GEOEEM WS 15/16

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Contents

0 Introduction

Common to each method is the fact that the current flow is used in the subsurface. The aim is the determination of the conductivity distribution of the subsurface from the Earth's surface down to several 100 km depth.
Application areas:

- Near surface exploration (0 - 300 m depth):
  - Application for the environment: Waste site exploration, search for suitable landfill sites, ...
  - Groundwater exploration
  - Archaeology
  - Exploration for deposits, engineering applications (e.g. cativity detection,...)
- Exploration of deep structures (> 300 m)
  - Geothermal fields, oil and gas exploration
  - tectonic questions, shear zones
  - deep crust and upper mantle

0.1 Classification of methods

Classifications possible as:

- According to the source (artificial or natural)
- Inclusion of magnetic field or not?
- Direct current or alternating current?

DC-resistivity methods: Direct current resistivity (DC), Induced polarization (IP), Self potential (SP)

Electromagnetic methods:

- *Frequency domain*: Magnetotellurics (MT), Audiomagnetotellurics (AMT), Controlled source AMT (CSAMT), Radiomagnetotellurics (RMT)
- *Time domain:* Transient electromagnetics (TEM), Long offset transient electromagnetics (LOTEM)

Electromagnetic methods using high frequencies ($f > 10$ MHz): Ground penetrating radar (GPR)

1 Conductivity

The conductivity σ of the minerals in the nature covers a range of 25 decades! For example:

$$10^{-18} S/m \rightarrow \text{Diamand}$$
$$10^{7} S/m \rightarrow \text{Copper}$$

Instead of the conductivity, the resistivity $\rho = \frac{1}{\sigma} \Omega m$ is often used.

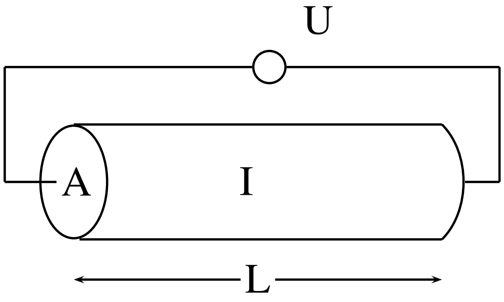


Figure 1.1: Schematic derivation of Ohm's law

Definition: Ohm's law

Let us consider a rock sample of length L, resistivity ρ and cross section A.
A current $I[A]$ flows by applying a voltag $U[V]$ to the rock sample:

$$I = \frac{AU}{\rho L}$$
$$\Leftrightarrow \rho \underbrace{\frac{I}{A}}_{\text{current density } j} = \underbrace{\frac{U}{L}}_{\text{electric Field } E}$$
$$\vec{j}\rho = \vec{E} \tag{1.1}$$

We measure I and U, A and L are known, so we can calculate ρ.

1.1 Mechanisms of electrical conductivity

Metallic conductivity: Current flows by free electrons $\rho \equiv T$

Electrolytic conductivity: Charge carriers are cations and anions: ρ decreases with temperature T.

Semi-conductors: Charge carriers must be activated by heat, light or EM-radiation. Strongly dependent on temperature T. Important for mantle (deep earth structures)

Boundary layer conductivity: Occurs due to the interaction of the pore liquid with the rock matrix. This is the source of SP-anomalies!

2 DC-resistivity method

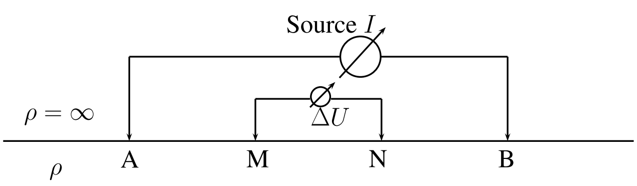


Figure 2.1: Four point measurement

Resistivity ρ of the subsurface derived from I (which is known), ΔU (which is measured) and the geometrical factor K (which is also known).

Frequently used electrode arrays

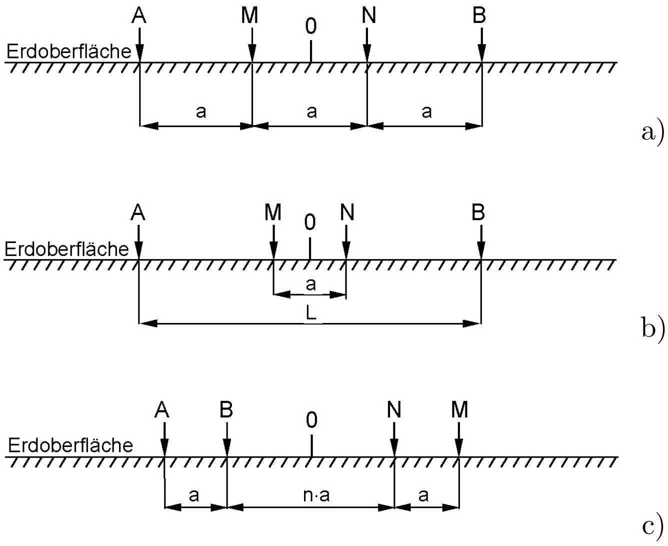


Figure 2.2: a) Wenner, Half-Wenner; b) Schlumberger, Half-Schlumberger; c) Dipole-dipole, source???

Industrial standard of measuring is via an *Multielectrode array.*

2.1 Basic equations of DC-resistivity

The first assumption of DC-resistivity methods and the major difference to EM-methods is the assumption of stationary currents:

$$\frac{\partial}{\partial t} = 0$$

The fields do not depend on time.
Looking at the *Maxwell's equations*:

$$\nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t} = 0 \tag{2.1}$$

This means irrotational electric field and from that follows, that the electric field vector can be derived by a scalar potential:

$$\vec{E} = -\nabla V \tag{2.2}$$

Insert equation (2.2) into eq. (1.1):

$$\vec{j} = -\sigma \nabla V \tag{2.3}$$

Continuity equation:

$$\nabla \cdot \vec{j} + \frac{\partial q}{\partial t} = 0 \tag{2.4}$$

Now new charges are generated in the course of time

$$\nabla \cdot \vec{j} = 0 \tag{2.5}$$

which is valid outside of the source.

If we insert eq. (2.3) into (2.5):

$$-\nabla \cdot (\sigma \nabla V) = 0$$
$$\nabla \sigma \nabla V + \sigma \nabla^2 V = 0$$

$\nabla \sigma = 0$ for areas with constant conductivity, so:

$$\nabla^2 V = 0 \tag{2.6}$$

which is called the *Laplace-equation*, the basic equation of DC-resistivity.
Derivation of solutions of this elliptic partial differential equation using different boundary conditions: Assume a current source with strength I at point $\vec{r}_0$, then the spatial current distribution can be given as:$\nabla \cdot \vec{j} = I\delta(\vec{r} - \vec{r}_0$ and so:

$$\nabla \cdot (\sigma \nabla V) = -I\delta(\vec{r} - \vec{r}_0) \tag{2.7}$$

This equation can be solved numerically for arbitrary distribution of conductivity ratio.

2.1.1 Potential of a current electrode

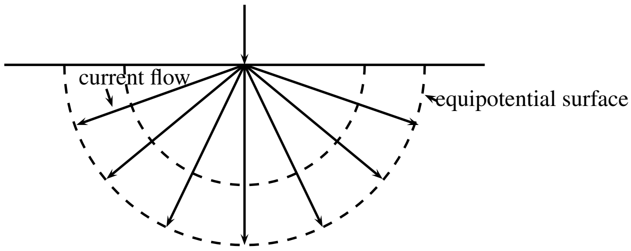


Figure 2.3: Single current source

Using *Ohm's law*: $\vec{E} = \rho \vec{j} = \rho \frac{I}{2\pi r^2}$, where $2\pi r^2$ is the surface of the half sphere. Using $E = -\frac{dV}{dr}$ follows the potential of a homogeneous half space:

$$V = \frac{\rho I}{2\pi r} \tag{2.8}$$

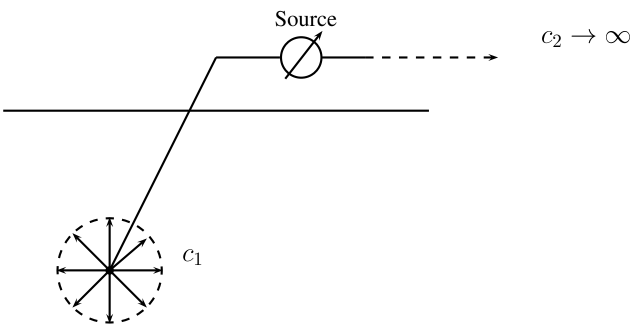


Figure 2.4: Mise-à-la-Masse method

In the case of the *Mise-à-la-Masse method* the potential of the homogeneous full space is:

$$V = \frac{\rho I}{4\pi r} \tag{2.9}$$

The same result can be derived by using the Laplace-equation (2.6) and the use of spherical coordinates:

$$\nabla^2 V = \frac{d^2 V}{dr^2} + \frac{2}{r}\frac{dV}{dr}$$

From the symmetry of the system the potential is a function of the distance to the source r only. Multiplying by r^2 and integrating, we get:

$$\frac{dV}{dr} = \frac{c_1}{r^2}$$

Integrating over r again leads to the solution:

$$V = -\frac{c_1}{r} + c_2 \qquad\qquad c_1, c_2 = const.$$

To determine the constants we have to use boundary conditions: From $\lim_{r\to\infty} V(r) = 0$ follows that $c_2 = 0$. Using the current density: $j = \frac{I}{A} \Leftrightarrow I = jA$:

$$I = 4\pi r^2 j = -4\pi r^2 \sigma \frac{dV}{dr} = -4\pi\sigma c_1$$

From this equation we can derive c_1:

$$V = \frac{I\rho}{4\pi r} \tag{2.10}$$

Boundary equations

Boundary with different conductivities.

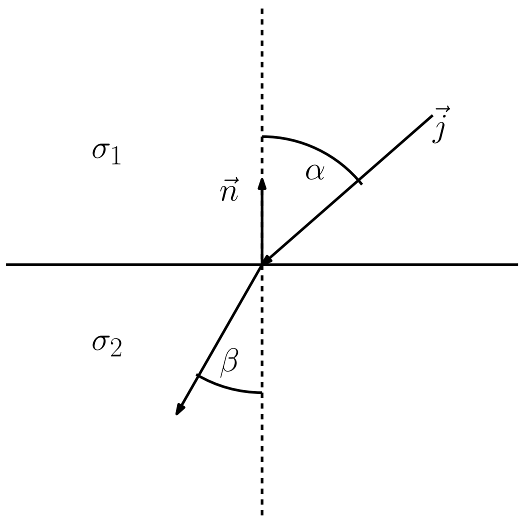


Figure 2.5: Boundary with dip angles GEOING s5

Two boundary conditions which must hold at any contact between two regions of different conductivity.

- Potential is continuous across the boundary
- j_n is also continuous.

$$V^1 = V^2, \quad \left(\frac{\partial V}{\partial x}\right)^1 = \left(\frac{\partial V}{\partial x}\right)^2, \quad j_n^1 = j_n^2$$

$$E_t^1 = E_t^2, \quad \sigma_1 E_n^1 = \sigma_2 E_n^2$$

$$\sigma_1 \frac{E_n^1}{E_t^1} = \sigma_2 \frac{E_n^2}{E_t^2}$$
$$\sigma_1 \cot\alpha = \sigma_2 \cot\beta$$
$$\frac{\tan\alpha}{\tan\beta} = \frac{\sigma_1}{\sigma_2}$$

Current line is bent towards to the normal if the resistivity of the second medium ρ_2 is larger than the one of the first medium ρ_1.

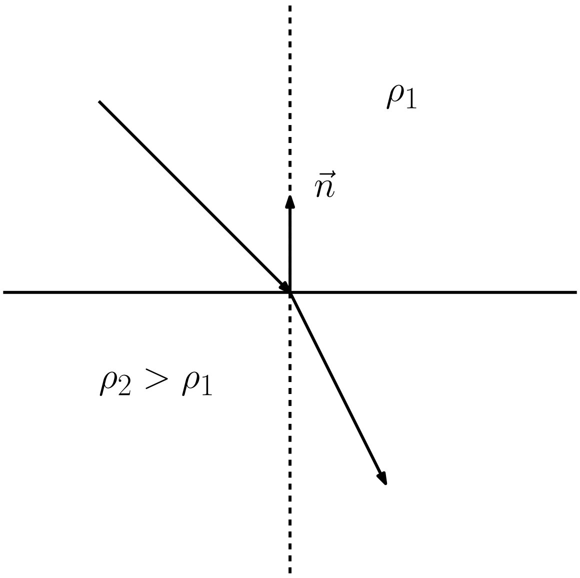


Figure 2.6: Bending towards normal

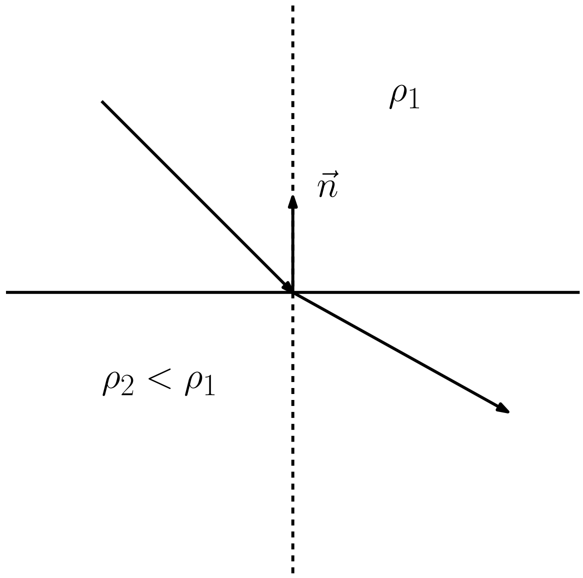


Figure 2.7: Bending away from normal

2.1.2 Potential distribution at the surface of a horizontally stratified earth (Solution of the Laplace equation (2.6))

Starting with a *model*:

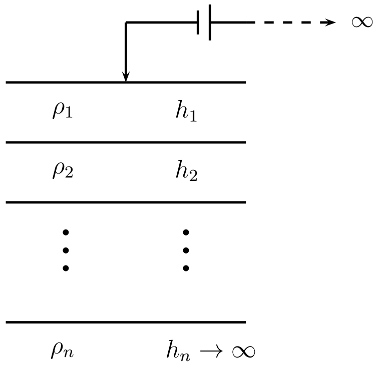


Figure 2.8: Model of n layer structure

The subsurface consists of finite number of layers with the last layer having infinite layer thickness ($h_n \to \infty$). We assume that ρ_i is isotropic (no dependence of the direction of measurement). The field is generated by a point source with the current I is a direct current.
Starting from the Laplace equation with potential V:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{2.11}$$

In cylindrical coordinates (r, θ, z):

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^V}{\partial z^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0 \tag{2.12}$$

The solution is symmetrical to the vertical axis, so $\frac{\partial V}{\partial \theta} = \frac{\partial^2 V}{\partial \theta^2} = 0$, so $V(r, \theta, z) = V(r, z)$. So the Laplace equation to be solved reduces to:

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^V}{\partial z^2} = 0 \tag{2.13}$$

Solution of (2.13). Ansatz:

$$V(r,z) = U(r)W(z) \tag{2.14}$$

So the solution is the product of a function of r alone and a function of z alone. We substitute (2.14) into (2.13) and multiply all terms with $1/UW$:

$$\underbrace{\frac{1}{UW}\frac{d^2U}{dr^2} + \frac{1}{UW}\frac{DU}{dr}}_{\text{depends on } r} + \underbrace{\frac{1}{W}\frac{d^2W}{dz^2}}_{\text{depends on } z} = 0 \tag{2.15}$$

This equation is satisfied, if

$$\frac{1}{U}\frac{d^2U}{dr^2} + \frac{1}{Ur}\frac{DU}{dr} = -\lambda^2 \tag{2.16}$$

$$\frac{1}{W}\frac{d^2W}{dz^2} = \lambda^2 \tag{2.17}$$

where λ is a real constant.

Solution of (2.17)

Using the Ansatz:

$$W = Ce^{-\lambda z} \quad , \quad W = Ce^{\lambda z} \tag{2.18}$$

where C and λ are arbitrary constants.

Solution of (2.16)

Using the Ansatz:

$$U = CJ_0(\lambda r) \tag{2.19}$$

with $J_0(\lambda r)$ the *Bessel-function* of order zero.

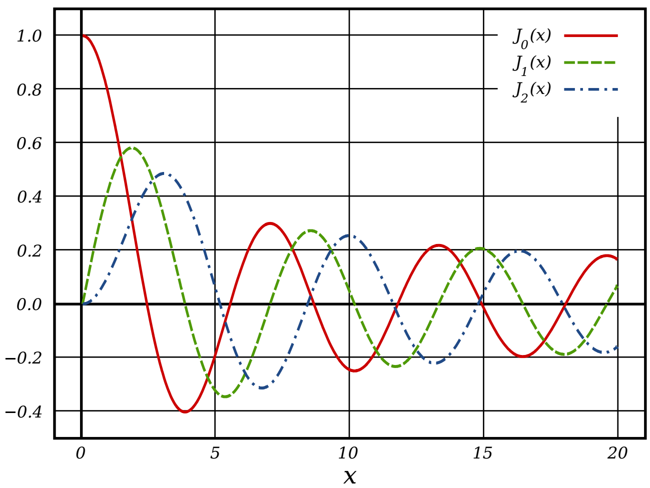


Figure 2.9: Bessel-functions source: $https://de.wikipedia.org/wiki/Besselsche_Differentialgleichung$

We combine the two solutions ((2.18) and (2.19)) for the solution of (2.13):

$$V = Ce^{-\lambda z}J_0(\lambda r) \quad , \quad V = Ce^{\lambda z}J_0(\lambda r) \tag{2.20}$$

λ varies from 0 to ∞ and C varies in dependence of λ. Than we write a general solution of the potential (2.13):

$$V = \int_0^\infty \left(\phi(\lambda)e^{-\lambda z} + \psi(\lambda)e^{\lambda z}\right) J_0(\lambda r) d\lambda \tag{2.21}$$

Where $\phi(\lambda)$ and $\psi(\lambda)$ are arbitrary functions of λ.

Potential of homogeneous halfspace

Starting of with the potential in cylindrical coordinates:

$$V = \frac{I\rho}{2\pi\sqrt{r^2+z^2}} \tag{2.22}$$

Looking at the *Lipschitz-Integral*:

$$\int_0^\infty e^{-\lambda z} J_0(\lambda r) d\lambda = \frac{1}{\sqrt{r^2+z^2}} \tag{2.23}$$

Now using (2.23) we write (2.22) as:

$$V = \frac{\rho_1 I}{2\pi} \int_0^\infty e^{-\lambda z} J_0(\lambda r) d\lambda \tag{2.24}$$

The general solution (2.21) can now be written:

$$V = \frac{\rho_1 I}{2\pi} \int_0^\infty \left(e^{-\lambda z} + \theta(\lambda) e^{-\lambda z} + X(\lambda) e^{\lambda z}\right) J_0(\lambda r) d\lambda \tag{2.25}$$

Where $\theta(\lambda)$ and $X(\lambda)$ are arbitrary functions of λ, and $\phi(\lambda) = \frac{\rho_1 I}{2\pi}(1+\theta(\lambda))$ and $\psi(\lambda) = \frac{\rho_1 I}{2\pi} X(\lambda)$. The solutions of the form (2.25) are valid in all layers but $\theta(\lambda)$ and $X(\lambda)$ can be different for each layer i:

$$V_i = \frac{\rho_1 I}{2\pi} \int_0^\infty \left(e^{-\lambda z} + \theta_i(\lambda) e^{-\lambda z} + X_i(\lambda) e^{\lambda z}\right) J_0(\lambda r) d\lambda \tag{2.26}$$

Adaption of the solution to the boundary conditions

Assuming we are at the layer boundaries of $z = h_i$.

A) Potential (2.26) is continious at each boundary plane in the subsurface:

$$V_i(r, h_i) = V_{i+1}(r, h_i) \tag{2.27}$$

This equation can only be satisfied if the integrands on both sides are equal:

$$\theta_i(\lambda) e^{-\lambda h_i} + X_i(\lambda) e^{\lambda h_i} = \theta_{i+1}(\lambda) e^{-\lambda h_i} + X_{i+1}(\lambda) e^{\lambda h_i} \tag{2.28}$$

B) At each boundary plane j_z the boundary condition must be fulfilled that:

$$j_z = -\frac{1}{\rho}\frac{\partial V}{\partial z} \tag{2.29}$$

and so

$$\frac{1}{\rho_i}\left((1+\theta_i(\lambda)) e^{\lambda h_i} - X_i(\lambda) e^{\lambda h_i}\right) = \frac{1}{\rho_{i+1}}\left((1+\theta_{i+1}(\lambda)) e^{\lambda h_i} - X_{i+1}(\lambda) e^{\lambda h_i}\right) \tag{2.30}$$

To satisfy this condition we differentiate the expression for the potential in the first layer (2.22) with respect to z and then substitute $z = 0$:

$$\frac{1}{\rho_1}\frac{\partial V_1(r,0)}{\partial z} = 0 \quad , \text{ for } \ r \neq 0 \tag{2.31}$$

We thus obtain the equation:

$$\int_0^\infty \left(-1-\theta_1(\lambda)+X_1(\lambda)\right) J_0(\lambda_r) d\lambda = 0 \tag{2.32}$$

$$\Rightarrow \theta_1(\lambda) = X_1(\lambda) \tag{2.33}$$

C) Near the current source the potential must approach to infinity

$$V_\infty = \frac{\rho I}{2\pi} \frac{1}{\sqrt{r^2+z^2}}$$

which is approaching asymtotically to the potential for a layer extending to infinite height.
D) $V \to 0$ if $z \to \infty$

$$\Rightarrow X_n = 0 \tag{2.34}$$

, because otherwise $e^{\lambda z}$ would drive the potential to an infinite value at an infinite depth.
The set of equations (2.28) - (2.34) provides a system of $2n$ equation in $2n$ unknown functions $\theta(\lambda)$ and $X(\lambda)$. To obtain the solution subsitute (2.33) into (2.28) and (2.30) and subsitute (2.34) into (2.28) and (2.30).
For brevity, we introduce the notations:

$$u_i = e^{\lambda h_i}, v_i = \frac{1}{u_i}, p_i = \frac{\rho_i}{\rho_{i+1}}$$

The system of equations then become:

$$\begin{aligned}
(u_1+v_1)\theta_1 - u_2\theta_2 - v_2X_2 &= 0\\
(v_1-u_1)\theta_1 + p_1u_1\theta_2 - p_1v_1X_2 &= (1-p_1)u_1\\
\vdots \qquad\qquad & \qquad \vdots\\
u_{n-1}\theta_{n-1} + v_{n-1}X_{n-1} - u_{n-1}\theta_n &= 0\\
-u_{n-1}\theta_{n-1} + v_{n-1}X_{n-1} + p_{n-1}u_{n-1}\theta_n - p_{n-1}v_{n-1}X_n &= (1-p_{n-1})u_{n-1}
\end{aligned}$$

Solution of the equations by applying *Cramer's rule*. For example: Solution of a two layer case (layer 1: ρ_1, h_1, layer 2: ρ_2):

$$\theta_1 = \frac{ku}{1-ku} \qquad\qquad \theta_2 = \frac{k(1+u)}{1-ku}$$

$$X_1 = \theta_1 \qquad\qquad X_2 = 0$$

with $u = e^{-2\lambda h_1}$ and the *reflection coefficient of DC* $k = \frac{\rho_2-\rho_1}{\rho_2+\rho_1}$
Interesting is the potential at the surface of the earth, with $z = 0$ and (2.33):

$$V_0 = V_1(r,z) = \frac{\rho_1 I}{2\pi} \int_0^\infty (1+2\theta_1) J_0(\lambda r) d\lambda \tag{2.35}$$

$$= \frac{\rho_1 I}{2\pi} \int_0^\infty K(\lambda) J_0(\lambda r) d\lambda \tag{2.36}$$

where $K(\lambda)$ is the *Slichter-function.*
We consider the Lipschitz-integral:

$$\int_0^\infty e^{-\lambda z} J_0(\lambda r) d\lambda = \frac{1}{\sqrt{r^2+z^2}} \overset{i=0}{\equiv} \int_0^\infty J_0(\lambda r) d\lambda = \frac{1}{r} \tag{2.37}$$

(2.26) can now be written in the form:

$$V_0(r) = \underbrace{\frac{I}{2\pi}\left(\frac{\rho_1}{r}\right.}_{\text{first layer}} + \int_0^\infty (T(\lambda) - \rho_1) J_0(\lambda r) d\lambda \Bigg) \tag{2.38}$$

with $T(\lambda) = \rho_1(1 + 2\theta_1(\lambda))$ Example/reminder for four point measurement:

$$V_1 = \frac{I\rho}{2\pi}\left(\frac{1}{AM} - \frac{1}{BM}\right)$$

2.1.3 Derivation of a formula for the apparent resistivity

Take an arbitrary DC-Array (compare Fig. 2.1). Then

$$\Delta U = \frac{I\rho}{2\pi}\left(\frac{1}{AM} - \frac{1}{AN} - \frac{1}{BM} + \frac{1}{BN}\right)$$

$$\rho_a = k\frac{\Delta U}{I}$$

where k is the geometrical factor. If we look at experimental data with an error of 1% for the distances between the electrodes, the error in ρ_a would be 2%. But 10% error in the lateral direction of the electrodes results only in 1% error in ρ_a.
In case of the Schlumberger array ($L = AM + MN/2$ and $a = MN$, $a \ll L$) we get a voltage decrease in U:

$$\begin{aligned} U &= 2\left(V_0(\frac{L}{2} - \frac{a}{2}\right) - V_0\left(\frac{L}{2} + \frac{a}{2}\right) \\ &\approx -2a\frac{\partial V_0}{\partial r}\bigg|_{r=L/2} \end{aligned}$$

and the geometrical factor in case of Schlumberger $k = \frac{\pi}{a}\left(\left(\frac{L}{2}\right)^2 - \left(\frac{a}{2}\right)^2\right)$

$$\rho_a(L/2) = K\frac{U}{I} = \frac{2\pi}{I}\left(\frac{L}{2}\right)^2\frac{\partial V_0}{\partial r}$$

with $\frac{d}{dx}J_0(x) = -J_1(x)$. From eq. 2.25!!!!:

$$\rho_a(L/2) = \rho_1 + \left(\frac{L}{2}\right)^2 \underbrace{\int_0^\infty (T(\lambda) - \rho_1) J_1(\lambda L/2) \lambda d\lambda}_{\text{Stefanescu-Integral}} \tag{2.39}$$

The calculations of the model response $\rho_a(L/2)$ from given model parameters (ρ_i, h_i) is a forward problem.
Given:

Figure 2.10: Given parameters

Now two steps are neccessary:

- Calculation of $T(\lambda)$
- Integration of (2.39) $\rightarrow$ Stefanescu-Integral

2.1.4 Calculation of the resistivity transform $T(\lambda)$

For a method for the determination of $T(\lambda)$ see chapter 2.1.2. Then we calculate the solution of the equation system θ_i and X_i and determine $T(\lambda)$ using $T(\lambda) = \rho_1(1 + 2\theta_1(\lambda))$. This procedure is too time consuming for a high number of layers.
Now we derive a recursion formula using the boundary conditions A to D from 2.1.2. At first a new definition:

$$T_i(\lambda) = \rho_1 \frac{1 + \theta_1(\lambda) + X_i(\lambda)e^{2\lambda t_i - 1}}{1 + \theta_1(\lambda) - X_i(\lambda)e^{2\lambda t_i - 1}} \tag{2.40}$$

with $i = 1, 2, \ddots, n$, $t_i = h_1 + h_2 + \ddots + h_i$ and $t_0 = 0$.
Because of (2.33): $\theta_1(\lambda) = X_1(\lambda)$, we get: $T_1(\lambda) = T(\lambda)$.
Because of (2.34): $X_n = 0$, we get: $T_n(\lambda) = \rho_n$.

From the boundary conditions (2.28) and (2.30):

a) $\theta_i(\lambda)e^{-\lambda t_i} + X_i(\lambda)e^{\lambda t_i} = \theta_{i+1}(\lambda)e^{-\lambda t_i} + X_{i+1}(\lambda)e^{\lambda t_i}$

b) $\frac{1}{\rho_i}\left((1 + \theta_i(\lambda))\, e^{-\lambda t_i} - X_i(\lambda)\right) = \frac{1}{\rho_i}\left((1 + \theta_{i+1}(\lambda))\, e^{-\lambda t_i} - X_{i+1}(\lambda)\right)$
The next steps are:

- Add $e^{-\lambda t_i}$ on both sides of (2.28)
- Divide each side over the corresponding side of (2.30)
- Cancel the left part of the new equation by $X_i(\lambda)$

Then:

$$\rho_i \frac{K_i(\lambda) + e^{2\lambda t_i}}{K_i(\lambda) - e^{2\lambda t_i}} = T_{i+1}(\lambda)$$

with $K_i(\lambda) = \frac{1+\theta_i(\lambda)}{X_i(\lambda)}$
Now the next step is to Solve the eq. (2.40), insert it and short translation and solving it for $T(\lambda)$:

$$T_i(\lambda) = \frac{T_{i+1} + \rho_i \tanh(\lambda h_i)}{1 + \frac{T_{i+1}(\lambda)}{\rho_i}\tanh(\lambda h_i)} \tag{2.41}$$

which is the *recursion formula of PEKERIS.*
Now start with $T_n = \rho_n$, calculate step by step $T_i(\lambda)$ until $T_1(\lambda) = T(\lambda) \rightarrow$ *resistivity transform*
Illustration of eq. (2.41):

$$\lambda = \frac{1}{L/2} \qquad\qquad T_n(\lambda) = \rho_n$$

Two extreme values:

$$L/2 \rightarrow 0 \qquad\qquad L/2 \rightarrow \infty$$

Two layer model

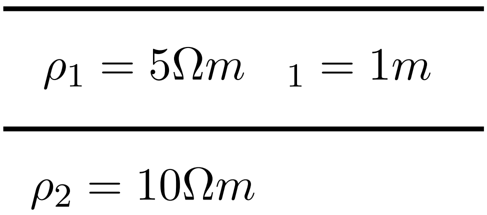


Figure 2.11: Two layer model

$T_n(\lambda) = T_2(\lambda) = 10\Omega m$

1. $L/2 \rightarrow 0 \Rightarrow \lambda \rightarrow \infty \Rightarrow \tanh(\infty) \rightarrow 1$

$$T(\lambda) = T_1(\lambda) = \frac{T_2 + \rho_1 \tanh(\lambda h_1)}{1 + \frac{T_2(\lambda)}{\rho_1} \tanh(\lambda h_1)} = \frac{10+5}{1+10/5} = 5\Omega m$$

2. $L/2 \rightarrow \infty \Rightarrow \lambda \rightarrow 0 \Rightarrow \tanh(0) \rightarrow 0$

$$T_1(\lambda) = \frac{10+0}{1+0} = 10\Omega m$$

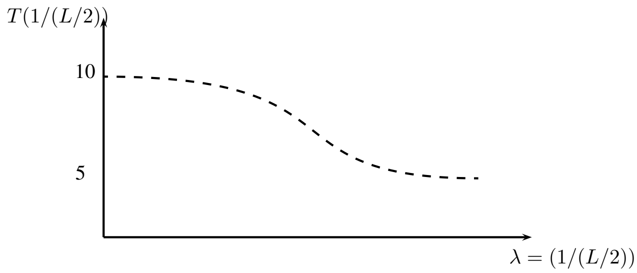


Figure 2.12: ??????

2.1.5 Solution of the Stefanescu-Integral

An analytical solution is not possible! One of the possibilities uses the linear filter method.

2.1.5.1 Basic Equations of the linear filter method: The fast HANKEL-transformation

The calculation of a function $g(r)$ from $\rho(\lambda)$ by

$$g(r) = \int_0^\infty \rho(\lambda) J_\nu(\lambda r) \lambda d\lambda \tag{2.42}$$

is defined as the *HANKEL-transformation*. It expresses any given function as the weighted sum of an infinite number of Bessel-functions.
The *inverse Hankel-transformation*:

$$\rho(\lambda) = \int_0^\infty g(r) r J_\nu(\lambda r) dr \tag{2.43}$$

Then the Stefanescu Integral has the form of a Hankel-transformation.

The following method to calculate the integral (2.42) is called the fast Hankel-transform. It provides function values for $g(r)$ at discrete points.
To solve this *four steps are neccessary*:
1) The variables are transformed into logarithmic values.

$$x = \ln(r/r_0) \qquad\qquad y = -\ln(\lambda/r_0)$$
$$\Rightarrow r = e^x \qquad\qquad \lambda = e^{-y}$$

with r_0 the reference length. Then:

$$r\lambda = e^{x-y} \qquad\qquad dr = r dx \qquad\qquad d\lambda = -\lambda dy$$

Insert this into eq. (2.42) and (2.43):

$$rg(r) = -\int_{-\infty}^{\infty} \rho(\lambda)\lambda J_\nu(e^{x-y})e^{x-y}dy$$

$\lambda \to 0, y \to \infty$ and $\lambda \to \infty, y \to -\infty$.

$$\lambda\rho(\lambda) = \int_{-\infty}^{\infty} g(r)rJ_\nu(e^{x-y})e^{x-y}dx$$

$r \to 0, x \to -\infty$ and $r \to \infty, x \to \infty$.
From this follow the *Convolution integrals*:

$$\begin{aligned} F(y) &= \int_{-\infty}^{\infty} G(x)H(x-y)dx \\ G(x) &= \int_{-\infty}^{\infty} F(y)H(x-y)dy \end{aligned} \tag{2.44}$$

The requirement for the fast Hankel transformation is *not* only that the integral (2.43) has the form of a Hankel transformation but it can be *transferred* to a *convolution integral* (2.44).

2) The function F is represented in the form (by using the sampling theorem):

$$F(y) = \sum_{j=-\infty}^{\infty} F(y_j)\,\mathrm{sinc}\left(\pi\frac{y-y_j}{\Delta y}\right) \tag{2.45}$$

with $\mathrm{sinc}(z) = \frac{\sin(z)}{z}$ and $y_j = y_0 + j\Delta y$ with y_0 arbitrary.
Sampling is the process of converting a signal into a numeric sequence. A band limited function can only be perfectly reconstructed from a countable sequence of samples, if the band limit B is not greater than half of the sampling rate. This leads to a formula for the reconstruction of the original function from it's samples:

$$\rho_{ny} = \frac{1}{2\Delta t}$$

3) Inserting (2.45) into (2.44) gives:

$$G(x) = \sum_{j=-\infty}^{\infty} c_j(x)F(y_j) \tag{2.46}$$

with $c_j(x) = \int\limits_{-\infty}^{\infty} \mathrm{sinc}\left(\pi\frac{y-y_j}{\Delta y}\right) H(x-y)dy$.
The calculation of a general function is reduced to the transformation of a sinc function.

4) The calculation of $G(x)$ is limited to the calculation of function values at discrete points: $x_k = x_0 + k\Delta x$ with $k = ..., -1, 0, 1, ...$ and x_0 arbitrary and $\Delta x = \Delta y$.
Inserting in the equation of $c_j(x)$:

$$c_j(x_k) = c_0(x_k)$$

and so it follows:

$$G(x_k) = \sum_{j=-\infty}^{\infty} c_{k-j} F(y_j) \tag{2.47}$$

with $c_{k-j} = c_0(x_k - j)$.
That means only the coefficients c_k will be calculated from the funtion.

$$c_0(x) = \int_{-\infty}^{\infty} H(x-y)\,\mathrm{sinc}\left(\pi \frac{y-y_0}{\Delta y}\right) dy \tag{2.48}$$

at the points x_k. The transformation is thereby reduced to the transformation of a single sinc function. Two conclusions result from the properties of the *sinc response* for the application of the fast Hankel-transformation:

(a) Only c_k with values over a lower (k_n) and upper (k_0) limit are calculated:

$$k_n < c_k < k_0$$

The amount of c_k can be defined as a filter, so that (2.47) becomes:

$$G(x_k) = \sum_{j=k-k_0}^{k-k_n} c_{k-j} F(y_j) \tag{2.49}$$

(b) Due to the oscillations, zero values of $c_0(x)$ result at distances of Δx for large and small x values. By subtle choice of x_0 it can be reached, that values x_k will be close to zero points for large and small k and the function values dissapear. That means the filter will be shorter.

2.1.5.2 Calculation of a filter

There are several possibilities to calculate values of $c_0(x)$. The first paper was published by Ghosh (1971). There exists functions $F(y)$ for which the eq. (2.44): $G(x) = \int F(y)H(x-y)dy$ can be solved analytically and $G(x)$ can be calculated. For such cases we apply the Fourier-transformation.

$$\tilde{F}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(y) e^{-iky} dy$$

and $\tilde{G}(k)$ analogous. Then (2.44) has the form of a convolution integral:

$$G(x) = F(y) * H(x-y)$$
$$G(k) = F(k) \cdot H(k)$$

Also eq. eqref eq:2.34 has the form of a convolution integral:

$$c_j(x) = \int_{-\infty}^{\infty} \mathrm{sinc}\left(\pi \frac{y-y_j}{\Delta y}\right) H(x-y) dy \qquad y_0 = 0$$

Therefore:

$$\tilde{c}_0(k) = \tilde{H}(k)\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \text{sinc}\left(\pi\frac{y}{\Delta y}\right) e^{-kyi} dy$$

with

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \text{sinc}\left(\pi\frac{y}{\Delta y}\right) e^{-kyi} dy = \begin{cases} \frac{\Delta y}{2\pi} & , for|k| < \pi/\Delta y \\ 0 & , otherwise \end{cases}$$

follows:

$$\tilde{c}_0(k) = \begin{cases} \frac{\tilde{G}(k)}{\tilde{F}(k)}\frac{\Delta y}{2\pi} & , for|k| < \pi/\Delta y \\ 0 & , otherwise \end{cases}$$

2.1.5.3 Example of a resistivity filter

eq. 2.27

$$\rho_a(L/2) = \rho_1 + (L/2)^2 \int\limits_0^{\infty} (T(\lambda) - \rho_1) J_1(\lambda L/2)\lambda d\lambda$$

$$x = \ln(L/2), y = -\ln(\lambda)$$

$L/2$ in m, $\lambda is 1/m$

$$G(x) = \int F(y)H(x-y)dy$$

with

$$\begin{aligned} G(x) &= \rho_a(e^x) - \rho_1 \\ F(y) &= \rho_a(e^{-y}) - \rho_1 \\ H(x) &= J_1(e^x) \end{aligned}$$

We can now calculate the resistivity filter $c_0(k) = G(k)/F(k)$ by applying the fast HT. Filter with 3 values per decade → Ghosh, with 6 values per decade → O'Neill, with 10 → Johansen.
For all filters:

$$\sum_{k=k_n}^{k_0} c_k = 1$$

Using eq. 2.37:

$$\rho_a(e^{x_k}) = \sum_{j=k-k_0}^{k_n} c_{k-j} T(e^{-y_j})$$

2.2 Principle of equivalence

We can now calculate theoretical apparent resistivities for a 1D model. For example with the Schlumberger-Array:

$$\rho_a(x) = \sum_{k_{min}}^{k_{max}} \underbrace{T(\lambda)}_{Pekeris} \underbrace{\rho_k}_{filtercoeff.}$$

using $\rho_1 = 10\Omega m, h_1 = 1m, \rho_2 = 10\Omega m, h_2 = 5m, \rho_3 = 50\Omega m$ in a three layer case results in

Figure 2.13: Apparent resistivity in a three layer case

Different types of ρ_a-curves

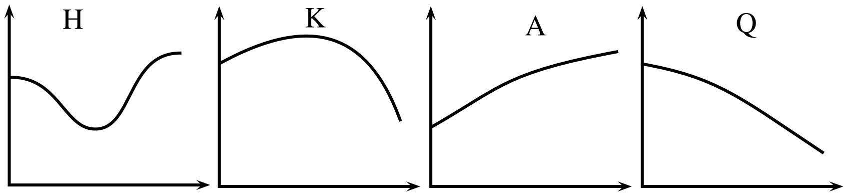


Figure 2.14: Different types of curves (in a three layer case).

K-Type: No difference between ρ_a-curves of different model, if $\rho_2 \cdot h_2$ is identical.

H-Type: h_2/ρ_2 is identical

Example:

Model 1 $\rho_1 = 1\Omega m, h_1 = 1m$
$\rho_2 = 20\Omega m, h_2 = 1m$
$\rho_1 = 1\Omega m$

Model 2 $\rho_1 = 1\Omega m, h_1 = 1m$
$\rho_2 = 40\Omega m, h_2 = 0.5m$
$\rho_1 = 1\Omega m$

in both cases: $\rho_2 \cdot h_2 = 20 \cdot 1 = 40 \cdot 0.5$.
Equivalent models should be calculated as the result of interpretation:

2.3 Interpretation of resistivity data: Inversion

The aim of the inversion is the minimization of the error function or cost function ψ_d between observed and calculated apparent resistivity data. Minimize:

$$\psi_d = \|\vec{y} - f(\vec{m})\|^2 \tag{2.50}$$

$\vec{y}$ is the vector of measured data (e.g. $\vec{y} = (\rho_a(L/2 = 5m), \rho_a(L/2 = 10m), ...)$.
$f(\vec{m}$ is the vector of calculated data.
ψ_d is the norm of differences between measured and observed data.

2.3.1 Strategies for the inversion

Different methods to minimize the difference between measured and calculated data:

- Trial and error
- method of Zohdy
- Automatic inversion by linearisation of the forward operator $f(\vec{m})$

2.3.1.1 Trial and Error

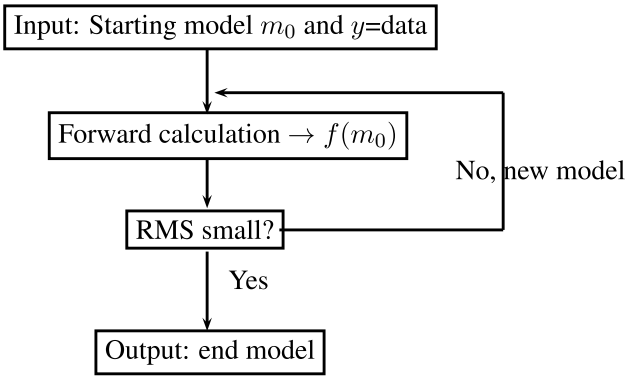


Figure 2.16: Trial and error scheme

with

$$RMS = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\frac{(\rho_a^m(i) - \rho_a^c(i))^2}{(\rho_a^c(i))^2}} \tag{2.51}$$

ρ_a^m measured data, ρ_a^c calculated data.

2.3.1.2 ZOHDY-technique

This method is suitable for the inversion of DC-resistivity data measured by a four electrode array (Schlumberger, Wenner, ...).Utilize the principle of equivalence: The fitting of the measured data by using a resistivity model with a *large* number of layers has the same quality if less layers are used.

Example:

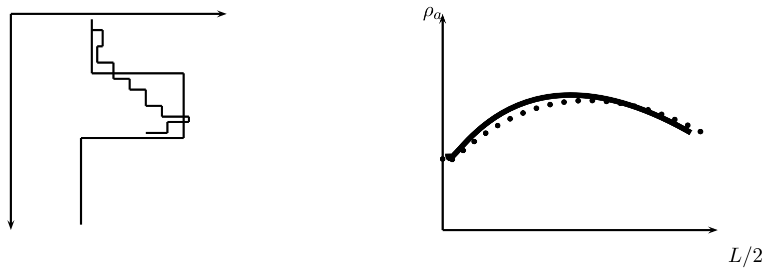


Figure 2.17: Example layers in inversion

Procedure

1) Starting model with m layers, where m is the number of measured data. $\rho_i = \rho_{a_i}$, $t_i = (L/2)_i$. Thickness of the i'th layer: $h_i = (L/2)_{i+1} - (L/2)_i$. Then do the forward and RMS calculation. For example RMS is 62%.
2) Reduction of the thickness of each layer as 0.9 and then do forward and RMS calculation. Repetition of the procedure until no improvement of RMS is possible. For example: 10 iterations reduce RMS to 12%.
3) Determination of resistivity of each layer:

$$\rho_{i+1}(j) = \rho_i(j) \frac{\rho_{a_{i,obs}}(j)}{\rho_{a_{i,c}}(j)}$$

with i the number of iterations and j the number of layer = $L/2$ number. $\rho_i(j)$: resistivity of the j'th layer of the i'th iteration. $\rho_{a_{i,c}}$: calculated apparent resistivity data for the j'th layer and i'th iteration.
Now do the forward calculation and calculate the RMS. Similar to step 2) repetition of the procedure (example 12% to 1%).

Disadvantage:

No good result, if data points have relatively large noise, because every data point is a layer.

2.3.1.3 Inversion by linearization of the forward operator

The aim of the inversion is to minimize the cost function ψ_d. A measure for the error:

$$X^2 = \frac{1}{N} \sum_{i=1}^{N} \frac{(y_i - f_i)^2}{\sigma_i^2} \tag{2.52}$$

where n is the number of measured data and calculated data. σ is the standard deviation. *Mathematically:* $\min \|y - f(m)\|^2$ (To solve use e.g. Gauss Newton method). The problem is not linear, therefore linearise $f(m)$ or ψ_d.
The linearisation can be done by a Taylor expansion of the forward operator $f(m)$ for small model changes Δm close to the starting model m_0:

$$f(m_0 + \delta m) = f(m_0) + \frac{\partial f(m_0)}{\partial m_0} \Delta m \approx f(m_0) + \underline{\underline{J}} \Delta m \tag{2.53}$$

where $\underline{\underline{J}}$ is the jacobian or sensitivity matrix. It describes the influence of model parameters on the model response. eq. 2.38 can now be written as:

$$\psi_d(m_a) = \|y - f(m_0)\|^2 = (y - f(m_0))^T (y - f(m_0)) \tag{2.54}$$

using the Taylor expansion:

$$\psi_d(m_0+\Delta m) = \|y - f(m_0+\Delta m)\|^2 = (y - f(m_0+\Delta m))^T (y - f(m_0+\Delta m)) \tag{2.55}$$

Set eq. 2.40 in eq. 2.42

$$\psi_d(m_0+\Delta m) = \|y - f(m_0) - \underline{\underline{J}}\Delta m\|^2 = (y - f(m_0) - \underline{\underline{J}}\Delta m)^T (y - f(m_0) - \underline{\underline{J}}\Delta m) \tag{2.56}$$

Calculation of the extreme of ψ_d:

$$\frac{\partial \psi_d(m_0+\Delta m)}{\partial \Delta m} = 0 = \frac{\partial}{\partial \Delta m}(y - f(m_0) - \underline{\underline{J}}\Delta m)^T (y - f(m_0) - \underline{\underline{J}}\Delta m)$$

with $\Delta d = y - f(m_0)$:

$$\begin{aligned} 0 &= \frac{\partial}{\partial \Delta m}(\Delta d - \underline{\underline{J}}\Delta m)^T(\Delta d - \underline{\underline{J}}\Delta m) \\ &= \frac{\partial}{\partial \Delta m}(\Delta d^T \Delta d - \Delta d^T \underline{\underline{J}}\Delta m - \Delta m J^T \Delta d + \Delta m^T J^T J \Delta m) \\ &= 2J^T \Delta d - 2J^T J \Delta m \\ \Leftrightarrow J^T J \Delta m &= J^T \Delta d \end{aligned}$$

Normal equation: Solution for this equation according to Δm

$$\Delta m = (J^T J)^{-1} J^T \Delta d \tag{2.57}$$

For the linear case the minima of ψ_d can be reached after one iteration. For the non-linear case $m_1 = m_0 + \Delta m$, $m_2 = m_1 + \Delta m$, so the solution will be iteratively improved!
Problem: No solution of (2.57) if $(J^T J)$ is singular, or in other words $\det(J^T J) = 0$. To stabilize it

$$\Delta m (J^T J + \beta I)^{-1} J^T \Delta d$$

with β the damping factor and I the identity matrix. The solution according to the eq. is known as *Marquardt-Levenberg method.*

2.4 Solution of the 2D DC forward problem

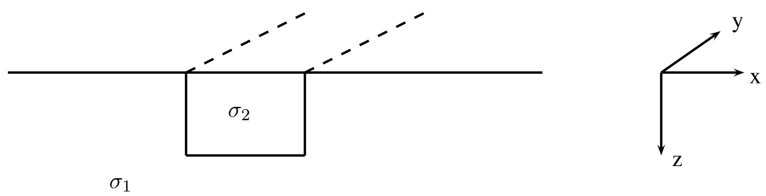


Figure 2.18: asdfasdf

Basic equations:

Ohm's law:

$$\begin{aligned} j &= \sigma E \\ E &= -\nabla V \\ j &= -\sigma \nabla V \end{aligned}$$

By using the charge retention over a volume, the continuity equation can now be written as:

$$\nabla j = \frac{\partial q}{\partial t}\delta(x)\delta(y)\delta(z) \tag{2.58}$$

The charge density q is represented at a point in the cartesian coordinates (x, y, z) with the Dirac-distribution $\delta(x)$.

$$-\nabla\left(\sigma(x, y, z)\nabla V(x, y, z)\right) = \frac{\partial q}{\partial t}\delta(x_s)\delta(y_s)\delta(z_s) \tag{2.59}$$

The *Poisson equation*, with x_s, y_s, z_s the coordinates of the source point. By using the vector equation $\nabla \cdot (\phi A) = \nabla\phi A + \phi\nabla \cdot A)$ with $A = \nabla V$ and $\phi = \sigma$. Then the equation (2.58) will be:

$$\nabla\sigma(x, y, z)\nabla V(x, y, z) + \sigma(x, y, z)\nabla^2 V(x, y, z) = -\frac{\partial q}{\partial t}\delta(x_s)\delta(y_s)\delta(z_s) \tag{2.60}$$

No change of electrical conductivity in y-direction so we have a 2D problem. $\Rightarrow \frac{\partial}{\partial y}\sigma(x, y, z) = 0$. Application of this condition to (2.58) and (2.60):

$$-\nabla \cdot (\sigma\nabla V) = \frac{\partial q}{\partial t}\delta(x_s)\delta(y_s)\delta(z_s) \tag{2.61}$$

$$\nabla\sigma \cdot \nabla V + \sigma\nabla^2 V = -\frac{\partial q}{\partial t}\delta(x_s)\delta(y_s)\delta(z_s) \tag{2.62}$$

Using the vector equation: $\nabla A \cdot \nabla B = \frac{1}{2}(-A\nabla^2 B + \nabla^2(AB) - B\nabla^2 A)$ with $A = \sigma$ and $B = V$ we get:

$$\nabla^2(\sigma(x, z)V(x, y, z)) + \sigma(x, z)\nabla^2 V(x, y, z) - V(x, y, z)\nabla^2\sigma(x, z) = -2\frac{\partial q}{\partial t}\delta(x_s)\delta(y_s)\delta(z_s)$$

Spatial distribution of the potential $V \rightarrow$ 3D, spatial distribution of the conductivity $\sigma \rightarrow$ 2D. Therefore the solution in this form is not possible.
The y-dependence of the potential can now be eliminated by the *Fourier-cosine transformation*:

$$\tilde{V}(x, K_y, z) = \int_0^\infty V(x, y, z)\cos(K_y y)dy$$

3D $V(x, y, z)$ is due to point source at (x_s, y_s, z_s) over a 2D conductivity structure is reduced to a 2D transformed potential $\tilde{V}(x, K_y, z)$, with K_y the wave number.
For $\tilde{V}(x, K_y, z)$ the solution

$$\nabla^2(\sigma(x, z)\tilde{V}(x, K_y, z)) + \sigma(x, z)\nabla^2\tilde{V}(x, K_y, z) - \tilde{V}(x, K_y, z)\nabla^2\sigma(x, z) - 2K_y\sigma(x, z)\tilde{V}(x, K_y, z) = -2Q\delta(x_s)\delta(z_s) \tag{2.63}$$

is looked for with $Q\delta(x_s)\delta(z_s) = \frac{1}{2}\frac{\partial q}{\partial t}$
The relationship between the stationary current density Q and the current:

$$Q = \frac{I}{2\Delta A}$$

where ΔA is the area around the current electrodes.
The eq. (2.63) is solved for different wave numbers. Afterwards do the inverse transformation:

$$V(x, y, z) = \frac{2}{\pi}\int_0^\infty \tilde{V}(x, K_y, z)\cos(K_y y)dK_y$$

Numerical solution of (2.63) with *boundary conditions* (2D forward modelling). The boundary conditions are:
a) $V(x, y, z)$ is continuous between two media with different conductivity σ.

b) $V(x,y,z) \rightarrow 0$ if $z \rightarrow \infty$
c) j_n is also continuous
Now discretization of the subsurface and solution of (2.63) with (for example) finite differences:

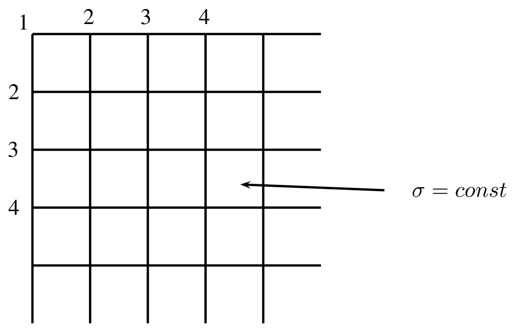


Figure 2.19: Finite differences grid

Calculation of the potential at the knots of the mesh and afterwards $\rho_a = K\frac{\Delta V}{I}$

Electromagnetic methods

3 Electromagnetic induction

3.1 Principle of EM-induction as an example of transformer

The simplifications are 1 winding, no μ_r core.

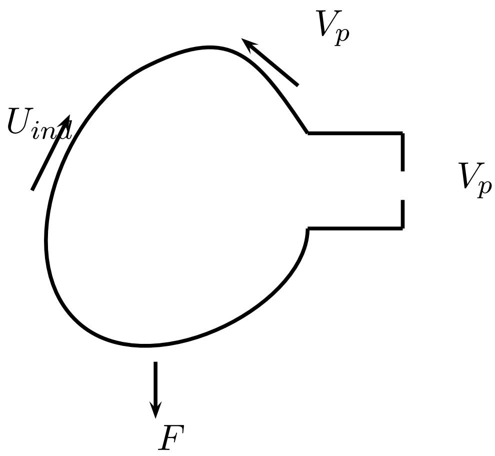


Figure 3.1: EM Spule

Primary coil

Alternating current voltage: $V_p = V_{p0}\sin(\omega t)$ produces magnetic field flux F, F produces induced voltage $U_{ind} = -V_p = \frac{dF}{dt}$ (Lentz law). The induced voltage is orientated in the opposite direction of $\frac{dF}{dt}$.

$$F = \int V_p dt = -V_{p0}\cos(1/\omega)$$

$$V_p \sin(\omega t) = \frac{dF}{dt} = AL\dot{I}$$

$$\Rightarrow I = -I_0 \cos(\omega t)$$

Secondary coil

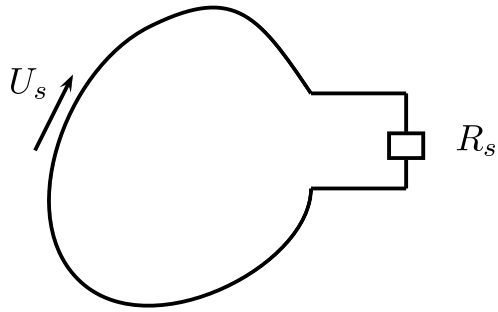


Figure 3.2: Secondary coil

F produces secondary induced voltage:

$$U_s = -\frac{dF}{dt} = -V_{p0}\sin(\omega t)$$

Current drain due to Ohmic resistance

$$I_s = \frac{U_s}{R_s} = -V_{p0}\sin(\omega t)/R_s$$

I_s produces additional magnetic flux:

$$F_s = \mu_0 HA = \mu_0 \frac{I}{l} A = -\mu_0 \frac{r}{2} V_{p0}\sin(\omega t)/R_s$$

which generates an additional voltage:

$$V_{p_s} = -\frac{dF}{dt} = (\omega)\frac{r}{2} V_{p0}\cos(\omega t)/R_s$$

3.2 Induction in the conductive subsurface

Primary current $\rightarrow$ current system in the ionosphere or artificial sources.
Secondary coil $\rightarrow$ conductive subsurface

Geomagnetic Depth Sounding

Aim: Derivation of in-situ conductivity from the observation of time varying electromagnetic fields at the earth surface.
Primary source region: Ionosphere, magnetosphere, where primary currents are flowing. Secondary source region: Conductive earth layers where secondary currents are flowing.
We observe at the earth surface:

a) Geomagnetic time variations $B(t)$ consisting of external B^e and of interior B^I part.
Tendency: In the horizontal components constructive interaction. Destructive interaction for the vertical component.

b) Telluric $E(t)$ variations for induced currents in the subsurface
Tendency: Strong telluric currents at near surface conductivity contrasts.

3.3 Basic Elements

3.3.1 Notation and units

1. **Position vector:** In spherical coordinates (r, θ, λ) with r the distance from the Earth center, θ the polar distance and λ the length or longitude.

   In plane coordinates: z is the depth, x the North direction and y the East direction.

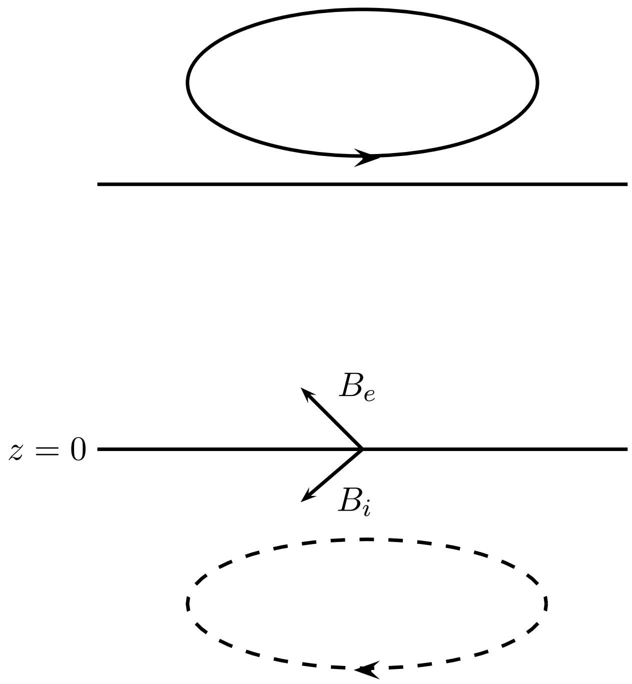


Figure 3.3: Geomagnetic sounding

2. Physical base items:

$\vec{B}$: magnetic induction in nT $= 10^{-9}$ Vsm^{-2}.

$\vec{E}$: electric field in mV/km $= 10^{-6}$ V/m.

$\vec{j}$: electric current density in A/m^2.

η: electric charge density in As/m^3.

3. Material constants:

ϵ, μ: electric permittivity and magnetic permittivity

$\mu_0 = 4\pi \cdot 10^{-7}$ Vs/Am

$\epsilon_0 = 8.85 \cdot 10^{-12}$ As/Vm

σ: conductivity in S/m

ρ: resistivity in Ω m

4. Material equations:

$\vec{D} = \epsilon\epsilon_0\vec{E}$: electrical displacement

$\vec{B} = \mu\mu_0\vec{H}$: $\vec{H}$ the magnetic field strength

5. Ranges:

Global Earth magnetic field: $3 - 6 \cdot 10^4$ nT

Earth magnetic variations: 1-100 nT

Telluric variations: 0.1 - 10 mV/km

Earth electric soil potential: 10 mV

$\mu = 1 + K$ with $K < 10^{-2}$ for rocks. Therefore $\mu = 1$ for the following derivations. $\epsilon = 1 - 80$ (water).

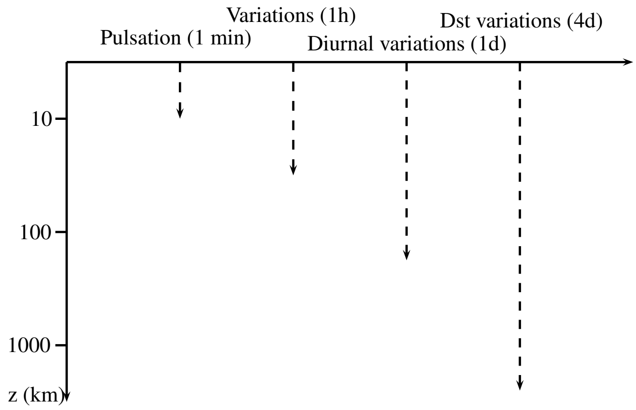


Figure 3.5: Variations

No.	Conductivity of	Charge carrier	T-dependce
1	Gases	Ions, dust particles aerosols	-
2	Semi conductors	electrons	with T increasing according to e^{-A/k_BT}, A: activation e
3	Electrolyt	Ions	p dependence of concentration of ions
4	Metal	free electrons	Drecreasing with increasing T

1) Atmosphere: $\rho \sim 10^{15}\Omega$m
2) Crystal: $\rho \sim 10^7\Omega$m
3) Sea water: $\rho \sim 0.25\Omega$m
4) Earth's core: $\rho \sim 10^{-5}\Omega$m

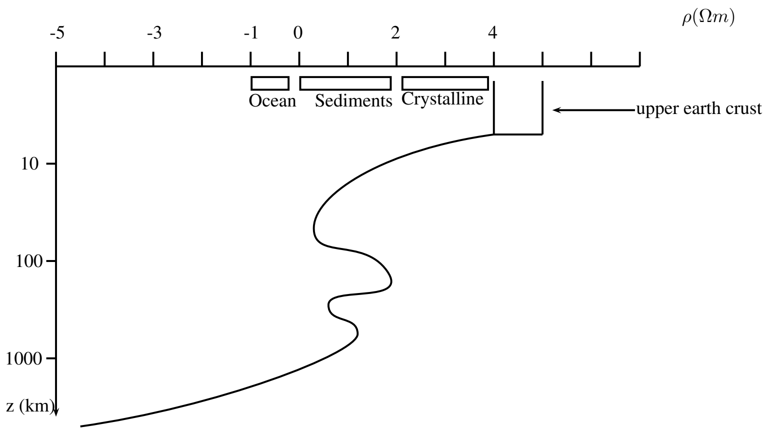


Figure 3.4: Resistivity structure on Earth with depth

The upper earth crust has conductive anomalies in different regions.

4 Basic equations for EM methods

Decay time of free space charges follows an exponential law $\eta(t) = \eta_0 e^{-t/\tau}$, where

$$\tau = \frac{\epsilon\epsilon_0}{\sigma}$$

- For air: $\sigma \approx 10{-}14$S/m, $\epsilon = 1$ thus $\tau \approx 15$min. - For subsurface: $\sigma \approx 10{-}5$S/m, $\epsilon = 1..80$ thus $\tau \approx 10^{-4}$s.
Equtions for - Static fields – no induction - in time scale of τ slowly oscillating fields – quasi-stationary fields - Induction by induced currents much larger than displacement currents - in time scale of τ quickly oscillating fields – displacement currents

```
| Static          | Slow           | Quick
```

$\nabla \times \vec{B}$ | $\mu_0\vec{j}$ | $\mu_0\vec{j}$ | $\mu_0\epsilon\epsilon_0\dot{E}$ $\nabla \times \vec{E}$ | 0 | $-\dot{\vec{B}}$ | $-\dot{\vec{B}}$

4.1 General Maxwell equations for electric and magnetic field quantities

$$\nabla \times \vec{E} = -\dot{\vec{B}} \tag{3.a}$$

Ampere's law states that every closed loop of current will have an associated magnetic field of magnitude proportional to the total current flow:

$$\nabla \times \vec{B} = \mu_0(\sigma\vec{E} + \partial_t\vec{E}) \tag{3.b}$$

$$\nabla \cdot \vec{B} = 0 \tag{3.c}$$

$$\nabla \cdot \vec{E} = \eta/\epsilon_0 \tag{3.d}$$

Rotation of (2.3a) gives

$$\nabla \times \nabla \times \vec{E} = -\nabla \times \vec{B}$$

Time-deriv of (2.3b) gives

$$\nabla \times \partial_t\vec{B} = \mu_0(\sigma\dot{\vec{E}} + \epsilon_0\ddot{\vec{E}}).$$

Without sources in the conductive subsurface $\nabla \cdot \vec{j} = \nabla \cdot (\sigma\vec{E}) = 0$. Using the double rotation identity, it follows for $\sigma \gg \omega\epsilon$:

$$\nabla^2\vec{\mathcal{F}} = \mu_0\sigma\dot{\vec{\mathcal{F}}}$$

where $\vec{\mathcal{F}}$ can be eithe electric or magnetic field. This is a diffusion equation for the fields coming from the external sources and diffusing through the earth. Thus EM fields propagate diffusively; our measurements yield *volume* soundings (response functions are volumetric averages of the sample medium).
OTOH, for quickly oscillating fields:

$$\nabla^2\vec{\mathcal{F}} = \mu_0\epsilon\epsilon_0\ddot{\vec{\mathcal{F}}}.$$

For the quasi-static fields in non-conductors:

$$\nabla \times \vec{B} = 0$$

$\vec{B}$ can be represented as gradient of a scalar potential:

$$\vec{B} = -\nabla U$$

with $\nabla^2 U = 0$.
Application areas:

- Atmosphere:
  - For $T \gg 15$min, field variations diffuse through the atmosphere.
  - In smaller time scales, they travel as waves.

4.2 Fourier transform of the field quantities

Switch to (position-space) frequency domain:

$$\vec{B}(\vec{r}, t) \to \tilde{\vec{B}}(\vec{r}, \omega)$$

where $\omega = \frac{2\pi}{T}$ is the angular frequency, $\tilde{\vec{B}}$ a complex Fourier amplitude.

$$\tilde{f}(\omega) = \int_{\mathbb{R}} \mathrm{d}t\, f(t)e^{-i\omega t}$$

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{d}t \, \tilde{f}(\omega) e^{i\omega t}$$

Induction equations in $(\vec{r}, \omega)$ domain with $n = 1$:

$$\nabla^2 \tilde{\vec{\mathcal{F}}} = i\omega\mu_0\sigma\tilde{\vec{\mathcal{F}}}$$

Wave equations:

$$\nabla^2 \vec{\mathcal{F}} = -\mu_0\epsilon\epsilon_0\omega^2\tilde{\vec{\mathcal{F}}}.$$

We scale (2.10) with the skin-depth formula:

$$p = \sqrt{\frac{2}{\omega\mu_0\sigma}}$$

which increases with $\sqrt{\rho T}$

$$\nabla^2 \vec{\mathcal{F}} = (\tfrac{1+i}{p})^2 \tilde{\vec{\mathcal{F}}}$$

The wave equation can be expressed in terms of the speed of light $c = (\mu_0\epsilon\epsilon_0)^{-1/2}$

$$\nabla^2 F\tilde{ield}$$

Estimation of p in km: - T in s (Pulsations): $p = \frac{1}{2}\sqrt{\rho/\Omega\mathrm{m} \cdot T/\mathrm{s}}$
Now switch also to wavenumber domain (in cartesian coordinates):

$$\vec{r} = x\vec{e}_{\mathrm{x}} + y\vec{e}_{\mathrm{y}} + z\vec{e}_{\mathrm{z}}$$

Assume in the following

$$\vec{k} = k_{\mathrm{x}}\vec{e}_{\mathrm{x}} + k_{\mathrm{y}}\vec{e}_{\mathrm{y}}$$

$$k = \sqrt{{k_{\mathrm{x}}}^2 + {k_{\mathrm{y}}}^2}$$

2D Fourier transform of the induction eqn:

$$\nabla^2 \hat{\tilde{\vec{\mathcal{F}}}} = -k^2\hat{\tilde{\vec{\mathcal{F}}}} + \frac{\mathrm{d}^2\vec{\mathcal{F}}}{\mathrm{d}z^2}$$

$$\frac{\mathrm{d}^2 \; \hat{\tilde{\vec{\mathcal{F}}}}_x}{\mathrm{d}z^2} = (i\omega\mu_0\sigma + k^2)\hat{\tilde{\vec{\mathcal{F}}}}_x$$

We introduce the *vertical wave number*

$$(1+14)K = \sqrt{i\omega\mu_0\sigma + k^2} = \sqrt{(\tfrac{1+i}{p})^2 + k^2}$$

in a layer of conductivity σ. $C = K^{-1}$ is called *complex penetration depth.*
Make further approximations: for $(p \cdot k)^2 \gg 1$,

$$K \approx k(1 + \tfrac{i}{(pk)^2}$$

which means the field diffuses through the layer like it would through a non-conductor. For $(p \cdot k)^2 \ll 1$,

$$K \approx \frac{1+i}{p}(1 - \tfrac{i}{4}(pk)^2)$$

which means the field diffuses like a quasi-homogeneous field ($k = 0$).

4.3 Example: oscillating current in height $H = 300\mathrm{km}$

- For $T = 36\mathrm{s}$, $p = 30\mathrm{km}$ so $p \cdot k = 0.15$, i.e. diffusion is dictated by p
- For $T = 1\mathrm{h}$, $p = 300\mathrm{km}$ so $p \cdot k = 1.5$, which permits neither approximation
- For $T = 24\mathrm{h}$, $p = 1500\mathrm{km}$, diffusion is dictated by k.

5 Induction in homogeneous half space

Solutions in $z > 0$ and $z < 0$ can be combined by continuity equations.

1. Solution for tangential-electric source field (e.g. plane layer currents in the ionosphere) $E_\mathrm{z} = 0$
   - Induced currents are also plane layer currents
   - $E_\mathrm{z} = 0$ in the whole space (*tangential electric* fields).

For $z > 0$ solution of eq. 2.13 $\frac{\mathrm{d}^2 \hat{E}_\mathrm{x}}{\mathrm{d}z^2} = (i\omega\mu_0\sigma + k^2)\hat{E}_\mathrm{x}$

$$\hat{E}_\mathrm{x}(z, k, w) = A_\mathrm{x} e^{-Kz} + B_\mathrm{x} e^{Kz}$$

$$\hat{E}_\mathrm{y}(z, k, w) = A_\mathrm{y} e^{-Kz} + B_\mathrm{y} e^{Kz}$$

Constraint: $E_\mathrm{x} \to 0$ as $z \to \infty$
Derivation of the ma/detic field from $\vec{E}$ using

$$\nabla \times \tilde{\vec{E}} = -i\omega \tilde{\vec{B}}$$

$$\nabla\times\tilde{\vec{E}} = \left(\partial_y \tilde{E}_\mathrm{z} - \partial_z \tilde{E}_\mathrm{y}\right)\vec{e}_\mathrm{x} + \left(\partial_z \tilde{E}_\mathrm{x} - \partial_x \tilde{E}_\mathrm{z}\right)\vec{e}_\mathrm{y} + \left(\partial_x \tilde{E}_\mathrm{y} - \partial_y \tilde{E}_\mathrm{x}\right)\vec{e}_\mathrm{z} = \left(-\partial_z \tilde{E}_\mathrm{y}\right)\vec{e}_\mathrm{x} + \left(\partial_z \tilde{E}_\mathrm{x}\right)\vec{e}_\mathrm{y} + \left(\partial_x \tilde{E}_\mathrm{y} - \partial_y \tilde{E}_\mathrm{x}\right)\vec{e}$$

$$\nabla \times \hat{\vec{E}} = -\frac{\mathrm{d}\hat{E}_\mathrm{y}}{\mathrm{d}z}\vec{e}_\mathrm{x} + \frac{\mathrm{d}\hat{E}_\mathrm{x}}{\mathrm{d}z}\vec{e}_\mathrm{y} + (ik_\mathrm{x}\hat{E}_\mathrm{y} - ik_\mathrm{y}\hat{E}_\mathrm{x})$$

$$\hat{B}_\mathrm{x} = \frac{1}{i\omega}\frac{\mathrm{d}\hat{E}_\mathrm{y}}{\mathrm{d}z}$$

$$\hat{B}_\mathrm{y} = -\frac{1}{i\omega}\frac{\mathrm{d}\hat{E}_\mathrm{x}}{\mathrm{d}z}$$

$$\hat{B}_\mathrm{z} = \tfrac{1}{\omega}(k_\mathrm{y}\hat{E}_\mathrm{x} - k_\mathrm{x}\hat{E}_\mathrm{y})$$

For $-H < z < 0$: solution of (2.5) (Helmholtz)

$$\hat{E}_\mathrm{x} = a_\mathrm{x} e^{-K_0 z} + b_\mathrm{x} e^{K_0 z}$$

$$\hat{E}_\mathrm{y} = a_\mathrm{y} e^{-K_0 z} + b_\mathrm{y} e^{K_0 z}$$

with

$$K_0 = \sqrt{i\omega\mu_0\sigma + k^2 - k_\mathrm{EM}}$$

and

$$k_\mathrm{EM} = \omega\sqrt{\mu_0\epsilon\epsilon_0} = \frac{\omega}{c}$$

K_0 is the wavenumber of an EM wave propagating with the velocity of light c

$$c = \frac{\lambda_\mathrm{EM}}{T} = \frac{\omega}{k_\mathrm{EM}}.$$

Example contributions to K_0:

$$\sigma_0 = 10^{-14}\mathrm{S/m} \Rightarrow \quad p_0 \geq \frac{1}{2}10^2\mathrm{km}$$

for $T > 1$s.

$$\lambda = 2\pi R_\mathrm{E} \Rightarrow \quad k \geq R_E^{-1} \approx \frac{1}{6000}\mathrm{km}^{-1}$$

$p_0 k \geq \frac{10^7}{12000}$, thus diffusion through the atmosphere is like through a conductor with $\sigma = 0$.

Continuity-Equation

For $z = -0$: $E_x = a_x + b_x$ and $B_y = \frac{k_0}{i\omega}(a-b)$
and $z = 0$: $E_x = A_x$ and $B_y = \frac{k}{i\omega}A_x$
Considering the continuity of the tangential $\vec{E}$ and $\vec{B}$:

$$\rightarrow A = a + b \qquad\qquad \text{and} \qquad\qquad KA = k_0(a-b)$$

E_x and E_y are continous functions, therefore B_z is also continious. Forming:

$$\begin{aligned} ae^{-\alpha} &= a(\cosh(\alpha) - \sinh(\alpha)) \\ be^{\alpha} &= b(\cosh(\alpha) + \sinh(\alpha)) \\ \Rightarrow ae^{-\alpha} + be^{\alpha} &= \underbrace{(a+b)}_{A}\cosh(\alpha) + \underbrace{(b-a)}_{-kA/k_0}\sinh(\alpha) \end{aligned}$$

Using eq. 3.15

$-H < z < 0$	$z > 0$
$\hat{E}_x = A_x(\cosh(k_0 z) - \frac{k}{k_0}\sinh(k_0 z))$	$= A_x e^{-kz}$
$\hat{E}_y = A_y(\cosh(k_0 z) - \frac{k}{k_0}\sinh(k_0 z))$	$= A_y e^{-kz}$
$\hat{B}_x = \frac{-A_y}{i\omega}(k\cosh(k_0 z) - k_0\sinh(k_0 z))$	$= \frac{-k}{i\omega}A_y e^{-kz}$
$\hat{B}_y = \frac{-A_x}{i\omega}(k\cosh(k_0 z) - k_0\sinh(k_0 z))$	$= \frac{-k}{i\omega}A_x e^{-kz}$
$\hat{B}_z = \frac{1}{\omega}(k_y A_x - k_x A_y)(k\cosh(k_0 z) - k_0\sinh(k_0 z))$	$= \frac{1}{\omega}(k_y A_x - k_x A_y)e^{-kz}$

For the quasi-homogeneous diffusive fields for $z > 0$ with $\rho k \ll 1$ and $k = \frac{1+i}{\rho}$

$$E_x = \underbrace{Ae^{-z/\rho}}_{reduction of the amplitude} \underbrace{(\cos(z/\rho) - \sin(z/\rho))}_{rotation of phase}$$

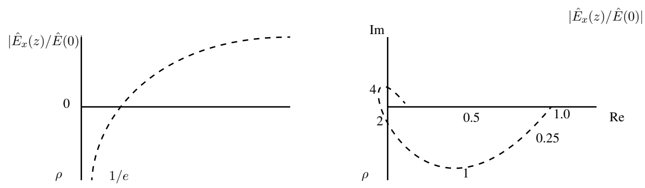


Figure 5.1: Skin effect spiral for homogeneous half space and quasi homogenous fields

Sounding of the halfspace relating to ρ using the observed fields at the earth surface $z = 0$

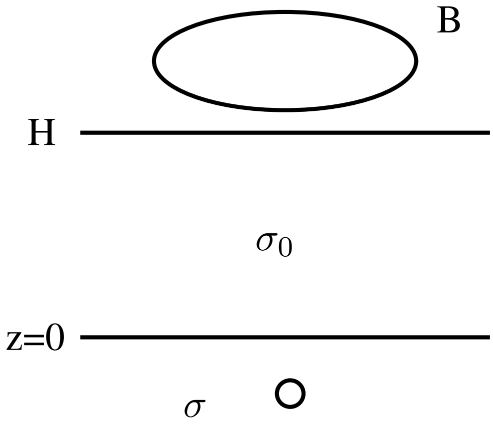
$\hat{E}_x = A_x$, $\hat{E}_y = A_y$, $\hat{B}_x = \frac{-k}{i\omega}A_y$, $\hat{B}_y = \frac{-k}{i\omega}A_x$, ...
Introducing the complex [enetration depth:

$$C(k,\omega) = k^{-1} = \frac{\rho}{2}(1-i) \qquad\qquad for|C|k \ll 1, \lambda \gg |C| \tag{5.17}$$

Determining of c:
$\hat{E}_x = i\omega C\hat{B}_y$, $\hat{E}_y = -i\omega C\hat{B}_x$, $z_{xy} = \frac{E_x}{B_y}$ - Magnetotelluric Sounding
$B_z = C(ik_x\hat{B}_x + ik_y\hat{B}_y)$ from the $z - H$-ratio - Geomagnetic Depth Sounding
Determination of ρ form C for a given wave number and frequency:
Is $|C|k \ll 1$ and $|C|^2 = \rho^2/2 = \frac{\rho}{\omega\mu_0}$ (using the Skin depth). Then formal: $k = 0$.

$$\rho = \omega\mu_0|C|^2 = \frac{\mu_0}{\omega}|z|^2$$

with $z = z_{xz} or z_{yx}$ and $\mu_0/\omega = 0.2$T for E in [mV/km] and B in [nT].



Example: MT-Sounding in Bramwald

Figure MT BRamwald

Figure 5.2: Example Sounding curves

$|C| = \frac{1}{\omega}\frac{E_x}{B_y} = \frac{2000s}{2\pi}\frac{10\cdot 10^{-6}V/m}{20\cdot 10^{-9}Vs/m^2} = 160\text{km}$
$\rho = 0.2 \cdot 2000\left(\frac{10}{20}\right)^2 = 100\Omega m$
Global GDS with Sq:

Figure MT GDS

Figure 5.3: Example Sounding curves GDS

$\lambda/2 = 2\pi R_E/4 = 10000km$, $k_y = 2\pi/20000 \approx 1/3000km^{-1}$
$|C| = B_z/(k_y B_y) = 750km$ for $T = 1d$.

2) Solution for tangential magnetic source fields by meridional currrents

$\Rightarrow$ induced currents are also meridional $\Rightarrow B_z = 0$
Solution of

$$\begin{aligned}\frac{d^2\hat{B}_x}{dz^2} &= k^2\hat{B}_x && in \;\; z > 0\\ &= k_0^2 && in \;\; H < z < 0\end{aligned}$$

$$\hat{B}_x = \begin{cases} ae^{-k_0 z} + be^{k_0 z} \\ Ae^{-kz} \end{cases}$$

Derivation of $\tilde{\vec{E}}$ from $\nabla\times\tilde{\vec{B}}$:

$$\nabla\times\tilde{\vec{B}} = \mu_0\sigma^\star\tilde{\vec{E}}$$

with

$$\sigma^\star = \begin{cases} \sigma_0(1 + i\omega C_0), C_0 = \frac{\epsilon\epsilon_0}{\sigma_0} \\ \sigma \end{cases}$$

$$\hat{E}_x = \frac{-1}{\mu_0 \sigma^\star} \frac{d\hat{B}_y}{dz}$$
$$\hat{E}_y = \frac{1}{\mu_0 \sigma^\star} \frac{d\hat{B}_x}{dz}$$
$$\hat{E}_z = \frac{1}{\mu_0 \sigma^\star}(ik_y \hat{B}_x - ik_x \hat{B}_y)$$

Continuity of $\mathbf{B_{x,y}}$ and $\mathbf{E_{x,y}}$ for $\mathbf{z = 0}$:
$A = a + b$, and

$$KA/\sigma = \frac{k_0(a-b)}{\sigma_0(1 + i\omega C_0)} \tag{5.18}$$

or

$$a - b = \gamma \frac{k}{k_0} A$$

with $\gamma = \frac{\sigma_0(1+i\omega C_0)}{\sigma}$.
Similar to eq. 3.17, full solution:

$$\hat{B}_x = A_x \begin{cases} \cosh(k_0 z) - \gamma \frac{k}{k_0} \sinh(k_0 z) & , z < 0 \\ e^{-kz} & , z > 0 \end{cases}$$

$$\hat{E}_y = \frac{-A_x}{\mu_0 \sigma} \begin{cases} k \cosh(k_0 z) - \frac{k_0}{\gamma} \sinh(k_0 z) & , z < 0 \\ ke^{-kz} & , z > 0 \end{cases}$$

For $z = 0$ (earth surface):

$$\hat{B}_x = A_x \qquad \hat{B}_y = A_y$$
$$\hat{E}_x = \frac{k}{\mu_0 \gamma} A_y \qquad \hat{E}_y = \frac{k}{\mu_0 \gamma} A_x$$
$$\hat{E}_z = \frac{1}{\mu_0 \sigma^\star}(ik_y A_x - ik_y A_y)$$

We form the admittance B_x/E_y ratio considering the complex penetration depth $C = k^{-1}$.

$$\hat{B}_x = -\mu_0 \sigma C \hat{E}_y$$
$$\hat{B}_y = \mu_0 \sigma C \hat{E}_x$$
$$\hat{E}_z = C(ik_x \hat{E}_x + ik_y \hat{E}_y) \begin{cases} 1/\gamma & z = -0 \\ 1 & z = +0 \end{cases}$$

Approximation for quasi-homogenous TM-fields, if $\rho k \ll 1$: $k_0 = k$ and $k = \sqrt{i\omega\mu_0\sigma}$

1.

$$\gamma = \begin{cases} \frac{\sigma_0}{\sigma} & for \;\; T \gg C_0 \\ \frac{i\omega\epsilon\epsilon_0}{\sigma} = -\frac{\omega^2 \mu_0 \epsilon\epsilon_0}{i\omega\mu_0\sigma} = -\left(\frac{k_E}{k}\right)^2 & for \;\; T \ll C_0 \end{cases}$$

2.

$$\mu_0 \sigma C = \frac{i\omega\mu_0\sigma C}{i\omega} = \frac{1}{i\omega C}$$

$\Rightarrow$

$$\hat{E}_x = i\omega C \hat{B}_y \tag{5.19}$$

Impedance of the surface fields does not depend on mode of the source field. Same sounding curves will be valid as derived from TE-source fields .

For the TE-source fields in the air

$\nabla \times \vec{B} = 0$ (see eq. 3.8) and $\vec{B} = -\nabla h$
Potential equation:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$$
$$FT \to -k^2\hat{U} + \frac{d^2\hat{U}}{dz^2} = 0$$

in $-H < z < 0$
Solution:

$$\hat{U}(z) = Ee^{-kz} + Ie^{kz} \quad , \; k = \sqrt{k_x^2 + k_y^2}$$

with E the potential coefficients of the external field part and I of the internal respectively.
Then

$$\hat{B}_x = -\frac{\partial \hat{U}}{\partial x} \to \hat{B}_x = -ik_x(Ee^{-kz} + Ie^{kz})$$
$$\hat{B}_y = -ik_y(Ee^{-kz} + Ie^{kz})$$
$$\hat{B}_z = -ik(Ee^{-kz} - Ie^{kz})$$

Tendency (see chapter 1): For horizontal components: addition of internal and external part. For the vertical component subtraction of internal from external.
Comparison with the "air solution" $(-H < z < 0)$ eq. 3.17 for $K_0 = k$

$$\hat{B}_x = -\frac{A_y}{i\omega}(k\cosh(kz) - k\sinh(kz))$$
$$= -\frac{A_y}{2i\omega}((k+k)e^{-kz} + (k-k)e^{kz})$$

(using $ae^{-\alpha} + be^{\alpha} = (a+b)\cosh\alpha + (b-a)\sinh\alpha$).

$$\Rightarrow E = \frac{-A_y}{2\omega k_x}(K+k) \tag{5.20}$$
$$I = \frac{-A_y}{2\omega k_y}(K-k) \tag{5.21}$$

Additional parameters for quasi-homogeneous source fields:

$$Q(k,\omega) = \frac{I(k,\omega)}{E(k,\omega)} = \frac{K-k}{K+k} = \frac{1-kC(k,\omega)}{1+kC(k,\omega)} \tag{5.22}$$

In summary: Sounding on the Earth's surface using
MT-impedance:

$$z(\omega, k) = i\omega C(k, \omega) : \hat{E}_x = z\hat{B}_y, \hat{E}_y = -z\hat{B}_x$$

GDS:z-H-ratio:

$$\hat{B}_z = ikB_y \frac{k}{k_y} = ikCB_y \frac{k}{k_y}$$

GDS Q(ω, **k**):

$$Q(\omega, k) = \frac{1 - kCQ(\omega, k)}{1 + kCQ(\omega, k)} \Rightarrow I = \frac{1 - kC}{1 + kC} E$$

6 Induction in 1D-Earth models

6.1 Layered models

General solution approach of eq. 3.13.

$$\frac{d^2\hat{E}_x}{dz^2} = (i\omega\mu_0\sigma + k^2)\hat{E}_x$$

in the ω, k domain for TE-fields ($E_z = 0$) for the m. layer.

$$\hat{E}_x(z) = A_m e^{-K_m z} + B_m e^{K_m z} \qquad\qquad z_m < z < z_{m+1}$$

with

$$K_m = \sqrt{i\omega\mu_0\sigma + k^2} \tag{6.1}$$

$$\hat{B}_z = k(Ee^{-kz} - Ie^{kz})$$

Using eq. 3.15 ($\hat{B}_y = -\frac{1}{i\omega}\frac{d\hat{E}_x}{dz}$):

$$\hat{B}_y = \frac{-1}{i\omega}\frac{d\hat{E}_x}{dz} = \frac{K_m}{i\omega}(A_m e^{-K_m z} + B_m e^{K_m z}) \tag{6.2}$$

For the homogeneous half space of the model:

$$\hat{E}_x(z) = A_m e^{-K_m z} \qquad\qquad \hat{B}_y(z) = \frac{K_m}{i\omega} A_m e^{-K_m z}$$

Analogous for $\hat{E}_y$ and $\hat{B}_x$ (see chapter 3.4).
Continuity of $\hat{E}_x$ and $\hat{B}_y$ at the layer boundaries are valid if their impedance ratio:

$$\frac{\hat{E}_x}{\hat{B}_y} = \frac{i\omega}{K_m G(z)} \tag{6.3}$$

with

$$G(z) = \frac{A_m e^{-K_m z} - B_m e^{K_m z}}{A_m e^{-K_m z} + B_m e^{K_m z}} \tag{6.4}$$

is continuous.