Algorithms: Homework 2

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Problem 1

Using Figure 8.3 as a model, illustrate the operation of RADIX-SORT on the following list of English words: NOD, HOG, SHY, BAN, BAR, JET, EBB, PAR, ASH, PET, TED, ROT, FIG.

0(*initial*)	*digit* 1	*digit* 2	*digit* 3
NOD	EB**B**	B**A**N	**A**SH
HOG	NO**D**	B**A**R	**B**AN
SHY	TE**D**	P**A**R	**B**AR
BAN	HO**G**	E**B**B	**E**BB
BAR	FI**G**	T**E**D	**F**IG
JET	AS**H**	J**E**T	**H**OG
EBB	BA**N**	P**E**T	**J**ET
PAR	BA**R**	S**H**Y	**N**OD
ASH	PA**R**	F**I**G	**P**AR
PET	JE**T**	N**O**D	**P**ET
TED	PE**T**	H**O**G	**R**OT
ROT	RO**T**	R**O**T	**S**HY
FIG	SH**Y**	A**S**H	**T**ED

Problem 2

Show how to sort n integers in the range 0 to $n^2 - 1$ in $\mathcal{O}(n)$ time.

Solution. Psuedocode for sorting n integers in $\mathcal{O}(n)$, X is an array storing all n integers:

SORT-N-LINEAR(X, n)

if $n == 1$
    return
for $i = 1$ **to** n
    // change $X[i]$ into base n
    TO-BASE-N$(X[i])$
// each integer will have at most $\log_n n^2 = 2$ digits
// radix sort uses counting sort to sort n at most 2 digits integers
RADIX-SORT(X)
for $i = 1$ **to** n
    // convert $X[i]$ back to its original value because we sorted them with base n
    CONVERT-BASE-N$(X[i])$

TO-BASE-N converts an integer into a base n number. E.g. 100 integers have the largest number of 9999, which can be converted to base 100, $9999 = 99 * 100^1 + 99 * 100^0$. Originally, radix sort will sort 4 digits for

9999. After converting its base to n, radix sort will need to sort **2 digits** 99_{100} and 99_{100}. Another example that the integer 400 will be converted to 4_{100}. The **digit** here is not limit from 0 to 9.

Solution 2:

```
SORT-N-LINEAR-v2(A, B, n)
1: if n == 1
2:     return
3: for i = 1 to n
4:     B[i][2] = A[i] % n
5:     B[i][1] = A[i]/n
6: RADIX-SORT(B)
7: for i = 1 to n
8:     A[i] = B[i][1] · n + B[i]
```

Line 1-2 constant time. Line 3-5 $\mathcal{O}(n)$, line 6 $\mathcal{O}(n)$, line 7-8 $\mathcal{O}(n)$. ■

Problem 3

Using Figure 8.4 as a model, illustrate the operation of BUCKET-SORT on the array A=<.49, .30, .56, .74, .68, .50, .69, .37, .41, .72>

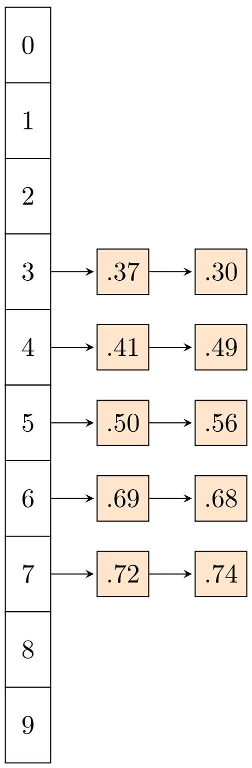

Figure 1: Insert data

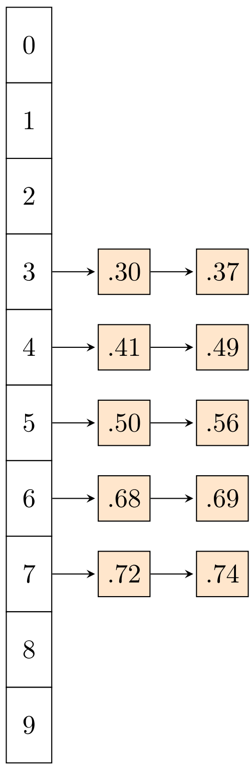

Figure 2: Sort each bucket

Solution. To get sorted data, chain all the nodes in Figure 2 from bucket 0 to bucket 9:
$< .30, .37, .41, .49, .50, .56, .68, .69, .72, .74 >$ ■

Problem 4

Suppose that you have a "black-box" worst-case linear time median subroutine. Give a simple, linear-time algorithm that solves the selection problem for an arbitrary order statistic.

Solution. Psuedocode for finding the k-th smallest element in array $X[1..n]$, and assume $n > 0$:

```
FIND-K-TH-SMALLEST(X, n, k)
half = 0
if n % 2 == 1 // n is odd
    half = ⌈n/2⌉
else // n is even
    half = n/2
m = BLACK-BOX(X)
if k == half // k-th smallest is median
    return m
// splits X into two arrays, Y contains elements less than m and Z contains elements greater than m
Y, Z = SPLIT(X, m)
if k < half
    return FIND-K-TH-SMALLEST(Y, half, k)
else // k > half
    // Find k − half-th smallest for Z since it's half the size of X
    return FIND-K-TH-SMALLEST(Z, half, k − half)
```

$T(n) = T(n/2) + \mathcal{O}(n)$(SPLIT). Apply master method $n^{\log_b a} = n^{\log_2 1} = n^0 = 1 < \mathcal{O}(n)$, may apply case 3. First, we find $\epsilon = 1$ where $f(n) = \Omega(n^{(\log_2 1)+\epsilon})$. Second, we find $c = 1/2$ where $af(n/b) = n/2 \leq 1/2n = cf(n)$ is true. Thus, we can conclude that the running time of this algorithm is $\Theta(f(n)) = \Theta(n)$.

Solution 2:

```
LINEAR-SELECT(A, p, r, i)
if p == r
    return A[p]
q = BLACK-BOX-MEDIAN(A, p, r)
// let A[q] be the pivot
exchange A[q], A[r]
q = PARTITION(A, p, r)
// the location of q
k = q − p + 1
if i == k
    return A[q]
else if i < k
    return LINEAR-SELECT(A, p, q − 1, i)
else
    return LINEAR-SELECT(A, q + 1, r, i − k)
```

This algorithm is also $\Theta(n)$. ■

Problem 5

Let $X[1..n]$ and $Y[1..n]$ be two arrays, each containing n numbers already in sorted order. Give an $\mathcal{O}(\lg n)$-time algorithm to find the median of all $2n$ elements in arrays X and Y.

Solution. Psuedocode for finding the median of arrays $X[1..n]$ and $Y[1..n]$, xi is X array's starting index and xj is X array's last index; yi is Y array's starting index and yj is Y array's last index:

```
FIND-MEDIAN(X, Y, x1, x2, y1, y2, n)
if n == 1
    // Both arrays have only one element, pick the smaller one as median
    return MIN(X[1], Y[1])
xmid = X[⌊(x1 + x2)/2⌋]
ymid = Y[⌊(y1 + y2)/2⌋]
if xmid == ymid
    return xmid
if xmid < ymid
    return FIND-MEDIAN(X, Y, xmid, x2, y1, ymid, n/2)
else
    return FIND-MEDIAN(X, Y, x1, xmid, ymid, y2, n/2)
```

$T(n) = T(n/2) + \mathcal{O}(1)$
Apply master method case 2 because $n^{\log_b a} = n^{\log_2 1} = n^0 = 1 = f(n)$.
$T(n) = \Theta(n^{\log_2 1} \lg n) = \Theta(\lg n)$. ■

Problem 6

Demonstrate the insertion of the keys 23, 38, 19, 40, 29, 43, 12, 27, 11 into a hash tale with collisions resolved by chaining. Let the table have 9 slots, and let the hash function be $h(k) = k \pmod 9$.

(1) insert $23 \to 23 \pmod 9 = 5$
(2) insert $38 \to 38 \pmod 9 = 2$
(3) insert $19 \to 19 \pmod 9 = 1$
(4) insert $40 \to 40 \pmod 9 = 4$
(5) insert $29 \to 29 \pmod 9 = 2,$ insert head
(6) insert $43 \to 43 \pmod 9 = 7$
(7) insert $12 \to 12 \pmod 9 = 3$
(8) insert $27 \to 27 \pmod 9 = 0$
(9) insert $11 \to 11 \pmod 9 = 2,$ insert head

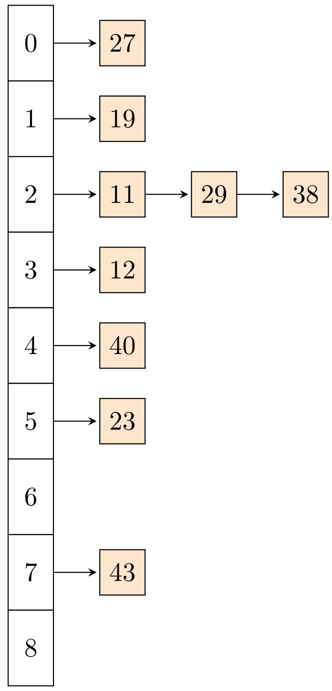

Figure 3: Hash table with chaining

Problem 7

Consider inserting the keys 10, 23, 31, 4, 12, 28, 17, 87, 58 into a hash table of length $m = 11$ using open addressing with the auxiliary hash function $h'(k) = k \pmod{m}$. Illustrate the result of inserting these keys using **linear probing**, using **quadratic probing** with $c_1 = 1$, and $c_2 = 3$, and using **double hashing** with $h_2(k) = 1 + (k \pmod{(m-1)})$.

(1) insert 10 $\rightarrow 10 \pmod{11} = 10$
(2) insert 23 $\rightarrow 23 \pmod{11} = 1$
(3) insert 31 $\rightarrow 31 \pmod{11} = 9$
(4) insert 4 $\rightarrow 4 \pmod{11} = 4$
(5) insert 12 $\rightarrow 12 \pmod{11} = 1$ $\times$ collision
insert 12 to slot 2 because slot 1 is occupied
(6) insert 28 $\rightarrow 28 \pmod{11} = 6$
(7) insert 17 $\rightarrow 17 \pmod{11} = 6$ $\times$
insert 17 to index 7 because slot 6 is occupied
(8) insert 87 $\rightarrow 87 \pmod{11} = 10$ $\times$
insert 87 to index 0 because slot 10 is occupied
(9) insert 58 $\rightarrow 58 \pmod{11} = 3$

0	87
1	23
2	12
3	58
4	4
5	x
6	28
7	17
8	x
9	31
10	10

Figure 4: linear probing

---

resolve collision: $h(k) + 1 * i + 3 * i^2 \pmod{11}$
(1) insert 10 $\rightarrow 10 \pmod{11} = 10$
(2) insert 23 $\rightarrow 23 \pmod{11} = 1$
(3) insert 31 $\rightarrow 31 \pmod{11} = 9$
(4) insert 4 $\rightarrow 4 \pmod{11} = 4$
(5) insert 12 $\rightarrow 12 \pmod{11} = 1$ $\times$ collision
$(1 + 1 + 3 * 1 * 1) \pmod{11} = (1 + 1 + 3) \pmod{11} = 5$
insert 12 to slot 5
− −
(6) insert 28 $\rightarrow 28 \pmod{11} = 6$
(7) insert 17 $\rightarrow 17 \pmod{11} = 6$ $\times$
$(6 + 1 + 3 * 1 * 1) \pmod{11} = (6 + 1 + 3) \pmod{11} = 10$ $\times$
$(6 + 2 + 3 * 2 * 2) \pmod{11} = (6 + 2 + 12) \pmod{11} = 9$ $\times$
$(6 + 3 + 3 * 3 * 3) \pmod{11} = (6 + 3 + 27) \pmod{11} = 3$
insert 17 to slot 3
− −
(8) insert 87 $\rightarrow 87 \pmod{11} = 10$ $\times$
$(10 + 1 + 3 * 1 * 1) \mod 11 = (-1 + 1 + 3) \mod 11 = 3$ $\times$
$(10 + 2 + 3 * 2 * 2) \mod 11 = (-1 + 2 + 12) \mod 11 = 2$
insert 87 to slot 2
− −
(9) insert 58 $\rightarrow 58 \pmod{11} = 3$ $\times$
$(3 + 1 + 3 * 1 * 1) \mod 11 = (3 + 1 + 3) \mod 11 = 7$
insert 58 to slot 7

0	x
1	23
2	87
3	17
4	4
5	12
6	28
7	58
8	x
9	31
10	10

Figure 5: quadratic probing

resolve collision: $h_2(k) = 1 + k \pmod{10}$
$new_k = (h_1(k) + i * h_2(k)) \pmod{11}$
(1) insert $10 \to 10 \pmod{11} = 10$
(2) insert $23 \to 23 \pmod{11} = 1$
(3) insert $31 \to 31 \pmod{11} = 9$
(4) insert $4 \to 4 \pmod{11} = 4$
(5) insert $12 \to 12 \pmod{11} = 1$ $\times$ collision
$h_2(12) = 1 + 12 \pmod{10} = 3$
$1 + 1 * 3 \pmod{11} = 4 \times$
$1 + 2 * 3 \pmod{11} = 7$
insert 12 to slot 7
$-----------------------$
(6) insert $28 \to 28 \pmod{11} = 6$
(7) insert $17 \to 17 \pmod{11} = 6 \times$
$h_2(17) = 1 + 17 \pmod{10} = 8$
$6 + 1 * 8 \pmod{11} = 3$
insert 17 to slot 3
$-----------------------$
(8) insert $87 \to 87 \pmod{11} = 10 \times$
$h_2(87) = 1 + 87 \pmod{10} = 8$
$10 + 1 * 8 \pmod{11} = 7 \times$
$10 + 2 * 8 \pmod{11} = 4 \times$
$10 + 3 * 8 \pmod{11} = 1 \times$
$10 + 4 * 8 \pmod{11} = 9 \times$
$10 + 5 * 8 \pmod{11} = 6 \times$
$10 + 6 * 8 \pmod{11} = 3 \times$
$10 + 7 * 8 \pmod{11} = 0$
insert 87 to slot 0
$-----------------------$
(9) insert $58 \to 58 \pmod{11} = 3 \times$
$h_2(58) = 1 + 58 \pmod{10} = 9$
$3 + 1 * 9 \pmod{11} = 1 \times$
$3 + 2 * 9 \pmod{11} = 10 \times$
$3 + 3 * 9 \pmod{11} = 8$
insert 58 to slot 8

0	87
1	23
2	x
3	17
4	4
5	x
6	28
7	12
8	58
9	31
10	10

Figure 6: double hashing

Problem 8

Solve the following assembly-line problem:
$e_1 = 1, e_2 = 1, x_1 = 7, x_2 = 3$
$a_{1,1} = 3, a_{1,2} = 2, a_{1,3} = 5, a_{1,4} = 4, a_{1,5} = 2$
$a_{2,1} = 4, a_{2,2} = 2, a_{2,3} = 4, a_{2,4} = 6, a_{2,5} = 4$
$t_{1,1} = 1, t_{1,2} = 2, t_{1,3} = 3, t_{1,4} = 1$
$t_{2,1} = 2, t_{2,2} = 3, t_{2,3} = 1, t_{2,4} = 2.$

	1	2	3	4	5	
$f_1[j]$	4	6	11	15	17	24
$f_2[j]$	5	7	11	17	20	23

Figure 7: f table

	2	3	4	5
$l_1[j]$	1	1	1	1
$l_2[j]$	1, 2	2	2	1

Figure 8: l table

Solution. $f^* = 23, l^* = 2$. The best solution for this assembly line is to start from $a_{1,1}$, and goes through $a_{1,2}, a_{1,3}, a_{1,4}, t_{1,4}, a_{2,5}, x_2$. ■

Problem 9

Find an optimal parenthesization of a matrix-chain product whose sequence of dimension is <5, 7, 3, 2, 4, 5>.
$A_1 = A_{5\times 7}, A_2 = A_{7\times 3}, A_3 = A_{3\times 2}, A_4 = A_{2\times 4}, A_5 = A_{4\times 5}, p_0 = 5, p_1 = 7, p_2 = 3, p_3 = 2, p_4 = 4, p_5 = 5$

$$
\begin{aligned}
m[1,2] &= 5 \cdot 7 \cdot 3 = 105 \ (k=1) \\
m[2,3] &= 7 \cdot 3 \cdot 2 = 42 \ (k=2) \\
m[3,4] &= 3 \cdot 2 \cdot 4 = 24 \ (k=3) \\
m[4,5] &= 2 \cdot 4 \cdot 5 = 40 \ (k=4)
\end{aligned}
$$

$$
m[1,3] = \mathbf{min} \begin{cases} m[1,1] + m[2,3] + p_0 \cdot p_1 \cdot p_3 & = 0 + 42 + 5 \cdot 7 \cdot 2 = 112 \ (k=1) \checkmark \\ m[1,2] + m[3,3] + p_0 \cdot p_2 \cdot p_3 & = 105 + 0 + 5 \cdot 3 \cdot 2 = 135 \end{cases}
$$

$$
m[2,4] = \mathbf{min} \begin{cases} m[2,2] + m[3,4] + p_1 \cdot p_2 \cdot p_4 & = 0 + 24 + 7 \cdot 3 \cdot 4 = 108 \\ m[2,3] + m[4,4] + p_1 \cdot p_3 \cdot p_4 & = 42 + 0 + 7 \cdot 2 \cdot 4 = 98 \ (k=3) \checkmark \end{cases}
$$

$$
m[3,5] = \mathbf{min} \begin{cases} m[3,3] + m[4,5] + p_2 \cdot p_3 \cdot p_5 & = 0 + 40 + 3 \cdot 2 \cdot 5 = 70 \ (k=3) \checkmark \\ m[3,4] + m[5,5] + p_2 \cdot p_4 \cdot p_5 & = 24 + 0 + 3 \cdot 4 \cdot 5 = 84 \end{cases}
$$

$$
m[1,4] = \mathbf{min} \begin{cases} m[1,1] + m[2,4] + p_0 \cdot p_1 \cdot p_4 & = 0 + 98 + 5 \cdot 7 \cdot 4 = 238 \\ m[1,2] + m[3,4] + p_0 \cdot p_2 \cdot p_4 & = 105 + 24 + 5 \cdot 3 \cdot 4 = 189 \\ m[1,3] + m[4,4] + p_0 \cdot p_3 \cdot p_4 & = 112 + 0 + 5 \cdot 2 \cdot 4 = 152 \ (k=3) \checkmark \end{cases}
$$

$$
m[2,5] = \mathbf{min} \begin{cases} m[2,2] + m[3,5] + p_1 \cdot p_2 \cdot p_5 & = 0 + 70 + 7 \cdot 3 \cdot 5 = 175 \\ m[2,3] + m[4,5] + p_1 \cdot p_3 \cdot p_5 & = 42 + 40 + 7 \cdot 2 \cdot 5 = 152 \ (k=3) \checkmark \\ m[2,4] + m[5,5] + p_1 \cdot p_4 \cdot p_5 & = 98 + 0 + 7 \cdot 4 \cdot 5 = 238 \end{cases}
$$

$$
m[1,5] = \mathbf{min} \begin{cases} m[1,1] + m[2,5] + p_0 \cdot p_1 \cdot p_5 & = 0 + 152 + 5 \cdot 7 \cdot 5 = 327 \\ m[1,2] + m[3,5] + p_0 \cdot p_2 \cdot p_5 & = 105 + 70 + 5 \cdot 3 \cdot 5 = 250 \\ m[1,3] + m[4,5] + p_0 \cdot p_3 \cdot p_5 & = 112 + 40 + 5 \cdot 2 \cdot 5 = 202 \ (k=3) \checkmark \\ m[1,4] + m[5,5] + p_0 \cdot p_4 \cdot p_5 & = 152 + 0 + 5 \cdot 4 \cdot 5 = 252 \end{cases}
$$

j \ i	1	2	3	4	5
5	202	152	70	40	0
4	152	98	24	0	
3	112	42	0		
2	105	0			
1	0				

Figure 9: m table

j \ i	1	2	3	4
5	3	3	3	4
4	3	3	3	
3	1	2		
2	1			

Figure 10: s table

Solution. Positions of parenthesis: $((A_1(A_2A_3))(A_4A_5))$ *or* $(((5\times 7)((7\times 3)(3\times 2)))((2\times 4)(4\times 5)))$ ■

Problem 10

Determine an LCS of <1, 1, 0, 1, 0, 1> and <0, 0, 1, 1, 0, 1, 1>.

	i	0	1	2	3	4	5	6	7
j		y_j	0	0	1	1	0	1	1
0	x_i	0	0	0	0	0	0	0	0
1	1	0	$\uparrow$ 0	$\uparrow$ 0	$\nwarrow$ 1	$\nwarrow$ 1	$\leftarrow$ 1	$\nwarrow$ 1	$\nwarrow$ 1
2	1	0	$\uparrow$ 0	$\uparrow$ 0	$\nwarrow$ 1	$\nwarrow$ 2	$\leftarrow$ 2	$\nwarrow$ 2	$\nwarrow$ 2
3	0	0	$\nwarrow$ 1	$\nwarrow$ 1	$\uparrow$ 1	$\uparrow$ 2	$\nwarrow$ 3	$\leftarrow$ 3	$\leftarrow$ 3
4	1	0	$\uparrow$ 1	$\uparrow$ 1	$\nwarrow$ 2	$\nwarrow$ 2	$\uparrow$ 3	$\nwarrow$ 4	$\nwarrow$ 4
5	0	0	$\nwarrow$ 1	$\nwarrow$ 2	$\uparrow$ 2	$\uparrow$ 2	$\nwarrow$ 3	$\uparrow$ 4	$\uparrow$ 4
6	1	0	$\uparrow$ 1	$\uparrow$ 2	$\nwarrow$ 3	$\nwarrow$ 3	$\uparrow$ 3	$\nwarrow$ 4	$\nwarrow$ 5

Figure 11: c table for LCS

Solution. Thus, LCS length is 5 and LCS is $<$ 1, 1, 0, 1, 1$>$. ■