Assignment-1(STAT 638)

Yutae Lee (626005947)

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Problem 2.1

Marginal and conditional probability: The social mobility data from Section 2.5 gives a joint probability distribution on (Y1, Y2)= (father's occupation, son's occupation). Using this joint distribution, calculate the following distributions:

a) the marginal probability distribution of a father's occupation;

In order to get marginal probability of father's occupation, you need to get the summation of son's occupation for each margin of father's occupation.

```
farm = 0.018 + 0.035 + 0.031 + 0.008 + 0.018
operatives = 0.002 + 0.112 + 0.064 + 0.032 + 0.069
craftsmen = 0.001 + 0.066 + 0.094 + 0.032 + 0.084
sales = 0.001 + 0.018 + 0.019 + 0.010 + 0.051
professional = 0.001 + 0.029 + 0.032 + 0.043 + 0.130

cat('Marginal Probability of the Father:\n\n')
```

```
## Marginal Probability of the Father:
```

```
cat('Farm:', farm ,'\nOperatives:',operatives,'\nCraftsmen:',craftsmen,'\nSales:',sales,'\nPr
ofessional:', professional)
```

```
## Farm: 0.11
## Operatives: 0.279
## Craftsmen: 0.277
## Sales: 0.099
## Professional: 0.235
```

b) the marginal probability distribution of a son's occupation;

In order to get marginal probability of son's occupation, you need to get the summation of father's occupation for each margin of son's occupation.

```
farm = 0.018 + 0.002 + 0.001 + 0.001 + 0.001
operatives = 0.035 + 0.112 + 0.066 + 0.018 + 0.029
craftsmen = 0.031 + 0.064 + 0.094 + 0.019 + 0.032
sales = 0.008 + 0.032 + 0.032 + 0.010 + 0.043
professional = 0.018 + 0.069 + 0.084 + 0.051 + 0.130


cat('Marginal Probability of the Father:\n\n')
```

```
## Marginal Probability of the Father:
```

```
cat('Farm:', farm ,'\nOperatives:',operatives,'\nCraftsmen:',craftsmen,'\nSales:',sales,'\nPr
ofessional:', professional)
```

```
## Farm: 0.023
## Operatives: 0.26
## Craftsmen: 0.24
## Sales: 0.125
## Professional: 0.352
```

c) the conditional distribution of a son's occupation, given that the father is a farmer;

In order to find the conditional distribution of son given that the father is a farmer, we need to first add up the marginal probability of which the father's occupation is farm:

```
f_farm = 0.018 + 0.035 + 0.031 + 0.008 + 0.018
```

Then divide each son's marginal occupation where father's occupation is farm by the total marginal probability of the father's occupation (farm)

```
son_farm_father_farm = 0.018
son_operatives_father_farm = 0.035
son_craftsmen_father_farm = 0.031
son_sales_father_farm = 0.008
son_professional_father_farm = 0.018
cond_farm = son_farm_father_farm/f_farm
cond_operatives = son_operatives_father_farm/f_farm
cond_craftsmen = son_craftsmen_father_farm/f_farm
cond_sales = son_sales_father_farm/f_farm
cond_professional = son_professional_father_farm/f_farm

cat('Conditional Probability of the Son (given that father is farmer):\n\n')
```

```
## Conditional Probability of the Son (given that father is farmer):
```

```
cat('Farm:', cond_farm ,'\nOperatives:',cond_operatives,'\nCraftsmen:',cond_craftsmen,'\nSale
s:',cond_sales,'\nProfessional:', cond_professional)
```

```
## Farm: 0.1636364
## Operatives: 0.3181818
## Craftsmen: 0.2818182
## Sales: 0.07272727
## Professional: 0.1636364
```

d) the conditional distribution of a father's occupation, given that the son is a farmer.

In order to find the conditional distribution of father given that the son is a farmer, we need to first add up the marginal probability of which the son's occupation is farm:

```
s_farm = 0.018+0.002+0.001+0.001+0.001
```

Then divide each father's marginal occupation where father's occupation is farm by the total marginal probability of the son's occupation (farm)

```
son_farm_father_farm = 0.018
son_farm_father_operatives = 0.002
son_farm_father_craftsmen = 0.001
son_farm_father_sales = 0.001
son_farm_father_professional = 0.001
cond_farm = son_farm_father_farm/s_farm
cond_operatives = son_farm_father_operatives/s_farm
cond_craftsmen = son_farm_father_craftsmen/s_farm
cond_sales = son_farm_father_sales/s_farm
cond_professional = son_farm_father_professional/s_farm

cat('Conditional Probability of the Father (given that son is farmer):\n\n')
```

```
## Conditional Probability of the Father (given that son is farmer):
```

```
cat('Farm:', cond_farm ,'\nOperatives:',cond_operatives,'\nCraftsmen:',cond_craftsmen,'\nSale
s:',cond_sales,'\nProfessional:', cond_professional)
```

```
## Farm: 0.7826087
## Operatives: 0.08695652
## Craftsmen: 0.04347826
## Sales: 0.04347826
## Professional: 0.04347826
```

Problem 2.2

Expectations and variances: Let Y1 and Y2 be two independent random variables, such that $E[Y_i] = \mu_i$ and $Var[Y_i] = \sigma_i^2$.

Using the definition of expectation and variance, compute the following quantities, where a1 and a2 are given constants:

a) $E[a_1Y_1 + a_2Y_2]$, $Var[a_1Y_1 + a_2Y_2]$;

First let's consider $E[a_1Y_1]$ alone. By the formula of expectation
$E[a_1Y_1] = \sum a_1y_1p(y_1) = a_1(\sum y_1p(y_1)) = a_1E[Y_1]$ when discrete and
$E[a_1Y_1] = \int a_1y_1p(y_1) = a_1(\int y_1p(y_1)) = a_1E[Y_1]$ when continuous.

In any case we can tell that $E[a_1Y_1] = a_1E[Y_1]$ pretty simply. <- #1

Now consider $E[Y_1 + Y_2]$. By the formula
$E[Y_1 + Y_2] = \sum y_1p(y_1) + y_2p(y_2) = \sum y_1p(y_1) + \sum y_2p(y_2) = E[Y_1] + E[Y_2]$ when discrete and
$E[Y_1 + Y_2] = \int y_1p(y_1) + y_2p(y_2) = \int y_1p(y_1) + \int y_2p(y_2) = E[Y_1] + E[Y_2]$ when continuous.

In any case we can tell that $E[Y_1 + Y_2] = E[Y_1] + E[Y_2]$. <- #2

Using the two properties above (#1,#2) we can say that
$E[a_1Y_1 + a_2Y_2] = a_1E[Y_1] + a_2E[Y_2] = a_1\mu_1 + a_2\mu_2$

Now let's find $Var[a_1Y_1 + a_2Y_2]$.

Using the formula:

$$Var[a_1Y_1 + a_2Y_2] = E[(a_1Y_1 + a_2Y_2)^2] - E[a_1Y_1 + a_2Y_2]^2$$

=

$$E[a_1^2Y_1^2 + 2a_1a_2Y_1Y_2 + a_2^2Y_2^2] + E[a_1Y_1]^2 + 2E[a_1Y_1]E[a_2Y_2] + E[a_2Y_2]^2$$

=

$$a_1^2E[Y_1^2] + 2a_1a_2E[Y_1Y_2] + a_2^2E[Y_2^2] + a_1^2E[Y_1]^2 + 2a_1a_2E[Y_1]E[Y_2] + a_2^2E[Y_2]^2$$

=

$$a_1^2(E[Y_1^2] + E[Y_1]^2) + a_2^2(E[Y_2^2] + E[Y_2]^2) + 2a_1a_2(E[Y_1Y_2] + E[Y_1]E[Y_2])$$

=

$$a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1a_2cov(Y1, Y2)$$

Since Y1 and Y2 are independent covariance has to be 0

So $Var[a_1Y_1 + a_2Y_2] = a_1^2\sigma_1^2 + a_2^2\sigma_2^2$.

b) $E[a_1Y_1 - a_2Y_2]$, $Var[a_1Y_1 - a_2Y_2]$.

Consider $E[Y_1 - Y_2]$. By the formula
$E[Y_1 - Y_2] = \sum y_1p(y_1) - y_2p(y_2) = \sum y_1p(y_1) - \sum y_2p(y_2) = E[Y_1] - E[Y_2]$ when discrete and
$E[Y_1 - Y_2] = \int y_1p(y_1) - y_2p(y_2) = \int y_1p(y_1) - \int y_2p(y_2) = E[Y_1] - E[Y_2]$ when continuous.

In any case we can tell that $E[Y_1 - Y_2] = E[Y_1] - E[Y_2]$.

Using this fact and the fact from the a) we can say that
$E[a_1Y_1 - a_2Y_2] = a_1E[Y_1] - a_2E[Y_2] = a_1\mu_1 - a_2\mu_2$

Using the formula:

$$Var[a_1Y_1 - a_2Y_2] = E[(a_1Y_1 - a_2Y_2)^2] - E[a_1Y_1 - a_2Y_2]^2$$

=

$$E[a_1^2Y_1^2 - 2a_1a_2Y_1Y_2 + a_2^2Y_2^2] + E[a_1Y_1]^2 - 2E[a_1Y_1]E[a_2Y_2] + E[a_2Y_2]^2$$

=

$$a_1^2E[Y_1^2] - 2a_1a_2E[Y_1Y_2] + a_2^2E[Y_2^2] + a_1^2E[Y_1]^2 - 2a_1a_2E[Y_1]E[Y_2] + a_2^2E[Y_2]^2$$

=

$$a_1^2(E[Y_1^2] + E[Y_1]^2) + a_2^2(E[Y_2^2] + E[Y_2]^2) - 2a_1a_2(E[Y_1Y_2] + E[Y_1]E[Y_2])$$

=

$$a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1a_2cov(Y1, Y2)$$

Since Y1 and Y2 are independent covariance has to be 0

So $Var[a_1Y_1 - a_2Y_2] = a_1^2\sigma_1^2 + a_2^2\sigma_2^2$.

Problem 2.3

Full conditionals: Let X, Y, Z be random variables with joint density (discrete or continuous) p(x, y, z) $\propto$ f(x, z)g(y, z)h(z). Show that

a) p(x|y, z) $\propto$ f(x, z), i.e. p(x|y, z) is a function of x and z;

$p(x|y,z)$ = $\frac{p(x,y,z)}{p(y,z)}$ = $\frac{cf(x,z)g(y,z)h(z)}{p(y,z)}$ where c is just a constant (because we know that $p(x,y,z)$propto f(x,z)g(y,z)h(z)\)) since g(y,z) is just a function of y,z, $\frac{cf(x,z)g(y,z)h(z)}{p(y,z)} = \frac{cf(x,z)g(y,z)h(z)}{g(y,z)} = cf(x,z)h(z)$ we can now tell that this is just a function x,z which means **p(x|y, z) $\propto$ f(x, z)**

b) p(y|x, z) $\propto$ g(y, z), i.e. p(y|x, z) is a function of y and z;

$p(y|x,z)$ = $\frac{p(y,x,z)}{p(x,z)}$ = $\frac{cf(x,z)g(y,z)h(z)}{p(x,z)}$ where c is just a constant (because we know that $p(x,y,z) \propto f(x,z)g(y,z)h(z)$) since f(x,z) is just a function of x,z, $\frac{cf(x,z)g(y,z)h(z)}{p(x,z)} = \frac{cf(x,z)g(y,z)h(z)}{f(x,z)} = cg(y,z)h(z)$ we can now tell that this is just a function x,z which means **p(y|x, z) $\propto$ g(y, z)**

c) X and Y are conditionally independent given Z.

To prove the conditional independence we need to prove $Pr(X \cap Y|Z) = Pr(X|Z)Pr(Y|Z)$

$p(X,Y|Z) = \frac{p(X,Y,Z)}{p(Z)} = \frac{cf(x,z)g(y,z)h(z)}{p(z)}$ just as a),b) we can say h(z) is just a function of z so $\frac{cf(x,z)g(y,z)h(z)}{p(z)} = \frac{cf(x,z)g(y,z)h(z)}{h(z)} = cf(x,z)g(y,z)$ which we can say that $p(x,y|z) \propto f(x,z)g(y,z)$ choose c = $\frac{1}{p(z)^2}$ then we can easily get $p(x,y|z) = p(x|z)p(y|z)$.

Problem 2.5

Urns: Suppose urn H is filled with 40% green balls and 60% red balls, and urn T is filled with 60% green balls and 40% red balls. Someone will flip a coin and then select a ball from urn H or urn T depending on whether the coin lands heads or tails, respectively. Let X be 1 or 0 if the coin lands heads or tails, and let Y be 1 or 0 if the ball is green or red.

a) Write out the joint distribution of X and Y in a table.

Since $P(Y,X) = P(Y|X)P(X)$:

$$P(Y=1,X=1) = P(Y=1|X=1)P(X=1) = 0.4 * 0.5 = 0.2$$

$$P(Y=1,X=0) = P(Y=1|X=0)P(X=0) = 0.6 * 0.5 = 0.3$$

$$P(Y=0,X=1) = P(Y=0|X=1)P(X=1) = 0.6 * 0.5 = 0.3$$

$$P(Y=0,X=0) = P(Y=0|X=0)P(X=0) = 0.4 * 0.5 = 0.2$$

Table is below (done through R):

```
x_1 <- c(0.2,0.3)
x_0 <- c(0.3,0.2)
row_names <- c("Y = 1(Green)", "Y = 0 (Red)")
df <- data.frame ("Row Names" = row_names, "X = 1 (Head)" = x_1, "X = 0 (Tails)" = x_0)
print(df)
```

```
##      Row.Names X...1..Head. X...0..Tails.
## 1 Y = 1(Green)          0.2           0.3
## 2  Y = 0 (Red)          0.3           0.2
```

b) Find $E[Y]$. What is the probability that the ball is green?

We know that:

$E[Y] = \sum y_i p(Y = y_i)$

First we need to find $p(Y = 1)$ and $p(Y = 0)$.

$p(Y = 1) = p(Y = 1, X = 0) + p(Y = 1, X = 1) = 0.3 + 0.2 = 0.5$

then $p(Y = 0) = 1 - P(Y = 1) = 0.5$

our $E[Y]$ would be $1 * 0.5 + 0 * 0.5 = 0.5$

The probability for green which is $p(Y = 1) = 0.5$

c) Find $Var[Y|X = 0]$, $Var[Y|X = 1]$ and $Var[Y]$. Thinking of variance as measuring uncertainty, explain intuitively why one of these variances is larger than the others.

Finding $Var[Y|X = 0]$:

$Var[Y|X = 0] = E[Y^2|X = 0] - E[Y|X = 0]^2$

$E[Y^2|X = 0] = E[Y^2 = 1|X = 0] + E[Y^2 = 0|X = 0] = 1^2 * 0.6 + 0^2 * 0.4 = 0.6$

$E[Y|X = 0] = E[Y = 1|X = 0] + E[Y = 0|X = 0] = 1 * 0.6 + 0 * 0.4 = 0.6$

$Var[Y|X = 0] = E[Y^2|X = 0] - E[Y|X = 0]^2 = 0.6 - 0.6^2 = 0.24$

Finding $Var[Y|X = 1]$:

$Var[Y|X = 1] = E[Y^2|X = 1] - E[Y|X = 1]^2$

$E[Y^2|X = 1] = E[Y^2 = 1|X = 1] + E[Y^2 = 0|X = 1] = 1^2 * 0.4 + 0^2 * 0.6 = 0.4$

$E[Y|X = 1] = E[Y = 1|X = 1] + E[Y = 0|X = 1] = 1 * 0.4 + 0 * 0.6 = 0.4$

$Var[Y|X = 1] = E[Y^2|X = 1] - E[Y|X = 1]^2 = 0.4 - 0.4^2 = 0.24$

Finding $Var[Y]$:

$Var[Y] = E[Y^2] - E[Y]^2$

We know $E[Y] = 0.5$ from b)

let's find $E[Y^2] = 1^2 * 0.5 + 0^2 * 0.5 = 0.5$

Thus $Var[Y] = 0.5 - 0.25 = 0.25$.

$Var[Y]$ is bigger than the other two. The reason behind is because unlike the other two $Var[Y]$ does not have conditional statement which does not have any information given.

d) Suppose you see that the ball is green. What is the probability that the coin turned up tails?

$P(X = 0|Y = 1) = \frac{P(Y=1|X=0)*P(X=0)}{P(Y=1)}$ = $\frac{0.3}{0.5} = 0.6$

The probability is 60%