Non-Linear Cointegration Approaches in Pairs Trading

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1 Introduction

Pairs trading is a strategy that aims to exploit the relationship between two closely associated assets by trading on deviations from the norm. The two stock processes should move closely for this strategy to work well, and the spread should be mean reverting. In this sense, the primary purpose of finding cointegration is to identify comoving securities from the large space of combinations of securities and find a stationary spread between the securities.: Vidyamurthy(2004) and Rad *et al.* (2015) are the main studies related to the cointegration approach. Vidyamurthy's framework consists of three steps, which are 1) preselecting candidate pairs, 2) checking the tradability of a pair, and 3) implementing optimal trading strategy in the trading period. Since the cointegration approach mainly focuses on step 2, which is to find a pair through cointegration tests in the formation period, I will mainly focus on implementing non-linear tests and models to find additional investment opportunities that non-linear methods may generate.

The cointegration approach benefits from gathering economically reliable equilibrium relationships of identified pairs (Krauss 2017). The objective is to identify the spread of the two stocks deviating from their equilibrium so that the investor can gain the level of mispricing of the market. However, cointegration implicitly assumes that the deviations' adjustment towards the long-run equilibrium is instantaneous at each period (Balke and Fomby 1997; Stigler 2020).

Regarding the non-linearity of pairs trading, the most popular methods are either copulas (Liew 2013) or machine learning models (Huck 2010). However, most of these models focus on introducing non-linearity in optimal trading strategy, not finding possible pairs for statistical arbitrage. Machine learning approaches to exact the pairs still depend on cointegration tests. For example, Sarmento (2020) applies OPTICS on PCA transformed return series to find clusters of stocks but still iteratively uses the Engle-Granger test with the stocks in the clusters to find stationary spread. Non-linear cointegration is implemented by loosening the assumptions. For example, Yan, *et al.*(2022) apply the delayed cointegration concept with continuous-time dynamics to allow delayed adjustment of the price deviation from the long-run equilibrium. In this paper, I explore the cointegration approach on pairs trading through the Threshold Vector Error Correction Model (TVECM) and utilize the functional-coefficient cointegration model that Xiao (2009) suggested.

2 Models

2.1 Threshold VECM

Vector Error Correction Model (VECM) formulates a linear form of cointegration through the Vector Autoregressive model. In Equation (1), there is neither constant nor time trend term because pairs trading requires a stationary spread without a time trend. X_t is a length two vector with stock prices at time t as its element.

$$\Delta X_t = \alpha \mathbf{b}' X_{t-1} + \sum_{i=1}^{p-1} \phi_i \Delta X_{t-i} + a_i \tag{1}$$

where $\mathbf{b} = [1, -\beta]$

The equation decomposes the current price change ΔX_t into $\alpha \mathbf{b}' X_{t-1}$ and the autoregressive term (δX_{t-i}). The former is interpreted as the impact of the preceding period's deviation from the long-term equilibrium, and the other lagged differenced terms as a short-run effect (Yan 2022). Specifically, $\mathbf{b}' X_{t-1}$ is a defined as stationary common trend between the time series in X. This stationarity of a common trend connects VECM into a pairs trading strategy, using -β as the hedge ratio between the cointegrated securities. In other words, the common trend corresponds to the spread between two assets that we want to trade on. α contributes to the common trend on the current price change. Since α is fixed, the adjustment towards the long-run equilibrium is also not dynamic, so the model contains a linear effect of cointegrated series. Johansen test is a framework to find a cointegrated relationship between k time series. $\mathbf{b}$ statistically meaningful.

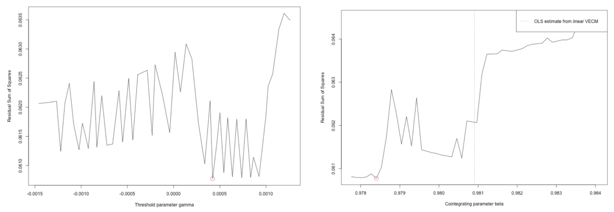
For the extension of VECM into a non-linear setting, I used a Threshold Vector Error Correction Model (TVECM), also called Regime-Switching Error Correction Model (Seo 2011). TVECM allows different adjustments depending on whether the deviation exceeds some critical threshold, thus taking into account possible costs such as transaction fees, bid-ask spread, or slippage by illiquidity. Furthermore, TVECM can capture asymmetries in the adjustment process, where deviations of varying signs are corrected in a dissimilar fashion. (Stigler, 2010).

Since I am analyzing a simple pairs trading strategy with two securities as a pair, I considered the bivariate relationship of the variables. Here, I used the bivariate TVECM with two states, Low (L) and High (H). The error-correction term and threshold γ define the two regimes, as seen in equation (2).

$$
\Delta X_t = \begin{cases} \alpha_H \mathbf{b}' x_{t-1} + \sum_{i=1}^{p=1} \phi_i \Delta X_{t-i} + a_i & \\ \qquad\qquad\qquad\qquad \text{if } \mathbf{b}' X_{t-1} > \gamma & \\ \alpha_L \mathbf{b}' x_{t-1} + \sum_{i=1}^{p=1} \phi_i \Delta X_{t-i} + a_i & \\ \qquad\qquad\qquad\qquad \text{if } \mathbf{b}' X_{t-1} < \gamma & \end{cases} \tag{2}
$$

where $\mathbf{b} = [1, -\beta]$

The model is estimated with a conditional Least Squares. Residual Sum of Squares (SSR) is measured for each β and γ through a 2-dimensional grid, and β and γ corresponding to minimal SSR is selected (Hansen and Seo 2002; Seo 2011; Stigler 2020). In figure 1, x-axis are either γ or β, while y-axis is SSR.



(a) Grid Search of γ (b) Grid Search of β

Figure 1: TVECM Greed Search

Seo (2006) suggested the sup-Wald test to test cointegration using TVECM. In this test, based on Equation (2), $\alpha_H = \alpha_L = 0$ is the null hypothesis of having no cointegration, while the alternative is that either α_H or α_L is different from zero (Stigler 2020). According to Stigler and Seo, the test is implemented with residual-based bootstrap to show its asymptotic consistency since the sup-Wald test may perform poorly in small samples. This computational complexity increases the computing time by up to several hours for a pair of stocks. Instead, I conducted a case study, so a few pairs were selected for the TVECM cointegration test. If there exists a pair that has a cointegration with threshold effect but no linear cointegration, this may imply that utilizing TVECM holds additional investment opportunities in pairs trading.

Also, even if there is linear cointegration in both the formation and trading periods, the cointegration may have non-linearity, such that using the β estimated from the formation period may incur a big challenge while trading. Sup-LM test is used to test the threshold effect with TVECM (Hansen and Seo 2002; Stigler 2020). A null hypothesis in this test is $H_0 : w_H = w_L, \alpha_H = \alpha_L$, and $\phi_H = \phi_L$, based on Equation (3).

$$
\Delta X_t = \begin{cases} \mathbf{b}' x_{t-1} + \sum_{i=1}^{p=1} \phi_{i,H} \Delta X_{t-i} + a_i & \\ \qquad\qquad\qquad\qquad \text{if } \mathbf{b}' X_{t-1} > \gamma & \\ \mathbf{b}' x_{t-1} + \sum_{i=1}^{p=1} \phi_{i,L} \Delta X_{t-i} + a_i & \\ \qquad\qquad\qquad\qquad \text{if } \mathbf{b}' X_{t-1} < \gamma & \end{cases} \tag{3}
$$

2.2 Functional-Coefficient Cointegration Models

Another model I utilized is the functional-coefficient cointegration model from Xiao (2009), which finds $\beta(z_t)$ for a cointegrated bivariate process. The underlying assumption is that the integrated processes have a cointegration model with β, which is affected by covariates (z_t), as equation (4). Here, the covariates are assumed to follow $I(0)$ processes. Xiao estimates $\beta(z_t)$ with Kernel estimator, with an assumption that the kernel K has support [-1,1] and is symmetric at zero and satisfies $\int K(u)du = 1$. I used a zero-centered Gaussian kernel with a scale (or sigma) of 2.5, truncated over the support [-1,1].

$$
y_t = \beta(z_t)' x_t + u_t \tag{4}
$$

The kernel estimator of $\beta(z)$ is determined by nonparametric estimation using a kernel. h is a bandwidth parameter, and I used Scott's method to calculate h.

$$
\hat{\beta}(z) = arg\,min \sum_{t=1}^{n} K\left(\frac{z_t - z}{h}\right)(y_t - x_t'\beta)^2 \tag{5}
$$

Regarding statistical tests, I implemented two tests suggested by Xiao: testing the stability of the cointegrating vector and cointegration. For the stability test, the null hypothesis is $\beta(z) = \beta$, whereas the alternative is varying coefficient $\beta(z)$. The cointegration test has a null hypothesis that y_t and x_t are cointegrated. If rejected, then $u_t = y_t - \beta(z_t)' x_t$ cannot be assumed as stationary.

To apply Xiao's model, the covariate process z_t should be assumed. The spread between the ETF pairs that track the same or similar market index would be affected by the same economic state of the market. S&P500 daily returns and differenced VIX series are used as the covariate. Both series meet the $I(0)$ process assumption of covariates.

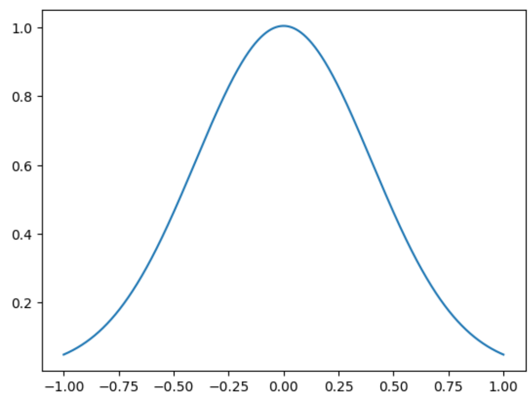


Figure 2: Truncated Gaussian Kernel

3 Analysis

3.1 Data

The time series data I analyzed is 14 U.S. Equity ETFs from 2011 to 2022. As the U.S. market portfolio is the most crucial market index and is often represented as a global economic state (Rapach 2013), I assumed that the ETFs tracking the market would have lower divergence and tracking error than the other ETFs, such as Global Equities. As such, the pairs generated with these candidate tickers will be more likely to have statistical arbitrage opportunities, and cointegration between the ETF series that follows the same index may be prominent. At the same time, through cointegration tests, we can see whether there are investment opportunities with pairs of ETFs that follow slightly different indices. For example, RSP and IWV had a trading opportunity in 2018, although RSP is following equal weighted S&P 500 and IWV is following Russel 3000. The data is from Nasdaq Qaundl, QuoteMedia database. Details are in Table 10.

To estimate the models, the formation period is fixed on two years, and one year of a trading period is assumed. Accounting for business days, each formation period has approximately 504 data points, and the trading period has 252 data points. Also, for each sample, the data is standardized in the form of $Z_{i,t} = log(X_{i,t}/X_{i,0})$ where i denotes ticker and t is time.

3.2 Linear Cointegration

The maximum order selected based on the AIC, BIC, and HQ of the VAR model is used to estimate VECM for the Johansen test. The pairs are selected when the trace test statistics reject the null hypothesis $H_0 : r = 0$ by 5%.

One example of linear cointegration is SPY and IVV. Both of them follow S&P500, and they are the two largest ETFs by assets under management (AUM), around \$ 359 B and \$295 B for each. This means they have a lesser chance of being diverged from the index relative to the other ETFs simply because they can hold more assets to approximate the weights of each stock in S&P500 accurately.

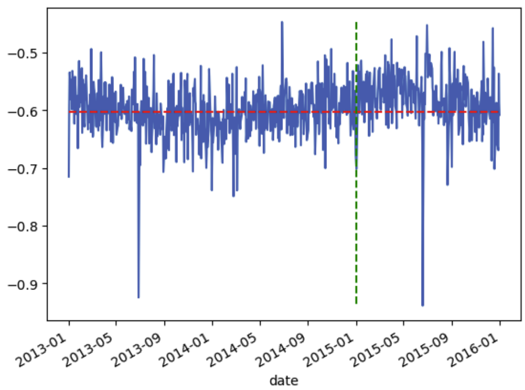


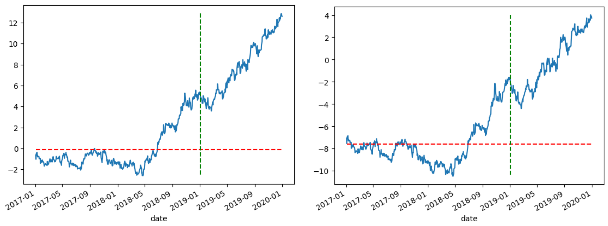
Figure 3: SPY and IVV, from 2013 to 2016

The threshold of the critical value of the Johansen test as 1% is set manually because 5% critical value is often

	10%	5%	1%	test
r=0	13.4294	15.4943	19.9349	16.353811
r≤ 1	2.7055	3.8415	6.6349	1.862442

Table 1: Johansen Test Result

insufficient to create a stationary spread. Trading a pair from the 5% threshold is risky because the model may pass the pairs with $\beta' x_t$ with the linear trend towards time. Figure 4 shows a spread of IWV and FEX, showing trace statistics higher than 5% critical value in the formation period (2017 to 2018). However, from Figure 3, the spread does not show mean-reverting properties and has a time trend. The hedge ratio β is estimated in two ways, one from VECM with the Johansen test and the other from $x_{IWV,t} = \beta x_{FEX,t} + u_t$ with OLS.



(a) β from VECM (b) β from OLS

Figure 4: Number of cointegrated pairs in Formation and Trading periods

Table 2 shows the number of selected cointegrated pairs through the Johansen test in the Formation and Trading period. The index of Table 2 is the year the pairs are assumed to be traded. For example, the formation period of the trading year 2017 is from 2015 to 2016. Common pairs denote the number of pairs that appear both in the Formation period and the succeeding Trading period. This means the trading strategy based on linear cointegration may generate actual profits in the trading period.

	Formation Period	Trading Period	Common Pairs
2013	9	12	4
2014	13	6	5
2015	7	7	5
2016	5	17	5
2017	3	16	3
2018	8	14	3
2019	23	39	9
2020	5	5	3
2021	6	24	4
2022	6	17	2
2023	4	56	3

Table 2: Number of cointegrated pairs in Formation and Trading periods

3.3 TVECM

3.3.1 Non-linearity in Formation Period

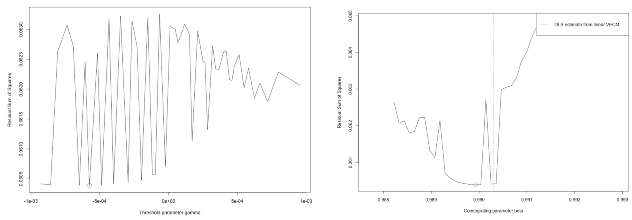
Using TVECM and the cointegration test, I analyzed additional investment opportunities that non-linearity may create. Since the cost of computing is very high for the TVECM cointegration test, I focused on pairs and trading periods, which turned out to be linearly cointegrated in the trading period (or out-of-sample). In contrast, the Johansen test did not indicate that the pairs are cointegrated in the formation period. These pairs are from Table in Appendix with values −1.

First, I analyzed two cases where linear cointegration could not capture the investment opportunity as a gradual adjustment occurred during a formation period. The pair selected is IWV and ITOT, with the formation period from 2015 to 2016. From the Johansen test, the null hypothesis is not rejected since the trace test statistics have a value of 2.95, but the critical value on the 10% level is 10.47.

	10%	5%	1%	test
r=0	10.4741	12.3212	16.3640	2.957476
r≤1	2.9762	4.1296	6.9406	1.388439

Table 3: Johansen Test Result

Below Figure 5 is a grid search result of the TVECM model with (IWV, ITOT) pair. $\beta = 0.99$ is selected to minimize the sum of squared residuals. Two regimes are assumed when the spread is higher than the $\gamma = -5e-4$ or lower than the threshold.



(a) Grid Search of γ (b) Grid Search of β

Figure 5: TVECM, IWV and ITOT, 2019

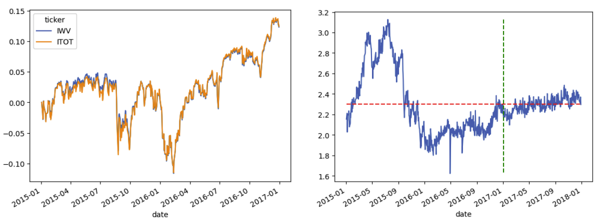
Table 4 is the result of the TVECM cointegration test suggested by Seo (2016). The P-value of the bootstrapped result is less than 0.01%, which indicates that the pair has cointegration with the threshold effect. Figure 4 shows that the two price processes are moving very close to each other and have a mean-reverting property within a longer term. Since TVECM is adaptable to graduate adjustment towards long-run equilibrium, the result of the TVECM cointegration test seems adequate. This also implies that the ETF series that follow different underlying indexes can be used for pairs trading even in the case of seemingly divergent cases. The price and spread between IWV and ITOT are plotted in Figure 6.

3.3.2 Non-linearity in Trading Period

Figure 7 is spread between VV and SPY. VV is a Vanguard Large-Cap Index fund, while SPY is an S&P500

	sup-Wald	10%	5%	2.5%	1%	nboot
Seo Test	9.391	8.239	8.4667	8.8685	9.1096	30.0

Table 4: TVECM Cointegration Test, Critical Values



(a) Standardized prices of IWV and ITOT (b) Spread between IWV and ITOT

Figure 6: Prices and Spread Processes

tracking ETF. 2017 to 2018 is a formation period, while 2019 is used as a trading period. In the figure, the dotted horizontal line is the mean value of the spread in the formation period. Although the common trend is more volatile and less frequently mean-reverting, the graph shows that the pair would generate investment opportunities if non-linearity is considered. Specifically, the spread shows different reverting characteristics, depending on the common trend being above the mean or not.

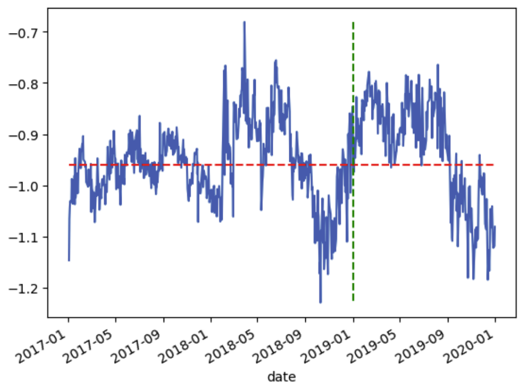


Figure 7: Non-linear trend in Trading Period, SPY and VV, from 2017 to 2019

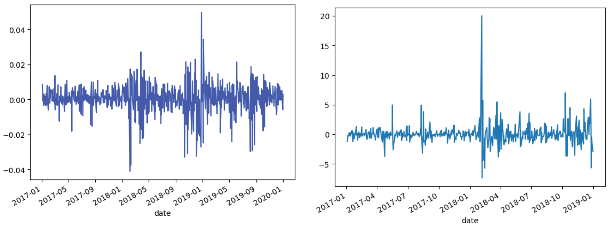
	Test Statistics	10%	5%	1%	threshold	nboot
H-Seo Test	29.64	14.91	15.31	17.23	0.00586	100.0

Table 5: Threshold Effect Test

3.4 Functional-Coefficient Model

To implement a functional-coefficient model, the data of SPY and VV from section 2.3.1. is first tested. For the covariates, as I mentioned in section 1.3, daily returns of S&P 500 and differenced series of the VIX Index are used. Figure 8 and Table 6 show that two covariates are (0) process.

The stability and cointegration test statistics are calculated in the formation period. Both covariates have shown



(a) S&P500 Index Daily Return (b) Differenced Daily VIX Index

Figure 8: Covariates

	ADF	1%	5%	10%
S&P500	-12.591785	-2.566391	-1.941078	-1.616743
VIX	-14.126148	-2.566391	-1.941078	-1.616743

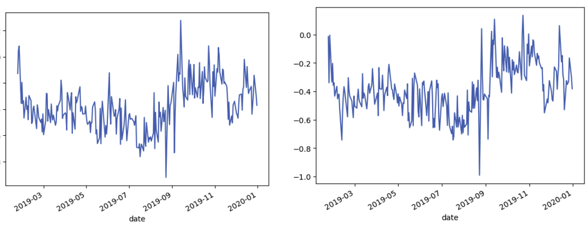
Table 6: Augmented Dickey-Fuller Test of Covariates

enough stability of varying $\beta(z)$. However, the cointegration test for S&P500 is rejected with 10%, which means that the residual term $u_t = y_t - \beta(z_t)'x_t$ can have a unit root.

Covariate	Type	Test Statistics	1%	5%	10%
S&P500	Stability	1447.15	7.75	12.93	9.22
	Cointegration	-1.26	-2.32	-1.64	-1.28
VIX	Stability	1862.94	8.03	8.97	11.70
	Cointegration	-1.43	-2.32	-1.64	-1.28

Table 7: Stability and Cointegration Tests

Figure 9 is generated spread in the trading period 2019. To prevent lookahead bias, $\beta(z)_t$ is generated using z value two weeks earlier. The augmented dickey-fuller test results in Table 8 indicate that the spread generated by $\beta(z)$ with 2-weeks lagged z has a stationary residual term.



(a) Spread using S&P500 (b) Spread using VIX

Figure 9: Sample Spread generated in Trading Period

	ADF	1%	5%	10%
S&P500	-3.19	-3.45	-2.87	-2.57
VIX	-2.40	-3.45	-2.87	-2.57

Table 8: Augmented Dickey-Fuller Test of Spread

However, a drawback of using the functional-coefficient method in real trading is that it requires rebalancing the position whenever new $\beta(z_t)$ is evaluated. This may imply

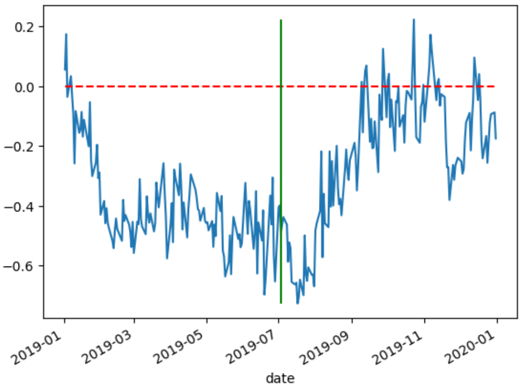
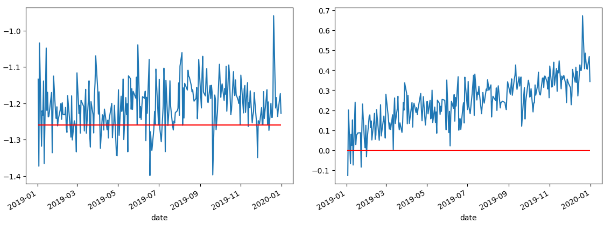


Figure 10: 6-Months Rebalancing

a huge cost of using the model in reality. Instead, it can be updated less frequently. Figure 10 assumes evaluating $\beta(z_t)$ once in a 6-months. The red vertical line indicates the rebalancing, while the blue horizontal line is the mean value of the spread. The graph still maintains a reverting shape, although the initial diversion from the mean does not converge until it rebalances in July.

Another drawback is coming from the formula $y_t = \beta(z_t)' x_t + u_t$. The equation does not contain any constant, such that the level effect of x and y will be incorporated in βz_t. This resulted in finding an inaccurate cointegrated relationship, such that the spread of out-of-sample often has a trend over time. Below Figure 11 is a comparison between the OLS beta and functional-coefficient model with S&P500 return as covariate series. Price data of SPY and IVV from 2017 and 2018 are used to calculate the hedge ratio, and the hedge ratio is evaluated in 2019 as the trading period. In this case, even though the cointegration test shows a 2% p-value, the functional-coefficient cointegration model does not work well in out-of-sample. In contrast, the spread using OLS has decent reverting towards its in-sample mean.



(a) OLS Beta (b) Functional-Coefficient

Figure 11: Out-of-sample (Trading Period) Spread between SPY and IVV

Covariate:S&P500	p-value	Test Statistics	1%	5%	10%
Stability	0.00	1918.92	15.92	9.51	9.90
Cointegration	0.02	-2.14	-2.33	-1.64	-1.28

Table 9: Stability and Cointegration Test of SPY and IVV

4 Conclusion

Pairs trading, a strategy predicated on deviations from the normative correlation between two correlated assets, hinges on the mean-reverting nature of the spread between these assets. This research delves into the cointegration approach, emphasizing non-linear tests and models to elucidate potential investment avenues. While the cointegration method offers insights into stable equilibrium relationships between pairs, it operates under the assumption of instantaneous adjustments to deviations. As such, I delved into the threshold vector error correction model (TVECM) and the functional-coefficient cointegration Model proposed by Xiao (2009). The TVECM allows for differentiated adjustments when deviations surpass a specified threshold, aptly accounting for costs such as transaction fees and bid-ask spreads. Furthermore, this model adeptly captures asymmetries in the adjustment process. Conversely, the functional-coefficient cointegration model discerns a cointegrated bivariate process with a coefficient influenced by covariates.

91 pairs generated from 14 U.S. equity ETFs are mainly investigated for possible cointegration. First, the findings from TVECM suggest promising investment avenues in pairs trading; specifically, the Seo-test of TVECM could discover a new pair that VECM could not capture. Additionally, H-Seo test results on SPY and VV indicate that non-linearity should be considered when a strategy is employed. Second, the functional-coefficient approach, despite its promising concept, might present obstacles in practical trading. Due to the changing hedge ratio, the strategy requires frequent rebalancing, which may result in high costs in practice. Also, the lack of a constant in its equation potentially results in erroneous cointegrated relationships in out-of-sample contexts. For further research, testing another kernel for a functional-coefficient model would be possible.

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Appendix

ticker	Description	Type
MGC	Vanguard Mega Cap 300 ETF	US Equity ETF
VONE	Vanguard Russell 1000 ETF	US Equity ETF
VOO	Vanguard S&P 500 ETF	US Equity ETF
RSP	Invesco - Guggenheim (Rydex) S&P Equal Weight ETF	US Equity ETF
SPY	SPDR S & P 500 ETF	US Equity ETF
VV	Vanguard Large-Cap ETF	US Equity ETF
IWV	iShares Russell 3000 ETF	US Equity ETF
FEX	First Trust Large Cap Core AlphaDEX ETF	US Equity ETF
IVV	iShares Core S&P 500 Index Fund ETF	US Equity ETF
ITOT	iShares Core S&P Total U.S. Stock Market ETF	US Equity ETF
IYY	iShares Dow Jones U.S. Total Market Index Fund...	US Equity ETF
IWB	iShares Russell 1000 ETF	US Equity ETF
VTI	Vanguard Total Stock Market ETF	US Equity ETF
SCHB	Schwab U.S. Broad Market ETF	US Equity ETF

Table 10: Candidate ETFs

	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023
(FEX, ITOT)	1	1	1	1	1	1	1	1	1	1	-1
(FEX, SCHB)	1	1	1	1	1	1	1	1	1	1	-1
(FEX, VTI)	1	1	1	1	1	1	1	1	1	1	-1
(ITOT, IWB)	-2	2	1	1	1	1	1	1	-1	-1	1
(ITOT, IYY)	1	-2	1	1	1	1	1	1	-1	-1	-1
(ITOT, SCHB)	1	1	1	-1	1	-1	2	2	2	-1	1
(ITOT, VTI)	1	1	1	-1	-1	-2	-1	1	-1	-2	2
(IVV, ITOT)	1	1	1	1	1	1	-1	1	1	1	1
(IVV, IWB)	1	1	1	-1	1	1	1	1	1	1	-1
(IVV, IYY)	1	1	1	1	1	1	-1	1	1	1	-1
(IVV, SCHB)	1	1	1	1	1	1	1	1	1	1	-1
(IVV, VTI)	1	1	1	1	1	1	-1	1	1	1	-1
(IWB, SCHB)	1	1	1	1	-1	1	1	1	-1	-1	-1
(IWB, VTI)	1	1	1	1	1	1	1	1	-1	1	1
(IWV, ITOT)	1	1	1	-1	-1	1	-1	1	-2	-1	-1
(IWV, IVV)	1	1	1	1	1	1	-1	1	1	1	1
(IWV, IWB)	1	1	1	1	1	1	-1	1	-1	1	-1
(IWV, SCHB)	-1	1	-1	-1	1	1	2	-1	-2	-1	-2
(IWV, VTI)	-1	-1	2	2	-1	-2	1	1	1	-1	1
(IYY, IWB)	1	1	1	1	-1	1	1	1	-1	-1	-1
(IYY, SCHB)	-2	1	1	1	-1	1	-1	1	-1	-1	-1
(IYY, VTI)	-2	1	1	1	1	1	1	1	-1	-1	-1
(MGC, FEX)	1	1	1	1	1	1	-2	1	1	1	-1
(MGC, ITOT)	-1	1	1	1	1	-1	-2	1	1	1	-1
(MGC, IVV)	1	1	1	1	1	1	-1	1	1	1	1
(MGC, IWB)	1	1	1	1	1	1	-1	1	1	1	1
(MGC, IWV)	1	1	1	1	-1	-1	-2	1	1	1	-1
(MGC, IYY)	1	1	1	1	1	1	-1	1	1	-1	-1
(MGC, RSP)	1	1	1	1	1	1	-2	1	1	1	-1
(MGC, SCHB)	1	1	1	1	-1	-1	2	1	1	1	-1
(MGC, SPY)	-1	1	1	1	-1	1	1	1	1	1	1
(MGC, VONE)	1	1	1	1	1	1	-1	1	1	1	-1
(MGC, VOO)	-1	1	-2	1	1	1	1	1	1	1	1
(MGC, VTI)	1	1	1	1	-1	-1	2	1	1	1	-1
(MGC, VV)	1	1	1	1	-1	1	-1	1	1	1	-1
(RSP, FEX)	1	1	1	1	1	1	1	1	1	1	-1
(RSP, ITOT)	1	1	1	1	1	1	-2	1	1	1	-1
(RSP, IVV)	1	1	1	1	1	1	-2	1	1	1	-1
(RSP, IWB)	1	1	1	1	1	1	-2	1	1	1	-1
(RSP, IWV)	1	1	1	1	1	1	-2	1	1	1	-1
(RSP, IYY)	1	1	1	1	1	1	1	1	1	1	-1
(RSP, SCHB)	1	1	-2	1	1	1	-2	1	1	1	-1

	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023
(RSP, SPY)	1	1	1	1	1	1	-2	1	1	1	-1
(RSP, VTI)	1	1	1	1	1	1	-2	1	1	1	-1
(RSP, VV)	1	1	1	1	1	1	-2	1	1	1	-1
(SPY, FEX)	1	1	1	1	1	1	1	1	1	1	-1
(SPY, ITOT)	1	1	1	1	1	1	-1	1	1	1	-1
(SPY, IVV)	2	2	2	2	2	2	2	-2	-1	-2	1
(SPY, IWB)	1	1	1	1	1	1	-1	1	1	1	1
(SPY, IWV)	1	1	1	1	1	1	-1	1	1	1	-1
(SPY, IYY)	1	1	1	1	1	1	-1	1	1	1	-1
(SPY, SCHB)	1	1	1	1	1	1	1	1	1	1	-1
(SPY, VTI)	1	1	1	1	1	1	-1	1	1	1	-1
(SPY, VV)	1	1	1	-1	1	-2	-1	1	1	1	-1
(VONE, ITOT)	-2	-2	1	1	1	1	1	1	-1	1	1
(VONE, IVV)	1	1	1	-1	1	1	-1	1	1	1	-1
(VONE, IWB)	2	2	2	2	-1	-1	2	2	2	-2	-1
(VONE, IWV)	1	-2	1	1	1	1	1	1	-1	1	1
(VONE, IYY)	-1	-2	1	1	1	1	1	1	1	1	-1
(VONE, RSP)	1	1	1	1	1	1	-2	1	1	1	-1
(VONE, SCHB)	-2	-2	1	1	1	1	1	1	-1	1	-1
(VONE, SPY)	1	1	1	-1	1	1	-1	1	1	1	1
(VONE, VOO)	1	1	1	-1	1	1	-1	1	1	1	-1
(VONE, VTI)	1	-2	1	1	1	1	1	1	-1	1	1
(VONE, VV)	2	-2	1	1	1	1	-1	1	1	1	1
(VOO, ITOT)	1	1	1	1	1	1	1	1	1	1	-1
(VOO, IVV)	2	2	2	2	2	2	2	2	2	2	2
(VOO, IWB)	1	1	1	1	1	1	1	1	1	1	-1
(VOO, IWV)	1	1	1	1	1	-1	-1	1	1	1	1
(VOO, IYY)	1	1	1	1	1	1	-1	1	1	1	1
(VOO, RSP)	1	1	1	1	1	1	-2	1	1	1	-1
(VOO, SCHB)	1	1	1	1	1	1	1	1	1	1	-1
(VOO, SPY)	-1	2	2	2	-1	2	2	-2	-1	-2	1
(VOO, VTI)	1	1	1	1	1	-1	1	1	1	1	-1
(VOO, VV)	1	1	1	-1	1	-2	2	1	1	1	1
(VTI, SCHB)	1	1	-1	-1	2	1	-1	-1	2	2	2
(VV, FEX)	1	1	1	1	1	1	1	1	1	1	-1
(VV, ITOT)	1	-2	1	1	1	1	-1	1	-1	-1	1
(VV, IVV)	1	1	1	-1	1	-2	-1	1	1	1	1
(VV, IWB)	1	1	1	1	1	1	1	1	1	-1	1
(VV, IWV)	1	1	1	1	1	-1	-1	1	-1	1	1
(VV, IYY)	-1	1	1	1	1	1	1	1	-1	-1	-1
(VV, SCHB)	1	1	1	1	1	-1	1	1	-1	-1	1
(VV, VTI)	1	1	1	1	1	-1	-1	1	-1	1	1

Table 11: Detailed Result of Table 2, Linear Cointegration

From the above Table, each number indicates

- 2: True Positive, Cointegrated both on Formation period and Trading period
- 1: True Negative, Not cointegrated both on Formation and Trading period
- -1: False Negative, Cointegrated in the Trading period, but not in the Formation period
- -2: False Positive. Cointegrated on Formation but not in the Trading period