Numerical Analysis Test 1

Name: ______________________________

1. [Bisection Method] (a) Assume that $f(x)$ is continuous on $[a, b]$ and that $f(a)f(b) < 0$. Show that there's an $\alpha \in [a, b]$ such that $f(\alpha) = 0$.

(b) Describe the search method of the Bisection Method, and assuming that the conditions of (a) are true, how often will it work?

(c) Assuming the conditions in (a) are true, at what specific speed will (b) converge? What do we call that rate of convergence?

2. [Fixed Point Methods] (a) Suppose that $a < g(a) < b$, $a < g(b) < b$), and $|g'(x)| \leq k < 1$ and that $g'(x)$ is continuous for $x \in [a, b]$. Show that there must be a point $\alpha \in [a, b]$ such that $g(\alpha) = \alpha$.

(b) Show that if $x_{n+1} = g(x_n)$ then the sequence $|x_n - \alpha| \to 0$.

3. [Accelerated Convergence, Fixed Point Methods] (a) Suppose that the conditions from Problem 1 hold, and in addition, $g'(\alpha) = 0$, and $g''(x)$ is continuous on $[a, b]$. Show that

$$\lim_{n\to\infty} \frac{(x_{n+1} - \alpha)}{(x_n - \alpha)^2} = g''(\alpha)/2$$

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(b) What type of convergence do we call this?

4. [Newton's Method] (a) Describe Newton's Method.

(b) Show that if $f(\alpha) = 0$, $f'(\alpha) \neq 0$, and $f''(x)$ is continuous on $[a, b]$ then Newton's method satisfies $g'(\alpha) = 0$. Moreover it satifies $|g'(x)| < K < 1$ on some interval about α, and therefore must converge in an interval about α. You may use the result of problem 2. From this conclude that it converges quadratically on some interval about α.

5. [Lagrange Polynomials] (a) Describe the conditions which Lagrange polynomials solve, and the basic formula for the Lagrange polynomials.

(b) Are the Lagrange Polynomials the only polynomials of order n which will solve this problem on $n+1$ points? Why

(c) State the Lagrange Error Formula

(d) Explain the consequences of the formula, if your points are distributed in the interval $[0, 1]$. How about the interval $[-10, 10]$.

6. [Hermite Polynomials] (a) Describe the conditions that the Hermite polynomials solve.

(b) Use a limiting case of the Lagrange polynomials to build the Hermite polynomials. Can you get an error formula from this?