ECE5022: Digital Signal Processing

Lecture 7: Spectral Analysis & Frequency-Domain Signal Processing

May 3, 2022

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Outline

- Power Spectral Density
- Spectral Analysis
- STFT Signal Reconstruction
- STFT-Based Signal Processing
- Introduction to HW #2

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Instantaneous Power

신호 → Voltage / current 로 표현 가능.

- If a signal is a WSS random process, we can estimate its instantaneous power as its expected value of its square i.e.,

Auto-corr function

$$E\{x^2(t)\}$$

$$= E\{x(t)x(t+\tau)\}\Big|_{\tau=0} = R_{XX}(\tau)\Big|_{\tau=0} = R_{XX}(0)$$

- Where $R_{XX}(\tau)$ is the autocorrelation of the random variable x

Power Spectral Density for WSS Process

- Given $R_{XX}(\tau)$, the ***power spectral density*** is

$$S_{XX}(\Omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\Omega\tau}d\tau$$

F.T. with respect to the time

- Given $S_{XX}(\Omega)$, the autocorrelation function is

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\Omega)e^{j\Omega\tau}d\Omega$$

Fourier Transform Pair

PSD for a DT WSS Random Process

- If a DT random process $X[n]$ is WSS, then

$$S_{XX}(\omega) = \sum_{k=-\infty}^{\infty} R_{XX}[k]e^{-j\omega k}$$

auto corr 의 DTFT 결과.
R_{XX}

Discrete-Time Fourier Transform Pair

$$R_{XX}[k] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{XX}(\omega)e^{j\omega k}\, d\omega$$

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- Power Spectral Density
- Spectral Analysis ← random processes
- STFT Signal Reconstruction
- STFT-Based Signal Processing
- Introduction to HW #3

DFT ← Deterministic Signal

가장 큰 차이!

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Power Spectral Density

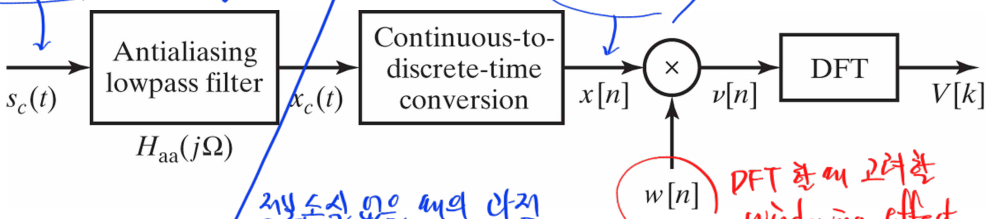
DTFT FT

$\omega = \Omega T \Leftrightarrow \Omega = \frac{\omega}{T}$

$P_{ss}(\Omega)$

no information loss

$P_{xx}(\omega)$



정보손실 없을 때의 과정.

DFT 할 때 고려할 windowing effect.

- Given $P_{ss}(\Omega)$, the power spectral density of $s_c(t)$

$$P_{xx}(\omega) = \frac{1}{T} P_{ss}\left(\frac{\omega}{T}\right)$$

샘플링의 scaling factor

Ω

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유한 범위 signal 일 때. 고려해서 변형.

Power Spectral Density Estimation

- The DTFT of the windowed signal

(random process) signal을 가지고 (windowed 된) DTFT를 하면

length L

$$V(e^{j\omega}) = \sum_{n=0}^{L-1} w[n]x[n]e^{-j\omega n}$$

$P_{ss}(\Omega)$

$P_{xx}(\omega) = \frac{1}{T} P_{ss}\left(\frac{\omega}{T}\right)$

$V[n] = W[n]\,X[n]$

- An estimate of the power spectrum of $P_{ss}(\Omega)$

$$I(\omega) = \frac{1}{LU}\left|V(e^{j\omega})\right|^2$$

U 정규화

PSD.=Periodogram.

normalization factor

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PSD 추정(신호의 spectral analysis) 하는 단계

→ Signal 샘플링 후 finite duration (유한 구간 샘플) 취한다.

→ window 적용

→ DTFT 취한다.

→ 정규화 한다.

끝!

Periodogram

→ the most popular method of PSD estimation.

- Power Spectral Density Estimate

$$I(\omega) = \frac{1}{LU}\left|V(e^{j\omega})\right|^2$$

w 에 대한 함수 얻음.

이게 연속시간 신호에 대한 원래신호 PSD 된다.

→ can be used for noise estimation.

- Periodogram
  - If the window sequence $w[n]$ is a rectangular window
- Modified periodogram
  - If the window sequence $w[n]$ is not a rectangular window

but. 다른거 쓰는게 일반적.

Periodogram and Autocorrelation

- It can be shown that (Problem 10.33)

$$I(\omega) = \frac{1}{LU}\left|V(e^{j\omega})\right|^2 = \frac{1}{LU}\sum_{m=-(L-1)}^{L-1} c_{vv}[m]e^{-j\omega n}$$

where

$$c_{vv}[m] = \sum_{n=0}^{L-1} w[n]x[n]w[n+m]x[n+m]$$

is the time autocorrelation of the windowed sequence

→ estimate of the autocorrelation.

The DFT of the Periodogram

- Discrete samples of the periodogram

$$i[k] = I(\omega)\Big|_{\omega=\frac{2\pi k}{N}} = \frac{1}{LU}\left|V[k]\right|^2$$

- Can be computed by a digital computer

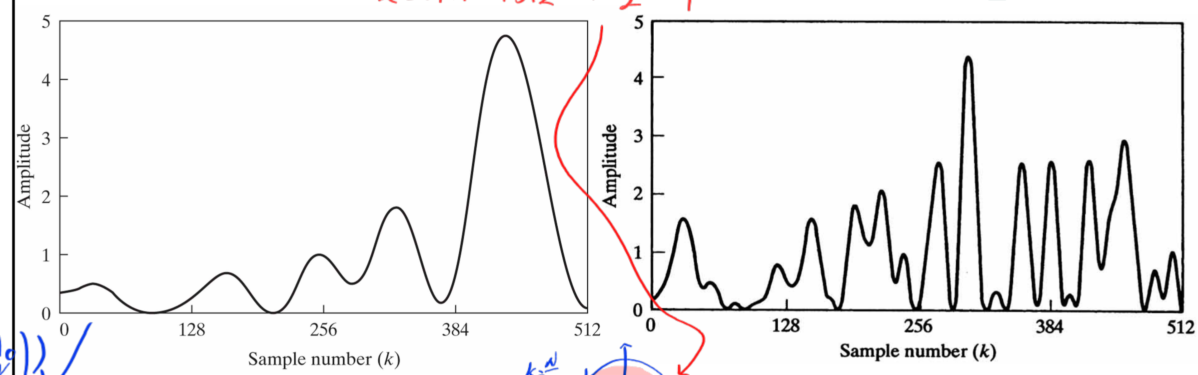
Periodogram of Sample Function of White Noise.

Periodogram Examples (White Noise)

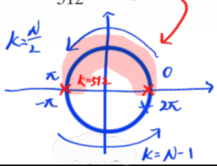
ex) $N=1024$ 인 경우: $k=0,1,\cdots,512,\cdots,1023$. ↳ $\frac{N}{2} \Leftarrow \pi$ (maximum frequency)

- Periodograms of white-noise sequences

$k=0,1,\cdots,512 \Leftarrow \frac{N}{2}+1$ points



L=16, N=1024 L=64, N=1024



(window 길이) L 줄이면 스무딩 됨.

그렇다니 이건.

White Noise에도 Flat한 response는 얻을 수 없음.

why? → windowing effect

Periodogram Examples (White Noise)

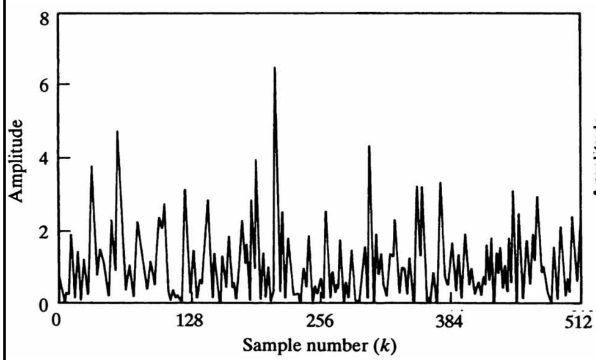
$\frac{1}{LV}|V[k]|^2$

v[k] is a Random Variable

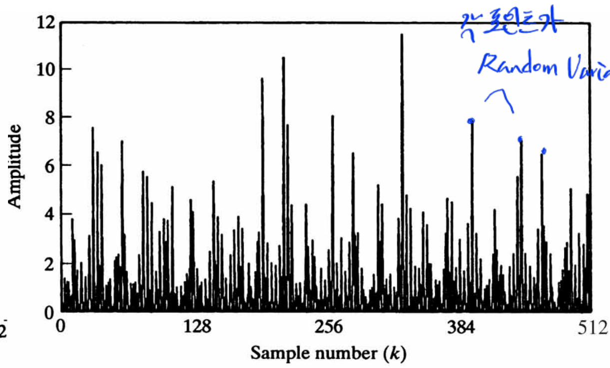
- Periodograms of white-noise sequences

각 포인트가 Random Variables.

평균, 표준편차 알 수 있음. (계산가능)



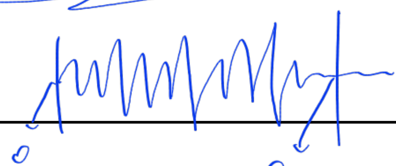
L=256, N=1024



L=1024, N=1024

Periodogram Averaging

- We need to find ***smoother*** estimate of the power spectrum
- Increasing the window length L ***does not*** help
- We can find the average of periodogram estimates
- This procedure is called Bartlett's procedure



(N : DFT 길이로 보통 쓰니까) Q로.

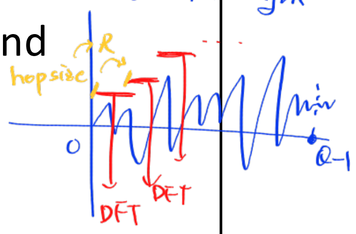
Bartlett

random process → PSD

이번시간에는 이론적인 분석.

Taking segments of Length L samples

N: DFT Length



- Given a data sequence of length Q, we can find segments of length L sequences by

window length

$$x_r[n] = x[rR + n]w[n], \quad 0 \le n \le L-1$$

frame index, hop size

- The periodogram of the r^{th} segment is

$\frac{1}{LV}$ frame

$$I_r(\omega) = \left|X_r(e^{j\omega})\right|^2, \quad X_r(e^{j\omega}) = \sum_{n=0}^{L-1} x[n]e^{j\omega n}$$

can be ignored

: K개 N.
window length L
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K개 periodogram

N = Q points DFT.

짧은 segment 쓰면 주파수 해상도 하락. K 감소. (N<Q)

Finding the Average of Periodograms

- Given the periodogram estimates for K segments, we can find the average as

$$\bar{I}(\omega) = \frac{1}{K}\sum_{r=0}^{K-1} I_r(\omega),$$

DTFT

$\bar{I}[k] = \frac{1}{K}\sum_{r=0}^{K-1}$

- If a rectangular window is used, then this periodogram averaging is called ***Bartlett's procedure***

Expected Value of Averaged Periodograms

- The expected value of averaged periodograms by definition

mean of the averaged periodogram

$$E\{\bar{I}(\omega)\} = \frac{1}{K}\sum_{r=0}^{K-1} E\{I_r(\omega)\}$$

random variable

Rect window

- If we assume that $I_r(\omega)$ are identically distributed independent (*i.i.d*) random variables, then

$x_r[n] = x[rR+n]\cdot w[n]$ 에서

Estimator

$$E\{\bar{I}(\omega)\} = E\{I_r(\omega)\}, \quad \text{for any } r$$

$I_r(w) \to PSD$

$L \to \infty$ 인 경우

estimator 값의 평균이 the property to estimator와 같다면 → unbiased estimator

Variance of Averaged Periodograms

- If $I_r(\omega)$ are *i.i.d* random variables, it can be shown that (Bertekas and Tsisiklis, 2008)

$$\text{var}\{\bar{I}(\omega)\} \cong \frac{1}{K}\text{var}\{I_r(\omega)\}$$

Variance : inversely proportional to K

$\lim_{K\to\infty} \text{var}\{\bar{I}(w)\} = 0$ → consistent estimator.

Bartlett's Procedure

- Average of periodograms

$$\bar{I}(\omega) = \frac{1}{K}\sum_{r=0}^{K-1} I_r(\omega)$$

- With a rectangular window

$$w[n] = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases}$$

Periodogram Averaging

- Given a data sequence of length Q
- Segment the data with K sequences of length L
- Compute the periodogram for each of K sequences
- Average the periodograms

$K \leftrightarrow L$ Tradeoff.

- It is an unbiased estimate for $L \rightarrow \infty$ (Sufficiently Large L)
- It is a consistent estimate for $K \rightarrow \infty$ (Sufficiently Large K)

$K \not\rightarrow \infty$.

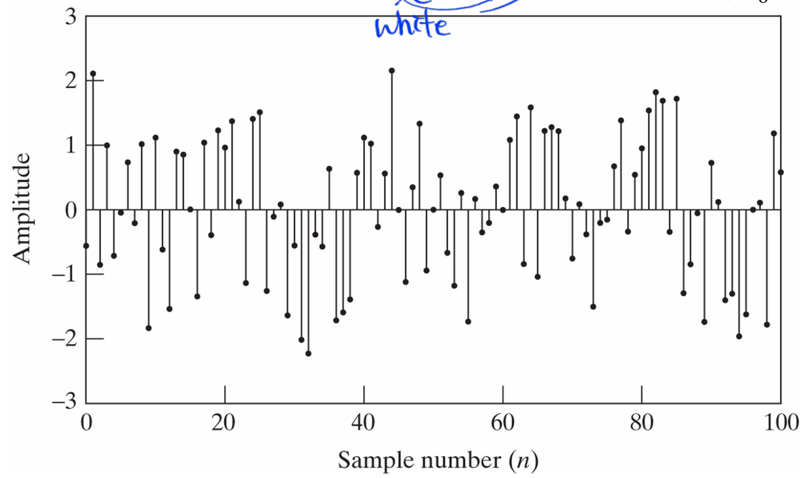
finite. 충분히 클 수는 있다.

White Gaussian random process = White Gaussian Noise (WGN)

Periodogram Analysis Example (I/IV)

uncorrelated white noise e[n]

- A sinusoid with additive white noise $x[n] = A\cos(\omega_0 n + \theta) + e[n]$ noise



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Periodogram Analysis Example (II/IV)

- The power spectrum of this signal is

$$P_{xx}(\omega) = \frac{A^2\pi}{2}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right] + \sigma_e^2$$

- The expected value of the average periodogram is

$$E\left\{\overline{I}(\omega)\right\} = \frac{A^2}{4LU}\left[C_{ww}(e^{j(\omega-\omega_0)}) + C_{ww}(e^{j(\omega+\omega_0)})\right] + \sigma_e^2$$

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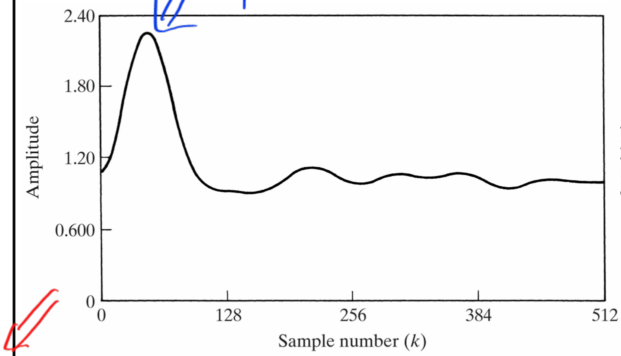
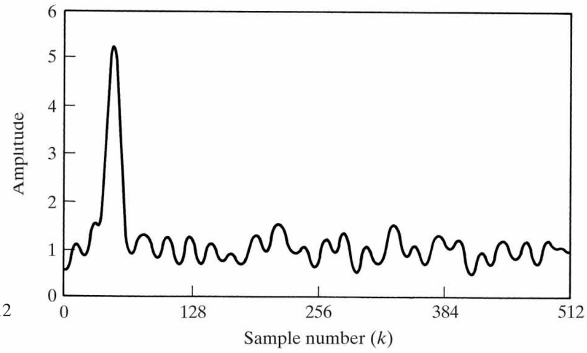
Periodogram Analysis Example (III/IV)

ideal case

$Q = 1024$



K=127, L=16, N=1024



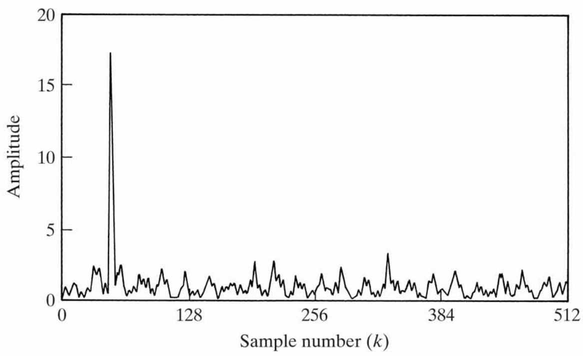
K=31, L=64, N=1024

Periodogram averaging 했으니 더 flatten.

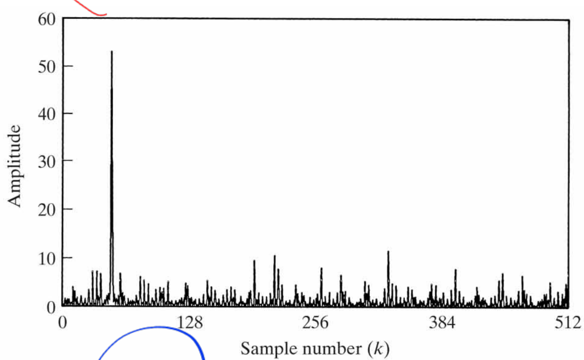
$L \downarrow \Rightarrow K \uparrow$

window 사이즈 줄이면 segment 개수 많이 얻을 수 있음.

Periodogram Analysis Example (IV/IV)



K=7, L=256, N=1024



K=1, L=Q=1024, N=1024

Outline

- Power Spectral Density
- Spectral Analysis
- STFT Signal Reconstruction → Problem 4. of HW2.
- STFT-Based Signal Processing
- Introduction to HW #3

Signal Reconstruction

- Given the sampled DTFT, the DFT, we can find

windowed Signal

DFT

$$\underbrace{x_r[n+m]w[m]}_{\text{windowed Signal}} \underset{\text{inverse}}{\overset{\text{DFT}}{\rightleftharpoons}} \frac{1}{N}\sum_{k=0}^{N-1} X_r[n,k]e^{j(2\pi/N)km}$$

inverse

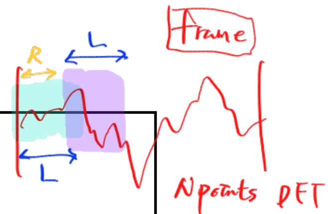
- The original sequence $x_r[n]$ can be found by

$$x_r[n+m] = \frac{1}{Nw[m]}\sum_{k=0}^{N-1} X_r[n,k]e^{j(2\pi/N)km}$$

$x_r[n+m]$ = r th segment of the signal

→ w[m] ≠ 0 라 가정

frame: 보통 비디오 seq.의 image 의미.
→ 음성신호에도 적용 가능.



STFT Signal Reconstruction

- Given the STFT:

frame index ($r=0$, $x_0[n] = x[n]w[n]$ 2 $(0 \sim L-1)$
$r=1$, $x_1[n] = x[n+R]w[n]$ 2 $x[R] \sim x[L+R-1]$
R만큼 hop.

$$X_r[k] = \sum_{m=0}^{L-1} x_r[m] e^{-j(2\pi/N)km}, \quad x_r[m] \equiv x[rR+m]w[m]$$

$m = 0, 1, \cdots, R-1$

No overlap. 보통은 겹치게 함.

- If $R=L=N$, then we can recover $x[n]$ by

$n \to n - rR$

$$x[rR+n] = \frac{x_r[n]}{w[n]} \Rightarrow x[n] = \frac{x_r[n-rR]}{w[n-rR]}, \quad rR \le n \le [(r+1)R-1]$$

windowed signal x을 window w로 나눔.

STFT Signal Reconstruction I/III

- Consider the parameters
  - N: DFT length
  - L: Window length
  - R: Sampling interval in time dimension (Hop size)
- Suppose that $R \le L \le N$

STFT

$$x_r[n] = x[rR+n]w[n] = \frac{1}{N} \sum_{k=0}^{N-1} X_r[k] e^{j(2\pi/N)kn}, \quad 0 \le n \le L-1$$

STFT Signal Reconstruction II/III

- Suppose that we can reconstruct $x[n]$ by

reconstructed signal

$x_r[n]$: signal at the rth frame.

$$\hat{x}[n] = \sum_{r=-\infty}^{\infty} x_r[n-rR]$$

⇒ We want $\hat{x}[n] = x[n]$

- We can show that

$x_r[n] = x[n+rR]w[n]$

$$\hat{x}[n] = \sum_{r=-\infty}^{\infty} x[rR+n-rR]w[n-rR] = x[n]\sum_{r=-\infty}^{\infty} w[n-rR] = x[n]\tilde{w}[n]$$

we want $\tilde{w}[n]$

- Define $\tilde{w}[n] = \sum_{r=-\infty}^{\infty} w[n-rR] \Rightarrow \hat{x}[n] = x[n]\tilde{w}[n]$

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STFT Signal Reconstruction III/III

- Condition for perfect reconstruction

$$\boxed{\tilde{w}[n] = \sum_{r=-\infty}^{\infty} w[n-rR] = C,} \quad -\infty < n < \infty$$

constant

shifted version of the window by integer multiple of R

- Examples
  - Rectangular window with R=L
  - Rectangular window with R=L/2 (L even)
  - Bartlett and Hann windows

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Bartlett and Hann Windows I/II

- Bartlett (Triangular Window)

$$w[n]=\begin{cases} 2n/M, & 0\le n\le M/2 \\ 2-2n/M, & M/2<n\le M \\ 0, & \text{otherwise} \end{cases}$$

- Hann Window → cos window

$$w[n]=\begin{cases} 0.5-0.5\cos\left(\dfrac{2\pi n}{M}\right), & 0\le n\le M \\ 0, & \text{otherwise} \end{cases}$$

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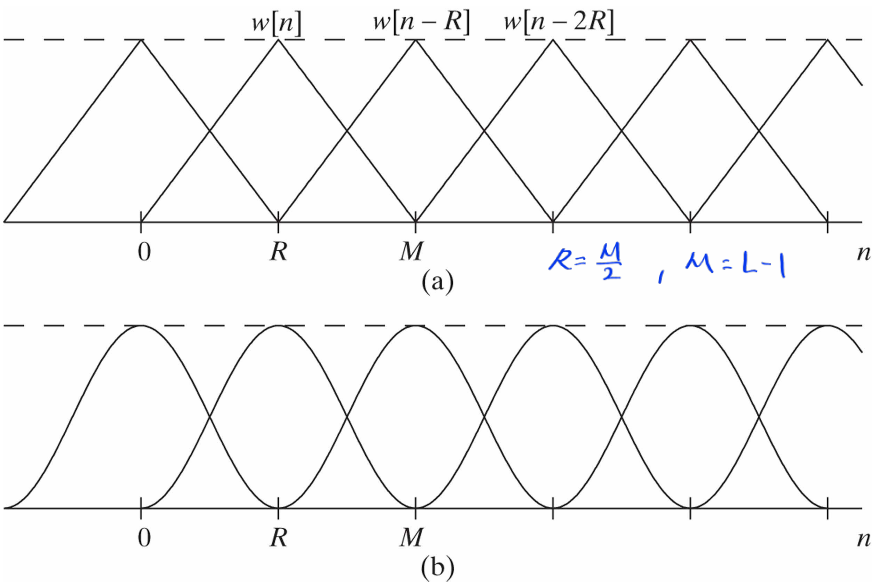
Bartlett and Hann Windows II/II



$R=\frac{M}{2}$, $M=L-1$

L: window Length

M: window 범위 1~m 표시.

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Windows for Reconstruction

- The envelope sequence is periodic, thus can be represented as an inverse of the DFT

$$\tilde{w}[n] = \sum_{r=-\infty}^{\infty} w[n-rR] = \underbrace{\frac{1}{R}\sum_{k=0}^{R-1} W(e^{j(2\pi k/R)})e^{j(2\pi k/R)n}}_{\text{IDFT.}}$$

- The condition for perfect reconstruction is

$$W(e^{j(2\pi k/R)}) = 0, \quad k = 1, 2, \ldots R-1$$

skip for now.

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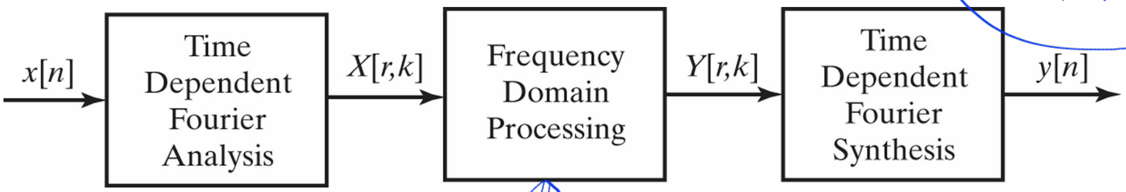
Signal Processing based on STFT

- Fourier analysis, processing, and synthesis

if $Y[r,k] = X[r,k]$

then $y[n] = x[n]$.

$\rightarrow Y[r,k] = H[r,k] \cdot X[r,k]$



HW2

processing by blocks of the

STFT-Based Signal Processing I/II

- Consider an input block with a rectangular window

$$x_r[n] = x[rR+n]w[n] = x[rR+n]$$

- Suppose $R = L$

$$x_r[n] = x[rL+n]$$

- Since the blocks do not overlap

$$x[n] = \sum_{r=-\infty}^{\infty} x_r[n-rL]$$

STFT-Based Signal Processing II/II

- Given the block of input

$$x_r[n] = x[rL+n]$$ frame index

- The output DFT for this input can be defined as

$$\boxed{Y_r[k] = H_r[k]X_r[k]}$$ ← circular convolution

- With proper zero padding, the output is linear convolution:

$$y_r[n] = \frac{1}{N}\sum_{r=-\infty}^{\infty} Y_r[k]e^{j(2\pi/N)kn}$$

Filter Bank Representation I/III

- Consider the STFT: with out Sampling in time

$X_r[k]$
$= X[n,k]\big|_{n=rR}$
$= X[rR,k]$

$$X[n,k] = \sum_{m=0}^{L-1} x[n+m]w[m]e^{-j(2\pi/N)km}$$

window

$\sum_{m=-\infty}^{\infty} x[n]\cdot h[n-m]$

- If we define $h_k[n] \equiv w[-n]e^{j(2\pi/N)kn}$

$\sum_{m=0}^{L-1} x[n+m]h_k[-m]$

$\tilde{m} = -m$

- Then the STFT becomes

$\sum_{\tilde{m}=0}^{-L+1} x[n-\tilde{m}]h_k[\tilde{m}] \Rightarrow \sum_{-m=0}^{-L+1} x[n-m]h_k[m]$

$\tilde{m} \leftarrow m$

$$X[n,k] = \sum_{m=0}^{L-1} x[n+m]w[m]e^{-j(2\pi/N)km} = x[n]*h_k[n]$$

Filter Bank Representation II/III

- The STFT:

$$X[n,k] = \sum_{m=0}^{L-1} x[n+m]w[m]e^{-j(2\pi/N)km} = x[n] * h_k[n]$$

- There exist N filters for $k = 0, 1, \ldots, N-1$
- The frequency response of the kth filter

$$H_k(e^{j\omega}) = W(e^{j[2\pi k/N - \omega]})$$

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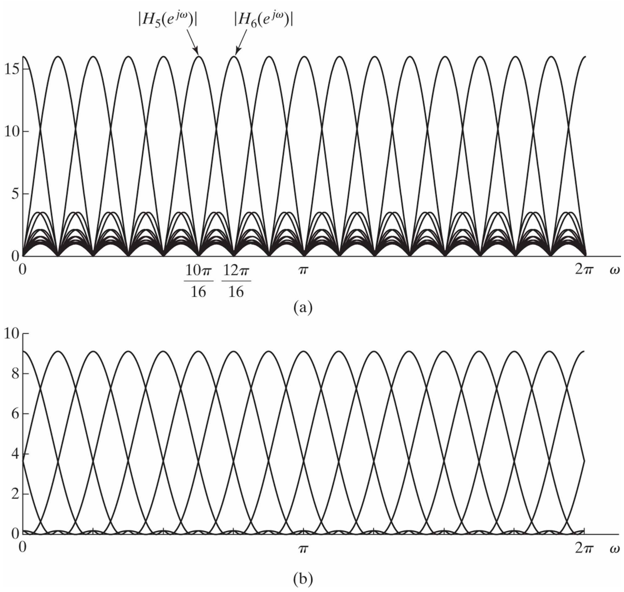
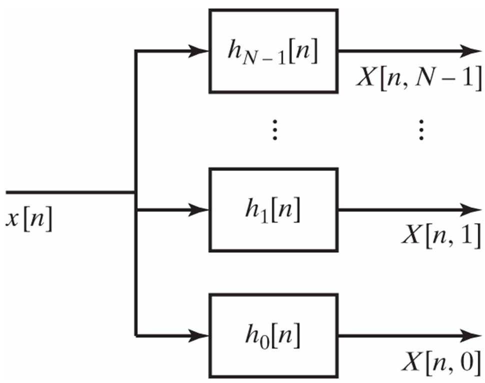
Filter Bank Representation III/III





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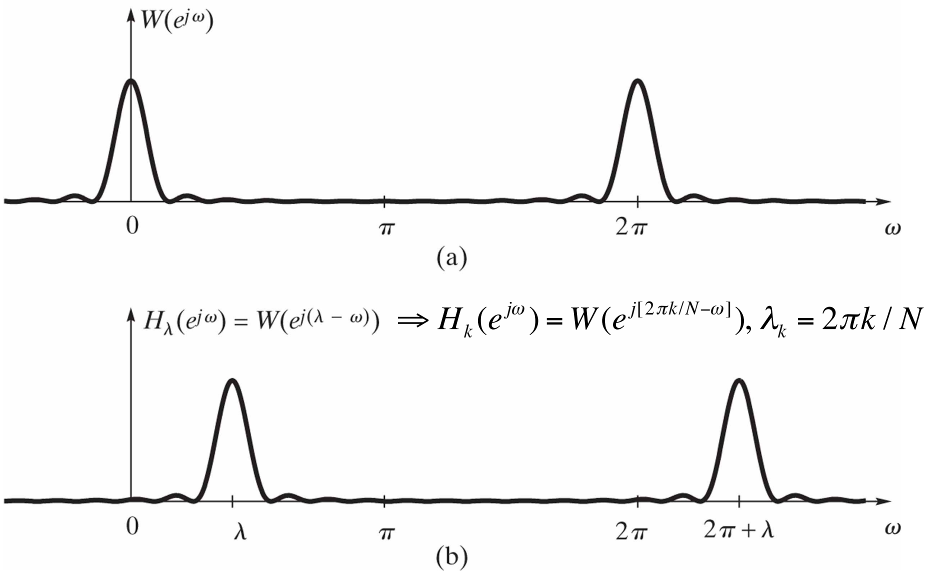
Filter Bank Filters



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Homework #2 (Machine Problem)

Homework #2 consists of the followings

1. Write your own Matlab/Python function for short-time Fourier transform (analysis)
2. Write your own Matlab/Python function for plotting spectrogram

Practical Considerations I/III

- The original formulation of STFT

$$X[n,\lambda) = \sum_{m=-\infty}^{\infty} x[n+m]w[m]e^{-j\lambda m}$$

  - Non-causal
  - Defined for continuous frequency λ
  - Defined for every sample index n
- Sampling in time and frequency of the STFT
  - Use length-*N* DFT and hop size *R*

Practical Considerations II/III

- After sampling in frequency

$$X[n,k]=\sum_{m=0}^{L-1} x[n+m]w[m]e^{-j(2\pi/N)km}$$

- After sampling in time

$$X_r[k]=X[rR,k]=\sum_{m=0}^{L-1} x[rR+m]w[m]e^{-j(2\pi/N)km}$$

Non-Causal

Typical STFT Formulation I/II

- With the Discrete-Time Fourier Transform (DTFT)

$$X[n,\omega)=\sum_{m=-\infty}^{\infty} x[m]w[m-n]e^{-j\omega m}$$

- With the Discrete Fourier Transform (DFT)

$$X[n,k]=\sum_{m=-\infty}^{\infty} x[m]w[m-n]e^{-j(2\pi/N)km}$$

Typical STFT Formulation II/II

- After sampling in time by the factor of R

$$X[rR,k] = \sum_{m=-\infty}^{\infty} x[m]w[m-rR]e^{-j(2\pi/N)km}$$

- Finally

$$X_r[k] = \sum_{m=-\infty}^{\infty} x[m]w[m-rR]e^{-j(2\pi/N)km}$$

Questions?