Midterm Review Session

2.40

40) **Discrete entropies**

Let X and Y be two independent integer-valued random variables. Let X be uniformly distributed over $\{1, 2, \ldots, 8\}$, and let $\Pr\{Y = k\} = 2^{-k}, \quad k = 1, 2, 3, \ldots$

a) Find $H(X)$
b) Find $H(Y)$
c) Find $H(X+Y, X-Y)$.

40) a) For a uniform distribution, $H(X) = \log m = \log 8 = 3$.
b) For a geometric distribution, $H(Y) = \sum_k k2^{-k} = 2$. (See solution to problem 2.1
c) Since $(X, Y) \to (X+Y, X-Y)$ is a one to one transformation, $H(X+Y, X-Y) = H(X, Y) = H(X) + H(Y) = 3 + 2 = 5$.

3.5

5) **Sets defined by probabilities.**

Let $X_1, X_2, \ldots$ be an i.i.d. sequence of discrete random variables with entropy $H(X)$. Let

$$C_n(t) = \{x^n \in \mathcal{X}^n : p(x^n) \geq 2^{-nt}\}$$

denote the subset of n-sequences with probabilities $\geq 2^{-nt}$.

a) Show $|C_n(t)| \leq 2^{nt}$.
b) For what values of t does $P(\{X^n \in C_n(t)\}) \to 1$?

5) a) Since the total probability of all sequences is less than 1, $|C_n(t)| \min_{x^n \in C_n(t)} p(x^n) \leq 1$, and hence $|C_n(t)| \leq 2^{nt}$.
b) Since $-\frac{1}{n} \log p(x^n) \to H$, if $t < H$, the probability that $p(x^n) > 2^{-nt}$ goes to 0, and if $t > H$, the probability goes to 1.

3.8

8) **Products.** Let

$$X = \begin{cases} 1, & \frac{1}{2} \\ 2, & \frac{1}{4} \\ 3, & \frac{1}{4} \end{cases}$$

Let $X_1, X_2, \ldots$ be drawn i.i.d. according to this distribution. Find the limiting behavior of the product

$$(X_1 X_2 \cdots X_n)^{\frac{1}{n}} .$$

8) *Products.* Let

$$P_n = (X_1 X_2 \ldots X_n)^{\frac{1}{n}} . \tag{156}$$

Then

$$\log P_n = \frac{1}{n} \sum_{i=1}^{n} \log X_i \to E \log X, \tag{157}$$

with probability 1, by the strong law of large numbers. Thus $P_n \to 2^{E \log X}$ with prob. 1. We can easily calculate $E \log X = \frac{1}{2} \log 1 + \frac{1}{4} \log 2 + \frac{1}{4} \log 3 = \frac{1}{4} \log 6$, and therefore $P_n \to 2^{\frac{1}{4} \log 6} = 1.565$.

4.13

13) **The past has little to say about the future.** For a stationary stochastic process $X_1, X_2, \ldots, X_n, \ldots$, show that

$$\lim_{n\to\infty} \frac{1}{2n} I(X_1, X_2, \ldots, X_n; X_{n+1}, X_{n+2}, \ldots, X_{2n}) = 0. \tag{192}$$

Thus the dependence between adjacent n-blocks of a stationary process does not grow linearly with n.

13)

$$\begin{aligned}
&I(X_1, X_2, \ldots, X_n; X_{n+1}, X_{n+2}, \ldots, X_{2n}) \\
&\quad= H(X_1, X_2, \ldots, X_n) + H(X_{n+1}, X_{n+2}, \ldots, X_{2n}) - H(X_1, X_2, \ldots, X_n, X_{n+1}, X_{n+2}, \ldots, X_{2n}) \\
&\quad= 2H(X_1, X_2, \ldots, X_n) - H(X_1, X_2, \ldots, X_n, X_{n+1}, X_{n+2}, \ldots, X_{2n}) \qquad (271)
\end{aligned}$$

since $H(X_1, X_2, \ldots, X_n) = H(X_{n+1}, X_{n+2}, \ldots, X_{2n})$ by stationarity.
Thus

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{2n} I(X_1, X_2, \ldots, X_n; X_{n+1}, X_{n+2}, \ldots, X_{2n}) \\
&\quad= \lim_{n\to\infty} \frac{1}{2n} 2H(X_1, X_2, \ldots, X_n) - \lim_{n\to\infty} \frac{1}{2n} H(X_1, X_2, \ldots, X_n, X_{n+1}, X_{n+2}, \ldots, X_{2n}) \qquad (272) \\
&\quad= \lim_{n\to\infty} \frac{1}{n} H(X_1, X_2, \ldots, X_n) - \lim_{n\to\infty} \frac{1}{2n} H(X_1, X_2, \ldots, X_n, X_{n+1}, X_{n+2}, \ldots, X_{2n}) \qquad (273)
\end{aligned}$$

Now $\lim_{n\to\infty} \frac{1}{n} H(X_1, X_2, \ldots, X_n) = \lim_{n\to\infty} \frac{1}{2n} H(X_1, X_2, \ldots, X_n, X_{n+1}, X_{n+2}, \ldots, X_{2n})$ since both converge to the entropy rate of the process, and therefore

$$\lim_{n\to\infty} \frac{1}{2n} I(X_1, X_2, \ldots, X_n; X_{n+1}, X_{n+2}, \ldots, X_{2n}) = 0. \tag{274}$$

4.31

31) **Markov.**
Let $\{X_i\} \sim$ Bernoulli(p). Consider the associated Markov chain $\{Y_i\}_{i=1}^n$ where
Y_i = (the number of 1's in the current run of 1's). For example, if $X^n = 101110\ldots$, we have $Y^n = 101230\ldots$.

a) Find the entropy rate of X^n.
b) Find the entropy rate of Y^n.

31) **Markov solution.**

a) For an i.i.d. source, $H(\mathcal{X}) = H(X) = H(p)$.
b) Observe that X^n and Y^n have a one-to-one mapping. Thus, $H(\mathcal{Y}) = H(\mathcal{X}) = H(p)$.