 1 | $s_1$ | $s_2$ | $s_3$ | $s_4$ |
|---|---|---|---|---|
| H | −1, +1 | −1, +1 | +1, −1 | +1, −1 |
| T | +1, −1 | −1, +1 | +1, −1 | −1, +1 |



# Randomized choices

- **Definition.** Given player $i$'s (finite) pure strategy set $S_i$, a mixed strategy for player $i$, $\sigma_i : S_i \to [0,1]$ assignes to each pure strategy $s_i \in S_i$, a probability $\sigma_i(s_i) \geq 0$ that it will be played, where $\Sigma_{s_i \in S_i} \sigma_i(s_i) = 1$.
- Suppose that player $i$ has $M$ pure strategies in set $S_i = \{s_{1i}, \ldots, s_{Mi}\}$. Player $i$'s set of possible mixed strategies can therefore be associated with the points of the following simplex:

$$\Delta(S_i) = \{(\sigma_{1i}, \ldots, \sigma_{Mi}) \in R^M : \sigma_{mi} \geq 0 \text{ for all } m = 1,\ldots, M \text{ and } \Sigma_{m=1}^{M} \sigma_{mi} = 1\}$$

- This simplex is called the *mixed extension* of $S$.
- Note that a pure strategy can be viewd as a special case of a mixed strategy in which the probability distribution over the element of $S_i$ is degenerate.

# Randomized choices

- When players randomize over their pure strategies, the induced outcome is itself random, leading to a probability distribution over the terminal nodes of the game. Since each player $i$'s normal form payoff function $u_i(s)$ is of the von Neumann-Morgenstern type, player $i$'s payoff given a profile of mixed strategies $\sigma = (\sigma_1, \ldots, \sigma_I)$ for $I$ players is her expected utility $E_\sigma[u_i(s)]$, the expectation being taken with respect to the probabilities induced by $\sigma$ on pure strategy profiles $s = (s_1, \ldots, s_I)$. That is, letting $S = S_1 \times \cdots \times S_I$, player $i$'s von Neumann-Morgenstern utility from mixed strategy profile $\sigma$ is

$$E_\sigma[u_i(s)] = \Sigma_{s \in S}[\sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_I(s_I)]u_i(s)$$

  which, with a slight abuse of notation, we denote by $u_i(\sigma)$. Note that because we assume that each player randomizes on her own, we take the realizations of players' randomizations to be independent of one another.

- The basic definition of the normal form representation is $\Gamma_N = [I, \{\Delta(S_i\}, \{u_i(\cdot)\}]$.

# Randomized choices

- **Another way to randomize**.
- Rather than randomizing over the pure strategies in $S_i$, she could randomize separately over the possible actions at each of her information sets $H \in \mathscr{H}_i$. This way of randomizing is called a *behavior strategy*.
- **Definition**. Given an extensive form game $\Gamma_E$, a *behavior strategy* for player $i$ specifies, for every information set $H \in \mathscr{H}_i$ and action $a \in C(H)$, a probability $\lambda_i(a, H) \geq 0$, with $\Sigma_{a \in C(H)} \lambda_i(a, H) = 1$ for all $H \in \mathscr{H}_i$.

# Simultaneous-Move Games

# Dominant and dominated strategies

- Consider games, $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ whose strategy sets allow for only pure strategies.
- **Definition**. A strategy $s_i \in S_i$ is a *strictly dominant strategy* for player $i$ in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if for all $s_i' \neq s_i$, we have

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$$

  for all $s_{-i} \in S_{-i}$

| | | Prisoner 2 | |
|---|---|---|---|
| | | Don't Confess | Confess |
| Prisoner 1 | Don't Confess | −2, −2 | −10, −1 |
| | Confess | −1, −10 | −5, −5 |

Playing "confess" is each player's best strategy regardless of what the other player does. This type of strategy is known as a strictly dominant strategy.

- **Definition**. A strategy $s_i \in S_i$ is *strictly dominated* for player $i$ in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if there exists another strategy $s_i' \in S_i$ and $s_i' \neq s_i$ such that for all $s_{-i} \in S_{-i}$,

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$$

- In this case, we say that strategy $s_i'$ *strictly dominates* strategy $s_i$
- Strategy $s_i$ is a *strictly dominant strategy* for player $i$ in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if it strictly dominates every other strategy in $S_i$.

# Dominant and dominated strategies

- **Definition**. A strategy $s_i \in S_i$ is *weakly dominated* in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if there exists another strategy $s_i' \in S_i$ such that for all $s_{-i} \in S_{-i}$,

$$u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$$

  with strict inequality for some $s_i$.

- A strategy is a *weakly dominant strategy* for player $i$ in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if it weakly dominates every other strategy in $S_i$

Player 2

| | | $L$ | $R$ |
|---|---|---|---|
| | $U$ | 1, −1 | −1, 1 |
| Player 1 | $M$ | −1, 1 | 1, −1 |
| | $D$ | −2, 5 | −3, 2 |

D is strictly dominated

Player 2

| | | $L$ | $R$ |
|---|---|---|---|
| | $U$ | 5, 1 | 4, 0 |
| Player 1 | $M$ | 6, 0 | 3, 1 |
| | $D$ | 6, 4 | 4, 4 |

U and M are weakly dominated

# Dominant and dominated strategies

- Note for weakly dominated strategies. Unlike a strictly dominated strategy, a strategy that is only weakly dominated cannot be ruled out based solely on principles of rationality. Caution might therefore rule out weakly dominated strategies. More generally, weakly dominated strategies could be dismissed if players always believed that there was at least some positive probability that any strategies of their rivals could be chosen.

| | | Player 2 | |
|---|---|---|---|
| | | $L$ | $R$ |
| | $U$ | 1, −1 | −1, 1 |
| Player 1 | $M$ | −1, 1 | 1, −1 |
| | $D$ | −2, 5 | −3, 2 |

D is strictly dominated

| | | Player 2 | |
|---|---|---|---|
| | | $L$ | $R$ |
| | $U$ | 5, 1 | 4, 0 |
| Player 1 | $M$ | 6, 0 | 3, 1 |
| | $D$ | 6, 4 | 4, 4 |

U and M are weakly dominated

# Dominant and dominated strategies

- **Iterated deletion of strictly dominated strategies**
- Note the way players' common knowledge of each other's payoffs and rationality is used to solve the game on the right. Elimination of strictly dominated strategies requires only that each player be rational. What we have just done, however, requires not only that prisoner 2 be rational but also that prisoner I know that prisoner 2 is rational. Put somewhat differently, a player need not know anything about his opponents' payoffs or be sure of their rationality to eliminate a strictly dominated strategy from consideration as his own strategy choice; but for the player to elimination of his strategies from consideration because it is dominated if his opponents never play their dominated strategies does require this knowledge.

| | | Prisoner 2 | |
|---|---|---|---|
| | | Don't Confess | Confess |
| Prisoner 1 | Don't Confess | 0, −2 | −10, −1 |
| | Confess | −1, −10 | −5, −5 |

# Dominant and dominated strategies

- **Iterated deletion of strictly dominated strategies**
- The iterated deletion of weakly dominated strategies is harder to justify. As we have already indicated, the argument for deletion of a weakly dominated strategy for player *i* is that he contemplates the possibility that every strategy combination of his rivals occurs with positive probability. However, this hypothesis clashes with the logic of iterated deletion, which assumes, precisely, that eliminated strategies are not expected to occur.This inconsistency leads the iterative elimination of weakly dominated strategies to have the undesirable feature that it can depend on the order of deletion. The game on the right provides an example. If we first eliminate strategy U, we next eliminate strategy L, and we can then eliminate strategy M: (D. R) is therefore our prediction. If, instead, we eliminate strategy M first, we next eliminate strategy R, and we can then eliminate strategy U: now (D, L) is our prediction.

| | | Player 2 | |
|---|---|---|---|
| | | *L* | *R* |
| | *U* | 5, 1 | 4, 0 |
| Player 1 | *M* | 6, 0 | 3, 1 |
| | *D* | 6, 4 | 4, 4 |

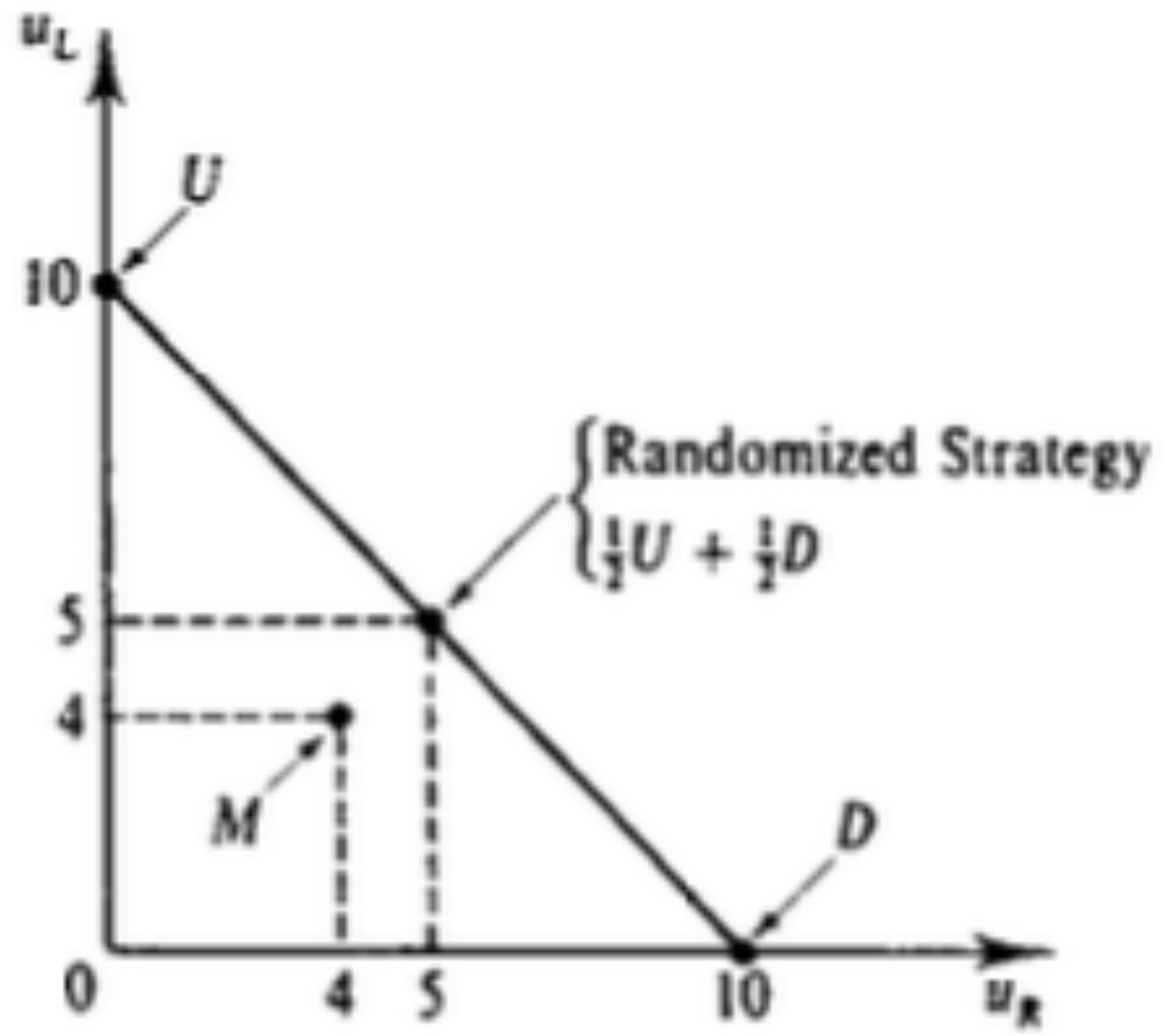
# Dominant and dominated strategies

- **Dominant and dominated strategies *with mixed strategies.***
- **Definition**. A strateay $\sigma_i \in \Delta(S_i)$ is *strictly dominated* for player $i$ in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if there exists another strategy $\sigma_i' \in \Delta(S_i)$ such that for all $\sigma_{-i} \in \Pi_{j \neq i} \Delta(S_j)$,

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i})$$

- We say that strategy $\sigma_i'$ *strictly dominates* strategy $\sigma_i$. A strategy $\sigma_i$ is a strictly dominant strategy for player $i$ in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if it strictly dominates every other strategy in $\Delta(S_i)$.

| | | Player 2 | |
|---|---|---|---|
| | | $L$ | $R$ |
| Player 1 | $U$ | 10, 1 | 0, 4 |
| | $M$ | 4, 2 | 4, 3 |
| | $D$ | 0, 5 | 10, 2 |



# Dominant and dominated strategies

- **Dominant and dominated strategies *with mixed strategies.***
- **Proposition**. Player $i$'s pure strategy $s_i \in S_i$ is strictly dominated in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if and only if there exists another strategy $\sigma_i' \in \Delta(S_i)$ such that for all $s_{-i} \in S_{-i}$,

$$u_i(\sigma_i', s_{-i}) > u_i(s_i, s_{-i})$$

## Rationalizable strategies

- **Definition**. In game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, strategy $\sigma_i$ is a *best response* for player $i$ to his rivals' strategies $\sigma_{-i}$ if for all $\sigma_i' \in \Delta(S_i)$

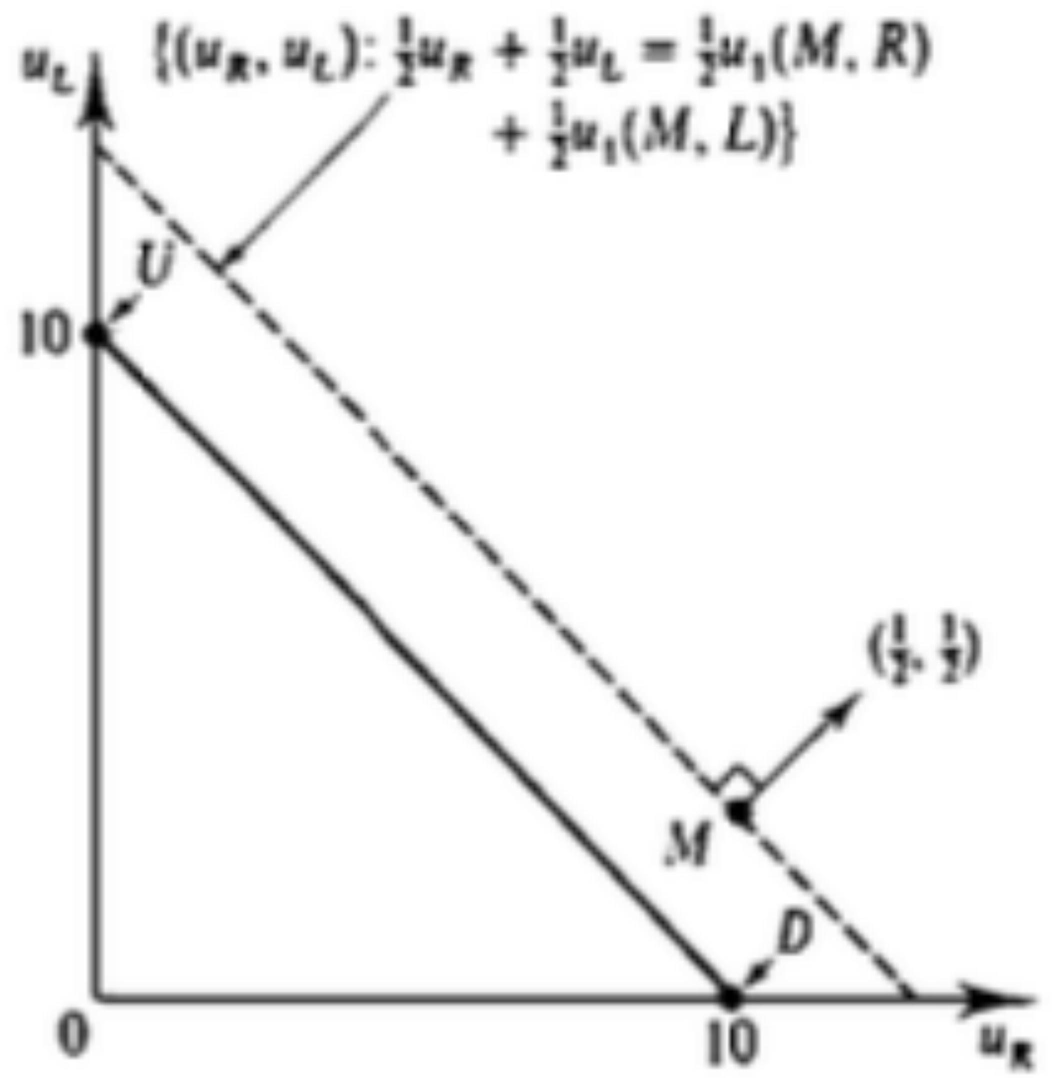
$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

  Strategy $\sigma_i$ is *never a best response* if there is no $\sigma_{-i}$ for which $\sigma_i$ is a best response.

- **Definition**. In game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, the strategies in $\Delta(S_i)$ that survive the iterated removal of strategies that are never a best response are known as player $i$'s rationalizable strategies.

Player 2

| Player 1 | $b_1$ | $b_2$ | $b_3$ | $b_4$ |
|---|---|---|---|---|
| $a_1$ | 0, 7 | 2, 5 | 7, 0 | 0, 1 |
| $a_2$ | 5, 2 | 3, 3 | 5, 2 | 0, 1 |
| $a_3$ | 7, 0 | 2, 5 | 0, 7 | 0, 1 |
| $a_4$ | 0, 0 | 0, −2 | 0, 0 | 10, −1 |



# Nash equilibrium

- **Definition**. A strategy profile $s = (s_1, \ldots, s_I)$ constitutes a Nash equilibrium of game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$, if for every $i = 1, \ldots, I$,

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$$

for all $s_i' \in S_i$.

| | | Player 2 | | |
|---|---|---|---|---|
| | | $l$ | $m$ | $r$ |
| | $U$ | 5, 3 | 0, 4 | 3, 5 |
| Player 1 | $M$ | 4, 0 | 5, 5 | 4, 0 |
| | $D$ | 3, 5 | 0, 4 | 5, 3 |

| | | Mr. Schelling | |
|---|---|---|---|
| | | Empire State | Grand Central |
| Mr. Thomas | Empire State | 100, 100 | 0, 0 |
| | Grand Central | 0, 0 | 100, 100 |

## Nash equilibrium

- **Discussion of the concept of Nash equilibrium**

(1) Nash equilibrium as a consequence of rational inference.

(2) Nash equilibrium as a necessary condition if there is a unique predicted outcome to a game.

(3) Focal points.

(4) Nash equilibrium as a self-enforcing agreement.

(5) Nash equilibrium as a stable social convention.

# Nash equilibrium

- **Mixed strategy Nash equilibrium**
- **Definition.** A mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ constitutes a *Nash equilibrium* of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if for every $i = 1, \cdots, I$,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$$

  for all $\sigma_i' \in \Delta(S_i)$.

- **Proposition**. Let $S_i^+ \subset S_i$ denote the set of pure strategies that player $i$ plays with positive probability in mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$. Strategy profile $\sigma$ is a Nash equilibrium in game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ if and only if for all $i = 1, \ldots, I$,

  (i) $u_i(s_i, \sigma_{-i}) = u_i(s_i', \sigma_{-i})$ for all $s_i, s_i' \in S_i^+$;

  (ii) $u_i(s_i, \sigma_{-i}) \geq u_i(s_i', \sigma_{-i})$ for all $s_i \in S_i^+$ and all $s_i' \notin S_i^+$.

- **Corollary**. Pure strategy profile $s = (s_1, \ldots, s_I)$ is a Nash equilibrium of game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if and only if it is a (degenerate) mixed strategy Nash equilibrium of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$.

# Nash equilibrium

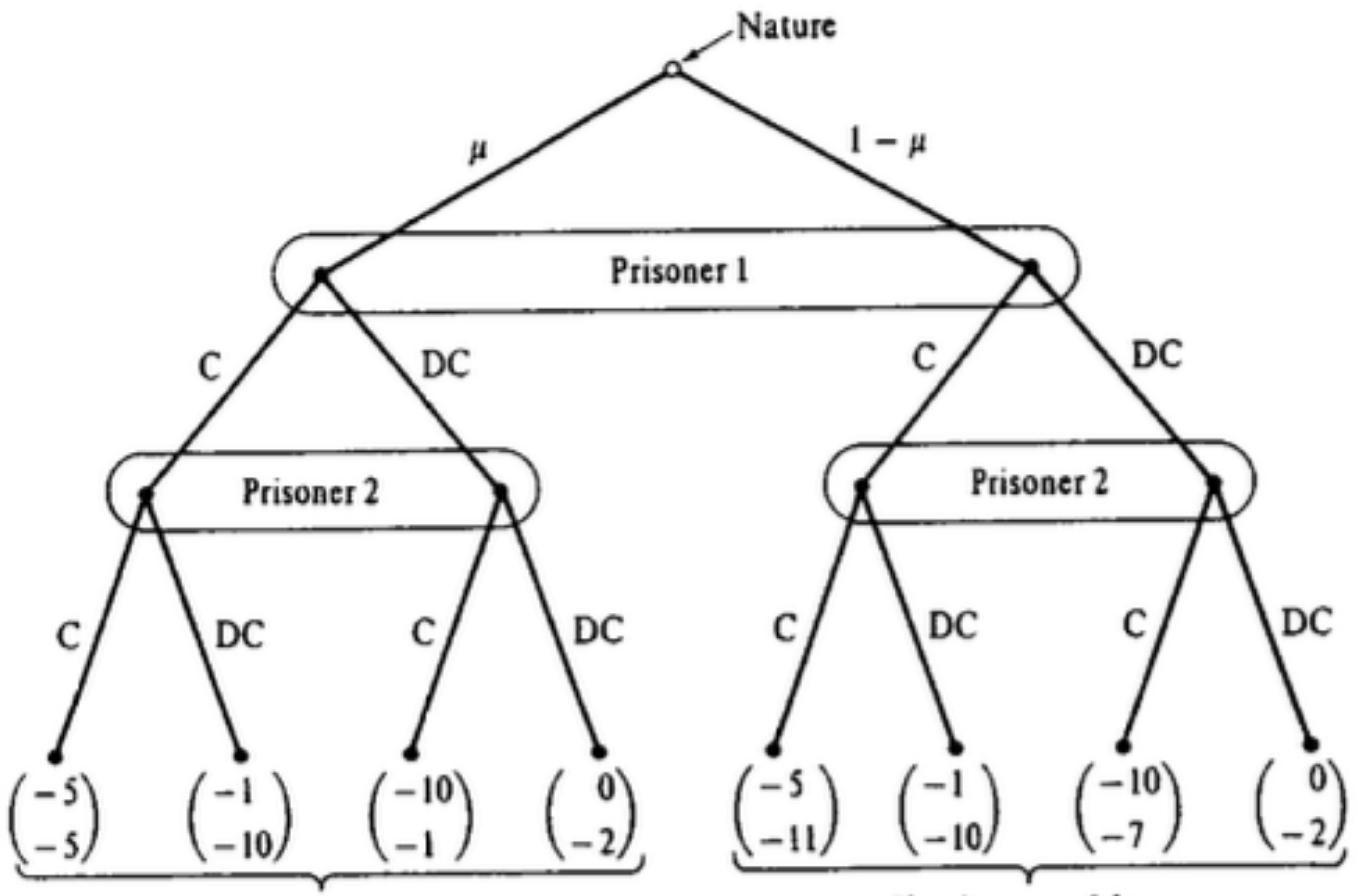
- **Existence of Nash equilibrium**
- **Proposition**. Every game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ in which the sets $S_1, \ldots, S_I$ have a finite number of elements has a mixed strategy Nash equilibrium.
- **Proposition**. A Nash equilibrium exists in game $\Gamma_N = [I, \{S_i\}, \{u_i(\cdot)\}]$ if for all $i = 1, \ldots, I$,

  (i) $S_i$ is a nonempty, convex, and the compact subset of some Euclidean space $\mathrm{R}^M$.

  (ii) $u_i(s_1, \ldots, s_I)$ is continuous in $(s_1, \ldots, s_I)$ and quasiconcave in $s_i$.

# Games of incomplete information: Bayesian Nash equilibrium

- Games of complete information. players know all relevant information about each other, including the payoffs that each receives from the various outcomes of the game.
- Games of incomplete information. players do not know all relevant information.




Simultaneous-Move Game:

Prisoner 2

| Prisoner 1 | DC | C |
|---|---|---|
| DC | 0, −2 | −10, −1 |
| C | −1, −10 | −5, −5 |

Simultaneous-Move Game:

Prisoner 2

| Prisoner 1 | DC | C |
|---|---|---|
| DC | 0, −2 | −10, −7 |
| C | −1, −10 | −5, −11 |

# Games of incomplete information: Bayesian Nash equilibrium

- In a Bayesian game
- Each player $i$ has a payoff function $u_i(s_i, s_{-i}, \theta_i)$ where $\theta_i \in \Theta_i$ is a random variable chosen by nature that is observed only by player $i$.
- The joint probability distribution of the $\theta_i$'s is given by $F(\theta_1, \ldots, \theta_I)$, which is assumed to be common knowledge among the players.
- Letting $\Theta = \Theta_1 \times \ldots \times \Theta_I$, a Bayesian game is summarized by the data $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$.
- A pure strategy for player $i$ in Bayesian game is a function $s_i(\theta_i)$, or *decision rule*, that gives the player's strategy choice for each realization of his type $\theta_i$. Player $i$'s pure strategy set $\mathcal{S}_i$ is the set of all such functions.
- Player $i$'s expected payoff given a profile of pure strategies for the $I$ players $(s_1(\cdot), \ldots, s_I(\cdot))$ is given by

$$\bar{u}_i(s_1(\cdot), \ldots, s_I(\cdot)) = E_\theta[u_i(s_1(\theta_1), \ldots, s_I(\theta_I), \theta_i)]$$

## Games of incomplete information: Bayesian Nash equilibrium

- **Definition**. A (pure strategy) Baysian Nash equilibrium for the Bayesian game $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$ is a profile of decision rules $(s_1(\cdot), \ldots, s_I(\cdot))$ that constitutes a Nash equilibrium of game $\Gamma_N = [I, \{\mathcal{S}_i\}, \{\bar{u}_i(\cdot)\}]$. That is, for every $i = 1, \ldots, I$

$$\bar{u}_i(s_1(\cdot), s_{-i}(\cdot)) \geq \bar{u}_i(s_1'(\cdot), s_{-i}(\cdot))$$

for all $s_i'(\cdot) \in \mathcal{S}_i$.

- **Proposition**. A profile of decision rules $(s_1(\cdot), \ldots, s_I(\cdot))$ is a Bayesian Nash equilibrium in Bayesian game $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$ if and only if, for all $i$ and all $\bar{\theta}_i \in \Theta_i$ occuring with positive probability

$$E_{\theta_{-i}}[u_i(s_i(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i) \,|\, \bar{\theta}_i)] \geq E_{\theta_{-i}}[u_i(s_i'(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i) \,|\, \bar{\theta}_i)]$$

for all $s_i' \in S_i$, where the expectation is taken over realizations of the other players' random variables conditional on player $i$'s realization of his signal $\bar{\theta}_i$.

# Possibility of mistakes: Trembling hand perfection

- For any normal form game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$, a *perturbed game* is defined by choosing for each player $i$ and strategy $s_i \in S_i$ a number $\varepsilon_i(s_i) \in (0,1)$, with $\Sigma_{s_i \in S_i} \varepsilon_i(s_i) < 1$ and player $i$'s perturbed strategy set is defined as

$$\Delta_\varepsilon(S_i) = \{\sigma_i : \sigma_i(s_i) \geq \varepsilon_i(s_i) \text{ for all } s_i \in S_i \text{ and } \Sigma_{s_i \in S_i} \sigma_i(s_i) = 1\}$$

- **Definition**. A Nash equilibrium $\sigma$ of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ is *(normal form) trembling-hand perfect* if there is some sequence of perturbed games $\{\Gamma_{\varepsilon^k}\}_{k=1}^{\infty}$ that converges to $\Gamma_N$ [in a sense that $\lim_{k\to\infty} \varepsilon_i^k(s_i) = 0$ for all $i$ and $s_i \in S_i$], for which there is some associated sequence of Nash equilibria $\{\sigma^k\}_{k=1}^{\infty}$ that converges to $\sigma$ [i.e., such that $\lim_{k\to\infty} \sigma^k = \sigma$].

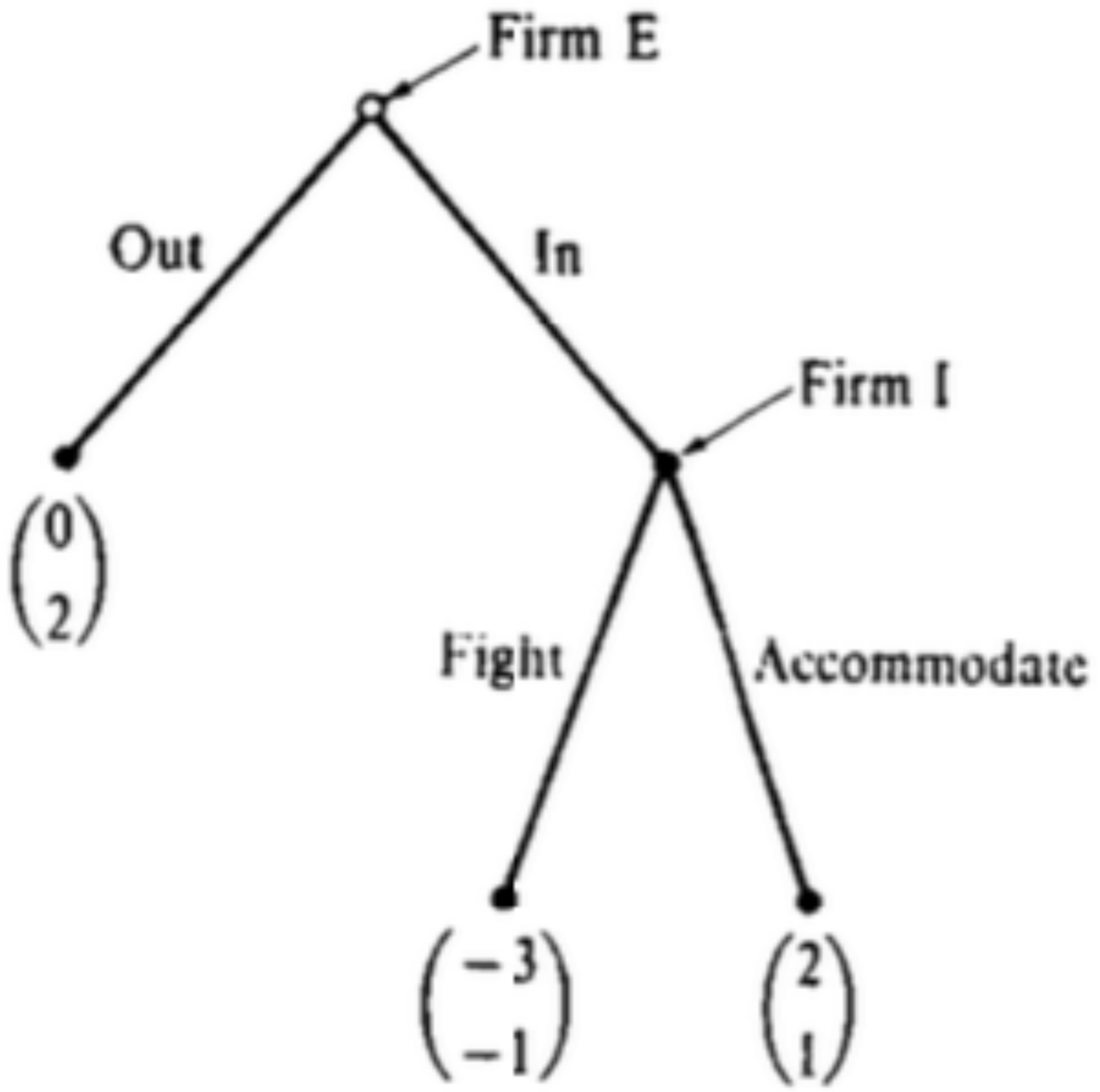
| | L | R |
|---|---|---|
| U | 1, 1 | 0, −3 |
| D | −3, 0 | 0, 0 |

# Possibility of mistakes: Trembling hand perfection

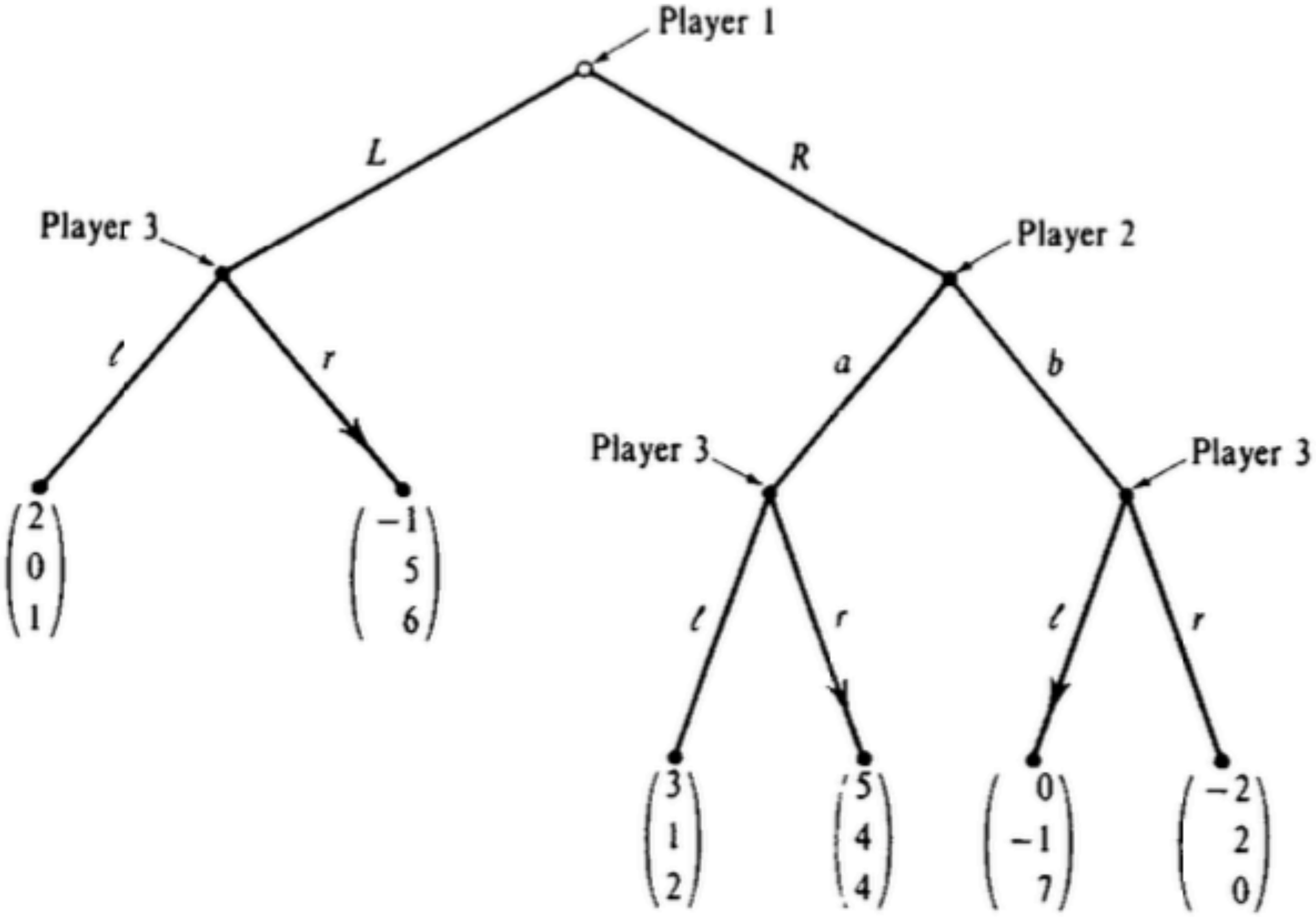
- **Proposition.** A Nash equilibrium $\sigma$ of game $\Gamma_N = [I, \{\Delta(S_i)\}, \{u_i(\cdot)\}]$ is (normal form) trembling-hand perfect if and only if there is some sequence of totally mixed strategies $\{\sigma^k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty}\sigma^k = \sigma$ and $\sigma_i$ is a best response to every element of sequence $\{\sigma_{-i}^k\}_{k=1}^{\infty}$ for all $i = 1,\ldots,I$.
- **Proposition**. If $\sigma = (\sigma_1, \ldots, \sigma_I)$ is a (normal form) trembling-hand perfect Nash equilibrium, then $\sigma_i$ is not a weakly dominated strategy for any $i = 1,\ldots,I$. Hence, in any (normal form) trembling-hand perfect Nash equilibrium, no weakly dominated pure strategy can be played with positive probability.

# Dynamic Games

# Sequential rationality, backward induction, and subgame perfection




| | | Firm I | |
|---|---|---|---|
| | | Fight if Firm E Plays "In" | Accommodate if Firm E Plays "In" |
| Firm E | Out | 0, 2 | 0, 2 |
| | In | −3, −1 | 2, 1 |



## Sequential rationality, backward induction, and subgame perfection

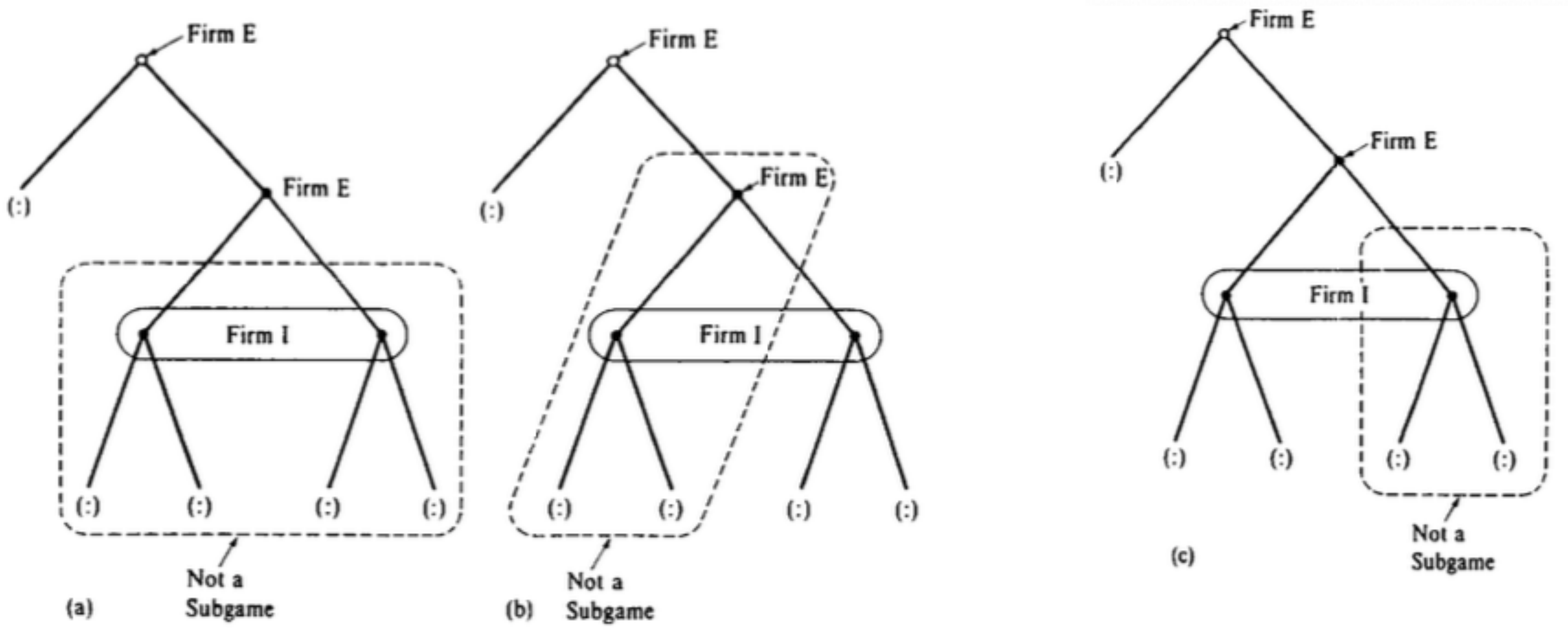
- **Proposition**. (Zermelo's Theorem) Every finite game of perfect information $\Gamma_E$ has a pure strategy Nash equilibrium that can be derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a uniqe Nash equilibrium that can be derived in this manner.

# Sequential rationality, backward induction, and subgame perfection

- **Definition**. A *subgame* of an extensive form game $\Gamma_E$ is a subset of the game having the following properties:

  (1) It begins with an information set containing a single decision node, contains all the decision nodes that are successors (both immediate and later) of this node, and contains *only* these nodes.

  (2) If decision node $x$ is in the subgame, then every $x' \in H(x)$ is also, where $H(x)$ is the information set that contains decision node $x$. (That is, there are no "broken" information sets.)



# Sequential rationality, backward induction, and subgame perfection

- **Definition**. A profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_I)$ in an $I$-player extensive form game $\Gamma_E$ is a *subgame perfect Nash equilibrium* (SPNE), if it induces a Nash equilibrium in every subgame of $\Gamma_E$.
- **Proposition.** Every finite game of perfect information $\Gamma_E$ has a pure strategy subgame perfect Nash equilibrium. Moreover, if no player h a s the same payoffs at any two terminal nodes, then there is a unique subgame perfect Nash equilibrium.
- **Proposition**. Consider an extensive form game I, and some subgame $S$ of $\Gamma_E$. Suppose that strategy profile $\sigma^S$ is an SPNE in subgame $S$, and let $\hat{\Gamma}_E$ be the reduced game formed by replacing subgame $S$ by a terminal node with payoffs equal to those arising from play of $\sigma^S$. Then

  (i) In any SPNE $\sigma$ of $\Gamma_E$ in which $\sigma^S$ is the play in subgame $S$, players' moves at information sets outside subgame $S$ must constitute an SPNE of reduced game $\hat{\Gamma}_E$.

  (ii) If $\hat{\sigma}$ is an SPNE of $\hat{\Gamma}_E$, then the strategy profile $\sigma$ that specifies the moves in $\sigma^S$ at information sets in subgame $S$ and that specifies the moves in $\hat{\sigma}$ at information sets no in $S$ in an SPNE of $\Gamma_E$.

# Sequential rationality, backward induction, and subgame perfection

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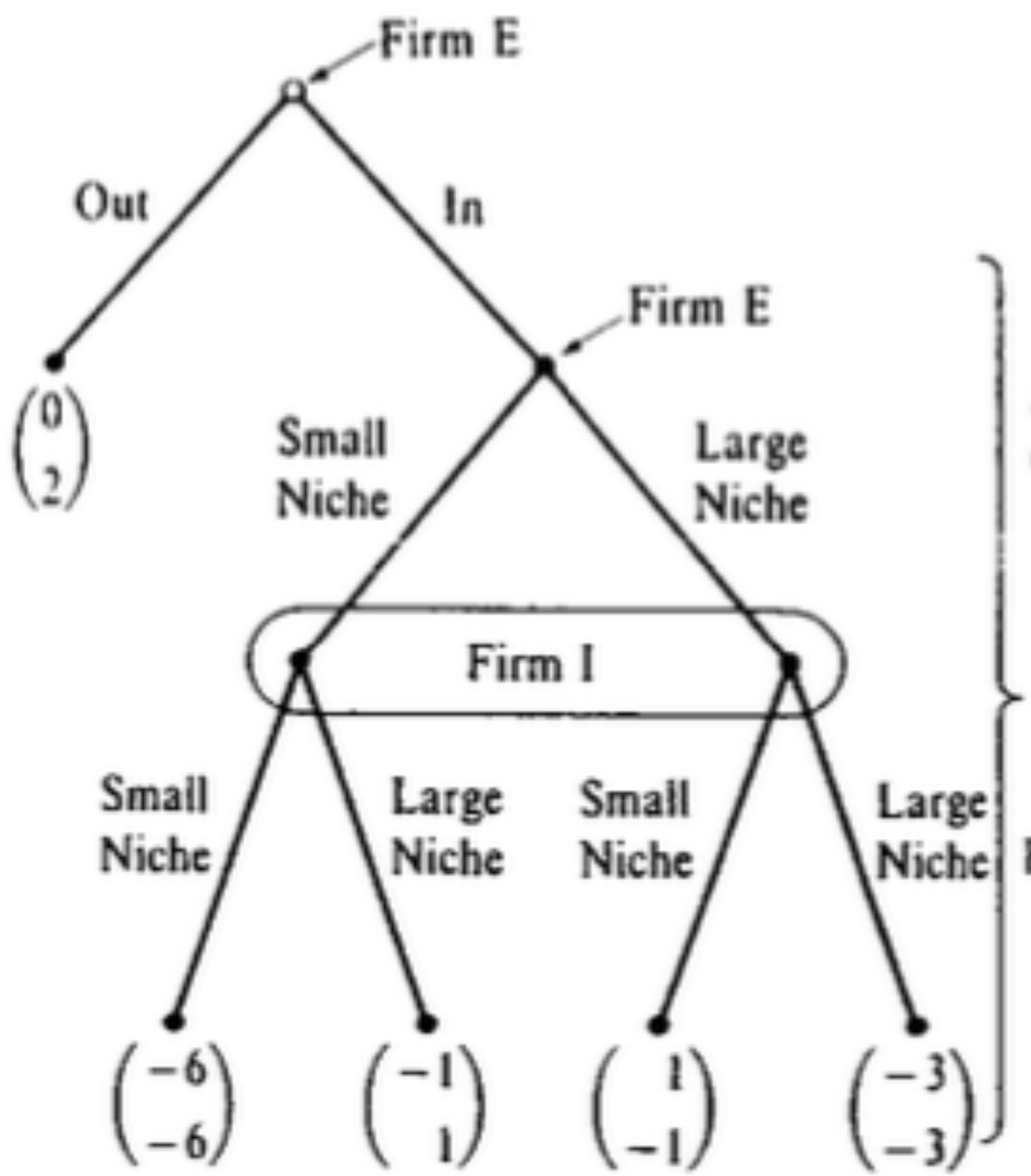
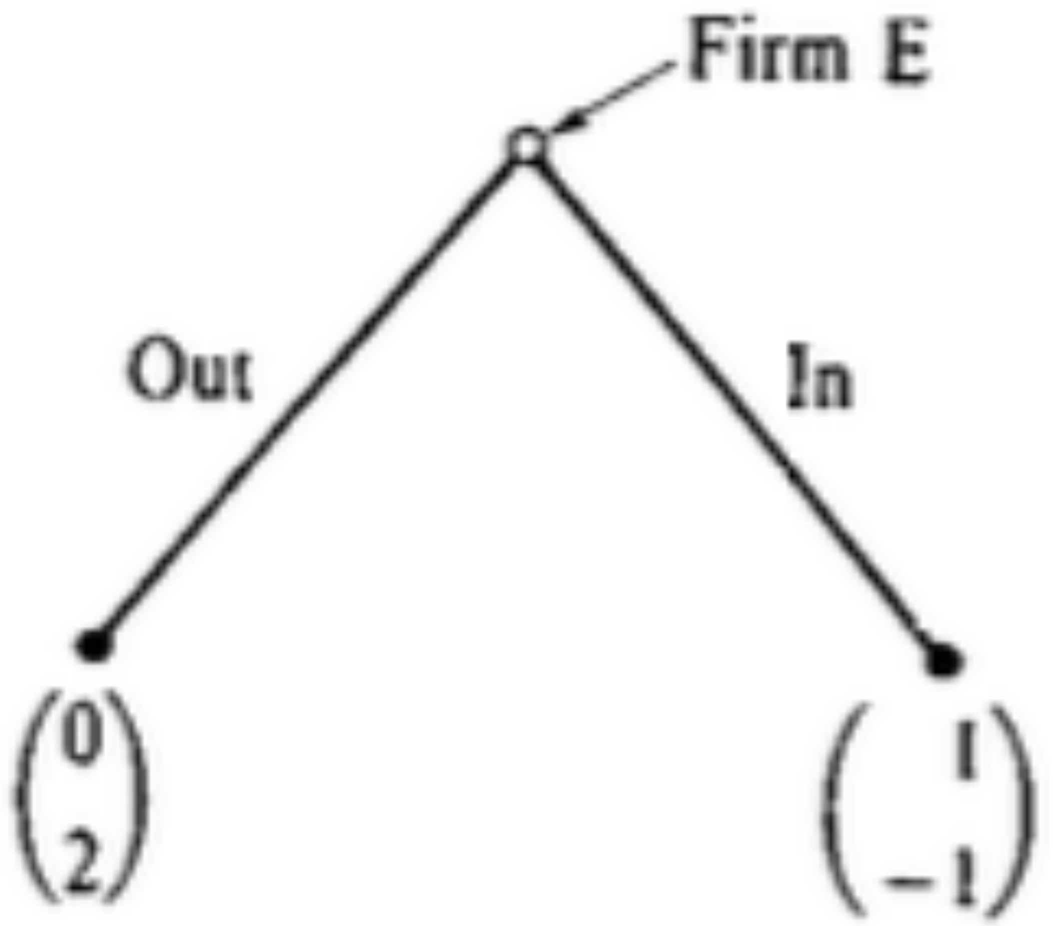
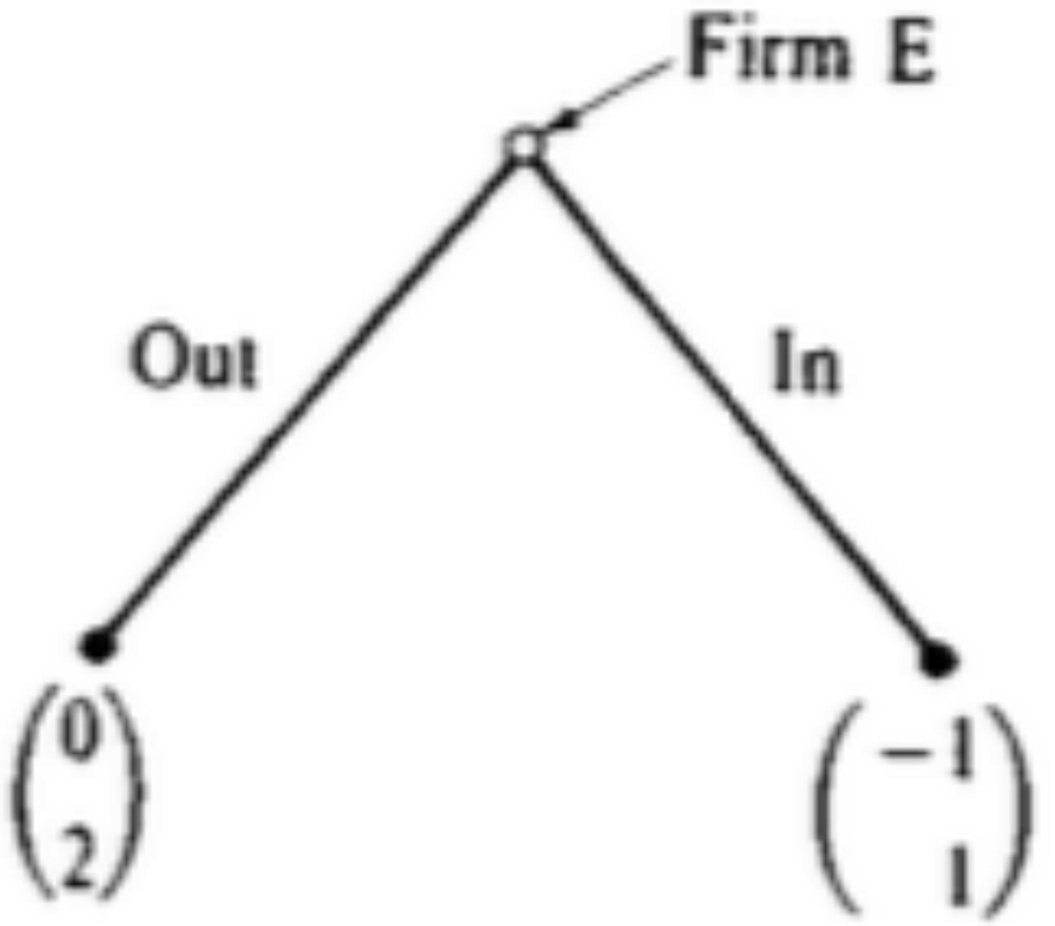



A Simultaneous-Move Game:

| | | Firm I: Small Niche | Firm I: Large Niche |
|---|---|---|---|
| Firm E | Small Niche | −6, −6 | −1, 1 |
| | Large Niche | 1, −1 | −3, −3 |



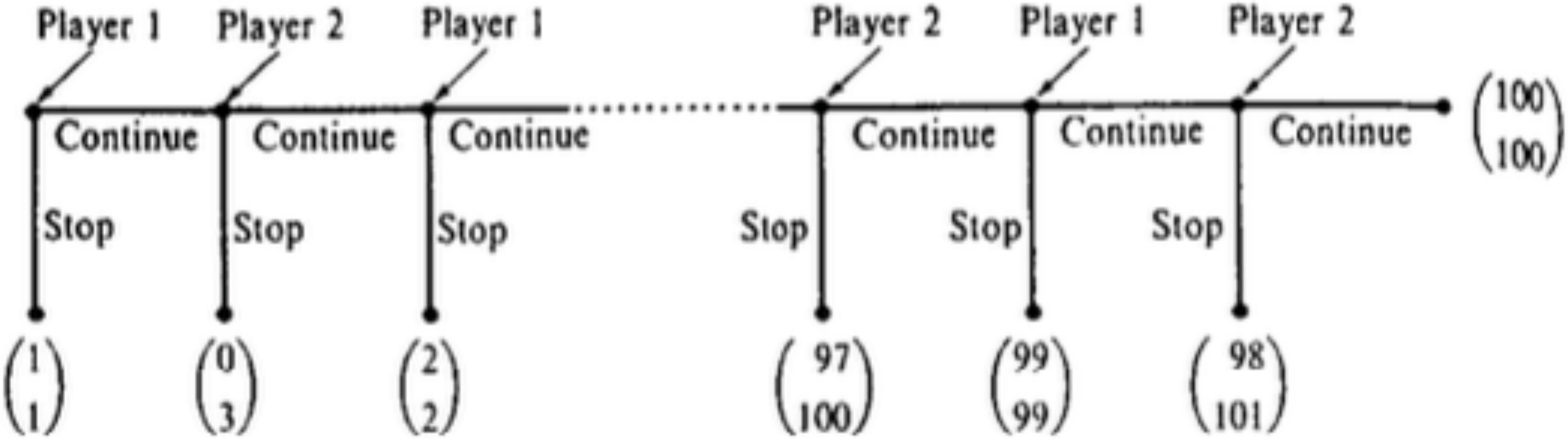



# Sequential rationality, backward induction, and subgame perfection

- **Proposition**. Consider an $I$-player extensive form game $\Gamma_E$ involvina successive play of $T$ simultaneous-move games, $\Gamma_N^t = \{I, \{\Delta(S_i^t\}, \{u_i^t(\cdot)\}]$ for $t = 1,\ldots, T$ with the players observing the pure strategies played in each game immediately after its play is concluded. Assume that each player's payoff is equal to the sum of her payoffs in the plays in the $T$ games. If there is a unique Nash equilibrium in each game say $\sigma^t = (\sigma_1^t, \ldots, \sigma_I^t)$, then there is a unique SPNE of $\Gamma_E$ and it consists of each player $i$ playing strategy $\sigma_i^t$ in each game $\Gamma_N^t$ regardless of what has appened previously.

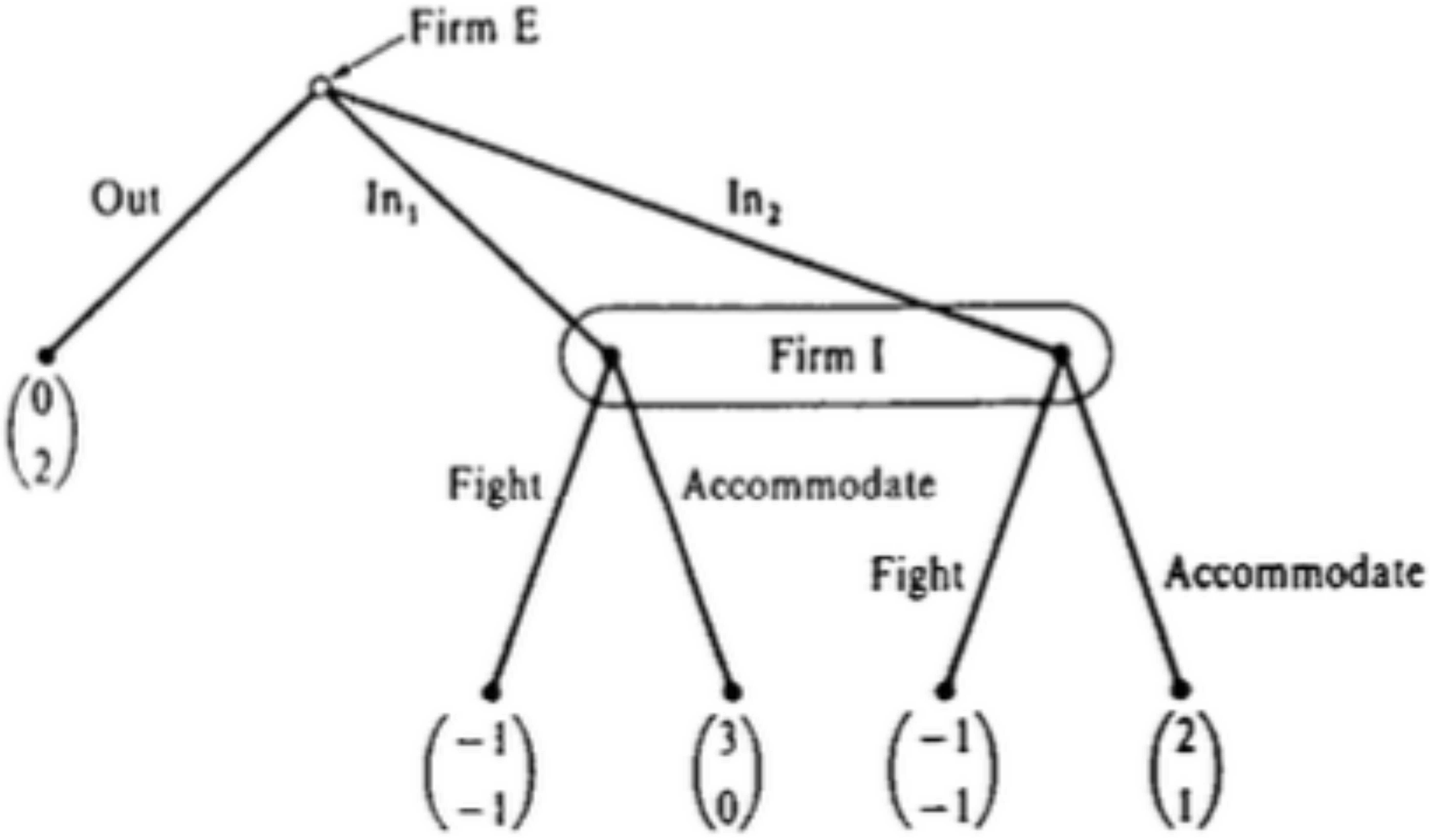
## Sequential rationality, backward induction, and subgame perfection

- An interesting tension present in the SPNE concept that is related to the bounded rationality issue that does not arise in the context of simultaneous move games.



# Beliefs and Sequential Rationality

- When SPNE concept fails to insure sequential rationality



# Beliefs and Sequential Rationality

- **Definition**. *A system of belief* $\mu$ in extensive form game $\Gamma_E$ is a specification of a probability $\mu(x) \in [0,1]$ for each decision node $x$ in $\Gamma_E$ such that $\Sigma_{x \in H} \mu(x) = 1$ for all information sets $H$.

- **Definition**. A strategy profile $\sigma = (\sigma_1, \ldots, \sigma_I)$ in extensive form game $\Gamma_E$ is *sequentially rational at information set H given a system of beliefs* $\mu$ if, denoting by $\iota(H)$ the player who moves at information set $H$, we have

$$E[u_{\iota(H)} \,|\, H, \mu, \sigma_{\iota(H)}, \sigma_{-\iota(H)}] \geq E[u_{\iota(H)} \,|\, H, \mu, \bar{\sigma}_{\iota(H)}, \sigma_{-\iota(H)}]$$

for all $\bar{\sigma}_{\iota(H)} \in \Delta(S_{\iota(H)})$. If strategy profile $\sigma$ satisfies this condition for *al*l information sets $H$, then we say that $\sigma$ is *sequentially rational given belief system* $\mu$.

# Beliefs and Sequential Rationality

- **Definition**. A profile of strategies and system of beliefs $(\sigma, \mu)$ is a weak perfect Bayesian equilibrium (weak PBE) in extensive form game $\Gamma_E$ if it has the following properties:

  (i) The strategy profile $\sigma$ is sequentially rational given belief system $\mu$.

  (ii) The system of beliefs $\mu$ is derived from strategy profile $\sigma$ through Bayes' rule whenever poossible. That is for any information set $H$ such that $\text{Prob}(H \mid \sigma) > 0$ (read as "the probability of reaching information set $H$ is positive under strategies $\sigma$"), we must have

$$\mu(x) = \frac{\text{Prob}(x \mid \sigma)}{\text{Prob}(H \mid \sigma)} \text{ for all } x \in H.$$

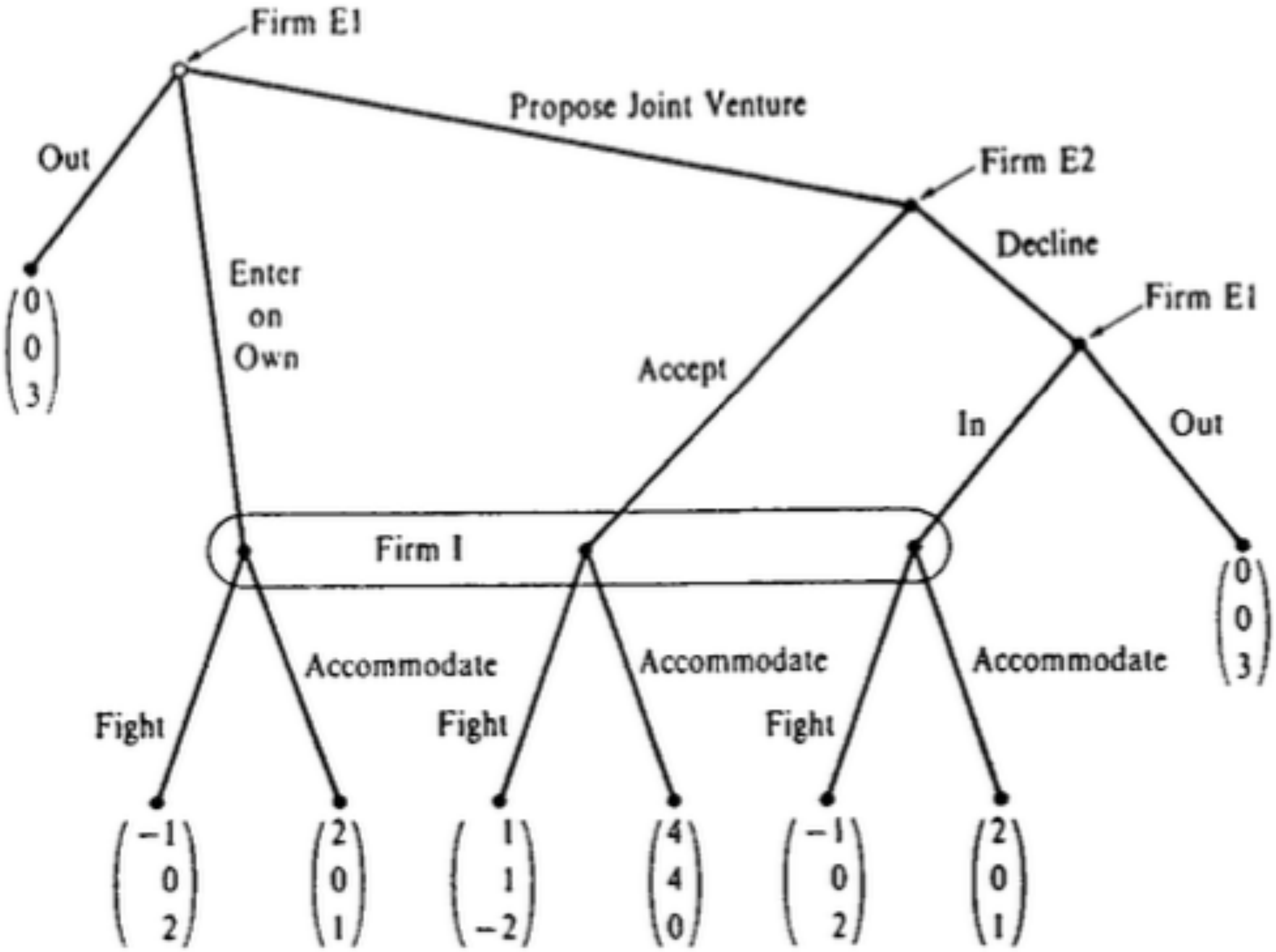
# Beliefs and Sequential Rationality

- **Proposition**. A strategy profile $\sigma$ is a Nash equilibrium of extensive form game $\Gamma_E$ if and only if there exists a system of beliefs $\mu$ such that

  (i) The strateay profile $\sigma$ is sequentially rational given belief system $\mu$ at all information sets $H$ such that $\text{Prob}(H \mid \sigma) > 0$.

  (ii) The system of beliefs $\mu$ is derived from strategy profile a through Bayes' rule whenever possible.

# Beliefs and Sequential Rationality

- 

# Beliefs and Sequential Rationality

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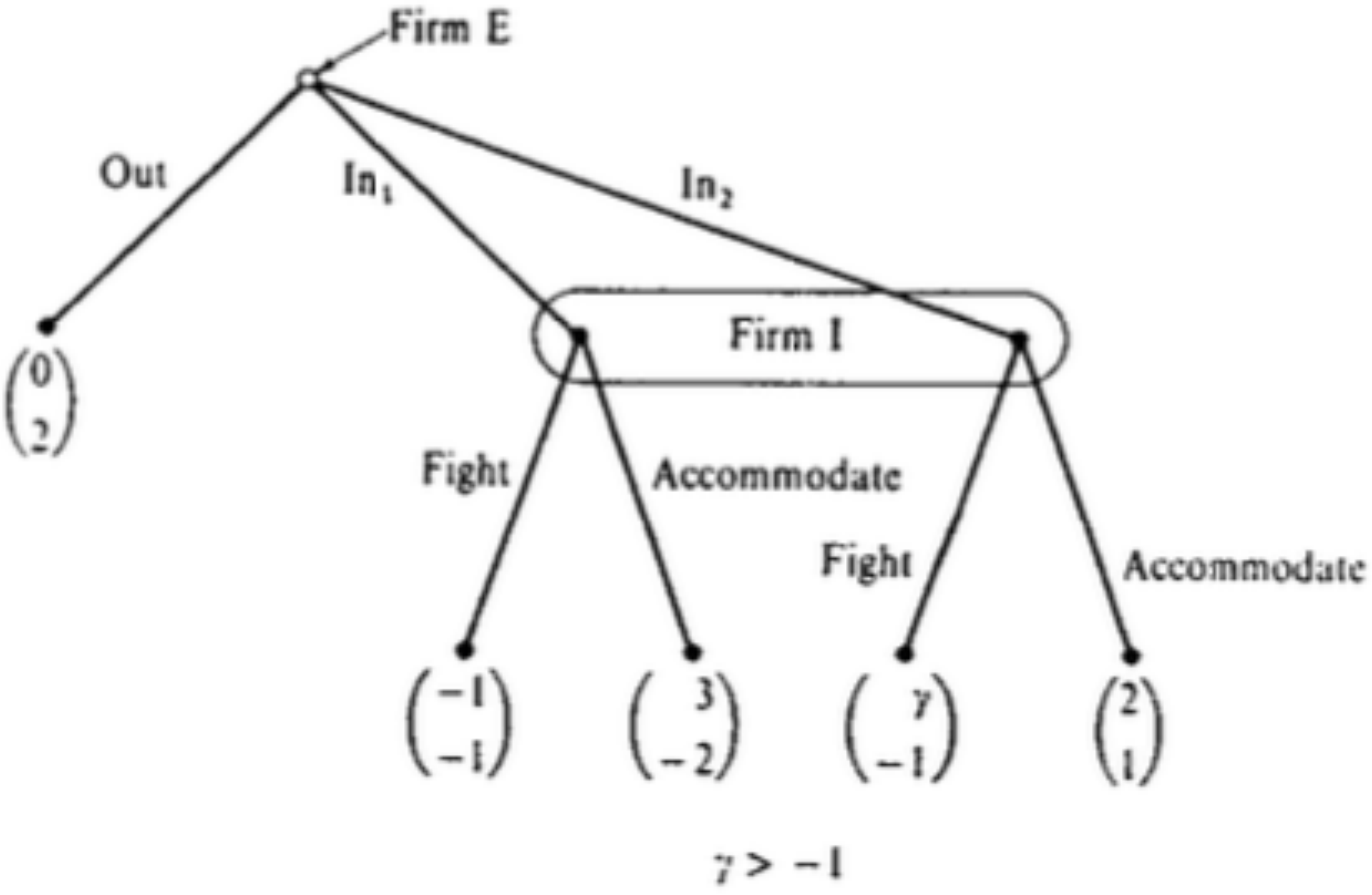
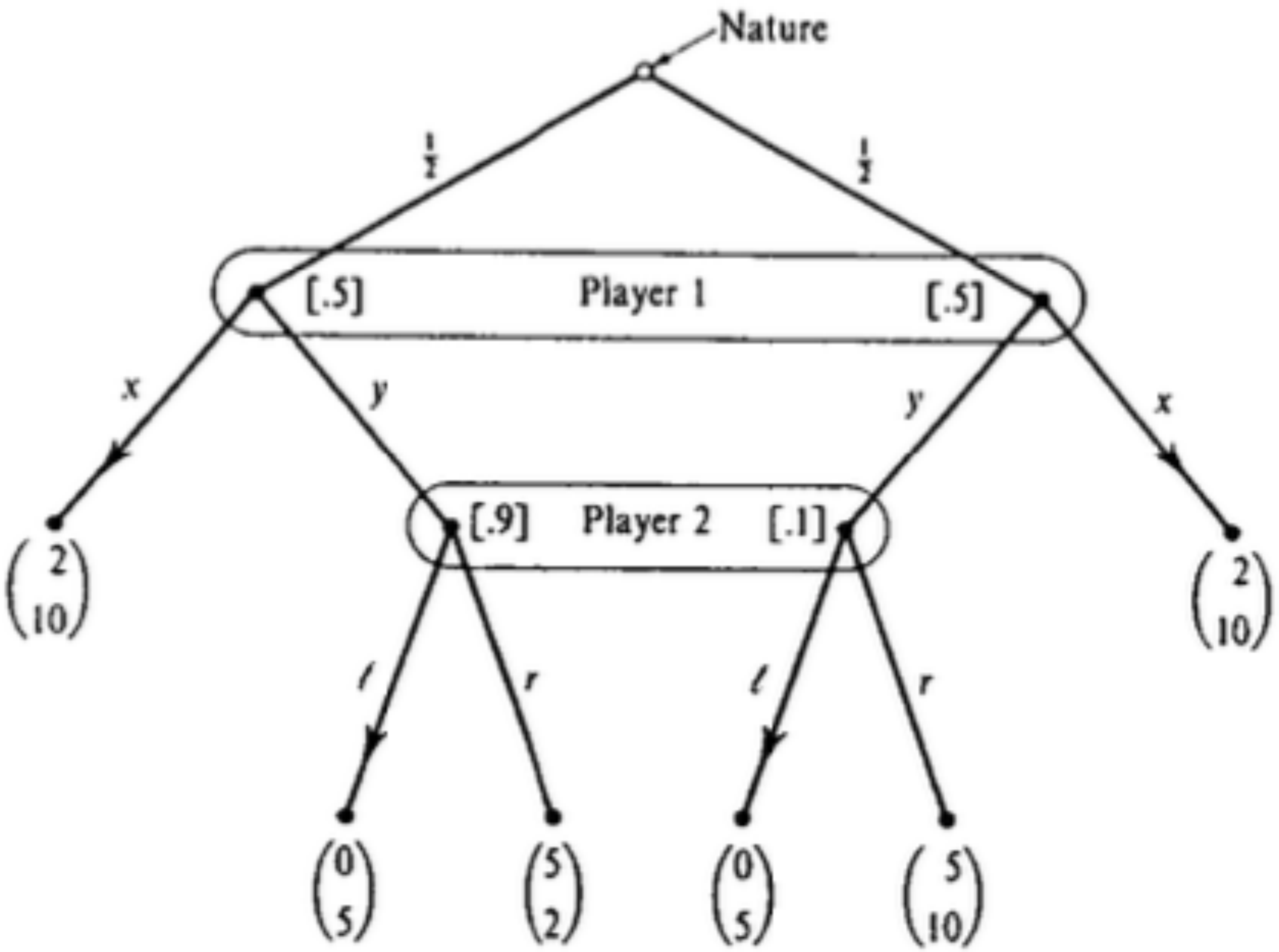


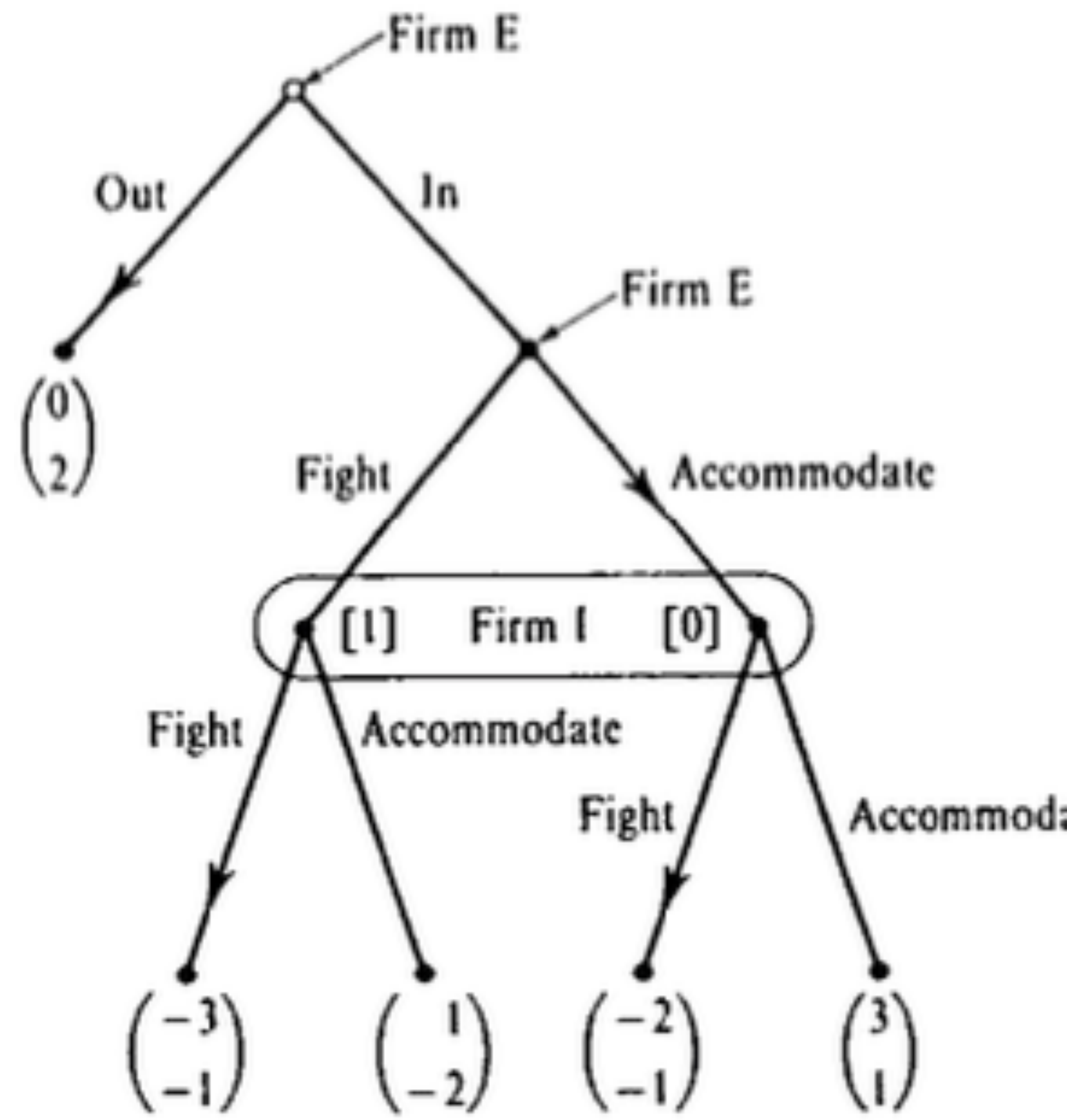
# Beliefs and Sequential Rationality

- Strengthenings of the weak perfect Bayesian equilibrium concept



# Beliefs and Sequential Rationality

- Perfect Bayesian equilibrium
- Definition. A strategy profile and systme of beliefs $(\sigma, \mu)$ is a sequential equilibrium of extensive form game $\Gamma_E$ if it has the following properties:
- (i) Strategy profile $\sigma$ is sequentially rational given belief system $\mu$.
- (ii) There exists a sequence of completely mixed strategies $\{\sigma^K\}_{k=1}^{\infty}$, with $\lim_{k\to\infty}\sigma^k = \sigma$, such that $\mu = \lim_{k\to\infty}\mu^k$, where $\mu^k$ denotes the beliefs derived from strategy profile $\sigma^k$ using Bayes' rule.



## Beliefs and Sequential Rationality

- Perfect Bayesian equilibrium
- Proposition. In every sequential equilibrium $(\sigma, \mu)$ of an extensive form game $\Gamma_E$, the equilibrium strategy profile $\sigma$ constitutes a subgame perfect Nash equilibrium of $\Gamma_E$.

# Reasonable Beliefs and Forward Induction

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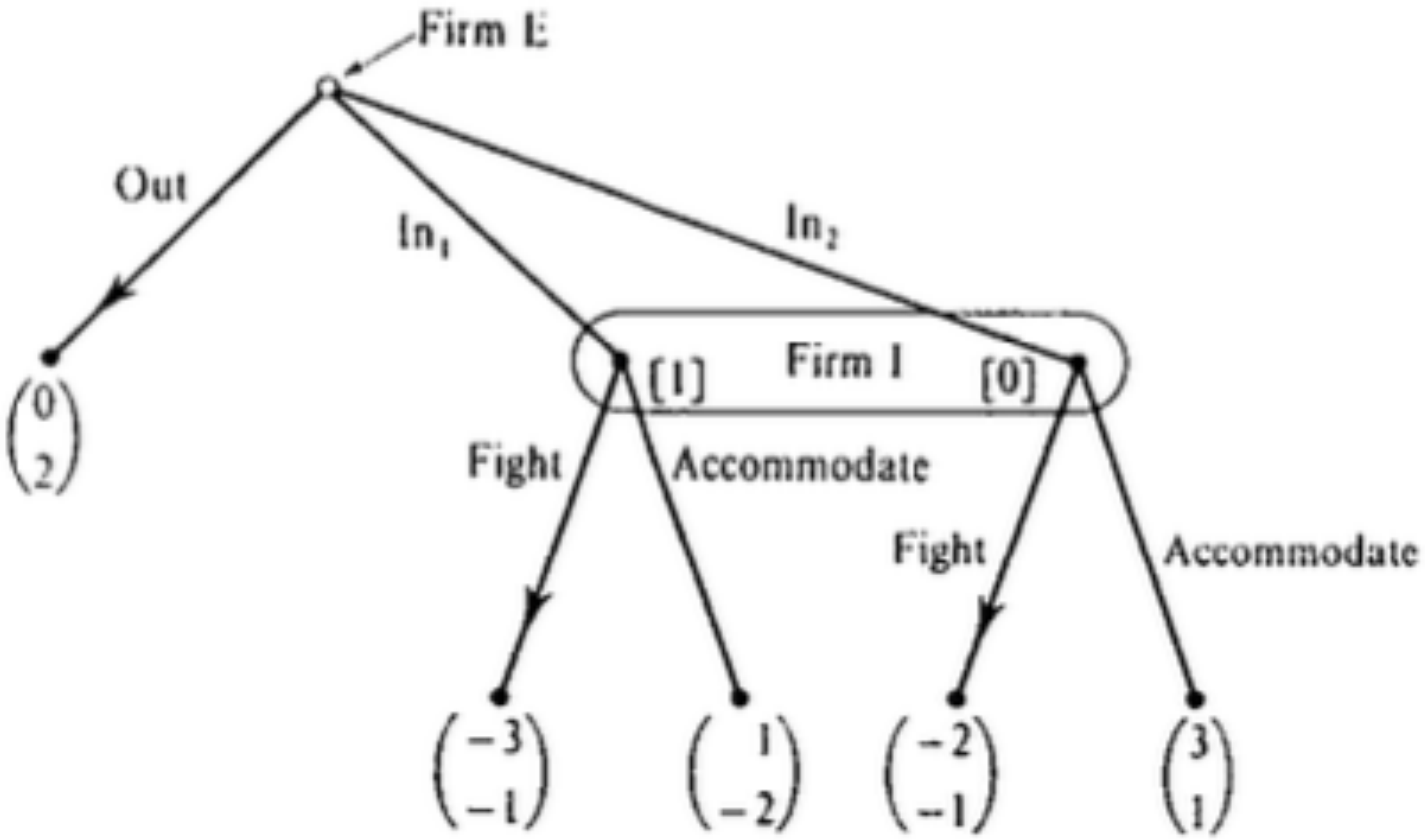


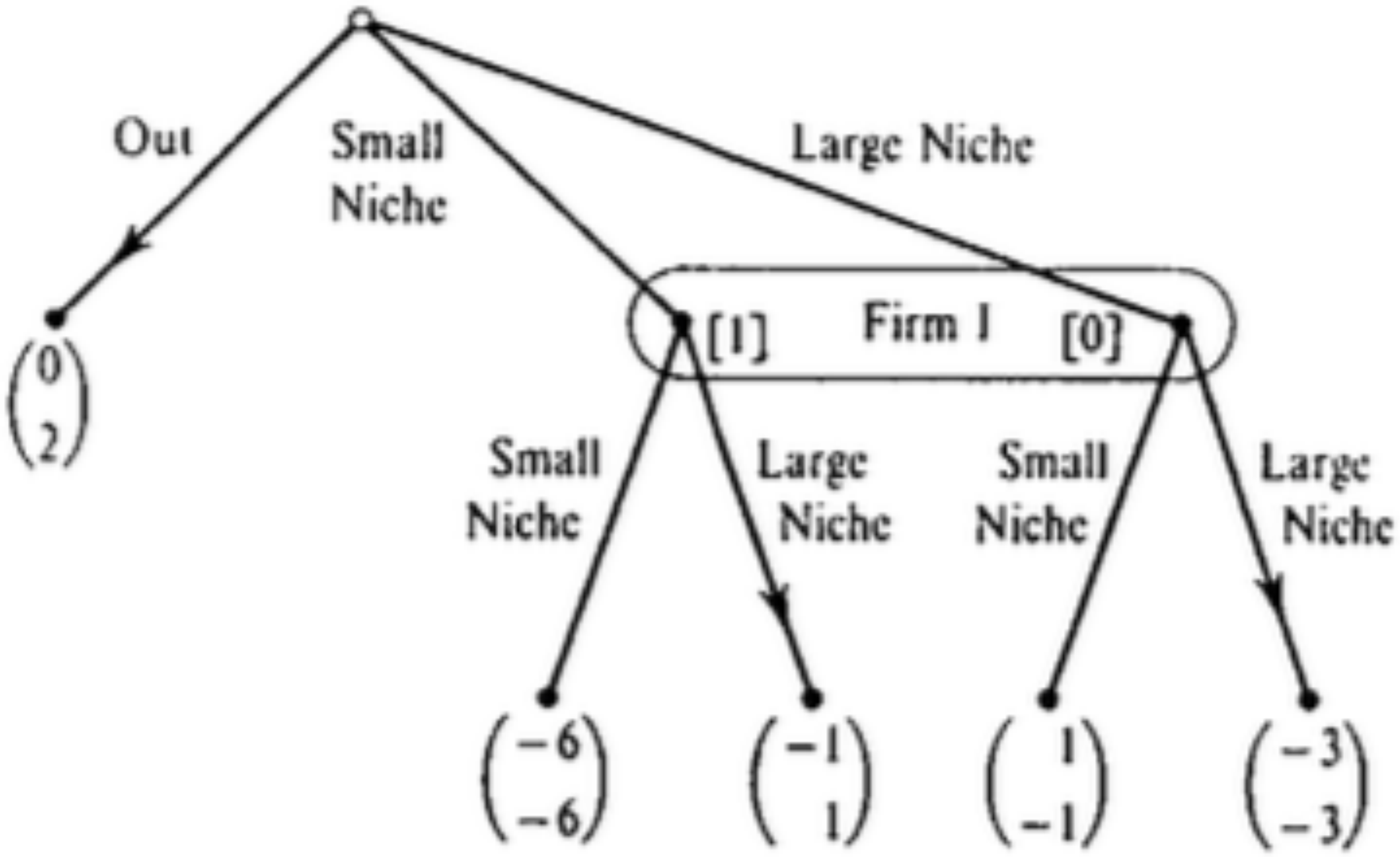

# Reasonable Beliefs and Forward Induction

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