Algorithms: Homework 2

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Problem 1

Using Figure 8.3 as a model, illustrate the operation of RADIX-SORT on the following list of English words: NOD, HOG, SHY, BAN, BAR, JET, EBB, PAR, ASH, PET, TED, ROT, FIG.

0(*initial*)	*digit* 1	*digit* 2	*digit* 3
NOD	EB**B**	B**A**N	**A**SH
HOG	NO**D**	B**A**R	**B**AN
SHY	TE**D**	P**A**R	**B**AR
BAN	HO**G**	E**B**B	**E**BB
BAR	F**I**G	T**E**D	**F**IG
JET	AS**H**	J**E**T	**H**OG
EBB	BA**N**	P**E**T	**J**ET
PAR	BA**R**	S**H**Y	**N**OD
ASH	PA**R**	F**I**G	**P**AR
PET	JE**T**	N**O**D	**P**ET
TED	PE**T**	H**O**G	**R**OT
ROT	RO**T**	R**O**T	**S**HY
FIG	SH**Y**	A**S**H	**T**ED

Problem 2

Show how to sort n integers in the range 0 to $n^2 - 1$ in $\mathcal{O}(n)$ time.

Solution. Psuedocode for sorting n integers in $\mathcal{O}(n)$, X is an array storing all n integers:

```
SORT-N-LINEAR(X, n)
1:  if n == 1
2:    return
3:  for i = 1 to n
4:    // change X[i] into base n
5:    TO-BASE-N(X[i])
6:  // each integer will have at most log_n n^2 = 2 digits
7:  // radix sort uses counting sort to sort n at most 2 digits integers
8:  RADIX-SORT(X)
9:  for i = 1 to n
10:   // convert X[i] back to its original value because we sorted them with base n
11:   CONVERT-BASE-N(X[i])
```

TO-BASE-N converts an integer into a base n number. E.g. 100 integers have the largest number of 9999, which can be converted to base 100, $9999 = 99 * 100^1 + 99 * 100^0$. Originally, radix sort will sort 4 digits for

9999. After converting its base to n, radix sort will need to sort **2 digits** 99_{100} and 99_{100}. Another example that the integer 400 will be converted to 4_{100}. The **digit** here is not limit from 0 to 9. ■

Problem 3

Using Figure 8.4 as a model, illustrate the operation of BUCKET-SORT on the array A=<.49, .30, .56, .74, .68, .50, .69, .37, .41, .72>

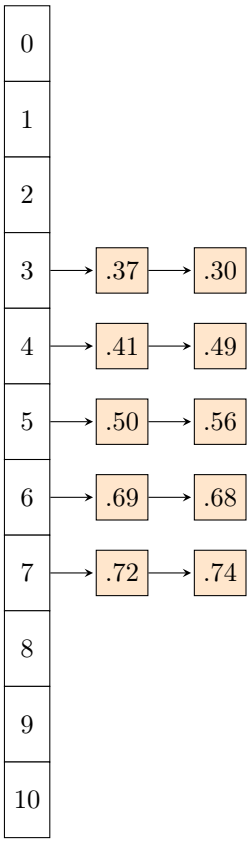

Figure 1: Insert data

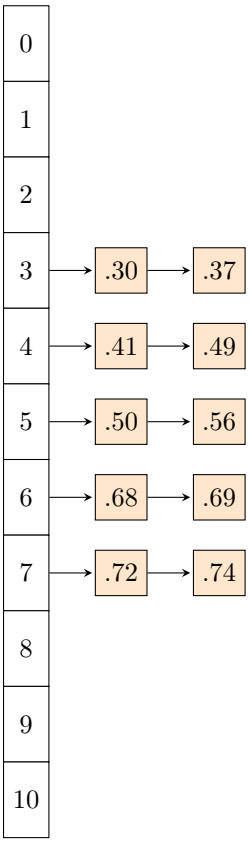

Figure 2: Sort each bucket

Solution. To get sorted data, chain all the nodes in Figure 2 from bucket 0 to bucket 10:
$< .30, .37, .41, .49, .50, .56, .68, .69, .72, .74 >$ ■

Problem 4

Suppose that you have a "black-box" worst-case linear time median subroutine. Give a simple, linear-time algorithm that solves the selection problem for an arbitrary order statistic.

Solution. Psuedocode for finding the k-th smallest element in array $X[1..n]$:

```
FIND-K-TH-SMALLEST(X, n, k)
1:  // returns the index of the median for array X in linear time
2:  b = BLACK-BOX(X)
3:  if k == n / 2 // k-th smallest is median
4:     return X[b]
5:  // splits array X into two arrays, Y contains elements less than b and Z contains elements greater than b
6:  Y, Z = SPLIT(X, b)
7:  if k < n/2
8:     return FIND-K-TH-SMALLEST(Y, n/2, k)
9:  else
10:    // Find n/2 − k-th smallest for Z since it's half the size of X
11:    return FIND-K-TH-SMALLEST(Z, n/2, n/2 − k)
```

$T(n) = T(n/2) + \mathcal{O}(n)$
Apply master method $n^{\log_b a} = n^{\log_2 1} = n^0 = 1 < \mathcal{O}(n)$, may apply case 3.
First, we find $\epsilon = 1$ where $f(n) = \Omega(n^{(\log_2 1)+\epsilon})$.
Second, we find $c = 1/2$ where $af(n/b) = n/2 \leq 1/2n = cf(n)$ is true.
Thus, we can conclude that the running time of this algorithm is $\Theta(f(n)) = \Theta(n)$. ■

Problem 5

Let $X[1..n]$ and $Y[1..n]$ be two arrays, each containing n numbers already in sorted order. Give an $\mathcal{O}(\lg n)$-time algorithm to find the median of all $2n$ elements in arrays X and Y.

Solution. Psuedocode for finding the median of arrays $X[1..n]$ and $Y[1..n]$, xi is X array's starting index and xj is X array's last index; yi is Y array's starting index and yj is Y array's last index:

```
FIND-MEDIAN(X, Y, xi, xj, yi, yj, n)
1:  if n == 1
2:     // Both arrays have only one element, pick the smaller one as median
3:     return MIN(X[1], Y[1])
4:  xmid = X[(xi + xj)/2] // X's median
5:  ymid = Y[(yi + yj)/2] // Y's median
6:  if xmid < ymid
7:     // median must be greater than xmid and less than ymid
8:     return FIND-MEDIAN(X, Y, xmid + 1, xj, yi, ymid, n/2)
9:  else
10:    // median must be less than xmid and greater than ymid
11:    return FIND-MEDIAN(X, Y, xi, xmid, ymid + 1, yj, n/2)
```

$T(n) = T(n/2) + \mathcal{O}(1)$
Apply master method case 2 because $n^{\log_b a} = n^{\log_2 1} = n^0 = 1 = f(n)$.
$T(n) = \Theta(n^{\log_2 1} \lg n) = \Theta(\lg n)$. ■

Problem 6

Demonstrate the insertion of the keys 23, 38, 19, 40, 29, 43, 12, 27, 11 into a hash tale with collisions resolved by chaining. Let the table have 9 slots, and let the hash function be $h(k) = k \pmod 9$.

Solution. ■

(1) insert $23 \to 23 \pmod 9 = 5$
(2) insert $38 \to 38 \pmod 9 = 2$
(3) insert $19 \to 19 \pmod 9 = 1$
(4) insert $40 \to 40 \pmod 9 = 4$
(5) insert $29 \to 29 \pmod 9 = 2$, insert head
(6) insert $43 \to 43 \pmod 9 = 7$
(7) insert $12 \to 12 \pmod 9 = 3$
(8) insert $27 \to 27 \pmod 9 = 0$
(9) insert $11 \to 11 \pmod 9 = 2$, insert head

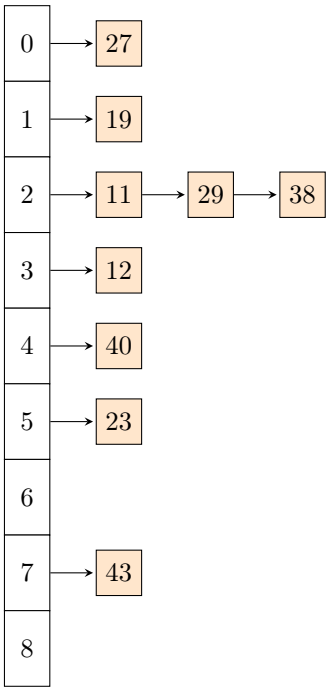

Figure 3: Hash table with chaining

Problem 7

Consider inserting the keys 10, 23, 31, 4, 12, 28, 17, 87, 58 into a hash table of length $m = 11$ using open addressing with the auxiliary hash function $h'(k) = k \pmod m$. Illustrate the result of inserting these keys using **linear probing**, using **quadratic probing** with $c1 = 1$, and $c2 = 3$, and using **double hashing** with $h_2(k) = 1 + (k \pmod{(m-1)})$.

(1) insert $10 \to 10 \pmod{11} = 10$
(2) insert $23 \to 23 \pmod{11} = 1$
(3) insert $31 \to 31 \pmod{11} = 9$
(4) insert $4 \to 4 \pmod{11} = 4$
(5) insert $12 \to 12 \pmod{11} = 1$ $\times$ collision
insert 12 to slot 2 because slot 1 is occupied
(6) insert $28 \to 28 \pmod{11} = 6$
(7) insert $17 \to 17 \pmod{11} = 6$ $\times$
insert 17 to index 7 because slot 6 is occupied
(8) insert $87 \to 87 \pmod{11} = 10$ $\times$
insert 87 to index 0 because slot 10 is occupied
(9) insert $58 \to 58 \pmod{11} = 3$

0	87
1	23
2	12
3	58
4	4
5	x
6	28
7	17
8	x
9	31
10	10

Figure 4: linear probing

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resolve collision: $h(k) + 1 * i + 3 * i^2 \pmod{11}$
(1) insert $10 \to 10 \pmod{11} = 10$
(2) insert $23 \to 23 \pmod{11} = 1$
(3) insert $31 \to 31 \pmod{11} = 9$
(4) insert $4 \to 4 \pmod{11} = 4$
(5) insert $12 \to 12 \pmod{11} = 1$ $\times$ collision
$(1 + 1 + 3 * 1 * 1) \pmod{11} = (1 + 1 + 3) \pmod{11} = 5$
insert 12 to slot 5
– –
(6) insert $28 \to 28 \pmod{11} = 6$
(7) insert $17 \to 17 \pmod{11} = 6$ $\times$
$(6 + 1 + 3 * 1 * 1) \pmod{11} = (6 + 1 + 3) \pmod{11} = 10$ $\times$
$(6 + 2 + 3 * 2 * 2) \pmod{11} = (6 + 2 + 12) \pmod{11} = 9$ $\times$
$(6 + 3 + 3 * 3 * 3) \pmod{11} = (6 + 3 + 27) \pmod{11} = 3$
insert 17 to slot 3
– –
(8) insert $87 \to 87 \pmod{11} = 10$ $\times$
$(10 + 1 + 3 * 1 * 1) \mod 11 = (-1 + 1 + 3) \mod 11 = 3$ $\times$
$(10 + 2 + 3 * 2 * 2) \mod 11 = (-1 + 2 + 12) \mod 11 = 2$
insert 87 to slot 2
– –
(9) insert $58 \to 58 \pmod{11} = 3$ $\times$
$(3 + 1 + 3 * 1 * 1) \mod 11 = (3 + 1 + 3) \mod 11 = 7$
insert 58 to slot 7

0	x
1	23
2	87
3	17
4	4
5	12
6	28
7	58
8	x
9	31
10	10

Figure 5: quadratic probing

resolve collision: $h_2(k) = 1 + k \pmod{10}$
$new_k = (h_1(k) + i * h_2(k)) \pmod{11}$
(1) insert $10 \to 10 \pmod{11} = 10$
(2) insert $23 \to 23 \pmod{11} = 1$
(3) insert $31 \to 31 \pmod{11} = 9$
(4) insert $4 \to 4 \pmod{11} = 4$
(5) insert $12 \to 12 \pmod{11} = 1$ $\times$ collision
$h_2(12) = 1 + 12 \pmod{10} = 3$
$1 + 1 * 3 \pmod{11} = 4$ $\times$
$1 + 2 * 3 \pmod{11} = 7$
insert 12 to slot 7
– –
(6) insert $28 \to 28 \pmod{11} = 6$
(7) insert $17 \to 17 \pmod{11} = 6$ $\times$
$h_2(17) = 1 + 17 \pmod{10} = 8$
$8 + 1 * 8 \pmod{11} = 5$
insert 17 to slot 5
– –
(8) insert $87 \to 87 \pmod{11} = 10$ $\times$
$h_2(87) = 1 + 87 \pmod{10} = 8$
$8 + 1 * 8 \pmod{11} = 5$ $\times$
$8 + 2 * 8 \pmod{11} = 2$
insert 87 to slot 2
– –
(9) insert $58 \to 58 \pmod{11} = 3$

0	x
1	23
2	87
3	58
4	4
5	17
6	28
7	12
8	x
9	31
10	10

Figure 6: double hashing

Problem 8

Solve the following assembly-line problem:
$e_1 = 1, e_2 = 1, x_1 = 7, x_2 = 3$
$a_{1,1} = 3, a_{1,2} = 2, a_{1,3} = 5, a_{1,4} = 4, a_{1,5} = 2$
$a_{2,1} = 4, a_{2,2} = 2, a_{2,3} = 4, a_{2,4} = 6, a_{2,5} = 4$
$t_{1,1} = 1 t_{1,2} = 2, t_{1,3} = 3, t_{1,4} = 1$
$t_{2,1} = 2, t_{2,2} = 3, t_{2,3} = 1, t_{2,4} = 2.$

Solution. Draw f table l table, l start

	0	1	2	3	4	5	6	7	8	9
$f_1[j]$	0	0	1	0	0	0	0	0	0	0
$f_2[j]$	0	2	5	6	6	8	8	9	10	10

Figure 7: F table

■

Problem 9

Find an optimal parenthesization of a matrix-chain product whose sequence of dimension is <5, 7, 3, 2, 4, 5>

Solution. Draw M table, S table, and the positions of parenthesis ■

Problem 10

Determine an LCS of <1, 1, 0, 1, 0, 1> and <0, 0, 1, 1, 0, 1, 1>.

Solution. Draw C table ■