AST 304, Fall 2024
Stars

Group Computational Project 3

In the previous project, you used your solver for ordinary differential equations to compute the mass-radius relation of a cold white dwarf. In this final project, you will calculate the radius and luminosity of low-mass stars, and explore how their structure and luminosity depends on the $\mathrm{p} + \mathrm{p}$ reaction rate.

1 Background

When modeling the structure of a white dwarf, you solved the Lagrangian equations for radius and pressure,

$$\frac{\mathrm{d}r}{\mathrm{d}m} = \frac{1}{4\pi r^2 \rho} \tag{1}$$

$$\frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4} \tag{2}$$

This system is solvable because the equation of state $\rho(P)$ does not depend on temperature. For low-mass, main sequence stars, the equation of state is approximately that of an ideal gas, and so now we do need to know the temperature throughout the star. We also need an equation for the luminosity as a function of Lagrangian mass m.

The Lagrangian equation for the nuclear luminosity L_{nuc} is

$$\frac{\mathrm{d}L_{\mathrm{nuc}}}{\mathrm{d}m} = \varepsilon_{\mathrm{nuc}}, \tag{3}$$

where $\varepsilon_{\mathrm{nuc}}$ is the nuclear heating rate per unit mass (units $\mathrm{W\,kg^{-1}}$). For the proton-proton chain of reactions, the heating rate is

$$\varepsilon_{\mathrm{nuc}} \approx \frac{2.4 \times 10^{-3} \rho X_{\mathrm{H}}^2}{T_9^{2/3}} \exp\left(-\frac{3.380}{T_9^{1/3}}\right) \mathrm{W\,kg^{-1}}. \tag{4}$$

In this equation, T_9 means temperature in units of GigaKelvin; for example, if $T = 10^7\,\mathrm{K}$, then $T_9 = 0.01$. The density ρ is in units of $\mathrm{kg\,m^{-3}}$. The mass fraction of hydrogen is denoted by X_{H}.

Stars with mass $M \lesssim 0.3\,M_\odot$ are fully convective. As a result, the pressure, temperature, and density throughout the star lie on an adiabat:

$$P = P_c \left(\frac{\rho}{\rho_c}\right)^\gamma, \tag{5}$$

with $\gamma = 5/3$. Here P_c and ρ_c are the pressure and density at the center of the star. We can invert eq. (5) to obtain an equation for the density,

$$\rho = \rho_c \left(\frac{P}{P_c}\right)^{1/\gamma}. \tag{6}$$

To get an equation for the temperature, we write the ideal gas equation of state as

$$\frac{\rho}{\rho_c} = \frac{P}{P_c}\frac{T_c}{T}, \tag{7}$$

substitute this into eq. (5), and solve for T:

$$T = T_c \left(\frac{P}{P_c}\right)^{1-1/\gamma}. \tag{8}$$

Hence, if we know the central pressure P_c, the central density ρ_c, and the central temperature T_c, we can get the density and temperature for any pressure using these adiabatic relations.

THUS, IN ORDER TO MODEL THE STRUCTURE OF A LOW-MASS, MAIN-SEQUENCE STAR, we solve the following equations:

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}m} &= \frac{1}{4\pi r^2 \rho} && (9)\\ \frac{\mathrm{d}P}{\mathrm{d}m} &= -\frac{Gm}{4\pi r^4} && (10)\\ \frac{\mathrm{d}L_{\mathrm{nuc}}}{\mathrm{d}m} &= \varepsilon_{\mathrm{nuc}} && (11)\end{aligned}$$

supplemented with adiabatic relations for the density and temperature,

$$\begin{aligned} \rho &= \rho_c \left(\frac{P}{P_c}\right)^{1/\gamma} && (12)\\ T &= T_c \left(\frac{P}{P_c}\right)^{1-1/\gamma}, && (13)\end{aligned}$$

and the heating rate,

$$\varepsilon_{\mathrm{nuc}} \approx \frac{2.4 \times 10^{-3} \rho X_{\mathrm{H}}^2}{T_9^{2/3}} \exp\left(-\frac{3.380}{T_9^{1/3}}\right) \mathrm{W\,kg^{-1}}. \tag{14}$$

THE CENTRAL PRESSURE, DENSITY, AND TEMPERATURE OBEY THE VIRIAL SCALINGS, $P_{\mathrm{c}} \propto GM^2/R^4$, $\rho_{\mathrm{c}} \propto 3M/(4\pi R^3)$, $T_{\mathrm{c}} \propto (GM/R)(\mu m_{\mathrm{u}}/k_{\mathrm{B}})$, but to use these relations, we need to know the numerical coefficients. Fortunately, we do know them for the case $P \propto \rho^{5/3}$; because the pressure scales with density the same as in a cold white dwarf star, we can use the results of the previous project. In that project, you should have found that

$$P_{\mathrm{c}} = 0.77\frac{GM^2}{R^4}, \tag{15}$$

$$\rho_{\mathrm{c}} = 5.99\left(\frac{3M}{4\pi R^3}\right) \tag{16}$$

You can verify that for an equation of state $P = K\rho^{5/3}$, the coefficients 0.77 and 5.99 are independent of K. As a result, since the pressure along an adiabat is $\propto \rho^{5/3}$, a fully convective star's central pressure and density will obey equations (15) and (16). Further, we can combine the equations for pressure and density to get an expression for the central temperature:

$$\begin{aligned} T_{\mathrm{c}} &= \frac{\mu m_{\mathrm{u}}}{k}\frac{P_{\mathrm{c}}}{\rho_{\mathrm{c}}} = \frac{0.77 \times 4\pi}{5.99 \times 3}\left(\frac{\mu m_{\mathrm{u}}}{k_B}\right)\left(\frac{GM}{R}\right) \\ &= 0.54\left(\frac{\mu m_{\mathrm{u}}}{k_B}\right)\left(\frac{GM}{R}\right). \end{aligned} \tag{17}$$

Equations (15)–(17) specify P_{c}, ρ_{c}, and T_{c} for a given mass, radius, and mean molecular weight.

LOWER-MASS PRE–MAIN-SEQUENCE STARS HAVE NEARLY CONSTANT SURFACE EFFECTIVE TEMPERATURES AS THEY CONTRACT. Table 1 gives theoretical effective temperatures (Chabrier et al. 2000) for several low-mass "zero-age main-sequence" (ZAMS) stars. We can use this to compute the luminosity from the surface for a given radius:

$$L = 4\pi R^2 \sigma T_{\mathrm{eff}}^4. \tag{18}$$

Suppose we start at some large radius, $R = 2R_\odot$, say. As the star contracts, the surface luminosity L decreases (less surface area) while the nuclear luminosity increases (T_c increases as R gets smaller). There will therefore be a radius when $L_{nuc} = L$, and at this radius the star stops contracting and joins the main-sequence.

Table 1: Effective temperatures for low-mass ZAMS stars, taken from Chabrier et al. (2000).

$M/M_\odot$	0.10	0.15	0.20	0.30
T_{eff}/K	2 800	3 150	3 300	3 400

> *Your mission is to find the radius at which $L_{nuc} = L$ for different masses, and hence determine the zero-age main-sequence for low-mass stars.*

Much of this project involves refactoring the routines you wrote for modeling the white dwarf to handle a different equation of state and the equation for the luminosity.

2 Instructions

1. Verify that
$$P_c = 0.77\frac{GM^2}{R^4}, \qquad \rho_c = 5.99\frac{3M}{4\pi R^3}. \tag{19}$$
You should have found these scalings when you made the table of white dwarf properties (part 2.7 of the instructions). First, make sure your values of δ_m, ξ, and η are sufficiently small that the coefficients agree to 2 significant digits across a range of masses. Then rerun your code with a different constant K in the equation of state $P = K\rho^{5/3}$—for example, multiply K by 0.5 and 2. You should find that eq. (19) remains satisfied.

2. Write a routine to compute the mean molecular weight for a fully ionized plasma. This routine is already started in the file `eos.py`,

```
def mean_molecular_weight(Z,A,X):
    """Computes the mean molecular weight for a fully ionized plasma with an
    arbitrary mixture of species

    Arguments
```

```
        Z, A, X (either scaler or array)
            charge numbers, atomic numbers, and mass fractions
            The mass fractions must sum to 1
    """
    Zs = np.array(Z)
    As = np.array(A)
    Xs = np.array(X)
    assert np.sum(Xs) == 1.0
```

I recommend using the Numpy intrinsic routine `sum`; an example is in `examples/sample_sum.py`.

When you have written the routine, test it.

We are computing stars at the start of their hydrogen burning phase—the so-called "Zero-Age Main-Sequence" (ZAMS). As a result, we can assume that their composition mirrors that of the current interstellar gas. A good approximation is to take the composition to be ^{1}H, ^{4}He, and ^{14}N, with mass fractions $X_{\mathrm{H}} = 0.706$, $X_{\mathrm{He}} = 0.275$, and $X_{\mathrm{N}} = 0.019$.

3. Write a routine that computes the density and temperature along an adiabat, equations (12) and (13). Given P, P_c, ρ_c, and T_c it should return ρ and T. A starter routine is in `eos.py`,

```
def get_rho_and_T(P,P_c,rho_c,T_c):
    """
    Compute density and temperature along an adiabat of index gamma given a
    pressure and a reference point (central pressure, density, and temperature).

    Arguments
        P (either scalar or array-like)
            value of pressure

        P_c, rho_c, T_c
            reference values; these values should be consistent with an ideal
            gas EOS

    Returns
        density, temperature
    """
```

When you have written the routine, test it.

4. Write a routine that computes $\varepsilon_{\mathrm{nuc}}(\rho, T, X_H)$ according to eq. (14). A starter routine is in `reactions.py`,

```
def pp_rate(T,rho,XH,pp_factor=1.0):
    """
    Specific heating rate from pp chain hydrogen burning. Approximate rate
    taken from Hansen, Kawaler, & Trimble.

    Arguments
        T, rho
            temperature [K] and density [kg/m**3]
        XH
            mass fraction hydrogen
        pp_factor
            multiplicative factor for rate
    Returns
        heating rate from the pp-reaction chain [W/kg]
    """
```

   Notice that we have added an additional factor `pp_factor` that has default value 1.0. This factor should multiply the rate; we'll use it to see how the star responds if the reaction rate is increased (see § 3).

   When you have written the routine, test it.

5. Write a routine that computes the central pressure, density, and temperature given a stellar mass and radius. This routine has been started, in `structure.py`,

```
def central_thermal(m,r,mu):
    """
    Computes the central pressure, density, and temperature from the polytropic
    relations for n = 3/2.

    Arguments
        m
```

   When you have written the routine, test it.

6. Copy your routines `stellar_derivatives`, `lengthscales`, `central_values` and `integrate` from the white dwarf project into the file `structure.py`, and modify them for this case.

(a) Add equation (11) to your existing stellar structure equations, eq. (9) and (10).

(b) When computing the stepsize, you will now need to include the scale $H_L \equiv L/|\mathrm{d}L/\mathrm{d}m|$.

(c) The boundary condition for L is that $L(m \to 0) \to 0$. Since you are starting slightly off-center at δ_m, you can set

$$L(\delta_m) = \varepsilon_{\rm nuc}(\rho = \rho_{\rm c}, T = T_{\rm c})\, \delta_m.$$

(d) The integration routine should return arrays of m, r, P, and L throughout the interior.

7. Test the convergence of your integration by varying the parameters ξ, η, and δ (see project two) to ensure that your integration has the desired precision.

8. Write a routine to interpolate the effective temperatures from Table 1. A starter routine is in `zams.py`,

```
def Teff(Mwant):
    """
    Interpolates effective temperatures given mass from tabulated [1] values
    for low-mass stars.

    [1] Chabrier, Baraffe, Allard, and Hauschildt. Evolutionary Models for Very
    Low-Mass Stars and Brown Dwarfs with Dusty Atmospheres. Astrophys. Jour.
    542:464--472, Oct. 2000.

    Parameters
        Mwant (float, scalar or array)
            Mass of the star(s) in units of solar masses
    Returns

       Teff (float, same type/shape as Mwant)
            Interpolated effective temperatures of the stars in Kelvin.
    """
```

I recommend using the Numpy routine `interp`, which linearly interpolates a table. An example is in `examples/sample_interpolation.py`. In addition you should write a routine to compute the surface luminosity given an effective temperature and radius. A starter routine for this is also in `zams.py`,

```
def surface_luminosity(Teff,radius):
    """
    Photospheric luminosity [W]

    Arguments
        Teff [K]
        radius [m]
    """
```

When you have written these routines, test them using the appropriate routines in tests.

9. Write a function that computes $L_{\mathrm{nuc}}(R) - 4\pi R^2 \sigma_{\mathrm{SB}} T_{\mathrm{eff}}^4$. Here $L_{\mathrm{nuc}}(R)$ is computed by integrating over the star's structure, and T_{eff} is found by interpolating from the table of masses and effective temperatures. The root of this equation is the star's main-sequence radius. Solve for this radius using a rootfind routine, either `scipy.optimize.bisect` or its more efficient cousin `scipy.optimize.brentq`.

10. Find the main-sequence radii, and hence find L and T_{eff}, for several masses in the range $0.1\,M_\odot \le M \le 0.3\,M_\odot$.

    For this set of main-sequence stars, make two plots:

    (a) $\log(L/L_\odot)$ against $\log(T_{\mathrm{eff}}/\mathrm{K})$. You will need to have the axis $\log(T_{\mathrm{eff}}/\mathrm{K})$ running "backwards" so that the plot resembles a Hertzsprung-Russell diagram.

    (b) $\log(T_c/\mathrm{K})$ against $\log(\rho_c/\mathrm{g\,cm^{-3}})$.

    Compare these plots against Figures 15 and 16 (*triangles*) in Paxton et al. (2011). *NB. The figures in Paxton et al. (2011) are in CGS units.*

11. Plot $T(r)$, $T(m)$, $L(r)$, and $L(m)$ for a star with $M = 0.3\,M_\odot$. At what radius does $L(r)$ reach 90% of its final value? What is the fraction of the star's mass enclosed by this radius?

3 Exploration—what would happen if?

Suppose the weak interaction were, say, 10^5 times stronger. This would make our universe a quite different place. Let's find out how it would affect stars. The slowest reaction in the PP chain is

$$^1\mathrm{H} + {}^1\mathrm{H} \to {}^2\mathrm{H} + e^+ + \nu_e,$$

so this increase in the weak interaction would make $\varepsilon_{\mathrm{nuc}}$ increase by the same factor, 10^5. Recompute your main-sequence stars with this increased reaction rate. (We can do this by setting the parameter `pp_factor` when calling `pp_rate`.) How do the stars change? Speculate: if the sun changed in a similar fashion to these lower-mass stars, how would surface temperatures on Earth change? Could we still be here?

References

G. Chabrier, I. Baraffe, F. Allard, and P. Hauschildt. Evolutionary Models for Very Low-Mass Stars and Brown Dwarfs with Dusty Atmospheres. *ApJ*, 542: 464–472, October 2000. doi: 10.1086/309513.

Bill Paxton, Lars Bildsten, Aaron Dotter, Falk Herwig, Pierre Lesaffre, and Frank Timmes. Modules for experiments in stellar astrophysics (MESA). *ApJS*, 192:3, January 2011.