Assignment #6

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Problem 6.1

a)

$$cov(\theta_A, \theta_B) = E[\theta_A \theta_B] * E[\theta_A]E[\theta_B]$$

Given that $\theta_A = \theta, \theta_B = \theta * \gamma$

We can see that:

$$cov(\theta_A, \theta_B) = E[\theta^2 * \gamma] - E[\theta]E[\theta * \gamma]$$

$$= E[\theta^2]E[\gamma] - E[\theta]^2 E[\gamma]$$

$$= (E[\theta^2] - E[\theta]^2)E[\gamma]$$

$$= var(\theta)E[\gamma] \neq 0$$

This indicates that θ_A, $theta_B$ are dependent.

When θ_B is a product of θ_A + Gamma distributed noise, then this joint prior would justified.

b)

The joint prior distribution is:

$$p(\theta, \gamma | \mathbf{y}_A, \mathbf{y}_B) \propto p(\theta, \gamma) * p(\mathbf{y}_A, \mathbf{y}_B | \theta, \gamma)$$
$$= p(\theta)p(\gamma) * p(\mathbf{y}_A | \theta) * p(\mathbf{y}_B | \theta)$$

$$\propto \left(\theta^{a_\theta - 1} e^{-b_\theta \theta}\right) * \left(\gamma^{a_\gamma - 1} e^{-b_\gamma \gamma}\right) * \left(\prod_{i=1}^{n_A} \theta^{y_{A_i}} e^{-\theta}\right) * \left(\prod_{i=1}^{n_B} (\gamma\theta)^{y_{B_i}} e^{-\gamma\theta}\right)$$
$$= \left(\theta^{a_\theta - 1} e^{-b_\theta \theta}\right) * \left(\gamma^{a_\gamma - 1} e^{-b_\gamma \gamma}\right) * \left(\theta^{\sum_{i=1}^{n_A} y_{A_i}} e^{-n_A \theta}\right) * \left((\gamma\theta)^{\sum_{i=1}^{n_B} y_{B_i}} e^{-n_B \gamma\theta}\right)$$
$$= \left(\theta^{a_\theta - 1} e^{-b_\theta \theta}\right) * \left(\gamma^{a_\gamma - 1} e^{-b_\gamma \gamma}\right) * \left(\theta^{n_A \bar{y}_A} e^{-n_A \theta}\right) * \left((\gamma\theta)^{n_B \bar{y}_B} e^{-n_B \gamma\theta}\right)$$

From this we can say that:

$$
\begin{aligned}
p(\theta, \mid \mathbf{y}_A, \mathbf{y}_B, \gamma) &\propto \left(\theta^{a_\theta - 1} e^{-b_\theta \theta}\right) * \left(\gamma^{a_\gamma - 1} e^{-b_\gamma \gamma}\right) * \left(\theta^{n_A \bar{y}_A} e^{-n_A \theta}\right) * \left((\gamma\theta)^{n_B \bar{y}_B} e^{-n_B \gamma \theta}\right) \\
&\propto \left(\theta^{a_\theta - 1} e^{-b_\theta \theta}\right) * \left(\theta^{n_A \bar{y}_A} e^{-n_A \theta}\right) * \left((\gamma\theta)^{n_B \bar{y}_B} e^{-n_B \gamma \theta}\right)
\end{aligned}
$$

$$
\propto \theta^{a_\theta + n_A \bar{y}_A + n_B \bar{y}_B - 1} \exp\left(-(b_\theta + n_A + n_B \gamma)\theta\right)
$$

$$
\propto dgamma\left(a_\theta + n_A \bar{y}_A + n_B \bar{y}_B, b_\theta + n_A + n_B \gamma\right)
$$

c)

$$
\begin{gathered}
p(\gamma, \mid \mathbf{y}_A, \mathbf{y}_B, \theta) \\
\propto \left(\theta^{a_\theta - 1} e^{-b_\theta \theta}\right) * \left(\gamma^{a_\gamma - 1} e^{-b_\gamma \gamma}\right) * \left(\theta^{n_A \bar{y}_A} e^{-n_A \theta}\right) * \left((\gamma\theta)^{n_B \bar{y}_B} e^{-n_B \gamma \theta}\right) \\
\propto \left(\gamma^{a_\gamma - 1} e^{-b_\gamma \gamma}\right) * \left((\gamma\theta)^{n_B \bar{y}_B} e^{-n_B \gamma \theta}\right) \\
\propto \left(\gamma^{a_\gamma - 1} e^{-b_\gamma \gamma}\right) * \left(\gamma^{n_B \bar{y}_B} e^{-n_B \gamma \theta}\right) \\
\propto \gamma^{a_\gamma + n_B \bar{y}_B - 1} \exp\left(-(b_\gamma + n_B \theta)\gamma\right)
\end{gathered}
$$

$$
\propto \text{dgamma}\left(a_\gamma + n_B \bar{y}_B, b_\gamma + n_B \theta\right)
$$

d)

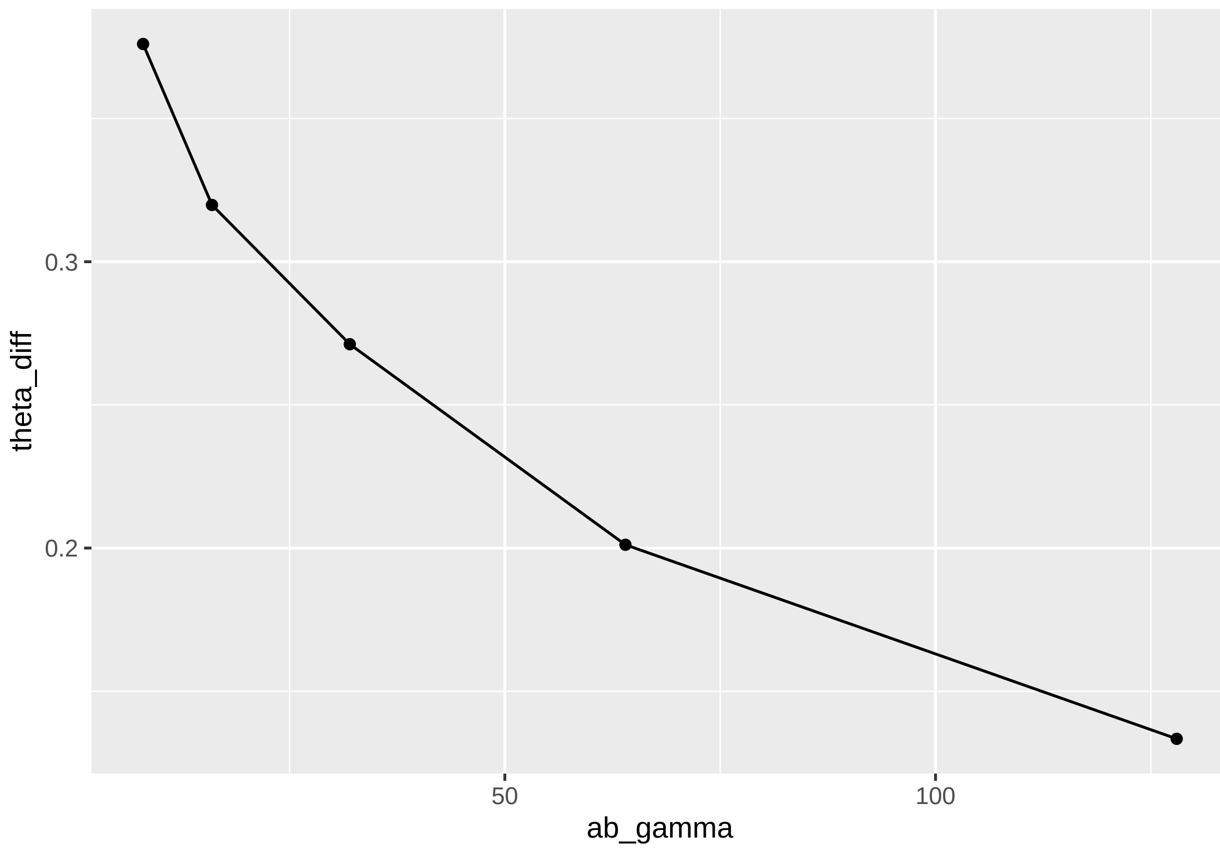
```
library(ggplot2)
```

```
## Warning:    'ggplot2'  R   4.1.3
```

```
Y_a <- scan(url('https://www2.stat.duke.edu/courses/Fall09/sta290/datasets/Hoffdata/menchild30bach.dat')
Y_b <- scan(url('https://www2.stat.duke.edu/courses/Fall09/sta290/datasets/Hoffdata/menchild30nobach.da
n_a = length(Y_a)
n_b = length(Y_b)
ybar_a = mean(Y_a)
ybar_b = mean(Y_b)
a_theta = 2
b_theta = 1
S = 5000
ab_gamma = c(8, 16, 32, 64, 128)
theta_diff = sapply(ab_gamma, function(abg) {
  a_gamma = b_gamma = abg
  THETA = numeric(S)
  GAMMA = numeric(S)
  theta = ybar_a
  gamma = ybar_a / ybar_b
  for (s in 1:S) {
    theta = rgamma(
      1,
      a_theta + n_a * ybar_a + n_b * ybar_b,
      b_theta + n_a + n_b * gamma
    )
```

```
    gamma = rgamma(
      1,
      a_gamma + n_b * ybar_b,
      b_gamma + n_b * theta
    )
    THETA[s] = theta
    GAMMA[s] = gamma
  }

  THETA_A = THETA
  THETA_B = THETA * GAMMA
  mean(THETA_B - THETA_A)
})
ggplot(data.frame(ab_gamma = ab_gamma, theta_diff = theta_diff), aes(x = ab_gamma, y = theta_diff)) +
  geom_point() +
  geom_line()
```



The gamma distribution is centered around 1 and the magnitude is how strong is our belief in which θ_B/θ_A is 1. We can see that as the number of θ_B/θ_A increases, the difference decreases.