2-Algebraic Geometry

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Abstract

Clausen-Scholze's analytic stacks rely heavily on $D(X)$. The definition of analytic ring explicitly contains $D(R)$, and every geometric content is given by the 6-functor formalism. These developments parallel the evolution of noncommutative geometry into derived noncommutative geometry. This paper presents speculations about 2-Algebraic Geometry, drawing significant inspiration from Clausen-Scholze's work.

Contents

1 Introduction

In this paper, we explore the concept of 2-algebraic geometry, a vertical categorification of usual algebraic geometry. Our approach is heavily influenced by the work of Clausen and Scholze.

Our work differs from previous works in following aspects:

1. We use (presentable) stable categories instead of abelian categories
2. Our geometry is inspired by Clausen-Scholze's recent work on D-topology and !-descent

Throughout this work, we will ignore set-theoretic issues and simply assume existence of Grothendieck universes.

2 2-Algebra

categorical level	$(1,0)$	$(\infty,0)$	$(\infty,1)$
additive space	abelian group	spectra	stable category
additive map	$f(0)=0$ $f(a+b)=f(a)+f(b)$	(-)	admits right adjoint (cocompleteness)
addition multiplication distributive law	abelian group unital commutative distributive law	spectra E_∞ E_∞-**algebra**	stable category symmetric monoidal internal hom
algebraic map	$f(1)=1$ $f(ab)=f(a)f(b)$	(-)	symmetric monoidal

Speculation 2.1. *We need another enlargement of 2-rings, similar to enlarging commutative rings to E_∞-ring spectra, before defining 3-rings.*

2.1 2-Additive Space

We begin by introducing the notion of a 2-additive space, which is usually called as 2-abelian group in the literature.

Definition 2.2 (2-Additive Space)**.** A *2-additive space* is a stable category. A *2-additive space* is a presentable stable category.

A *2-additive map* $f : R \to S$ is an adjoint pair $f_* \vdash f^* : R \to S$, with both functor being exact.

Remark 2.3. While the direction of morphism might be confusing, it is correct algebraic direction. This will be clearer when we define 2-Ring.

We see that the usual condition of $f(0) = 0, f(a)+f(b) = f(a+b)$ is replaced with admitting a right adjoint, roughly equivalent to being cocomplete.

We have analogues of usual linear algebraic constructions. For example, kernel and cokernel can be defined as follows:

Theorem 2.4 (kernel and cokernel)**.** *For any 2-additive map $f : V \to W$, where both V, W is 2-preadditive, we have* $\ker f$ *as the full subcategory of V where $f^*(x) = 0$, and* $\operatorname{coker} f$ *as the full subcategory of W where $f_*(x) = 0$.*

Note that conditions of form $a = 0$ do not add any additional homotopy data.

Proof. Without loss of generality, we only prove the kernel case. First, we need to prove that RHS forms stable category. $0 \in \ker(F)$ for any $p : a \to b$, $f^*(\mathrm{cofib}(p)) = 0$ as f^* is left adjoint, $\mathrm{cofib}(p) \in \ker(F)$

Now we show that

We now prove the universal properties. □

Theorem 2.5. *For any 2-preadditive spaces S and T,* $\hom(S, T)$ *has a natural 2-preadditive space structure, which will be denoted as $[S, T]$. This is 2-additive whenever T is 2-sadditive.*

Theorem 2.6. *The category of 2-additive spaces forms a closed monoidal category.*

Proof. □

2.2 2-Ring

We now extend the notion of a ring to the 2-categorical setting. A 2-ring is a E_∞-algebra in the category of 2-additive spaces, which unwinds as:

Definition 2.7 (2-Ring)**.** A *2-ring* is a closed symmetric monoidal stable category, with tensor product and internal hom being exact in both parameters.

An *algebraic morphism* $f : R \to S$ is an adjoint pair $f^* \dashv f_* : S \to R$ such that f^* is symmetric monoidal.

The distributive law becomes cocompleteness of $\otimes$, which is witnessed by internal hom.

Definition 2.8 (2-field)**.** A *2-field* is a 2-ring such that $1 \neq 0$, $a \otimes b = 0$ implies $a = 0 \vee b = 0$, and every object is dualizable.

For example, $D^b(\mathrm{FinVect}_k)$ is a 2-field. Without the earlier conditions, any direct sum of 2-fields will be another 2-field.

Theorem 2.9. *For any 2-ring homomorphism $f : R \to S$, S is enriched over R, with $[-,-]_R$ being $f_*([-,-]_S)$.*

Note that we haven't restricted ourselves to presentable stable categories. This allows us to consider $D^b(R)$ as a 2-ring, but now our category do not have an initial object.

Theorem 2.10. *The initial object among presentable 2-Rings is the category of Spectra.*

We now turn our attention where $f : R \to S$ induces fully faithful f_*, namely S is a reflective subcategory of R.

Theorem 2.11.

Proof. $[M, f_* f^* N] = [f^* M, f^* N]$
$f_* 1_S \otimes M == f_* f^* M$

□

2.3

2.4 2-Module

Definition 2.12. A *module over a 2-Ring* R is a 2-additive space M, with a monoidal 2-additive map $R \to \mathrm{End}(M)$.

Theorem 2.13. *An R-module M is naturally R-enriched.*

Proof. $[M, M] \to R$ we need adjoint of $(r, a) \mapsto ra : R \times M \to M$. this is obvious by construction. now we have homset, □

Let $f : R \to \mathrm{End}(M)$, $[a, b]_R = f_*$

Definition 2.14 (quotient ring)**.**

2.5 base change

It is easy to define tensor product with desired categorical properties. It is hard to construct one, as we are working with $(\infty, 1)$-categories.

Our first goal is to define $A \otimes_R B$ for $f : R \to A$, $g : R \to B$.

We call a map R-linear if it extends to a functor between R-enriched category

Definition 2.15 (tensor product)**.** $A \otimes_R B$ is a object representing R-bilinear functors from $A \otimes B$.

2.5.1 construction of tensor product

recall

$$ra \otimes b = r(a \otimes b) = a \otimes rb, (a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b, a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$$

3 2-Geometry

Following Clausen-Scholze, we derive all geometric content, including the correct choice of Grothendieck topology, from the 6-functor formalism.

This section will be a near copy of the work of Clausen-Scholze.

We simply define affine test space $\mathrm{Spec}\, S$ as S in $2\,\mathrm{Ring}^{\mathrm{op}}$, and call geometric morphism $f^o p$

3.1 Defining $f_!$

Definition 3.1 (open immersion)**.** A 2-ring homomorphism $f : R \to S$ is an open immmersion $f^{\mathrm{op}} : \operatorname{Spec} S \to \operatorname{Spec} R$ when f^* admits further left adjoint $f_!$ satisfying further projection formula.

Remark 3.2 (Scholze)**.** Let $f : X \to S$ be open immersion of usual 1-schemes. $f : \operatorname{Spec} D(X) \to \operatorname{Spec} D(S)$ is not an open immersion in the above sense unless it $f : X \to S$ is also a closed immersion

Thus, the above definition diverges from that of algebraic geometry.

Definition 3.3 (proper)**.** A 2-ring homomorphism $f : R \to S$ is a *proper map* $f^{\mathrm{op}} : \operatorname{Spec} S \to \operatorname{Spec} R$ when f_* satisfies projection formula and admits further right adjoint $f^!$

Definition 3.4 (!-able map)**.** A $f^{\mathrm{op}} : \operatorname{Spec} S \to \operatorname{Spec} R$ is a *!-able map* if it can be written as $\operatorname{Spec} S \to \operatorname{Spec} S' \to \operatorname{Spec} R$ where the former is an open immersion and the latter is a proper map.

Theorem 3.5. *Three classes of map satisfies 2-out-of-3 property, and stable under base change,*

Theorem 3.6 (6-functor formalism of 2-rings)**.**

3.2 2-Algebraic Stack

Definition 3.7 (!-topology)**.** A !-able map $f : R \to S$ is a !-cover if $R = lim^! S^{\times_R n}$

Definition 3.8 (2-sheaf)**.** A *2-sheaf* X satisfies descent condition for every !-hypercovers such that

Definition 3.9 (2-scheme)**.**

4 Conclusion