Homework 7 Solutions

Chapter 11

1) *Stein's lemma.*

a) $f_1 = \mathcal{N}(0, \sigma_1^2)$, $f_2 = \mathcal{N}(0, \sigma_2^2)$,

$$D(f_1||f_2) = \int_{-\infty}^{\infty} f_1(x) \left[\frac{1}{2} \ln \frac{\sigma_2^2}{\sigma_1^2} - \left(\frac{x^2}{2\sigma_1^2} - \frac{x^2}{2\sigma_2^2} \right) \right] dx \quad (1167)$$

$$= \frac{1}{2} \left[\ln \frac{\sigma_2^2}{\sigma_1^2} + \frac{\sigma_1^2}{\sigma_2^2} - 1 \right]. \quad (1168)$$

b) $f_1 = \lambda_1 e^{-\lambda_1 x}$, $f_2 = \lambda_2 e^{-\lambda_2 x}$,

$$D(f_1||f_2) = \int_0^{\infty} f_1(x) \left[\ln \frac{\lambda_1}{\lambda_2} - \lambda_1 x + \lambda_2 x \right] dx \quad (1169)$$

$$= \ln \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} - 1. \quad (1170)$$

c) $f_1 = U[0,1]$, $f_2 = U[a, a+1]$,

$$D(f_1||f_2) = \int_0^1 f_1 \ln \frac{f_1}{f_2} \quad (1171)$$

$$= \int_0^a f_1 \ln \infty + \int_a^1 f_1 \ln 1 \quad (1172)$$

$$= \infty. \quad (1173)$$

In this case, the Kullback Leibler distance of ∞ implies that in a hypothesis test, the two distributions will be distinguished with probability 1 for large samples.

d) $f_1 = \text{Bern}\left(\frac{1}{2}\right)$ and $f_2 = \text{Bern}(1)$,

$$D(f_1||f_2) = \frac{1}{2} \ln \frac{\frac{1}{2}}{1} + \frac{1}{2} \ln \frac{\frac{1}{2}}{0} = \infty. \quad (1174)$$

The implication is the same as in part (c).

2) *A relation between $D(P \parallel Q)$ and Chi-square.*

There are many ways to expand $D(P||Q)$ in a Taylor series, but when we are expanding about $P = Q$, we must get a series in $P - Q$, whose coefficients depend on Q only. It is easy to get misled into forming another series expansion, so we will provide two alternative proofs of this result.

- Expanding the log.

Writing $\frac{P}{Q} = 1 + \frac{P-Q}{Q} = 1 + \frac{\Delta}{Q}$, and $P = Q + \Delta$, we get

$$\begin{aligned} D(P||Q) &= \int P \ln \frac{P}{Q} && (1175) \\ &= \int (Q+\Delta) \ln \left(1 + \frac{\Delta}{Q}\right) && (1176) \\ &= \int (Q+\Delta) \left(\frac{\Delta}{Q} - \frac{\Delta^2}{2Q^2} + \dots\right) && (1177) \\ &= \int \Delta + \frac{\Delta^2}{Q} - \frac{\Delta^2}{2Q} + \dots. && (1178) \end{aligned}$$

The integral of the first term $\int \Delta = \int P - \int Q = 0$, and hence the first non-zero term in the expansion is

$$\frac{\Delta^2}{2Q} = \frac{\chi^2}{2}, \tag{1179}$$

which shows that locally around Q, $D(P||Q)$ behaves quadratically like χ^2.

- By differentiation.
  If we construct the Taylor series expansion for f, we can write

$$f(x) = f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{2} + \dots \tag{1180}$$

Doing the same expansion for $D(P||Q)$ around the point Q, we get

$$D(P||Q)_{P=Q} = 0, \tag{1181}$$

$$D'(P||Q)_{P=Q} = (\ln \frac{P}{Q} + 1)_{P=Q} = 1, \tag{1182}$$

and

$$D''(P||Q)_{P=Q} = \left(\frac{1}{P}\right)_{P=Q} = \frac{1}{Q}. \tag{1183}$$

Hence the Taylor series is

$$\begin{aligned} D(P||Q) &= 0 + \int 1(P-Q) + \int \frac{1}{Q}\frac{(P-Q)^2}{2} + \dots && (1184) \\ &= \frac{1}{2}\chi^2 + \dots. && (1185) \end{aligned}$$

and we get $\frac{\chi^2}{2}$ as the first non-zero term in the expansion.

3) *Error exponent for universal codes.*

a) We have to minimize $D(p||q)$ subject to the constraint that $H(p) \geq R$. Rewriting this problem using Lagrange multipliers, we get

$$J(p) = \sum p \log \frac{p}{q} + \lambda \sum p \log p + \nu \sum p. \tag{1186}$$

Differentiating with respect to $p(x)$ and setting the derivative to 0, we obtain

$$\log \frac{p}{q} + 1 + \lambda \log p + \lambda + \nu = 0, \tag{1187}$$

which implies that

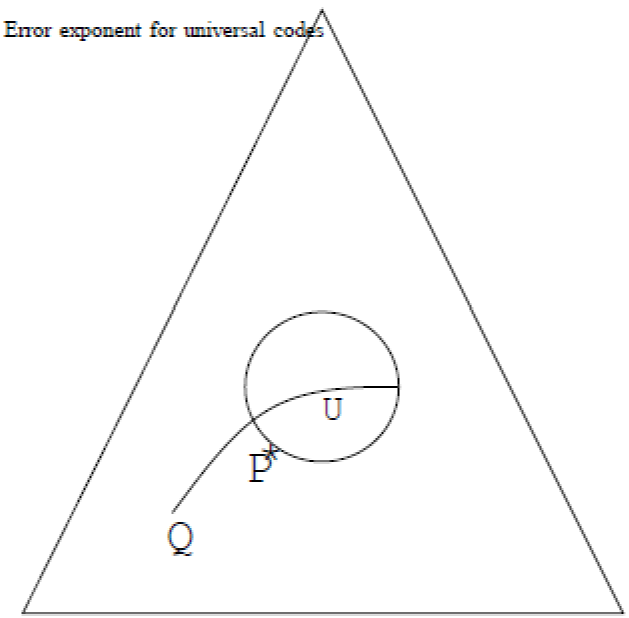
$$p^*(x) = \frac{q^\mu(x)}{\sum_a q^\mu(a)}. \tag{1188}$$

where $\mu = \frac{\lambda}{1-\lambda}$ is chosen to satisfy the constraint $H(p^*) = R$. We have to first check that the constraint is active, i.e., that we really need equality in the constraint. For this we set $\lambda = 0$ or $\mu = 1$, and we get $p^* = q$. Hence if q is such that $H(q) \geq R$, then the maximizing p^* is q. On the other hand, if $H(q) < R$, then $\lambda \neq 0$, and the constraint must be satisfied with equality.

Geometrically it is clear that there will be two solutions for λ of the form (1188) which have $H(p^*) = R$, corresponding to the minimum and maximum distance to q on the manifold $H(p) = R$. It is easy to see that for $0 \leq \mu \leq 1$, $p^*_\mu(x)$ lies on the geodesic from q to the uniform distribution. Hence, the minimum will lie in this region of μ. The maximum will correspond to negative μ, which lies on the other side of the uniform distribution as in the figure.

b) For a universal code with rate R, any source can be transmitted by the code if $H(p) < R$. In the binary case, this corresponds to $p \in [0, h^{-1}(R))$ or $p \in (1-h^{-1}(R), 1]$, where h is the binary entropy function.

Fig. 13. Error exponent for universal codes



7) *Fisher information and relative entropy*. Let $t = \theta' - \theta$. Then

$$\frac{1}{(\theta-\theta')^2} D(p_\theta||p_{\theta'}) = \frac{1}{t^2} D(p_\theta||p_{\theta+t}) = \frac{1}{t^2 \ln 2} \sum_x p_\theta(x) \ln \frac{p_\theta(x)}{p_{\theta+t}(x)}. \tag{1245}$$

Let

$$f(t) = p_\theta(x) \ln \frac{p_\theta(x)}{p_{\theta+t}(x)}. \tag{1246}$$

We will suppress the dependence on x and expand $f(t)$ in a Taylor series in t. Thus

$$f'(t) = -\frac{p_\theta}{p_{\theta+t}} \frac{dp_{\theta+t}}{dt}, \tag{1247}$$

and

$$f''(t) = \frac{p_\theta}{p_{\theta+t}^2} \left(\frac{dp_{\theta+t}}{dt}\right)^2 + \frac{p_\theta}{p_{\theta+t}} \frac{d^2 p_{\theta+t}}{dt^2}. \tag{1248}$$

Thus expanding in the Taylor series around $t = 0$, we obtain

$$f(t) = f(0) + f'(0)t + f''(0)\frac{t^2}{2} + O(t^3), \tag{1249}$$

where $f(0) = 0$,

$$f'(0) = -\frac{p_\theta}{p_\theta} \frac{dp_{\theta+t}}{dt}\bigg|_{t=0} = \frac{dp_\theta}{d\theta} \tag{1250}$$

and

$$f''(0) = \frac{1}{p_\theta}\left(\frac{dp_\theta}{d\theta}\right)^2 + \frac{d^2 p_\theta}{d\theta^2} \tag{1251}$$

Now $\sum_x p_\theta(x) = 1$, and therefore

$$\sum_x \frac{dp_\theta(x)}{d\theta} = \frac{d}{dt} 1 = 0, \tag{1252}$$

and

$$\sum_x \frac{d^2 p_\theta(x)}{d\theta^2} = \frac{d}{d\theta} 0 = 0. \tag{1253}$$

Therefore the sum of the terms of (1250) sum to 0 and the sum of the second terms in (1251) is 0. Thus substituting the Taylor expansions in the sum, we obtain

$$\begin{aligned}
\frac{1}{(\theta-\theta')^2} D(p_\theta||p_{\theta'}) &= \frac{1}{t^2 \ln 2} \sum_x p_\theta(x) \ln \frac{p_\theta(x)}{p_{\theta+t}(x)} && (1254)\\
&= \frac{1}{t^2 \ln 2}\left(0 + \sum_x \frac{dp_\theta(x)}{d\theta} t + \sum_x \left(\frac{1}{p_\theta}\left(\frac{dp_\theta}{d\theta}\right)^2 + \frac{d^2 p_\theta}{d\theta^2}\right)\frac{t^2}{2} + O(t^3)\right) && (1255)\\
&= \frac{1}{2\ln 2} \sum_x \frac{1}{p_\theta(x)} \left(\frac{dp_{\theta(x)}}{d\theta}\right)^2 + O(t) && (1256)\\
&= \frac{1}{\ln 4} J(\theta) + O(t) && (1257)
\end{aligned}$$

and therefore

$$\lim_{\theta' \to \theta} \frac{1}{(\theta-\theta')^2} D(p_\theta||p_{\theta'}) = \frac{1}{\ln 4} J(\theta). \tag{1258}$$

13) *Sanov's theorem*

- Since $n\overline{X}_n$ has a binomial distribution, we have

$$\Pr(n\overline{X}_n = i) = \binom{n}{i} q^i (1-q)^{n-i} \tag{1286}$$

and therefore

$$\Pr\{(X_1, X_2, \ldots, X_n) : p_{\mathbf{X}} \geq p\} \leq \sum_{i=\lfloor np \rfloor}^{n} \binom{n}{i} q^i (1-q)^{n-i} \tag{1287}$$

- $$\frac{\Pr(n\overline{X}_n = i+1)}{\Pr(n\overline{X}_n = i)} = \frac{\binom{n}{i+1} q^{i+1}(1-q)^{n-i-1}}{\binom{n}{i} q^i (1-q)^{n-i}} = \frac{n-i}{i+1}\frac{q}{1-q} \tag{1288}$$

This ratio is less than 1 if $\frac{n-i}{i+1} < \frac{1-q}{q}$,i.e., if $i > nq - (1-q)$. Thus the maximum of the terms occurs when $i = \lfloor np \rfloor$.

- From Example 11.1.3,

$$\binom{n}{\lfloor np \rfloor} \doteq 2^{nH(p)} \tag{1289}$$

and hence the largest term in the sum is

$$\binom{n}{\lfloor np \rfloor} q^{\lfloor np \rfloor}(1-q)^{n-\lfloor np \rfloor} = 2^{n(-p\log p-(1-p)\log(1-p))+np\log q+n(1-p)\log(1-q)} = 2^{-nD(p||q)} \tag{1290}$$

- From the above results, it follows that

$$\begin{aligned}
\Pr\{(X_1, X_2, \ldots, X_n) : p_{\mathbf{X}} \geq p\} &\leq \sum_{i=\lfloor np \rfloor}^{n} \binom{n}{i} q^i (1-q)^{n-i} && (1291)\\
&\leq (n - \lfloor np \rfloor)\binom{n}{\lfloor np \rfloor} q^i (1-q)^{n-i} && (1292)\\
&\leq (n(1-p)+1)2^{-nD(p||q)} && (1293)
\end{aligned}$$

where the second inequality follows from the fact that the sum is less than the largest term times the number of terms. Taking the logarithm and dividing by n and taking the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \frac{1}{n} \log \Pr\{(X_1, X_2, \ldots, X_n) : p_{\mathbf{X}} \geq p\} \leq -D(p||q) \tag{1294}$$

Similarly, using the fact the sum of the terms is larger than the largest term, we obtain

$$\begin{aligned}
\Pr\{(X_1, X_2, \ldots, X_n) : p_{\mathbf{X}} \geq p\} &\geq \sum_{i=\lceil np \rceil}^{n} \binom{n}{i} q^i (1-q)^{n-i} && (1295)\\
&\geq \binom{n}{\lceil np \rceil} q^i (1-q)^{n-i} && (1296)\\
&\geq 2^{-nD(p||q)} && (1297)
\end{aligned}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \log \Pr\{(X_1, X_2, \ldots, X_n) : p_{\mathbf{X}} \geq p\} \geq -D(p||q) \tag{1298}$$

Combining these two results, we obtain the special case of Sanov's theorem

$$\lim_{n\to\infty} \frac{1}{n} \log \Pr\{(X_1, X_2, \ldots, X_n) : p_{\mathbf{X}} \geq p\} = -D(p||q) \tag{1299}$$