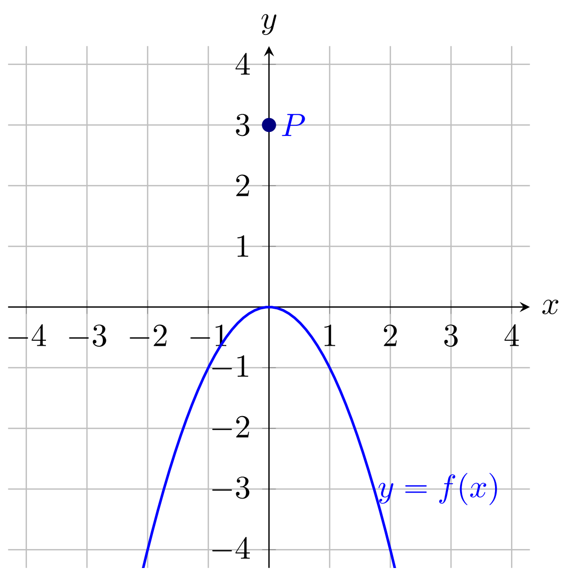
Exercise 1 *The figure below shows the graph of the function f, where $f(x) = -x^2$, and the point $P(0, 3)$.*

Find equations of all tangent lines to the curve $y = f(x)$ that pass through the point P.



We will solve this problem by completing several steps.

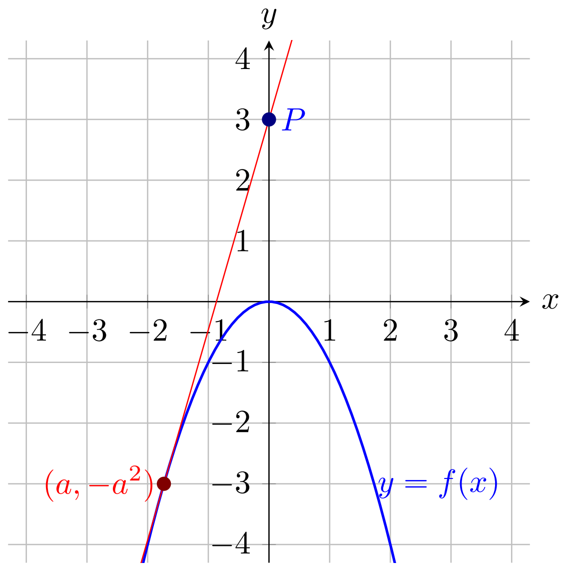
STEP 1

Any such tangent line will pass through the point $P(0, 3)$, and will have a slope, call it m. Therefore, its equation is given by

$$y - \boxed{3} = m(x - \boxed{0})$$

STEP 2

Now we have to find an expression for the slope m.



The line is tangent to the curve $y = f(x)$ at some point on the curve, $(a, f(a)) = (a, -a^2)$. Therefore, the slope of the tangent line, m, is given by

$$m = f'(a)$$

and, expressing $f'(a)$ in terms of a, we obtain that

$$m = \boxed{-2}\,a.$$

Therefore, an equation of the tangent line is given by

$$y - 3 = \boxed{-2}\,ax,$$

where a has to be determined.

Hint: *Observe that the point $(a, -a^2)$ lies on both the tangent line and on the curve $y = f(x)$.*

Therefore, the point $(a, -a^2)$ satisfies the equation of the tangent line!

Plugging in $x = a$ and $y = -a^2$ into the equation of the line, we obtain an equation

$$-a^2 - 3 = -2a(a)$$

Solve this quadratic equation for a!

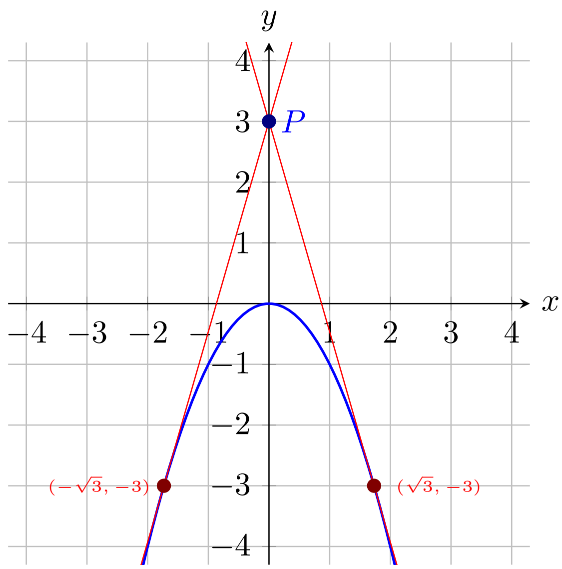
We find two solutions for a (written in increasing order)

$$a = \boxed{-\sqrt{3}} \text{ and } a = \boxed{\sqrt{3}}.$$

This means that there are two tangent lines passing through the point P,

one with the slope (written in increasing order) $m = \boxed{-2\sqrt{3}}$ *and the other with the slope* $m = \boxed{2\sqrt{3}}$.

Check the picture!



STEP 3

There are exactly two lines that are both tangent to the curve $y = f(x)$ and pass through the point P. The first line has a positive slope, and the second line has the negative slope, and their equations, written in that order, are:

$$y = \boxed{2\sqrt{3}x + 3}$$

$$y = \boxed{-2\sqrt{3}x + 3}$$