Numerical Analysis Test 2 Review:

[Euler's Method] Consider the initial value problem $y' = f(x,y)$, with $y(0) = y_0$, and Euler's method, $y_{n+1} = y_n + hf(x_n, y_n)$, where $x_{n+1} = x_n + h$. Assume that f satisfies a Lipshitz condition, or that $|f(x,y_1) - f(x,y_2)| \leq K|y_1 - y_2|$ for a fixed K and all x. Let $Y(x)$ be the actual solution to this problem and denote $Y_n = Y(x_n)$.

Find a basic bound between $e_{n+1} = Y_{n+1} - y_{n+1}$ and $e_n = Y_n - y_n$ (the local error).

Proof: Recall that $Y_n = Y(t_n)$ is the exact solution at t_n and y_n is the approximate solution at y_n. Using Taylor's Theorem to expand $Y(t)$ about t_n we have

$$e_{n+1} = Y_{n+1} - y_{n+1} = (Y(t_n) + hY'(t_n) + \frac{h^2}{2}Y''(\zeta_n)) - y_{n+1}.$$

Now since $Y'(t_n) = f(t_n, Y_n)$ and $y_{n+1} = y_n + hf(t_n, y_n)$ we have

$$e_{n+1} = (Y_n + hf(t_n, y_n) + \frac{h^2}{2}Y''(\zeta_n)) - (y_n + hf(t_n, y_n)).$$

Reorganizing gives us

$$e_{n+1} = (Y_n - y_n) + h(f(t_n, Y_n) - f(t_n, y_n)) + \frac{h^2}{2}Y''(\zeta_n) = e_n + h(f(t_n, Y_n) - f(t_n, y_n)) + \frac{h^2}{2}Y''(\zeta_n).$$

Now using the triangle inequality we get

$$|e_{n+1}| \leq |e_n| + h|f(t_n, Y_n) - f(t_n, y_n)| + \frac{h^2}{2}|Y''(\zeta_n)|.$$

If we let $M = max|Y''(\zeta_n)|$ and use the Lipshitz condition we get

$$|e_{n+1}| \leq |e_n| + hK|e_n| + \frac{h^2}{2}M = |e_n|(1 + hK) + \frac{h^2}{2}M,$$

which is a local error bound.

[Euler's Method, Global bound] Given the Local Bound, find a bound between e_{n+1} and e_1 (the global error).

Proof: We begin simply with the local bound and iterate on this bound.

$$\begin{aligned}
|e_{n+1}| &\leq |e_n|(1 + hK) + \frac{h^2}{2}M \\
&\leq (|e_{n-1}|(1 + hK) + \frac{h^2}{2}M)(1 + hK) + \frac{h^2}{2}M \\
&= |e_{n-1}|(1 + hK)^2 + \frac{h^2}{2}M(1 + (1 + hK)) \\
&\leq (|e_{n-2}|(1 + hK) + \frac{h^2}{2}M)(1 + hk)^2 + \frac{h^2}{2}M(1 + (1 + hK)) \\
&= |e_{n-2}|(1 + hK)^3 + \frac{h^2}{2}M(1 + (1 + hK) + (1 + hK)^2). \qquad (1)
\end{aligned}$$

Completing this iteration we get

$$|e_{n+1}| \leq |e_1|(1 + hK)^n + \frac{h^2}{2}M\sum_{k=0}^{n-1}(1 + hK)^k.$$

The right sum is a geometric series so we have

$$|e_{n+1}| \leq |e_1|(1+hK)^n + \frac{h^2}{2} M \frac{1-(1+hK)^n}{-hK}.$$