ECE5022: Digital Signal Processing

Lecture 10: Parametric Signal Modeling

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Outline

- Parametric Signal Modeling
- All-Pole Modeling of Signals
- Linear Prediction
- Estimation of Correlation Functions
- Solution of Autocorrelation Normal Equations

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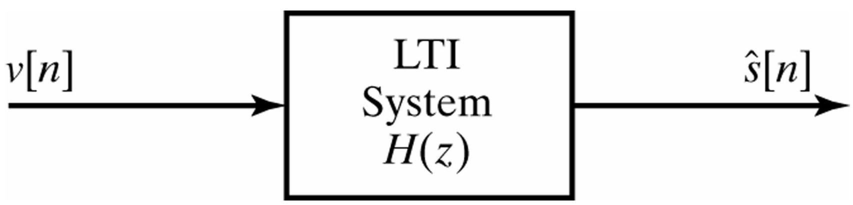
Speak & Spell (TI, 1978)

TV Commercial: https://youtu.be/mL2k1Gp8ywE
Demo: https://youtu.be/UwWaeEyhPP0

Linear System Model for a Signal

- Signal modeling as an output of an LTI System



- Input signal $v[n]$ and model parameters of the system $H(z)$ completely determine the signal

ARMA Modeling of a Signal

AR Model
MA Model

- Pole-Zero Modeling of the System Function

$H(z) = \frac{P(z)}{Q(z)}$

$$H(z)=\frac{\sum_{k=0}^{q} b_k z^{-k}}{1-\sum_{k=1}^{p} a_k z^{-k}}$$

moving average $\Rightarrow Q(z)=1$

auto regressive

- Autoregressive Moving Average (ARMA) Model
  - Poles (Autoregressive)
  - Zeros (Moving Average)

MA : $H(z)=\sum_{k=0}^{q} b_k z^{-k}$

$Y(z)=\sum_{k=0}^{q} b_k z^{-k} \cdot X(z)$

$\Rightarrow y[n]=b_0 x[n]+b_1 x[n-1]+b_2 x[n-2]+\cdots+b_q x[n-q]$

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Parametric Signal Modeling Applications

- Data Compression
- Spectrum Analysis
- Signal Prediction
- Filter Design
- System Identification
- Signal Detection & Classification

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All-Pole Modeling

- If we let $q = 0, b_0 = G$

$$H(z) = \frac{G}{1 - \sum_{k=1}^{p} a_k z^{-k}} = \frac{G}{A(z)}, \quad G = b_0$$

A(G) =

- Given an input $v[n]$, the output signal is

$$\hat{s}[n] = \sum_{k=1}^{p} a_k \hat{s}[n-k] + Gv[n]$$

All-Pole Model Error

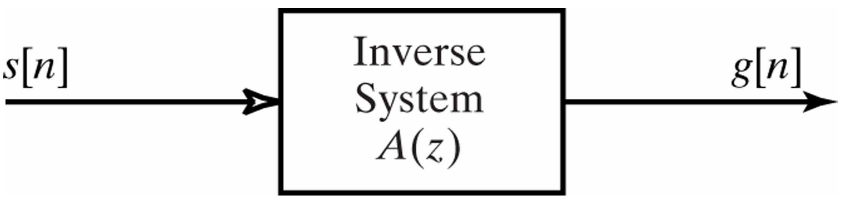
- The model parameters generate the model error
- The total squared error

$$\sum_{n=-\infty}^{\infty}\left(s[n]-\hat{s}[n]\right)^2=\sum_{k=1}^{p}\left(s[n]-\sum_{k=1}^{p}a_k\hat{s}[n-k]-Gv[n]\right)^2$$

- Least-squares problem
  – We can find the model parameters that minimize the total squared error

Inverse Model

- Inverse System of the All-Pole Model



$$A(z)=1-\sum_{k=1}^{p}a_k z^{-k}\Rightarrow g[n]=s[n]-\sum_{k=1}^{p}a_k s[n-k]$$

The Modeling Error I/II

- The output of the inverse system

$$g[n] = s[n] - \sum_{k=1}^{p} a_k s[n-k]$$

- Consider the modeling error

$$\hat{e}[n] = g[n] - Gv[n]$$
$$= s[n] - \sum_{k=1}^{p} a_k s[n-k] - Gv[n]$$
$$= e[n] - Gv[n] = \text{modeling error}$$

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The Modeling Error II/II

- The prediction error $e[n]$ is given by

$$e[n] = s[n] - \sum_{k=1}^{p} a_k s[n-k]$$

- If the modeling error is zero

$$\hat{e}[n] = e[n] - Gv[n] = 0 \Rightarrow e[n] = Gv[n]$$

- The parameter values can be chosen to minimize the modeling error

If $e[n] = 0$

$S[n] = \sum_{k=1}^{P} a_k S[n-k]$

Current Value of the signal

Weighted

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Minimizing the Modeling Error I/IV

- Find parameter values to minimize the modeling error

$$\mathbf{E} = \left\langle \hat{e}^2[n] \right\rangle = \left\langle (e[n] - Gv[n])^2 \right\rangle$$
$$= \left\langle e^2[n] \right\rangle + G^2 \left\langle v^2[n] \right\rangle - 2G \left\langle v[n]e[n] \right\rangle$$

- Take the partial derivative *w.r.t.* the i^th coefficient a_i

$$\frac{\partial \mathbf{E}}{\partial a_i} = \frac{\partial}{\partial a_i}\left[\left\langle e^2[n] \right\rangle - 2G \left\langle v[n]e[n] \right\rangle + G^2 \left\langle v^2[n] \right\rangle \right] = 0, \quad i = 1, 2, \cdots, p$$

Minimizing the Modeling Error II/IV

- With further derivations ($v[n]$ impulse, $s[n]$ causal)

$$\frac{\partial \mathbf{E}}{\partial a_i} = \frac{\partial}{\partial a_i}\left[\left\langle e^2[n] \right\rangle - 2G \left\langle v[n]s[n-i] \right\rangle \right] = \frac{\partial}{\partial a_i} \left\langle e^2[n] \right\rangle = 0$$

- Choosing the coefficients to minimize the mean squared ***modeling error*** $\langle \hat{e}^2[n] \rangle$ is equivalent to minimizing the mean squared ***prediction error*** $\langle e^2[n] \rangle$

Minimizing the Modeling Error III/IV

$(u^2)' = 2u'u$ $\quad (uv)' = u'v + uv' = 2u'u \quad (u=v)$

- Given the prediction error $e[n] = s[n] - \sum_{k=1}^{p} a_k s[n-k]$

$(e^2)' = 2e'e$

$$\frac{\partial}{\partial a_i}\langle e^2[n]\rangle = \frac{\partial}{\partial a_i}\left\langle \Bigg(\underbrace{s[n] - \sum_{k=1}^{p} a_k s[n-k]}_{=e[n]}\Bigg)\Bigg(\underbrace{s[n] - \sum_{k=1}^{p} a_k s[n-k]}_{e[n]}\Bigg)\right\rangle$$

$$= -2\langle s[n]s[n-i]\rangle + \frac{\partial}{\partial a_i}\left\langle \left(\sum_{k=1}^{p} a_k s[n-k]\right)\left(\sum_{k=1}^{p} a_k s[n-k]\right)\right\rangle$$

$$= -2\langle s[n]s[n-i]\rangle + 2\sum_{k=1}^{p} a_k \langle s[n-k]s[n-i]\rangle = 0$$

Minimizing the Modeling Error IV/IV

- Finally

$$\sum_{k=1}^{p} a_k \langle s[n-k]s[n-i]\rangle = \langle s[n]s[n-i]\rangle$$

- If we define

$$\phi_{SS}[i,k] = \langle s[n-k]s[n-i]\rangle$$

- Then

$$\sum_{k=1}^{p} a_k \phi_{SS}[i,k] = \phi_{SS}[i,0], \quad i = 1,2,\cdots,p$$

Equations for Solving a_k

- The set of equations

$$\sum_{k=1}^{p} a_k \phi_{SS}[i,k] = \phi_{SS}[i,0], \quad i = 1,2,\cdots,p$$

- p equations and p unknowns
- For wide-sense stationary (WSS) signals

$$\sum_{k=1}^{p} a_k \phi_{SS}[i-k] = \phi_{SS}[i], \quad i = 1,2,\cdots,p$$

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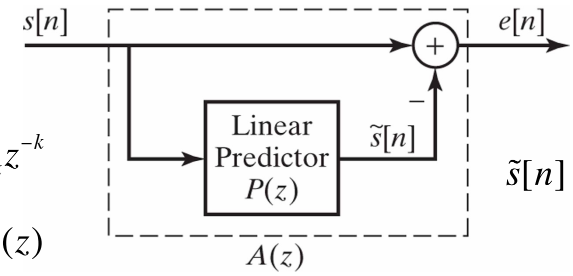
All-Pole Modeling as Linear Prediction

- Given the prediction error $e[n] = s[n] - \sum_{k=1}^{p} a_k s[n-k]$
- If the modeling error $\hat{e}[n] = e[n] - Gv[n]$ is zero

$$e[n] = s[n] - \sum_{k=1}^{p} a_k s[n-k] = Gv[n]$$

$$P(z) = \sum_{k=1}^{p} a_k z^{-k}$$
$$= 1 - A(z)$$



$$\tilde{s}[n] = \sum_{k=1}^{p} a_k s[n-k]$$

APM of Deterministic Signals

- Assume $s[n]$ is a deterministic signal
- If the deterministic signal is causal and stable

$$\mathbf{E} = \langle \hat{e}^2[n] \rangle = \sum_{n=-\infty}^{\infty} \hat{e}^2[n]$$

- The autocorrelation function can be defined as

$$\phi_{SS}[i,k] = \sum_{n=-\infty}^{\infty} s[n-k]s[n-i] = \sum_{n=-\infty}^{\infty} s[n]s[n-(i-k)] = r_{SS}[i-k]$$

Autocorrelation Normal Equations

- The deterministic autocorrelation function is defined as

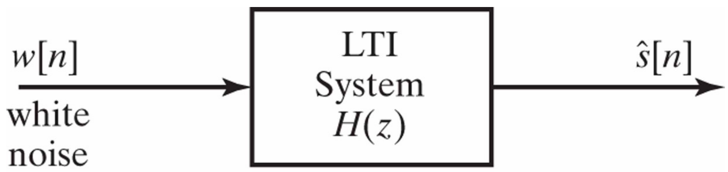
$$r_{SS}[m] = \sum_{n=-\infty}^{\infty} s[n+m]s[n] = \sum_{n=-\infty}^{\infty} s[n]s[n-m]$$

- The autocorrelation normal equation

$$\sum_{k=1}^{p} a_k r_{SS}[i-k] = r_{SS}[i], \quad i = 1, 2, \cdots, p$$

APM of Random Signals

- Assume that the input to the all-pole model is a wide-sense stationary, zero-mean, unit-variance, white noise sequence



Input Signal Properties

- White

$$E\{w[n+m]w[n]\} = \delta[m]$$

- Zero mean

$$E\{w[n]\} = 0$$

- Unit variance

$$E\{w^2[n]\} = 1$$

Mean-Squared Error

- The output signal

$$\hat{s}[n] = \sum_{k=1}^{p} a_k \hat{s}[n-k] + Gw[n]$$

- The mean squared value of the error

$$\mathbf{E} = \langle \hat{e}^2[n] \rangle = E\{\hat{e}^2[n]\}$$

- If $w[n]$ is wide-sense stationary

$$\phi_{SS}[i,k] = E\{s[n-k]s[n-i]\} = r_{SS}[i-k]$$

Autocorrelation Normal Equations

- The autocorrelation function is defined as

$$r_{SS}[m] = E\left\{s[n+m]s[n]\right\} = E\left\{s[n]s[n-m]\right\}$$

- The autocorrelation normal equation

$$\sum_{k=1}^{p} a_k r_{SS}[i-k] = r_{SS}[i], \quad i = 1,2,\cdots,p$$

- This is also called Yule-Walker equation

Minimum Mean-Squared Error

- If we write

$$\mathbf{E} = \left\langle \left(s[n] - \sum_{k=1}^{p} a_k s[n-k] \right)^2 \right\rangle$$

- It can be shown that (Problem 11.2)

$$\begin{aligned}\mathbf{E} &= \phi_{SS}[0,0] - \sum_{k=1}^{p} a_k \phi_{SS}[0,k] \\ &= r_{SS}[0] - \sum_{k=1}^{p} a_k r_{SS}[k]\end{aligned}$$

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Autocorrelation Functions

- We can find the model parameters if we know the autocorrelation functions
- It requires some knowledge of the signal to be modeled
- In most cases, no priori knowledge is available
- Signals are mostly nonstationary and seldom deterministic

Autocorrelation Function Estimation

- In practice, autocorrelation functions need to be estimated from given data
- Two main approaches
  - Autocorrelation method
  - Covariance method

Prediction Error

- The impulse response of the prediction error filter

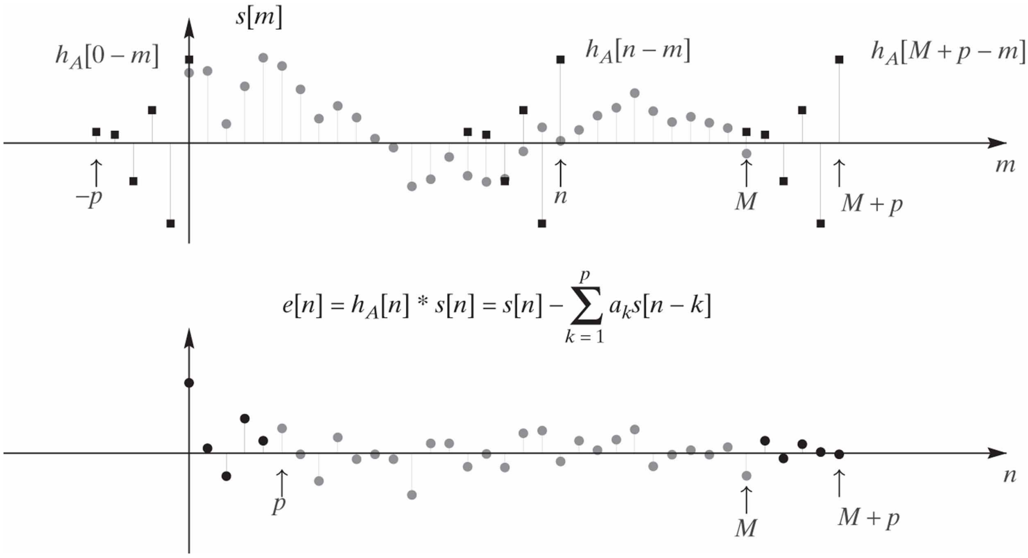
$$h_A[n] = \delta[n] - \sum_{k=1}^{p} a_k \delta[n-k]$$

- The prediction error sequence

$$e[n] = h_A[n] * s[n]$$

- If the input signal is nonzero for $n \in [0, M]$, then $e[n]$ is a length $M+p+1$ sequence

Prediction Error Illustration



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The Total Squared Prediction Error

- The total squared prediction error

$$E^{(p)} = \left\langle e^2[n] \right\rangle = \sum_{n=-\infty}^{\infty} e^2[n] = \sum_{n=0}^{M+p} e^2[n]$$

- The autocorrelation can be computed as

$$r_{SS}[|m|] = \sum_{n=-\infty}^{\infty} s[n]s[n+|m|] = \sum_{n=0}^{M-|m|} s[n]s[n+|m|]$$

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Autocorrelation Method

- Given a finite length sequence $s[n]$ for $n \in [0, M]$
- Compute the autocorrelation function as

$$r_{SS}[|m|] = \sum_{n=-\infty}^{\infty} s[n]s[n+|m|] = \sum_{n=0}^{M-|m|} s[n]s[n+|m|]$$

- Use the Yule-Walker equation to find a_k

$$\sum_{k=1}^{p} a_k r_{SS}[i-k] = r_{SS}[i], \quad i = 1, 2, \cdots, p$$

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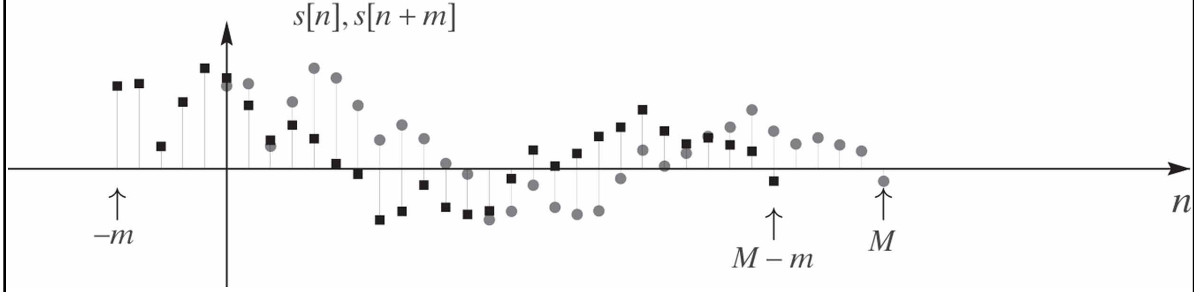
Autocorrelation Illustration



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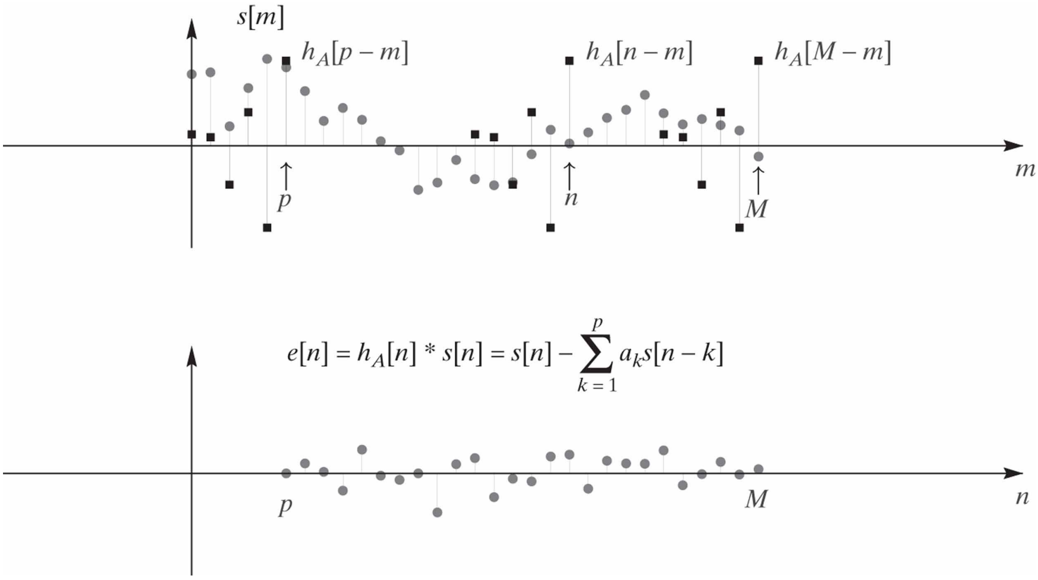
Computing the Prediction Error



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Autocorrelation Normal Equations

- The autocorrelation normal equation (or Yule-Walker equation) to find a_k for an all-pole model

$$\sum_{k=1}^{p} a_k \phi_{SS}[i,k] = \phi_{SS}[i,0], \quad i = 1,2,\cdots,p$$

- In case $\phi_{SS}[i,k] = r_{SS}[|i-k|]$

$$\sum_{k=1}^{p} a_k r_{SS}[|i-k|] = r_{SS}[i], \quad i = 1,2,\cdots,p$$

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Matrix Equation I

$$\begin{bmatrix} \phi_{SS}[1,1] & \phi_{SS}[1,2] & \phi_{SS}[1,3] & \cdots & \phi_{SS}[1,p] \\ \phi_{SS}[2,1] & \phi_{SS}[2,2] & \phi_{SS}[2,3] & \cdots & \phi_{SS}[2,p] \\ \phi_{SS}[3,1] & \phi_{SS}[3,2] & \phi_{SS}[3,3] & \cdots & \phi_{SS}[3,p] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{SS}[p,1] & \phi_{SS}[p,2] & \phi_{SS}[p,3] & \cdots & \phi_{SS}[p,p] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} \phi_{SS}[1,0] \\ \phi_{SS}[2,0] \\ \phi_{SS}[3,0] \\ \vdots \\ \phi_{SS}[p,0] \end{bmatrix}$$

$$\sum_{k=1}^{p} a_k \phi_{SS}[i,k] = \phi_{SS}[i,0], \quad i = 1,2,\cdots,p$$

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Matrix Equation II

$$\begin{bmatrix} r_{SS}[0] & r_{SS}[1] & r_{SS}[2] & \cdots & r_{SS}[p-1] \\ r_{SS}[1] & r_{SS}[0] & r_{SS}[1] & \cdots & r_{SS}[p-2] \\ r_{SS}[2] & r_{SS}[1] & r_{SS}[0] & \cdots & r_{SS}[p-3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{SS}[p-1] & r_{SS}[p-2] & r_{SS}[p-3] & \cdots & r_{SS}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} r_{SS}[1] \\ r_{SS}[2] \\ r_{SS}[3] \\ \vdots \\ r_{SS}[p] \end{bmatrix}$$

$$\sum_{k=1}^{p} a_k r_{SS}[|i-k|] = r_{SS}[i], \quad i = 1, 2, \cdots, p$$

Matrix Notation in a General Form

- The set of linear equations have the representation in vector-matrix form

$$\mathbf{\Phi a} = \psi$$

- Properties of the matrix $\mathbf{\Phi}$
  - Symmetric
  - Positive Definite (Invertible), and
  - Toeplitz Matrix

Solution of the APM Parameters

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$a = lpc(S, P)$

order

linear predictive coding

- Since the matrix $\mathbf{\Phi}$ is positive definite (invertible)

$$\Phi \mathbf{a} = \psi \Rightarrow \mathbf{a} = \Phi^{-1}\psi$$

- Computing the inverse of the matrix $\mathbf{\Phi}$ has the complexity of $O(n^3)$
- By exploiting the fact that the matrix $\mathbf{\Phi}$ is a Toeplitz Matrix, a simpler algorithm can be used to find the parameter vector $\mathbf{a}$ (Levinson-Durbin Recursion)

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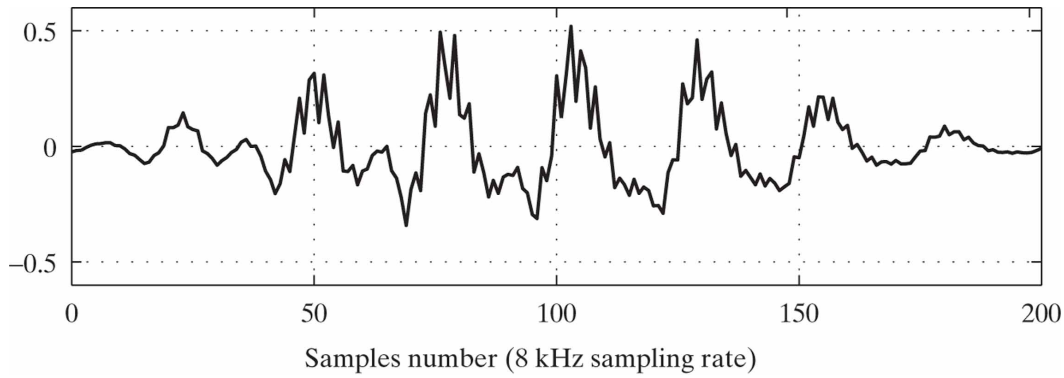
Linear Prediction and the Spectrum I



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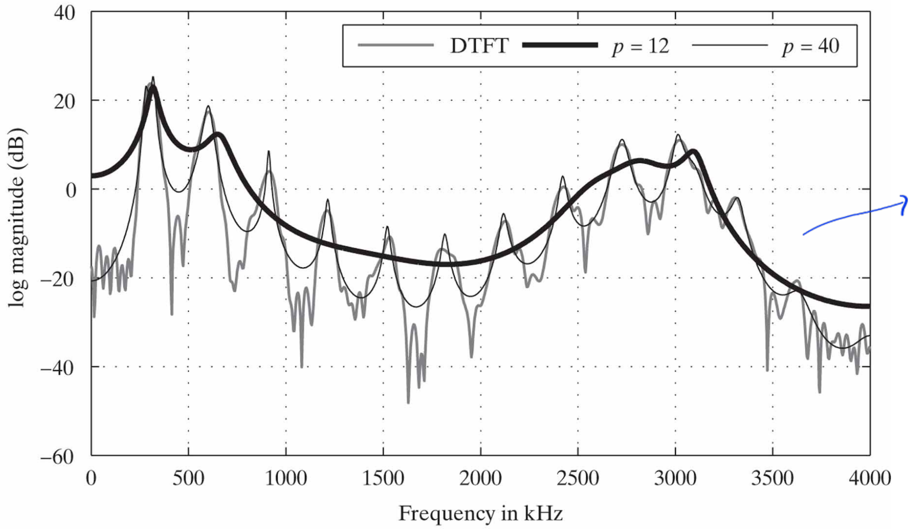
Linear Prediction and the Spectrum I



harmonic.

Parametric Modeling
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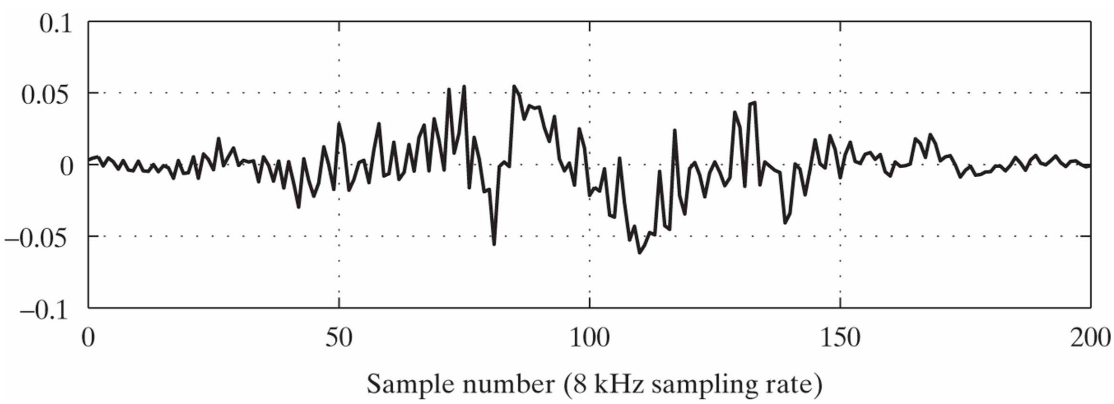
Linear Prediction and the Spectrum II



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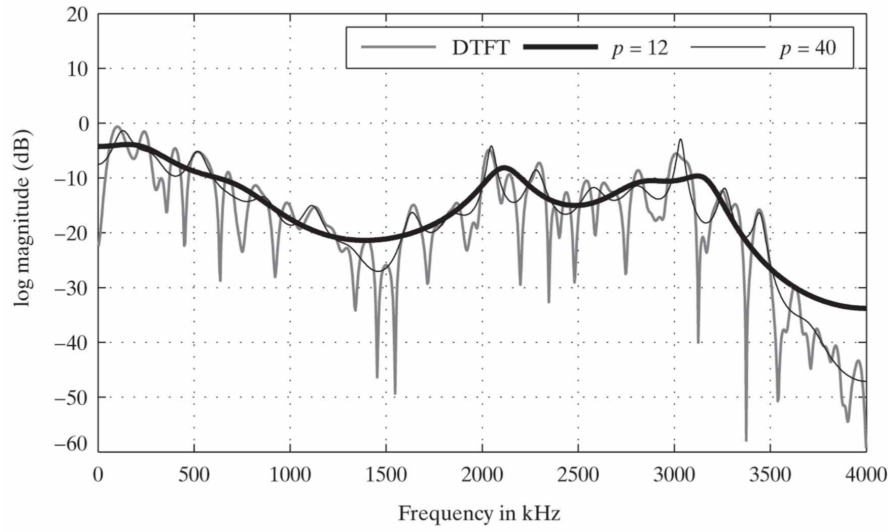
Linear Prediction and the Spectrum II



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Questions?

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