Lecture 19: Parallel Gaussian Channels

- Parallel Gaussian channel
- Water-filling

Christmas gift shopping list

Item	Unit Price	Quantity
	$2	?
	$10	?
	$5	?
	$329	?

Budget:
$500

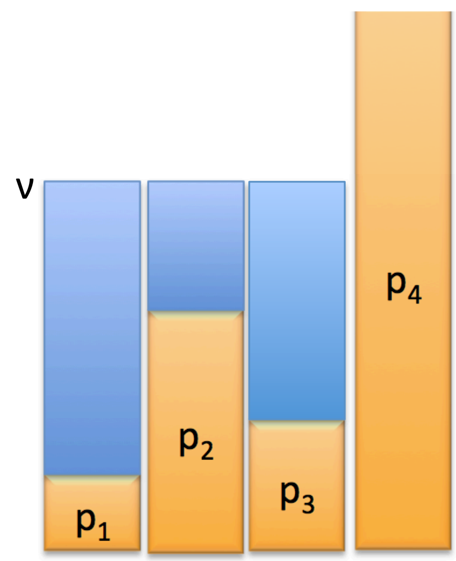
Budget allocation problem

- total budget: w, money allocated for nth item: w_n, unit price for nth item: p_n, can buy w_n/p_n items
- Goal: buy as many gift as possible, but also want to diversify. Diminishing return on the number of items bought $\log(1 + w_n/p_n)$
- budget allocation problem

$$
\begin{aligned}
\max_{w_n} \quad & \sum_{n=1}^{N} \log(1 + w_n/p_n) \\
\text{subject to} \quad & \sum_{n=1}^{N} w_n = W \\
& w_n \geq 0
\end{aligned}
$$

Optimal solution: water-filling

- $w_n = (\nu - p_n)^+$, $(x)^+ = x$ if $x \geq 0$, 0 otherwise
- ν determined by budget constraint: $\sum_{n=1}^{N} (\nu - p_n)^+ = w$



Parallel Gaussian channels

- Channel capacity of parallel Gaussian channel can be formulated into a similar problem

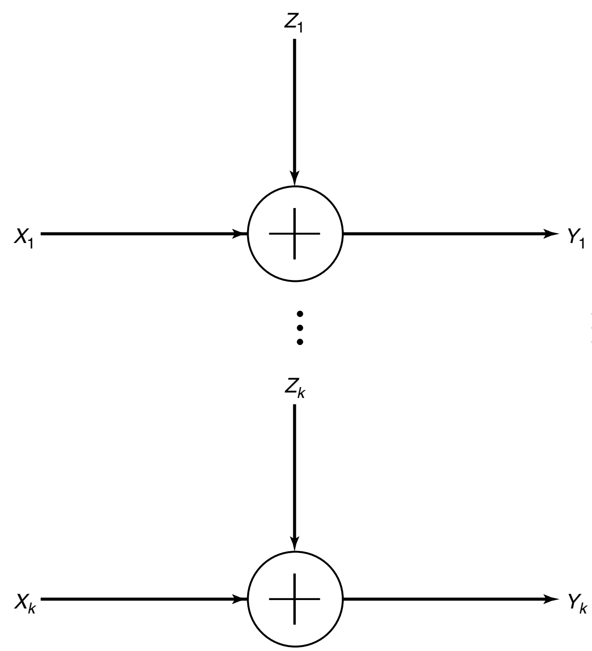


FIGURE 9.3. Parallel Gaussian channels.

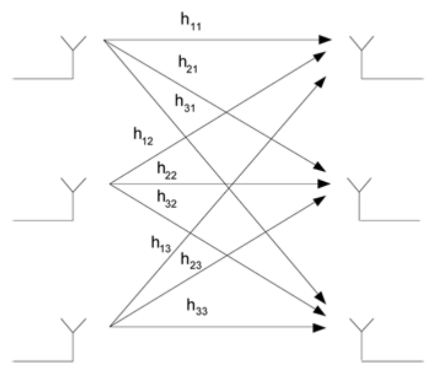
Where are parallel channels?

everywhere:

- OFDM (orthogonal frequency-division multiplexing), parallel channels formed in frequency domain

- MIMO (multiple-input-multiple-output) – multiple antenna system

- DSL (or discrete multi-tone systems)



Parallel independent channels

- k **independent** channels
- $Y_i = X_i + Z_i$, $i = 1, 2, \ldots, k$, $Z_i \sim \mathcal{N}(0, N_i)$
- total power constraint $E \sum_{i=1}^{k} X_i^2 \leq P$
- goal: distribute power among various channels to maximize the total capacity

Channel capacity

- channel capacity of parallel Gaussian channel

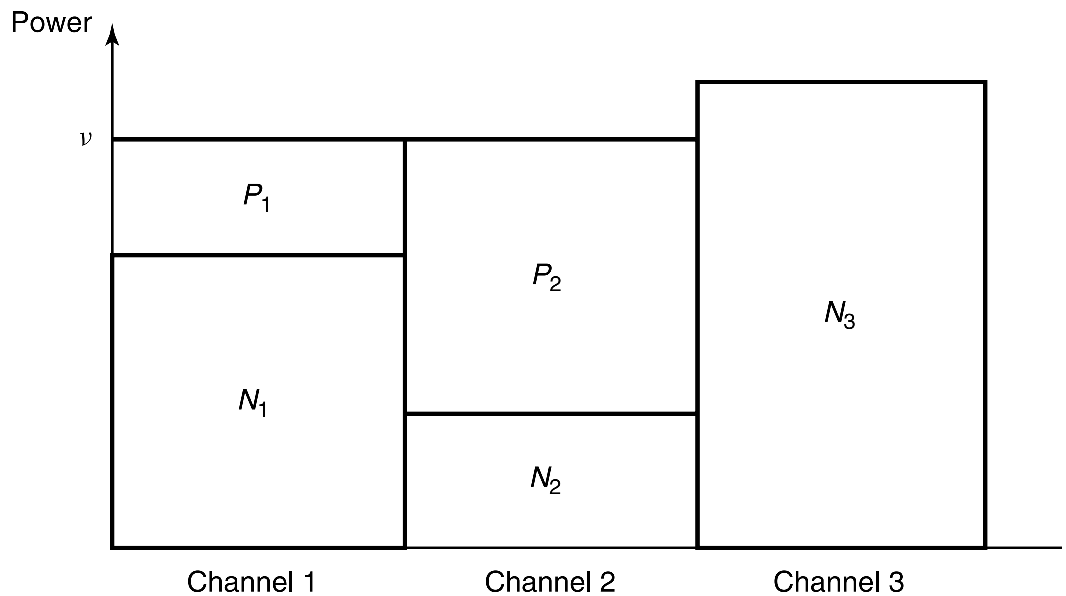
$$C = \max_{f(x_1,\ldots,x_k): \sum EX_i^2 \le P} I(X_1, \ldots, X_k; Y_1, \ldots, Y_k)$$

$$= \frac{1}{2} \log \left(1 + \frac{P_i}{N_i}\right)$$

- power allocation problem

$$\begin{aligned} \max_{P_i} \quad & \sum_{i=1}^{k} \log(1 + P_i/N_i) \\ \text{subject to} \quad & \sum_{i=1}^{k} P_i = P \\ & P_i \ge 0 \end{aligned}$$

Water-filling solution



Channels with colored Gaussian noise

- what if noise in different channels are correlated
- a model for channels with memory
- let K_z be noise covariance matrix
- let K_x be input covariance matrix
- power constraint: $\sum_i EX_i^2 \le P$, equivalently $\mathrm{tr}(K_x) \le P$

Channels capacity

- channel capacity is proportional to

$$\frac{1}{2}\log[(2\pi e)^n|K_x + K_z|]$$

- input covariance optimization problem

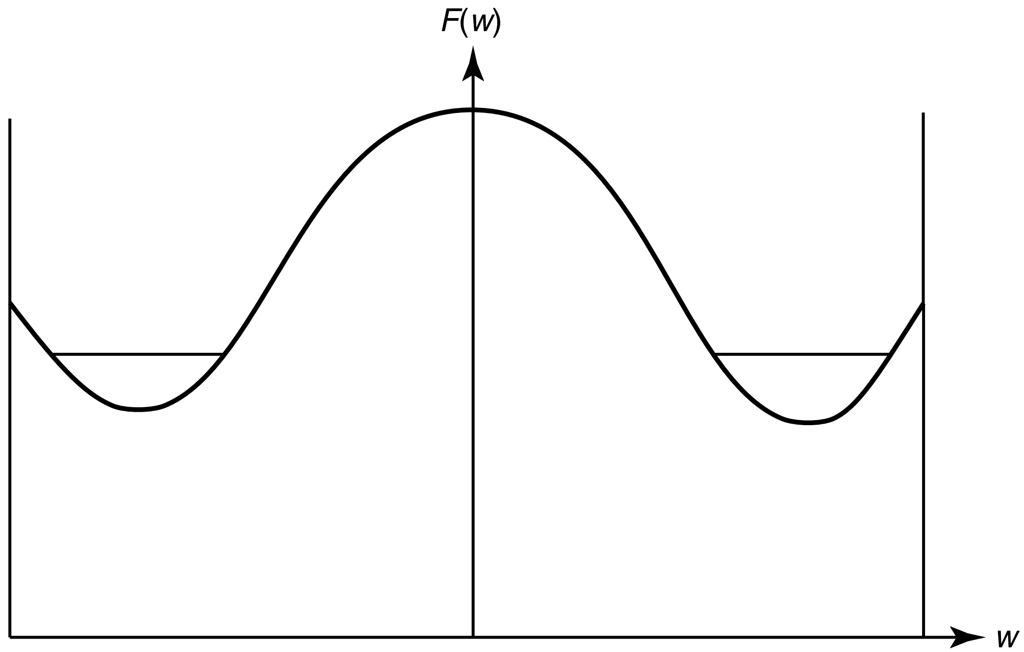
$$\begin{aligned} \max_{K_x} \quad & \frac{1}{2}\log[(2\pi e)^n|K_x + K_z|] \\ \text{subject to} \quad & \mathrm{tr}(K_x) = P \\ & K_x \succeq 0 \end{aligned}$$

- solution: $K_x = U\Lambda U^\top$, where $U =$ eigenvector of K_z, and $\Lambda =$ diagonal matrix

- $\lambda_i = (\nu - \lambda_{z,i})^+$, $\lambda_{z,i}$: eigenvalues of K_z

- ν found from: $\sum_{i=1}^k (\nu - \lambda_{z,i})^+ = P$

Continuous case



$$C = \int_{-\pi}^{\pi} \frac{1}{2} \log \left(1 + \frac{(\nu - N(f))^+}{N(f)}\right) df, \ \int (\nu - N(f))^+ df = P$$

Summary

Water filling:

- allocate power in parallel Gaussian channels
- optimal power allocation achieve power capacity
- allocate more power in less noisy channels
- very noisy channels are abandoned