Lecture 8

ECE5022: Digital Signal Processing

Lecture 8: Wiener Filter

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Outline

- Causal Wiener Filter
- Non-Causal Wiener Filter

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Wiener Filter

- A filter designed to produce an ***optimal*** estimate of a desired or target random process by linear time-invariant filtering of an observed noisy process
- Optimality criterion used for the Wiener filter is mininum mean-squared error
- Can be derived for
  - Causal discrete-time domain
  - Noncausal frequency domain

signal

→ noisy signal = target signal + noise

(MMSE)

계산하려면, have to know the target signal.

∴ target signal 안다는 가정 필요함!

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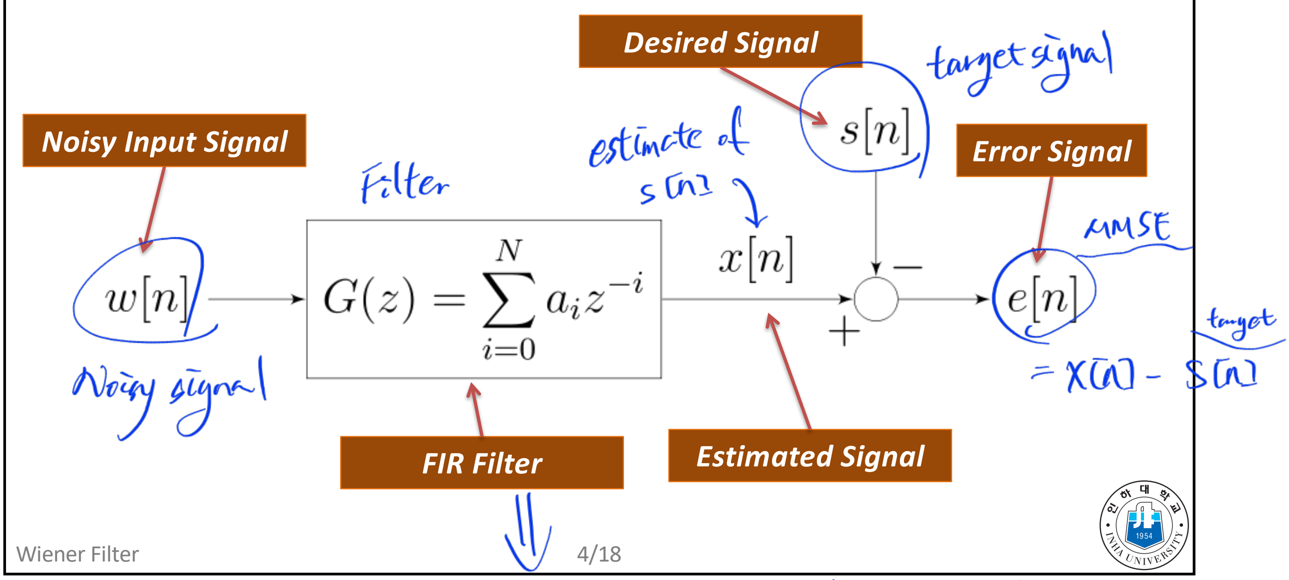
Discrete-Time Wiener Filter



FIR 버전의 wiener filter ⇒ $\{a_0, a_1, \cdots, a_N\}$

N+1 개 파라미터.

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Mean-Squared Error

- The mean squared value of the error

mean squared-error.

$*$ $E\{-\}$: expectation
↳ mean

$$\mathbf{E} = \left\langle e^2[n] \right\rangle = E\left\{e^2[n]\right\} = E\left\{\left(x[n]-s[n]\right)^2\right\}$$

$= E\{ x^2[n] - 2x[n] + s^2[n] \}$

$$= E\left\{x^2[n]\right\} + E\left\{s^2[n]\right\} - 2E\left\{x[n]s[n]\right\}$$

$$= E\left\{\left(\sum_{k=0}^{N} a_k w[n-k]\right)^2\right\} + E\left\{s^2[n]\right\} - 2E\left\{\left(\sum_{k=0}^{N} a_k w[n-k]\right)s[n]\right\}$$

$\Rightarrow$ quadratic function of $\{a_0, a_1, \cdots, a_N\}$
multivariate

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Minimizing the Mean-Squared Error

- Take the partial derivative *w.r.t.* the ith coefficient a_i

$$\frac{\partial \mathbf{E}}{\partial a_i} = \frac{\partial}{\partial a_i}\left[E\left\{\left(\sum_{k=0}^{N} a_k w[n-k]\right)^2\right\} + E\left\{s^2[n]\right\} - 2E\left\{\left(\sum_{k=0}^{N} a_k w[n-k]\right)s[n]\right\}\right]$$

$$= 2\sum_{k=1}^{N} a_k E\left\{w[n-k]w[n-i]\right\} - 2E\left\{s[n]w[n-i]\right\}$$

$$= 0, \quad i = 0, 1, \cdots, N$$

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Autocorrelation and Cross-Correlation

- Define the autocorrelation and cross-correlation as

$$r_{ww}[m] = E\{w[n]w[n+m]\}$$

m: difference

$$r_{ws}[m] = E\{w[n]s[n+m]\}$$

- If both $w[n]$ and $s[n]$ are wide-sense stationary

$$\sum_{k=0}^{N} a_k r_{ww}[i-k] - r_{ws}[i] = 0, \quad i = 0,1,\cdots,N$$

- This is called Wiener-Hopf equation

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Wiener-Hopf Equation

① $i=0$

- Wiener-Hopf Equation

② $i=1$

$$\sum_{k=0}^{N} a_k r_{ww}[i-k] = r_{ws}[i] \quad i = 0,1,\cdots,N$$

- Yule-Walker Equation

$$\sum_{k=1}^{p} a_k r_{ss}[i-k] = r_{ss}[i], \quad i = 1,2,\cdots,p$$

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Wiener-Hopf Equation in Matrix Form

- The Wiener-Hopf equation can be expressed in matrix form like the Yule-Walker equation

$$\begin{bmatrix} r_{ww}[0] & r_{ww}[1] & r_{ww}[2] & \cdots & r_{ww}[N] \\ r_{ww}[1] & r_{ww}[0] & r_{ww}[1] & \cdots & r_{ww}[N-1] \\ r_{ww}[2] & r_{ww}[1] & r_{ww}[0] & \cdots & r_{ww}[N-2] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r_{ww}[N] & r_{ww}[N-1] & r_{ww}[N-2] & \cdots & r_{ww}[0] \end{bmatrix} \begin{bmatrix} a_0 \\ a_2 \\ a_3 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} r_{ws}[0] \\ r_{ws}[1] \\ r_{ws}[1] \\ \vdots \\ r_{ws}[N] \end{bmatrix}$$

Matrix Notation in a General Form

- The Wiener-Hopf Equation has the representation in vector-matrix form

$$\Phi \mathbf{a} = \psi$$

- The solution can be found by

$$\mathbf{a} = \Phi^{-1} \psi$$

- The solution gives the minimum mean-squared error (MMSE) estimate of the signal $s[n]$

Levinson-Durbin Recursion

- Properties of the matrix $\mathbf{\Phi}$
  - Symmetric
  - Positive Definite (Invertible), and
  - Toeplitz Matrix
- Levinson-Durbin recursion can be used to quickly find the inverse of the matrix $\mathbf{\Phi}$

Outline

- Causal Wiener Filter
- Non-Causal Wiener Filter

Non-Causal Wiener Filter I/III

- Note that the Fourier transform of an autocorrelation function is defined as power spectral density
- We can derive a frequency domain filter based on the same constraint of minimizing the mean-squared error
- This gives us a non-causal Wiener filter in the frequency domain

Non-Causal Wiener Filter II/III

- Extend the FIR order of the Wiener-Hopf equation to infinity and replace i with n

$$\sum_{k=-\infty}^{\infty} a_k r_{ww}[n-k] = \sum_{k=-\infty}^{\infty} h[k] r_{ww}[n-k] = h[n] * r_{ww}[n] = r_{ws}[n]$$

- If we take the Fourier transform of the equation

$$H(e^{j\omega})\phi_{ww}(e^{j\omega}) = \phi_{ww}(e^{j\omega}) \Rightarrow H(e^{j\omega}) = \frac{\phi_{ws}(e^{j\omega})}{\phi_{ww}(e^{j\omega})}$$

Non-Causal Wiener Filter III/III

- Non-causal frequency domain Wiener filter

$$H(e^{j\omega}) = \frac{\phi_{ws}(e^{j\omega})}{\phi_{ww}(e^{j\omega})}$$

- If the noisy input is corrupted with additive uncorrelated noise, then

$$H(e^{j\omega}) = \frac{\phi_{ws}(e^{j\omega})}{\phi_{ww}(e^{j\omega})} = \frac{\phi_{ss}(e^{j\omega})}{\phi_{ss}(e^{j\omega}) + \phi_{nn}(e^{j\omega})}$$

Alternative Form

- The non-causal Wiener filter

$$H(e^{j\omega}) = \frac{\phi_{ws}(e^{j\omega})}{\phi_{ww}(e^{j\omega})} = \frac{\phi_{ss}(e^{j\omega}) / \phi_{nn}(e^{j\omega})}{\phi_{ss}(e^{j\omega}) / \phi_{nn}(e^{j\omega}) + \phi_{nn}(e^{j\omega}) / \phi_{nn}(e^{j\omega})}$$

- If we define

$$\xi(e^{j\omega}) \triangleq \frac{\phi_{ss}(e^{j\omega})}{\phi_{nn}(e^{j\omega})}$$

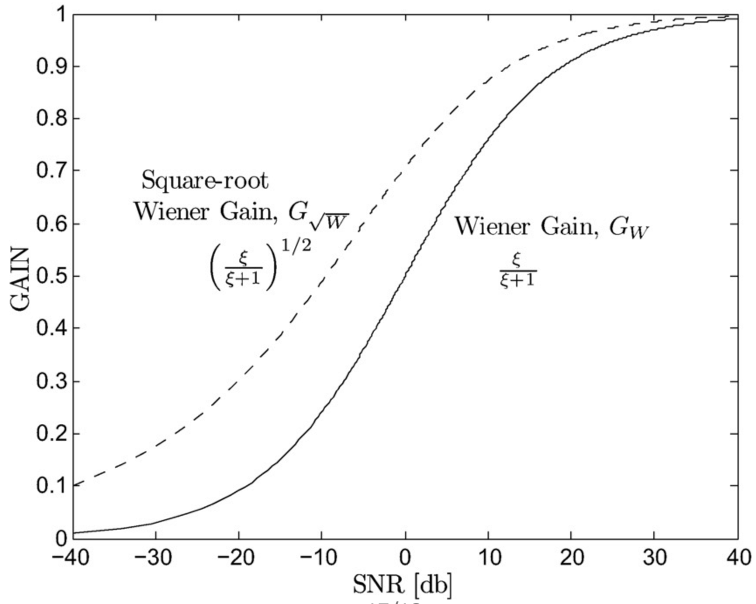
Signal-to-Noise Ratio

- Then

SNR for w

$$H(e^{j\omega}) = \frac{\xi(e^{j\omega})}{\xi(e^{j\omega}) + 1}$$

Wiener Gain



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Questions?

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