GEOEEM WS 15/16

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Contents

0 Introduction

Common to each method is the fact that the current flow is used in the subsurface. The aim is the determination of the conductivity distribution of the subsurface from the Earth's surface down to several 100 km depth.
Application areas:

- Near surface exploration (0 - 300 m depth):
  - Application for the environment: Waste site exploration, search for suitable landfill sites, ...
  - Groundwater exploration
  - Archaeology
  - Exploration for deposits, engineering applications (e.g. cativity detection,...)
- Exploration of deep structures (> 300 m)
  - Geothermal fields, oil and gas exploration
  - tectonic questions, shear zones
  - deep crust and upper mantle

0.1 Classification of methods

Classifications possible as:

- According to the source (artificial or natural)
- Inclusion of magnetic field or not?
- Direct current or alternating current?

DC-resistivity methods: Direct current resistivity (DC), Induced polarization (IP), Self potential (SP)

Electromagnetic methods:

- *Frequency domain*: Magnetotellurics (MT), Audiomagnetotellurics (AMT), Controlled source AMT (CSAMT), Radiomagnetotellurics (RMT)
- *Time domain:* Transient electromagnetics (TEM), Long offset transient electromagnetics (LOTEM)

Electromagnetic methods using high frequencies ($f > 10$ MHz): Ground penetrating radar (GPR)

1 Conductivity

The conductivity σ of the minerals in the nature covers a range of 25 decades! For example:

$$10^{-18} S/m \rightarrow \text{Diamand}$$
$$10^{7} S/m \rightarrow \text{Copper}$$

Instead of the conductivity, the resistivity $\rho = \frac{1}{\sigma} \Omega m$ is often used.

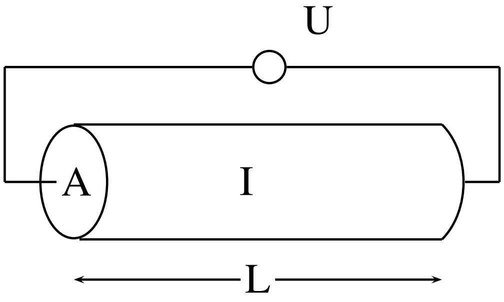

Figure 1.1: Schematic derivation of Ohm's law

Definition: Ohm's law

Let us consider a rock sample of length L, resistivity ρ and cross section A.
A current $I[A]$ flows by applying a voltag $U[V]$ to the rock sample:

$$I = \frac{AU}{\rho L}$$

$$\Leftrightarrow \rho \underbrace{\frac{I}{A}}_{\text{current density } j} = \underbrace{\frac{U}{L}}_{\text{electric Field } E}$$

$$\vec{j}\rho = \vec{E} \tag{1.1}$$

We measure I and U, A and L are known, so we can calculate ρ.

1.1 Mechanisms of electrical conductivity

Metallic conductivity: Current flows by free electrons $\rho \equiv T$

Electrolytic conductivity: Charge carriers are cations and anions: ρ decreases with temperature T.

Semi-conductors: Charge carriers must be activated by heat, light or EM-radiation. Strongly dependent on temperature T. Important for mantle (deep earth structures)

Boundary layer conductivity: Occurs due to the interaction of the pore liquid with the rock matrix. This is the source of SP-anomalies!

2 DC-resistivity method

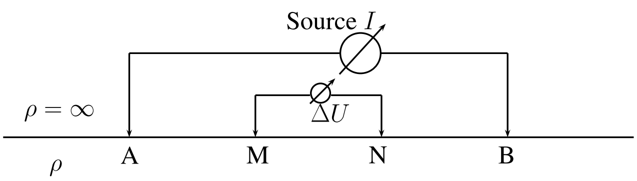

Figure 2.1: Four point measurement

Resistivity ρ of the subsurface derived from I (which is known), ΔU (which is measured) and the geometrical factor K (which is also known).

Frequently used electrode arrays

Industrial standard of measuring is via an *Multielectrode array*.

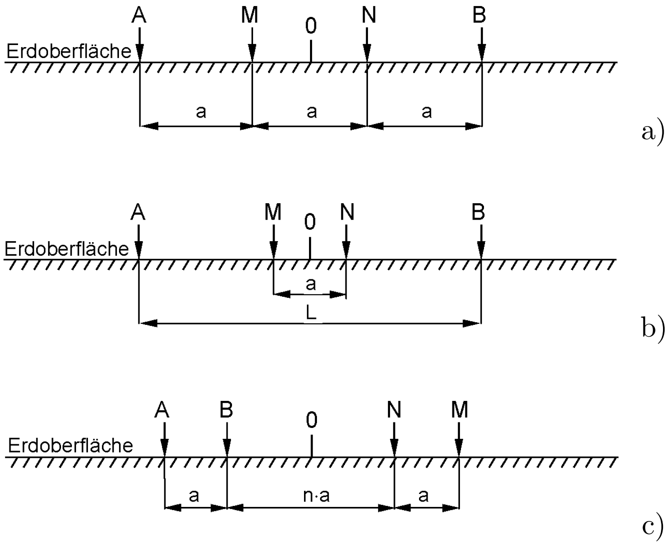

Figure 2.2: a) Wenner, Half-Wenner; b) Schlumberger, Half-Schlumberger; c) Dipole-dipole, source???

2.1 Basic equations of DC-resistivity

The first assumption of DC-resistivity methods and the major difference to EM-methods is the assumption of stationary currents:

$$\frac{\partial}{\partial t} = 0$$

The fields do not depend on time.
Looking at the *Maxwell's equations*:

$$\nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t} = 0 \tag{2.1}$$

This means irrotational electric field and from that follows, that the electric field vector can be derived by a scalar potential:

$$\vec{E} = -\nabla V \tag{2.2}$$

Insert equation (2.2) into eq. (1.1):

$$\vec{j} = -\sigma \nabla V \tag{2.3}$$

Continuity equation:

$$\nabla \cdot \vec{j} + \frac{\partial q}{\partial t} = 0 \tag{2.4}$$

Now new charges are generated in the course of time

$$\nabla \cdot \vec{j} = 0 \tag{2.5}$$

which is valid outside of the source.
If we insert eq. (2.3) into (2.5):

$$-\nabla \cdot (\sigma \nabla V) = 0$$
$$\nabla \sigma \nabla V + \sigma \nabla^2 V = 0$$

$\nabla \sigma = 0$ for areas with constant conductivity, so:

$$\nabla^2 V = 0 \tag{2.6}$$

which is called the *Laplace-equation*, the basic equation of DC-resistivity.
Derivation of solutions of this elliptic partial differential equation using different boundary conditions: Assume a current source with strength I at point $\vec{r}_0$, then the spatial current distribution can be given as:$\nabla \cdot \vec{j} = I\delta(\vec{r} - \vec{r}_0$ and so:

$$\nabla \cdot (\sigma \nabla V) = -I\delta(\vec{r} - \vec{r}_0) \tag{2.7}$$

This equation can be solved numerically for arbitrary distribution of conductivity ratio.

2.1.1 Potential of a current electrode

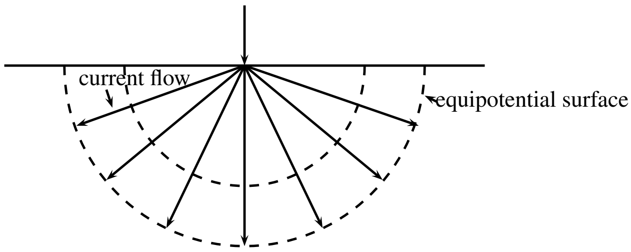

Figure 2.3: Single current source

Using *Ohm's law*: $\vec{E} = \rho\vec{j} = \rho\frac{I}{2\pi r^2}$, where $2\pi r^2$ is the surface of the half sphere. Using $E = -\frac{dV}{dr}$ follows the potential of a homogeneous half space:

$$V = \frac{\rho I}{2\pi r} \tag{2.8}$$

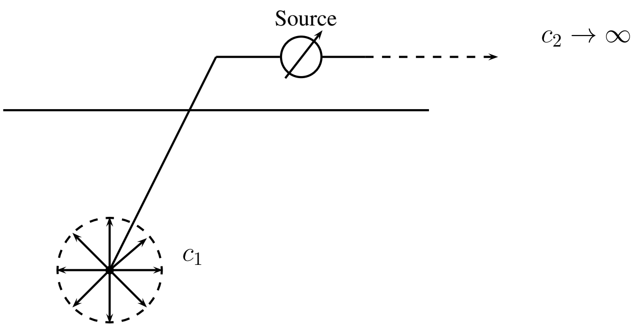

Figure 2.4: Mise-à-la-Masse method

In the case of the *Mise-à-la-Masse method* the potential of the homogeneous full space is:

$$V = \frac{\rho I}{4\pi r} \tag{2.9}$$

The same result can be derived by using the Laplace-equation (2.6) and the use of spherical coordinates:

$$\nabla^2 V = \frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr}$$

From the symmetry of the system the potential is a function of the distance to the source r only. Multiplying by r^2 and integrating, we get:

$$\frac{dV}{dr} = \frac{c_1}{r^2}$$

Integrating over r again leads to the solution:

$$V = -\frac{c_1}{r} + c_2 \qquad\qquad c_1, c_2 = const.$$

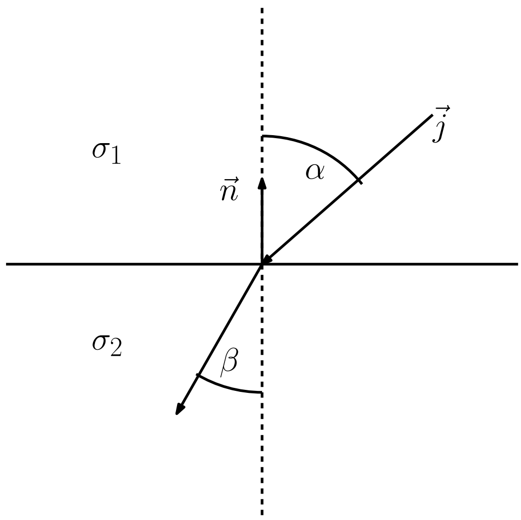

Figure 2.5: Boundary with dip angles GEOING s5

To determine the constants we have to use boundary conditions: From $\lim_{r\to\infty} V(r) = 0$ follows that $c_2 = 0$. Using the current density: $j = \frac{I}{A} \Leftrightarrow I = jA$:

$$I = 4\pi r^2 j = -4\pi r^2 \sigma \frac{dV}{dr} = -4\pi\sigma c_1$$

From this equation we can derive c_1:

$$V = \frac{I\rho}{4\pi r} \tag{2.10}$$

Boundary equations

Boundary with different conductivities.
Two boundary conditions which must hold at any contact between two regions of different conductivity.

- Potential is continuous across the boundary
- j_n is also continuous.

$$V^1 = V^2, \quad \left(\frac{\partial V}{\partial x}\right)^1 = \left(\frac{\partial V}{\partial x}\right)^2, \quad j_n^1 = j_n^2$$

$$E_t^1 = E_t^2, \quad \sigma_1 E_n^1 = \sigma_2 E_n^2$$

$$\sigma_1 \frac{E_n^1}{E_t^1} = \sigma_2 \frac{E_n^2}{E_t^2}$$
$$\sigma_1 \cot\alpha = \sigma_2 \cot\beta$$
$$\frac{\tan\alpha}{\tan\beta} = \frac{\sigma_1}{\sigma_2}$$

Current line is bent towards to the normal if the resistivity of the second medium ρ_2 is larger than the one of the first medium ρ_1.

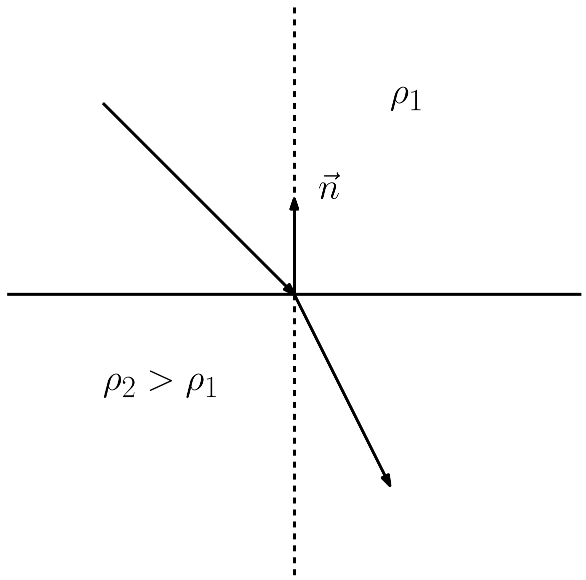

Figure 2.6: Bending towards normal

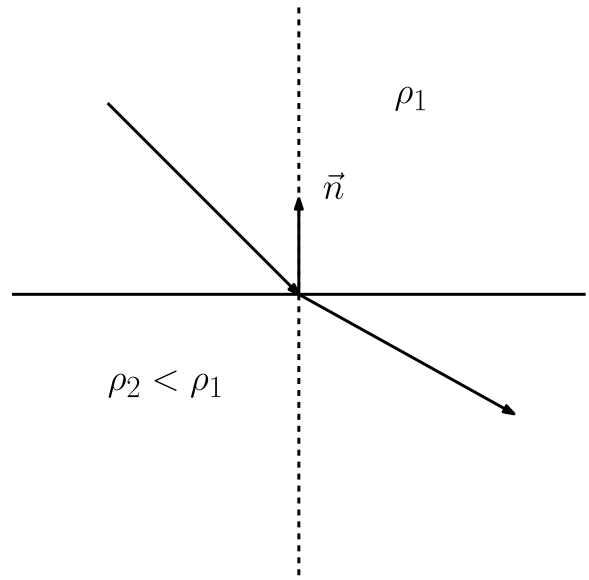

Figure 2.7: Bending away from normal

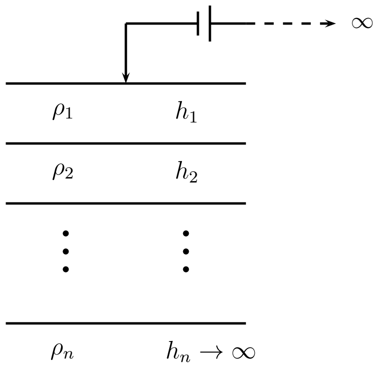

Figure 2.8: Model of n layer structure

2.1.2 Potential distribution at the surface of a horizontally stratified earth (Solution of the Laplace equation (2.6))

Starting with a *model*:
The subsurface consists of finite number of layers with the last layer having infinite layer thickness ($h_n \to \infty$). We assume that ρ_i is isotropic (no dependence of the direction of measurement). The field is generated by a point source with the current I is a direct current.
Starting from the Laplace equation with potential V:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{2.11}$$

In cylindrical coordinates (r, θ, z):

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^V}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0 \tag{2.12}$$

The solution is symmetrical to the vertical axis, so $\frac{\partial V}{\partial \theta} = \frac{\partial^2 V}{\partial \theta^2} = 0$, so $V(r, \theta, z) = V(r, z)$. So the Laplace equation to be solved reduces to:

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^V}{\partial z^2} = 0 \tag{2.13}$$

Solution of (2.13). Ansatz:

$$V(r,z) = U(r)W(z) \tag{2.14}$$

So the solution is the product of a function of r alone and a function of z alone. We substitute (2.14) into (2.13) and multiply all terms with $1/UW$:

$$\underbrace{\frac{1}{UW}\frac{d^2U}{dr^2} + \frac{1}{UW}\frac{DU}{dr}}_{\text{depends on } r} + \underbrace{\frac{1}{W}\frac{d^2W}{dz^2}}_{\text{depends on } z} = 0 \tag{2.15}$$

This equation is satisfied, if

$$\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{Ur}\frac{DU}{dr} = -\lambda^2 \tag{2.16}$$

$$\frac{1}{W}\frac{d^2W}{dz^2} = \lambda^2 \tag{2.17}$$

where λ is a real constant.

Solution of (2.17)

Using the Ansatz:

$$W = Ce^{-\lambda z} \quad , \quad W = Ce^{\lambda z} \tag{2.18}$$

where C and λ are arbitrary constants.

Solution of (2.16)

Using the Ansatz:

$$U = CJ_0(\lambda r) \tag{2.19}$$

with $J_0(\lambda r)$ the *Bessel-function* of order zero.

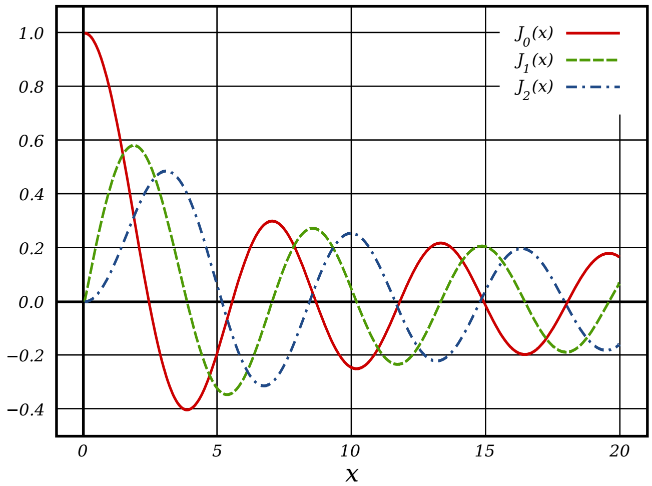

Figure 2.9: Bessel-functions source: $https://de.wikipedia.org/wiki/Besselsche_Differentialgleichung$

We combine the two solutions ((2.18) and (2.19)) for the solution of (2.13):

$$V = Ce^{-\lambda z}J_0(\lambda r) \quad , \quad V = Ce^{\lambda z}J_0(\lambda r) \tag{2.20}$$

λ varies from 0 to ∞ and C varies in dependence of λ. Than we write a general solution of the potential (2.13):

$$V = \int_0^\infty \left(\phi(\lambda)e^{-\lambda z} + \psi(\lambda)e^{\lambda z}\right) J_0(\lambda r)d\lambda \tag{2.21}$$

Where $\phi(\lambda)$ and $\psi(\lambda)$ are arbitrary functions of λ.

Potential of homogeneous halfspace

Starting of with the potential in cylindrical coordinates:

$$V = \frac{I\rho}{2\pi\sqrt{r^2 + z^2}} \tag{2.22}$$

Looking at the *Lipschitz-Integral*:

$$\int_0^\infty e^{-\lambda z} J_0(\lambda r) d\lambda = \frac{1}{\sqrt{r^2 + z^2}} \tag{2.23}$$

Now using (2.23) we write (2.22) as:

$$V = \frac{\rho_1 I}{2\pi} \int_0^\infty e^{-\lambda z} J_0(\lambda r) d\lambda \tag{2.24}$$

The general solution (2.21) can now be written:

$$V = \frac{\rho_1 I}{2\pi} \int_0^\infty \left(e^{-\lambda z} + \theta(\lambda) e^{-\lambda z} + X(\lambda) e^{\lambda z} \right) J_0(\lambda r) d\lambda \tag{2.25}$$

Where $\theta(\lambda)$ and $X(\lambda)$ are arbitrary functions of λ, and $\phi(\lambda) = \frac{\rho_1 I}{2\pi} (1 + \theta(\lambda))$ and $\psi(\lambda) = \frac{\rho_1 I}{2\pi} X(\lambda)$. The solutions of the form (2.25) are valid in all layers but $\theta(\lambda)$ and $X(\lambda)$ can be different for each layer i:

$$V_i = \frac{\rho_1 I}{2\pi} \int_0^\infty \left(e^{-\lambda z} + \theta_i(\lambda) e^{-\lambda z} + X_i(\lambda) e^{\lambda z} \right) J_0(\lambda r) d\lambda \tag{2.26}$$

Adaption of the solution to the boundary conditions

Assuming we are at the layer boundaries of $z = h_i$.

A) Potential (2.26) is continious at each boundary plane in the subsurface:

$$V_i(r, h_i) = V_{i+1}(r, h_i) \tag{2.27}$$

This equation can only be satisfied if the integrands on both sides are equal:

$$\theta_i(\lambda) e^{-\lambda h_i} + X_i(\lambda) e^{\lambda h_i} = \theta_{i+1}(\lambda) e^{-\lambda h_i} + X_{i+1}(\lambda) e^{\lambda h_i} \tag{2.28}$$

B) At each boundary plane j_z the boundary condition must be fulfilled that:

$$j_z = -\frac{1}{\rho} \frac{\partial V}{\partial z} \tag{2.29}$$

and so

$$\frac{1}{\rho_i} \left((1 + \theta_i(\lambda)) e^{\lambda h_i} - X_i(\lambda) e^{\lambda h_i} \right) = \frac{1}{\rho_{i+1}} \left((1 + \theta_{i+1}(\lambda)) e^{\lambda h_i} - X_{i+1}(\lambda) e^{\lambda h_i} \right) \tag{2.30}$$

To satisfy this condition we differentiate the expression for the potential in the first layer (2.22) with respect to z and then substitute $z = 0$:

$$\frac{1}{\rho_1} \frac{\partial V_1(r, 0)}{\partial z} = 0 \quad , \text{ for } \quad r \neq 0 \tag{2.31}$$

We thus obtain the equation:

$$\int_0^\infty (-1 - \theta_1(\lambda) + X_1(\lambda)) J_0(\lambda_r) d\lambda = 0 \tag{2.32}$$

$$\Rightarrow \theta_1(\lambda) = X_1(\lambda) \tag{2.33}$$

C) Near the current source the potential must approach to infinity

$$V_\infty = \frac{\rho I}{2\pi}\frac{1}{\sqrt{r^2+z^2}}$$

which is approaching asymtotically to the potential for a layer extending to infinite height.

D) $V \to 0$ if $z \to \infty$

$$\Rightarrow X_n = 0 \tag{2.34}$$

, because otherwise $e^{\lambda z}$ would drive the potential to an infinite value at an infinite depth.

The set of equations (2.28) - (2.34) provides a system of $2n$ equation in $2n$ unknown functions $\theta(\lambda)$ and $X(\lambda)$. To obtain the solution subsitute (2.33) into (2.28) and (2.30) and subsitute (2.34) into (2.28) and (2.30).

For brevity, we introduce the notations:

$$u_i = e^{\lambda h_i}, v_i = \frac{1}{u_i}, p_i = \frac{\rho_i}{\rho_{i+1}}$$

The system of equations then become:

$$\begin{aligned}
(u_1+v_1)\theta_1 - u_2\theta_2 - v_2X_2 &= 0\\
(v_1-u_1)\theta_1 + p_1u_1\theta_2 - p_1v_1X_2 &= (1-p_1)u_1\\
\vdots \qquad\qquad & \qquad \vdots\\
u_{n-1}\theta_{n-1} + v_{n-1}X_{n-1} - u_{n-1}\theta_n &= 0\\
-u_{n-1}\theta_{n-1} + v_{n-1}X_{n-1} + p_{n-1}u_{n-1}\theta_n - p_{n-1}v_{n-1}X_n &= (1-p_{n-1})u_{n-1}
\end{aligned}$$

Solution of the equations by applying *Cramer's rule*. For example: Solution of a two layer case (layer 1: ρ_1, h_1, layer 2: ρ_2):

$$\theta_1 = \frac{ku}{1-ku} \qquad \theta_2 = \frac{k(1+u)}{1-ku}$$

$$X_1 = \theta_1 \qquad X_2 = 0$$

with $u = e^{-2\lambda h_1}$ and the *reflection coefficient of DC* $k = \frac{\rho_2-\rho_1}{\rho_2+\rho_1}$

Interesting is the potential at the surface of the earth, with $z = 0$ and (2.33):

$$V_0 = V_1(r,z) = \frac{\rho_1 I}{2\pi}\int_0^\infty (1+2\theta_1)\, J_0(\lambda r)d\lambda \tag{2.35}$$

$$= \frac{\rho_1 I}{2\pi}\int_0^\infty K(\lambda)J_0(\lambda r)d\lambda \tag{2.36}$$

where $K(\lambda)$ is the *Slichter-function*.

We consider the Lipschitz-integral:

$$\int_0^\infty e^{-\lambda z}J_0(\lambda r)d\lambda = \frac{1}{\sqrt{r^2+z^2}} \overset{i=0}{=} \int_0^\infty J_0(\lambda r)d\lambda = \frac{1}{r} \tag{2.37}$$

(2.26) can now be written in the form:

$$V_0(r) = \frac{I}{2\pi}\Bigg(\underbrace{\frac{\rho_1}{r}}_{\text{first layer}} + \int_0^\infty (T(\lambda)-\rho_1)J_0(\lambda r)d\lambda\Bigg) \tag{2.38}$$

with $T(\lambda) = \rho_1(1+2\theta_1(\lambda))$ Example/reminder for four point measurement:

$$V_1 = \frac{I\rho}{2\pi}\left(\frac{1}{AM} - \frac{1}{BM}\right)$$

2.1.3 Derivation of a formula for the apparent resistivity

Take an arbitrary DC-Array (compare Fig. 2.1). Then

$$\Delta U = \frac{I\rho}{2\pi}\left(\frac{1}{AM} - \frac{1}{AN} - \frac{1}{BM} + \frac{1}{BN}\right)$$

$$\rho_a = k\frac{\Delta U}{I}$$

where k is the geometrical factor. If we look at experimental data with an error of 1% for the distances between the electrodes, the error in ρ_a would be 2%. But 10% error in the lateral direction of the electrodes results only in 1% error in ρ_a.
In case of the Schlumberger array ($L = AM + MN/2$ and $a = MN$, $a \ll L$) we get a voltage decrease in U:

$$\begin{aligned} U &= 2\left(V_0(\frac{L}{2} - \frac{a}{2}\right) - V_0\left(\frac{L}{2} + \frac{a}{2}\right) \\ &\approx -2a\frac{\partial V_0}{\partial r}\bigg|_{r=L/2} \end{aligned}$$

and the geometrical factor in case of Schlumberger $k = \frac{\pi}{a}\left(\left(\frac{L}{2}\right)^2 - \left(\frac{a}{2}\right)^2\right)$

$$\rho_a(L/2) = K\frac{U}{I} = \frac{2\pi}{I}\left(\frac{L}{2}\right)^2\frac{\partial V_0}{\partial r}$$

with $\frac{d}{dx}J_0(x) = -J_1(x)$. From eq. 2.25!!!!:

$$\rho_a(L/2) = \rho_1 + \left(\frac{L}{2}\right)^2 \underbrace{\int_0^\infty (T(\lambda) - \rho_1)J_1(\lambda L/2)\lambda d\lambda}_{\text{Stefanescu-Integral}} \tag{2.39}$$

The calculations of the model response $\rho_a(L/2)$ from given model parameters (ρ_i, h_i) is a forward problem.
Given:

Figure 2.10: Given parameters

Now two steps are neccessary:

- Calculation of $T(\lambda)$
- Integration of (2.39) $\rightarrow$ Stefanescu-Integral

2.1.4 Calculation of the resistivity transform $T(\lambda)$

For a method for the determination of $T(\lambda)$ see chapter 2.1.2. Then we calculate the solution of the equation system θ_i and X_i and determine $T(\lambda)$ using $T(\lambda) = \rho_1(1 + 2\theta_1(\lambda))$. This procedure is too time consuming for a high number of layers.
Now we derive a recursion formula using the boundary conditions A to D from 2.1.2. At first a new definition:

$$T_i(\lambda) = \rho_1\frac{1 + \theta_1(\lambda) + X_i(\lambda)e^{2\lambda t_i - 1}}{1 + \theta_1(\lambda) - X_i(\lambda)e^{2\lambda t_i - 1}} \tag{2.40}$$

with $i = 1, 2, \ddots, n$, $t_i = h_1 + h_2 + \ddots + h_i$ and $t_0 = 0$.
Because of (2.33): $\theta_1(\lambda) = X_1(\lambda)$, we get: $T_1(\lambda) = T(\lambda)$.
Because of (2.34): $X_n = 0$, we get: $T_n(\lambda) = \rho_n$.

From the boundary conditions (2.28) and (2.30):

a) $\theta_i(\lambda)e^{-\lambda t_i} + X_i(\lambda)e^{\lambda t_i} = \theta_{i+1}(\lambda)e^{-\lambda t_i} + X_{i+1}(\lambda)e^{\lambda t_i}$

b) $\frac{1}{\rho_i}\left((1+\theta_i(\lambda))\,e^{-\lambda t_i} - X_i(\lambda)\right) = \frac{1}{\rho_i}\left((1+\theta_{i+1}(\lambda))\,e^{-\lambda t_i} - X_{i+1}(\lambda)\right)$
The next steps are:

- Add $e^{-\lambda t_i}$ on both sides of (2.28)
- Divide each side over the corresponding side of (2.30)
- Cancel the left part of the new equation by $X_i(\lambda)$

Then:

$$\rho_i \frac{K_i(\lambda) + e^{2\lambda t_i}}{K_i(\lambda) - e^{2\lambda t_i}} = T_{i+1}(\lambda)$$

with $K_i(\lambda) = \frac{1+\theta_i(\lambda)}{X_i(\lambda)}$
Now the next step is to Solve the eq. (2.40), insert it and short translation and solving it for $T(\lambda)$:

$$T_i(\lambda) = \frac{T_{i+1} + \rho_i \tanh(\lambda h_i)}{1 + \frac{T_{i+1}(\lambda)}{\rho_i}\tanh(\lambda h_i)} \tag{2.41}$$

which is the *recursion formula of PEKERIS*.
Now start with $T_n = \rho_n$, calculate step by step $T_i(\lambda)$ until $T_1(\lambda) = T(\lambda) \rightarrow$ *resistivity transform*
Illustration of eq. (2.41):

$$\lambda = \frac{1}{L/2} \qquad\qquad T_n(\lambda) = \rho_n$$

Two extreme values:

$$L/2 \rightarrow 0 \qquad\qquad L/2 \rightarrow \infty$$

Two layer model

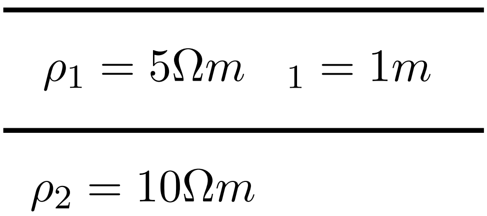

Figure 2.11: Two layer model

$T_n(\lambda) = T_2(\lambda) = 10\Omega m$

1. $L/2 \rightarrow 0 \Rightarrow \lambda \rightarrow \infty \Rightarrow \tanh(\infty) \rightarrow 1$

$$T(\lambda) = T_1(\lambda) = \frac{T_2 + \rho_1 \tanh(\lambda h_1)}{1 + \frac{T_2(\lambda)}{\rho_1}\tanh(\lambda h_1)} = \frac{10+5}{1+10/5} = 5\Omega m$$

2. $L/2 \to \infty \Rightarrow \lambda \to 0 \Rightarrow \tanh(0) \to 0$

$$T_1(\lambda) = \frac{10+0}{1+0} = 10\Omega m$$

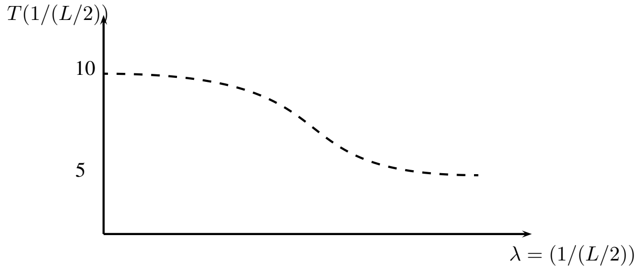

Figure 2.12: ??????

2.1.5 Solution of the Stefanescu-Integral

An analytical solution is not possible! One of the possibilities uses the linear filter method.

2.1.5.1 Basic Equations of the linear filter method: The fast HANKEL-transformation

The calculation of a function $g(r)$ from $\rho(\lambda)$ by

$$g(r) = \int_0^\infty \rho(\lambda) J_\nu(\lambda r) \lambda d\lambda \tag{2.42}$$

is defined as the *HANKEL-transformation*. It expresses any given function as the weighted sum of an infinite number of Bessel-functions.
The *inverse Hankel-transformation*:

$$\rho(\lambda) = \int_0^\infty g(r) r J_\nu(\lambda r) dr \tag{2.43}$$

Then the Stefanescu Integral has the form of a Hankel-transformation.
The following method to calculate the integral (2.42) is called the fast Hankel-transform. It provides function values for $g(r)$ at discrete points.
To solve this *four steps are neccessary*:
1) The variables are transformed into logarithmic values.

$$x = \ln(r/r_0) \qquad\qquad y = -\ln(\lambda/r_0)$$
$$\Rightarrow r = e^x \qquad\qquad \lambda = e^{-y}$$

with r_0 the reference length. Then:

$$r\lambda = e^{x-y} \qquad\qquad dr = rdx \qquad\qquad d\lambda = -\lambda dy$$

Insert this into eq. (2.42) and (2.43):

$$rg(r) = -\int_{-\infty}^{\infty} \rho(\lambda)\lambda J_\nu(e^{x-y})e^{x-y}dy$$

$\lambda \to 0, y \to \infty$ and $\lambda \to \infty, y \to -\infty$.

$$\lambda\rho(\lambda) = \int_{-\infty}^{\infty} g(r) r J_\nu(e^{x-y}) e^{x-y} dx$$

$r \to 0, x \to -\infty$ and $r \to \infty, x \to \infty$.
From this follow the *Convolution integrals*:

$$\begin{aligned} F(y) &= \int_{-\infty}^{\infty} G(x) H(x-y) dx \\ G(x) &= \int_{-\infty}^{\infty} F(y) H(x-y) dy \end{aligned} \tag{2.44}$$

The requirement for the fast Hankel transformation is *not* only that the integral (2.43) has the form of a Hankel transformation but it can be *transferred* to a *convolution integral* (2.44).

2) The function F is represented in the form (by using the sampling theorem):

$$F(y) = \sum_{j=-\infty}^{\infty} F(y_j) \operatorname{sinc}\left(\pi \frac{y - y_j}{\Delta y}\right) \tag{2.45}$$

with $\operatorname{sinc}(z) = \frac{\sin(z)}{z}$ and $y_j = y_0 + j\Delta y$ with y_0 arbitrary.
Sampling is the process of converting a signal into a numeric sequence. A band limited function can only be perfectly reconstructed from a countable sequence of samples, if the band limit B is not greater than half of the sampling rate. This leads to a formula for the reconstruction of the original function from it's samples:

$$\rho_{ny} = \frac{1}{2\Delta t}$$

3) Inserting (2.45) into (2.44) gives:

$$G(x) = \sum_{j=-\infty}^{\infty} c_j(x) F(y_j) \tag{2.46}$$

with $c_j(x) = \int_{-\infty}^{\infty} \operatorname{sinc}\left(\pi \frac{y-y_j}{\Delta y}\right) H(x-y) dy$.
The calculation of a general function is reduced to the transformation of a sinc function.

4) The calculation of $G(x)$ is limited to the calculation of function values at discrete points: $x_k = x_0 + k\Delta x$ with $k = ..., -1, 0, 1, ...$ and x_0 arbitrary and $\Delta x = \Delta y$.
Inserting in the equation of $c_j(x)$:

$$c_j(x_k) = c_0(x_k)$$

and so it follows:

$$G(x_k) = \sum_{j=-\infty}^{\infty} c_{k-j} F(y_j) \tag{2.47}$$

with $c_{k-j} = c_0(x_k - j)$.
That means only the coefficients c_k will be calculated from the funtion.

$$c_0(x) = \int_{-\infty}^{\infty} H(x-y) \operatorname{sinc}\left(\pi \frac{y - y_0}{\Delta y}\right) dy \tag{2.48}$$

at the points x_k. The transformation is thereby reduced to the transformation of a single sinc function. Two conclusions result from the properties of the *sinc response* for the application of the fast Hankel-transformation:

(a) Only c_k with values over a lower (k_n) and upper (k_0) limit are calculated:

$$k_n < c_k < k_0$$

The amount of c_k can be defined as a filter, so that (2.47) becomes:

$$G(x_k) = \sum_{j=k-k_0}^{k-k_n} c_{k-j} F(y_j) \tag{2.49}$$

(b) Due to the oscillations, zero values of $c_0(x)$ result at distances of Δx for large and small x values. By subtle choice of x_0 it can be reached, that values x_k will be close to zero points for large and small k and the function values dissapear. That means the filter will be shorter.

2.1.5.2 Calculation of a filter

There are several possibilities to calculate values of $c_0(x)$. The first paper was published by Ghosh (1971). There exists functions $F(y)$ for which the eq. (2.44): $G(x) = \int F(y)H(x-y)dy$ can be solved analytically and $G(x)$ can be calculated. For such cases we apply the Fourier-transformation.

$$\tilde{F}(k) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} F(y)e^{-iky}dy$$

and $\tilde{G}(k)$ analogous. Then (2.44) has the form of a convolution integral:

$$G(x) = F(y) \star H(x-y)$$
$$G(k) = F(k) \cdot H(k)$$

Also eq. eqref eq:2.34 has the form of a convolution integral:

$$c_j(x) = \int\limits_{-\infty}^{\infty} \text{sinc}\left(\pi \frac{y-y_j}{\Delta y}\right) H(x-y)dy \qquad y_0 = 0$$

Therefore:

$$\tilde{c}_0(k) = \tilde{H}(k) \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \text{sinc}\left(\pi \frac{y}{\Delta y}\right) e^{-kyi}dy$$

with

$$\frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \text{sinc}\left(\pi \frac{y}{\Delta y}\right) e^{-kyi}dy = \begin{cases} \frac{\Delta y}{2\pi} & , for |k| < \pi/\Delta y \\ 0 & , otherwise \end{cases}$$

follows:

$$\tilde{c}_0(k) = \begin{cases} \frac{\tilde{G}(k)}{\tilde{F}(k)} \frac{\Delta y}{2\pi} & , for |k| < \pi/\Delta y \\ 0 & , otherwise \end{cases}$$

2.1.5.3 Example of a resistivity filter

eq. 2.27

$$\rho_a(L/2) = \rho_1 + (L/2)^2 \int_0^\infty (T(\lambda) - \rho_1) J_1(\lambda L/2) \lambda d\lambda$$
$$x = \ln(L/2), y = -\ln(\lambda)$$

$L/2$ in m, $\lambda is 1/m$

$$G(x) = \int F(y) H(x-y) dy$$

with

$$G(x) = \rho_a(e^x) - \rho_1$$
$$F(y) = \rho_a(e^{-y}) - \rho_1$$
$$H(x) = J_1(e^x)$$

We can now calculate the resistivity filter $c_0(k) = G(k)/F(k)$ by applying the fast HT. Filter with 3 values per decade → Ghosh, with 6 values per decade → O'Neill, with 10 → Johansen.
For all filters:

$$\sum_{k=k_n}^{k_0} c_k = 1$$

Using eq. 2.37:

$$\rho_a(e^{x_k}) = \sum_{j=k-k_0}^{k_n} c_{k-j} T(e^{-y_j})$$