2-Algebraic Geometry

Songun Lee

August 6, 2024

Abstract

Clausen-Scholze's analytic stacks rely heavily on $D(X)$. The definition of analytic ring explicitly contains $D(R)$, and every geometric content is given by the 6-functor formalism. These developments parallel the evolution of noncommutative geometry into derived noncommutative geometry. This paper presents speculations about 2-Algebraic Geometry, drawing significant inspiration from Clausen-Scholze's work.

Contents

1 Introduction

In this paper, we explore the concept of 2-algebraic geometry, a vertical categorification of usual algebraic geometry. Our approach is heavily influenced by the work of Clausen and Scholze.

Throughout this work, we will ignore set-theoretic issues and assume that all categories appearing as objects are presentable.

2 2-Algebra

categorical level	$(1, 0)$	$(\infty, 0)$	$(\infty, 1)$
additive space	abelian group	spectra	stable category
additive map	$f(0) = 0$ $f(a + b) = f(a) + f(b)$	(-)	cocompleteness
addition multiplication distributive law	abelian group unital commutative distributive law	spectra E_∞ E_∞-**algebra**	stable category symmetric monoidal internal hom
algebraic map	$f(1) = 1$ $f(ab) = f(a)f(b)$	(-)	symmetric monoidal

Speculation 2.1. *We need another enlargement of 2-rings, similar to enlarging commutative rings to E_∞ ring spectra, before defining 3-rings.*

2.1 2-Additive Space

We begin by introducing the notion of a 2-additive space, which generalizes the concept of an additive space to higher categories.

Definition 2.2 (2-Additive Space)**.** A *2-additive space* is a stable category.

A *2-additive map* $f : R \to S$ is an adjoint pair $f_* \vdash f^* : R \to S$.

Any 2-additive space V has a dual space, which is the opposite category as a stable category.

Remark 2.3. While the direction of morphism might be confusing, it is correct algebraic direction. This will be clearer when we define 2-Ring.

The following table illustrates the progression of additive structures across different categorical levels:

We see that the usual condition of $f(0) = 0, f(a)+f(b) = f(a+b)$ is replaced with cocompleteness, which is equivalent to having an adjoint f^*.

We have analogues of usual linear algebraic constructions. For example, kernel and cokernel can be defined as follows:

Theorem 2.4. *For any 2-additive spaces S and T,* $\hom(S, T)$ *has a natural 2-additive space structure, which will be denoted as $[S, T]$.*

Theorem 2.5. *The category of 2-additive spaces forms a closed monoidal category, with $[S, T] = S^{\mathrm{op}} \otimes T$*

Proof. □

2.2 2-Ring

We now extend the notion of a ring to the 2-categorical setting. A 2-ring is a E_∞-algebra in the category of 2-additive spaces, which unwinds as:

Definition 2.6 (2-Ring)**.** A *2-ring* is a closed symmetric monoidal stable category.

An *algebraic morphism* $f : R \to S$ is an adjoint pair $f^* \dashv f_* : S \to R$ such that f^* is symmetric monoidal.

The distributive law becomes cocompleteness of $\otimes$, which is witnessed by internal hom.

The following table illustrates the progression of ring-like structures across different categorical levels:

Theorem 2.7. *For any 2-ring homomorphism $f : R \to S$, S is enriched over R, with $[-,-]_R$ being $f_*([-,-]_S)$.*

2.3 Initial Object

Theorem 2.8. *The initial object among 2-Rings is the category of Spectra.*

2.4 2-Module

Definition 2.9. A *module over a 2-Ring* R is a 2-additive space M, with a monoidal 2-additive map $R \to \mathrm{End}(M)$.

Theorem 2.10. *An R-module M is naturally R-enriched.*

3 2-Geometry

Following Clausen-Scholze, we derive all geometric content, including the correct choice of Grothendieck topology, from the 6-functor formalism.

This section will be a near copy of the work of Clausen-Scholze.

3.1 Defining $f_!$

Definition 3.1 (open immersion)**.** A 2-ring homomorphism $f : R \to S$ is an open immmersion $f^{\mathrm{op}} : \operatorname{Spec} S \to \operatorname{Spec} R$ when f^* is localization and admits further left adjoint that satisfies

Definition 3.2 (proper)**.** A 2-ring homomorphism $f : R \to S$ is a *proper map* $f^{\mathrm{op}} : \operatorname{Spec} S \to \operatorname{Spec} R$ when f_* admits further right adjoint, commutes with pullback, and satisfies projection formula.

Definition 3.3 (!-able map)**.** A $f^{\mathrm{op}} : \operatorname{Spec} S \to \operatorname{Spec} R$ is a *!-able map* if it can be written as $\operatorname{Spec} S \to \operatorname{Spec} S' \to \operatorname{Spec} R$ where the former is an open immersion and the latter is a proper map.

Theorem 3.4 (6-functor formalism of 2-rings)**.**

3.2 2-Algebraic Stack

Definition 3.5 (!-topology)**.** A !-able map $f : R \to S$ is a !-cover if $R = lim^! S^{\times_R n}$

Definition 3.6 (2-sheaf)**.** A *2-sheaf* X satisfies descent condition for every !-hypercovers such that

Definition 3.7 (2-scheme)**.**

4 Conclusion