Mass and Energy Conservation in Tempest using the Staggered Nodal FEM

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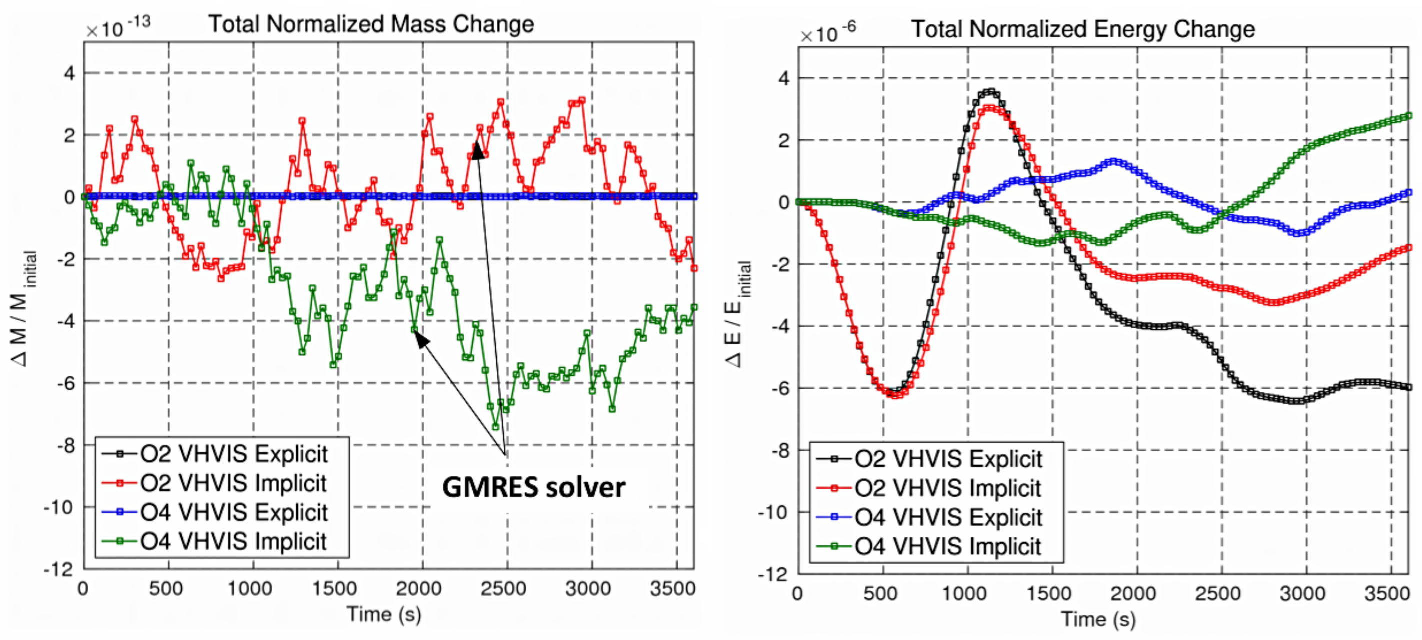
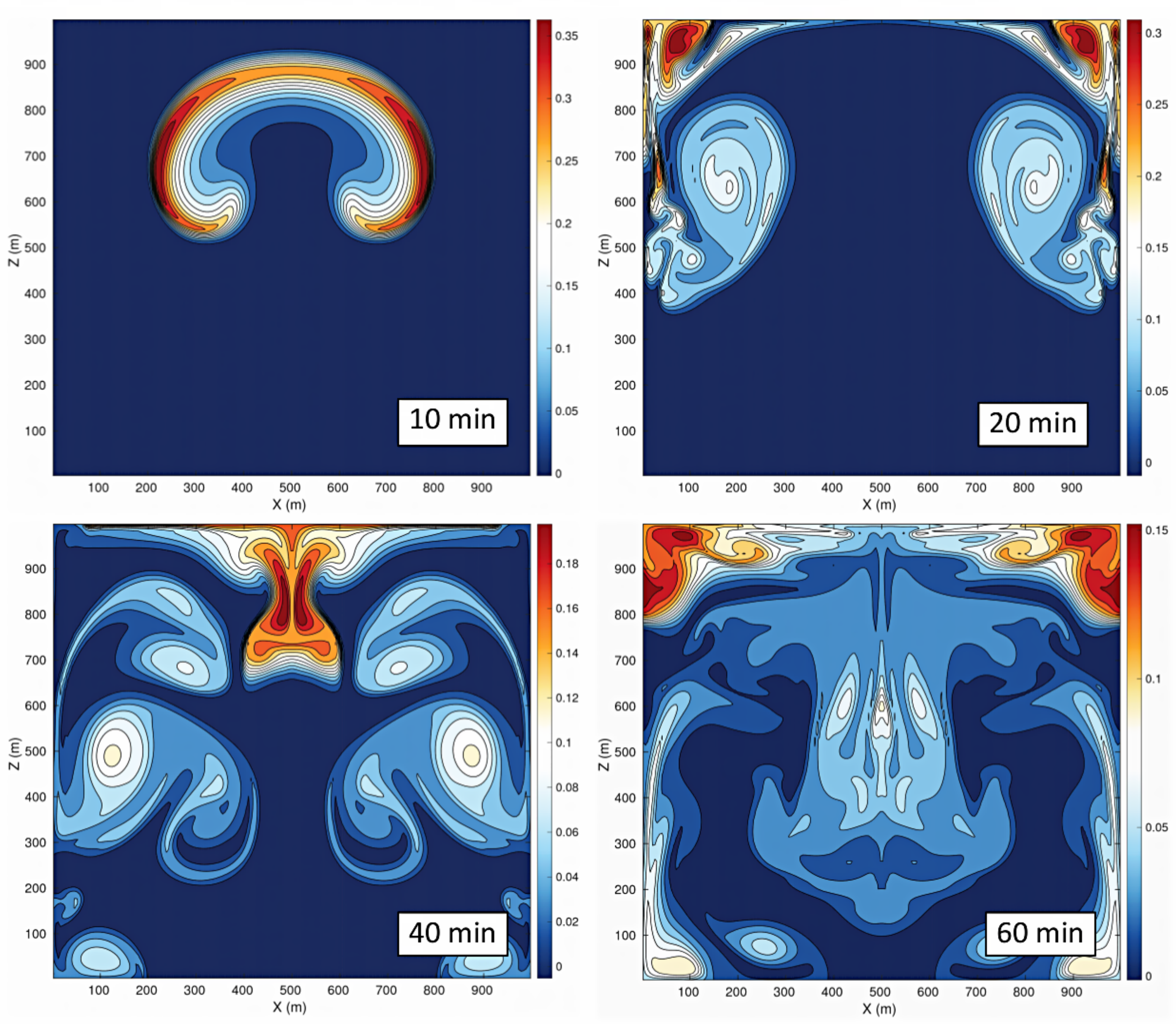
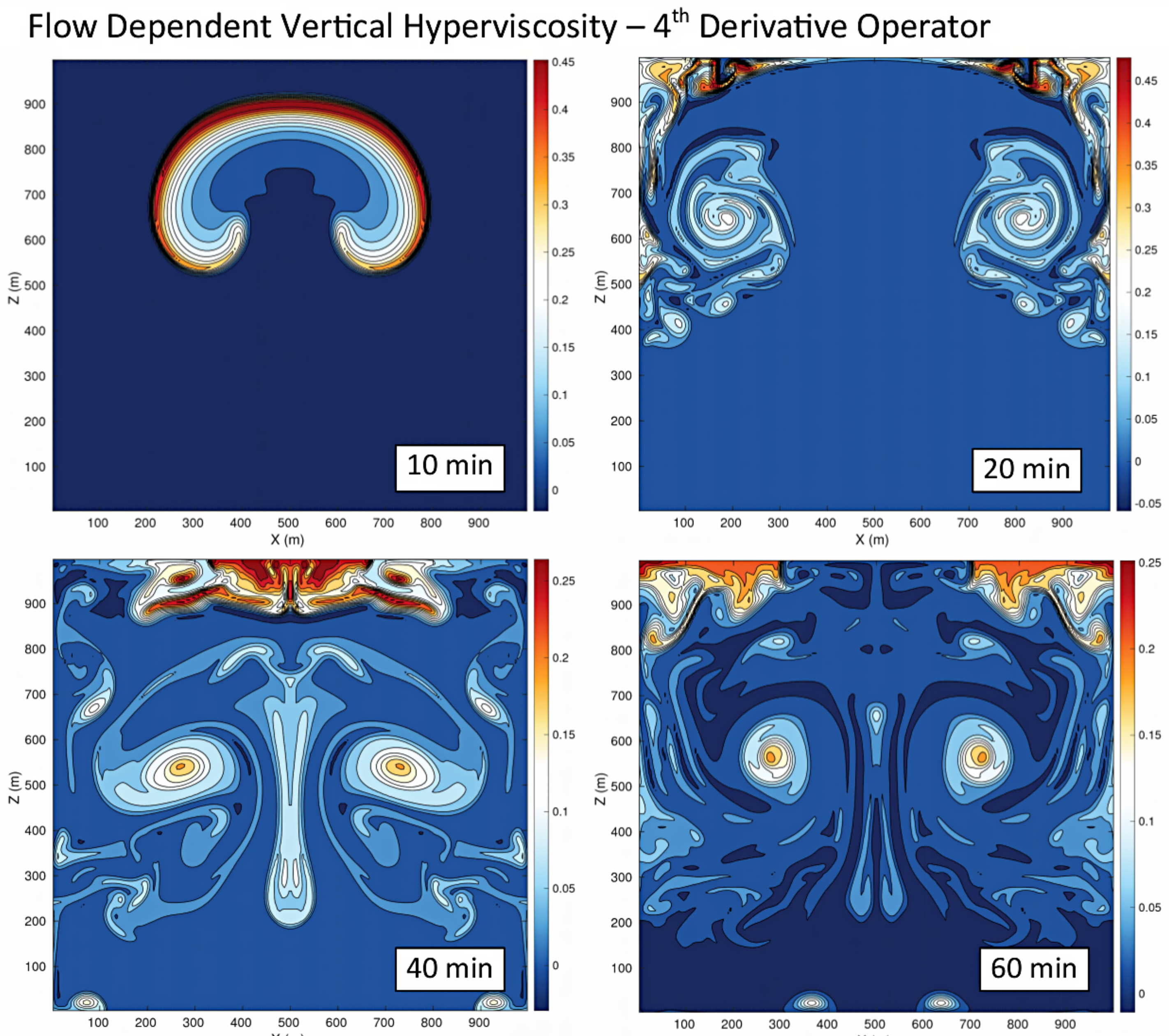
Rising Thermal Bubble 2-D



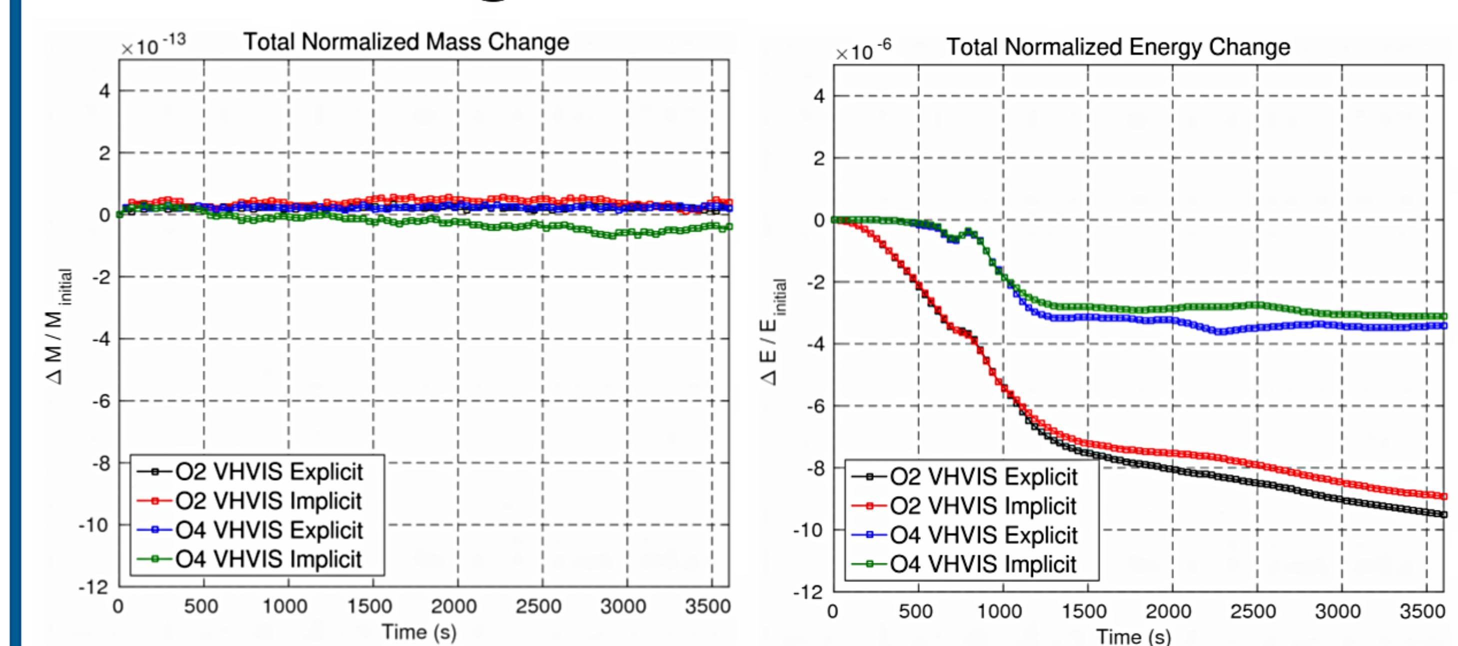
Flow Dependent Vertical Hyperviscosity – 2nd Derivative Operator



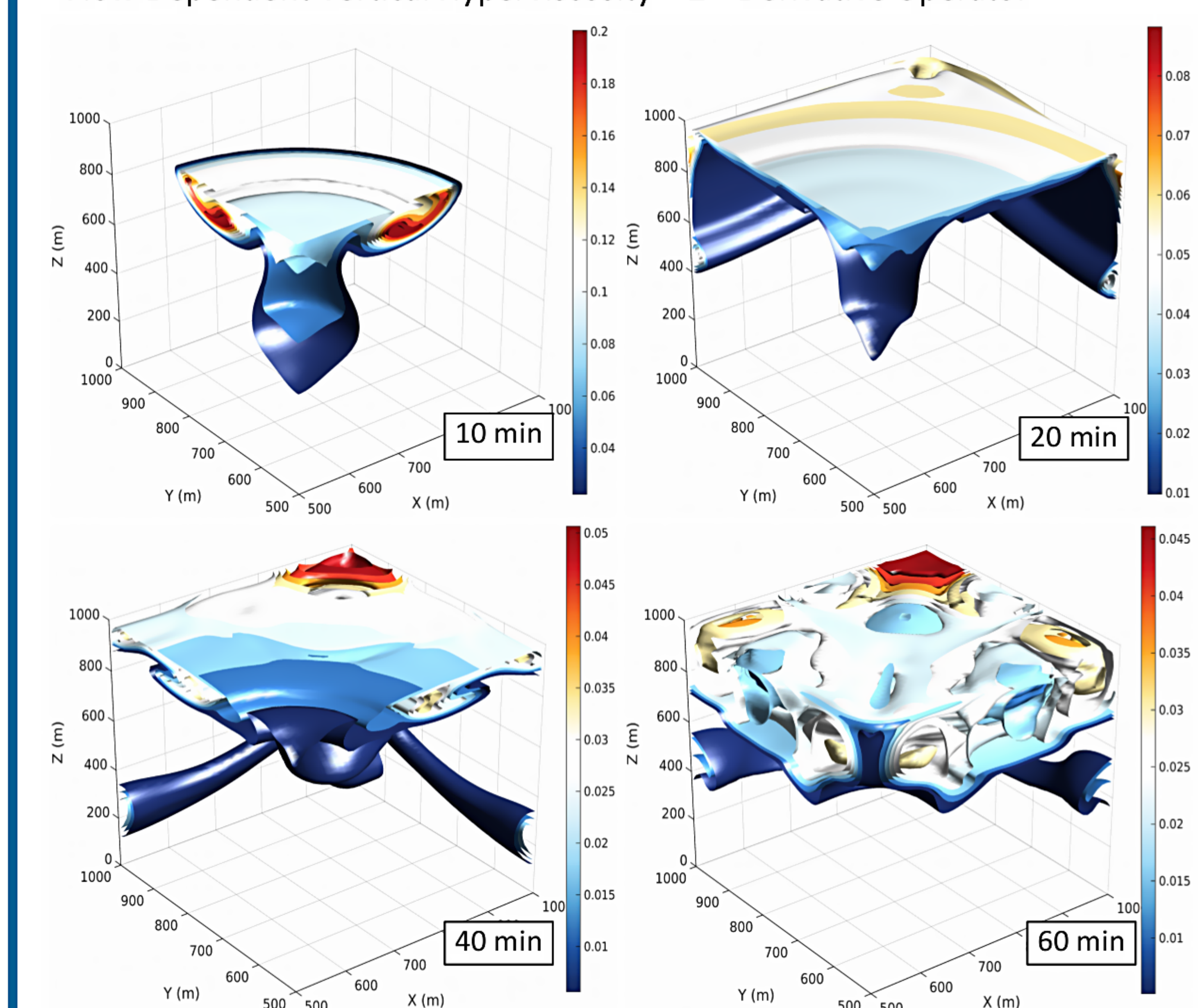
Flow Dependent Vertical Hyperviscosity – 4th Derivative Operator



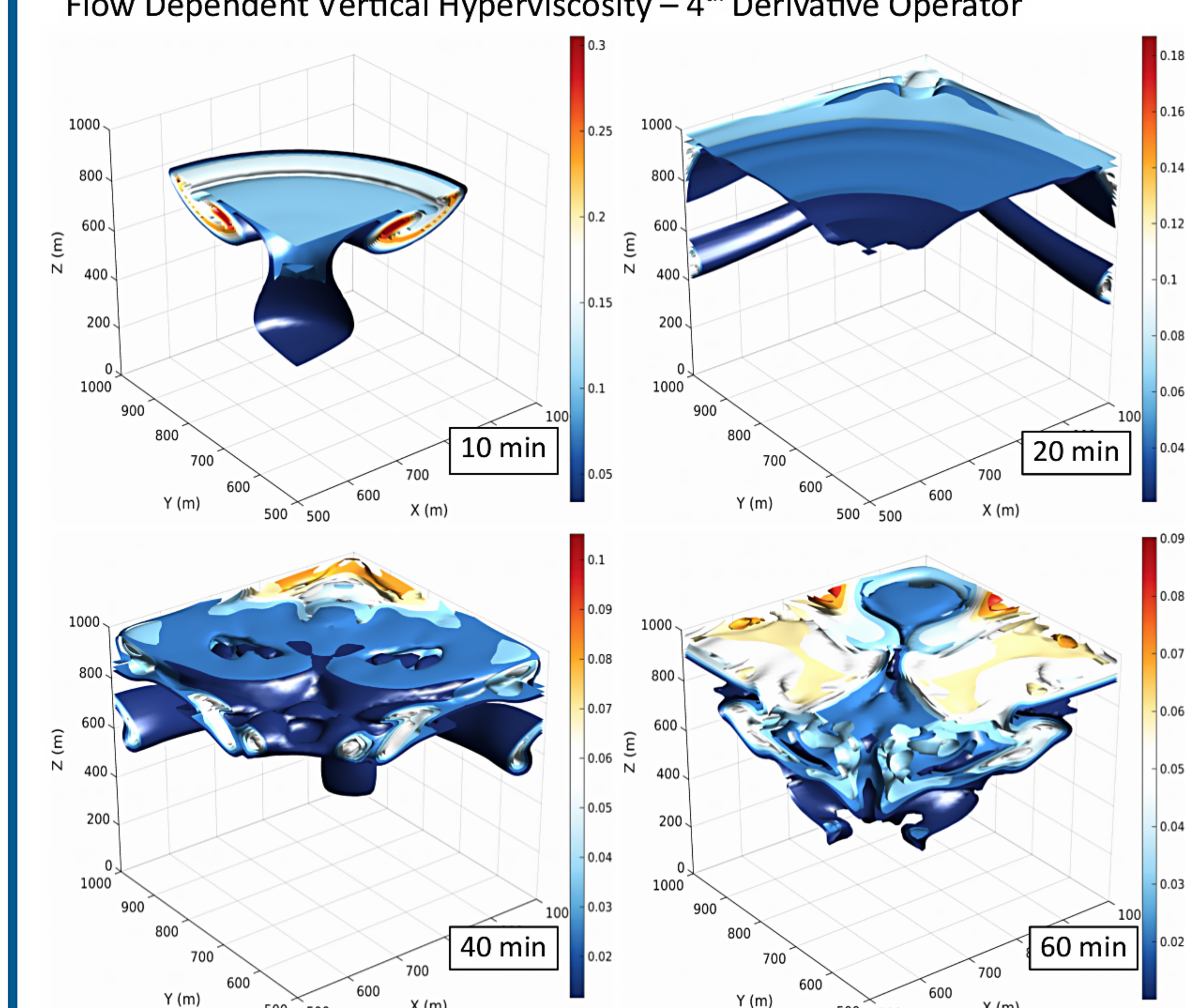
Rising Thermal Bubble 3-D



Flow Dependent Vertical Hyperviscosity – 2nd Derivative Operator



Flow Dependent Vertical Hyperviscosity – 4th Derivative Operator



Discussion

The Staggered Nodal Finite Element Method (SNFEM) allows for the construction of diffusion and hyperdiffusion discrete operators. Diffusion is used for stabilization and the application of explicit dissipation models to the inviscid system of equations.

Here we investigate the effects of combining diffusion with the nonlinear prognositic equations in 2-D and 3-D (in IMEX and fully explicit time integration) on total integrated mass and energy tendencies. We use the Rising Thermal Bubble as defined in (Giraldo, 2008) and extended into 3-D. While the advective equations are strictly mass and energy conservative, the addition of diffusion operators implies that energy should decrease to maintain stability.

However, numerical tests confirm that 2-D turbulent flow results in unique energy tendencies that appear to contradict linear stability criteria. Mass conservation is met, but sensitive to implicit solution strategy. Results from 2-D idealized test cases, while practical, must be considered carefully when evaluating the mimetic properties of a dynamical core.

Flow Dependent Diffusion

$$\frac{\partial \theta}{\partial t} = \cdots + \nu_z |u^{\xi}| \frac{\partial^{2k}\theta}{\partial \xi^{2k}} \qquad \begin{array}{ll} k=1: & \nu_z = (1/2)(\overline{\Delta\xi})^{-1} \\ k=2: & \nu_z = -(1/12)(\overline{\Delta\xi})^{-3} \end{array}$$

$u^{\xi} \rightarrow$ Contravariant vertical wind

$\theta \rightarrow$ Potential temperature

$\overline{\Delta\xi} \rightarrow$ Vertical grid spacing

2nd Derivatives SNFEM

$$\mathcal{L}(\nu)\theta \approx \nu \frac{\partial^2 \theta}{\partial \xi^2} :$$

$$\frac{\partial \theta}{\partial \xi} = 0 \quad \text{at } \xi = 0 \ , \ \xi = 1$$

$\Rightarrow$ Discretizing the weak form over a column in the continuous and discontinuous bases and applying the robust derivative (Ullrich, 2014), (Guerra, 2016) the SNFEM operators follow:

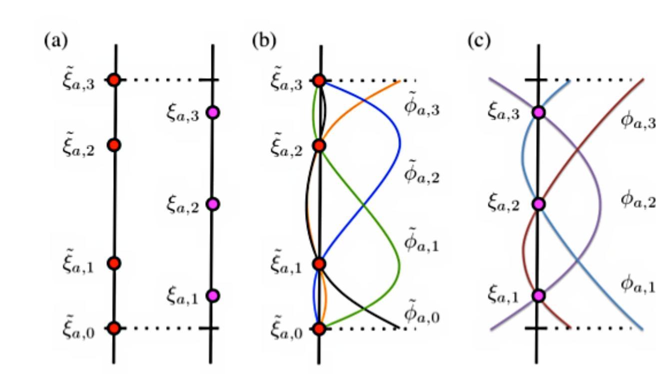
$$[(\mathcal{L}\theta)_{a,k}]^i_i = -\frac{1}{\int_0^1 \tilde{\phi}^2_{a,k} d\xi} \sum_{b=0}^{n_{ve}-1} \sum_{n=0}^{n_{vp}} \tilde{\theta}_{b,n} \int_0^1 \frac{\partial \tilde{\phi}_{a,k}}{\partial \xi} \frac{\partial \tilde{\phi}_{b,n}}{\partial \xi} d\xi$$

Continuous Interfaces

$$[(\mathcal{L}\theta)_{a,k}]^n_n = \frac{1}{\int_{\tilde{\xi}_{a,0}}^{\tilde{\xi}_{a,n_{vp}}} \phi^2_{a,k} d\xi} \left[(\hat{\mathcal{L}}^n_n \theta)_{a,k} + (\mathcal{D}^n_i \theta)_{a,v_{np}} \phi(\tilde{\xi}_{a,v_{np}}) - (\mathcal{D}^n_i \theta)_{a,0} \phi(\tilde{\xi}_{a,0}) \right]$$

$$\left[(\hat{\mathcal{L}}^n_n \theta)_{a,k} \right]^n_n = -\sum_{b=0}^{n_{ve}-1} \sum_{n=0}^{n_{vp-1}} \theta_{b,n} \int_{\tilde{\xi}_{a,0}}^{\tilde{\xi}_{a,n_{vp}}} \frac{\partial \phi_{a,k}}{\partial \xi} \frac{\partial \phi_{b,n}}{\partial \xi} d\xi$$

Discontinuous Levels



$n_{ve} \rightarrow$ Number of column elements

$n_{vp} \rightarrow$ Number of Gauss or Gauss-Lobatto nodes

Fig. 1. (a) Vertical placement of (left) Gauss-Lobatto nodes and (right) Gauss nodes within a vertical element with $n_{vp} = 3$. (b) Basis functions $\tilde{\phi}_{a,k}$ for Gauss-Lobatto nodes within element a. (c) Basis functions $\phi_{a,k}$ for Gauss nodes within element a.

References

1) Ullrich, P. A.: A global finite-element shallow-water model supporting continuous and discontinuous elements, Geosci. Model Dev., 7, 3017–3035, doi:10.5194/gmd-7-3017-2014, 2014.

2) Giraldo, F. X. and Restelli, M.: A study of spectral element and discontinuous Galerkin methods for the Navier Stokes equations in nonhydrostatic mesoscale atmospheric modeling: Equation sets and test cases, J. Comput. Phys., 227, 3849–3877, doi:10.1016/j.jcp.2007.12.009, 2008.

3) Guerra, J. E. and Ullrich, P. A.: A high-order staggered finite-element vertical discretization for non-hydrostatic atmospheric models, Geosci. Model Dev., 9, 2007-2029, doi:10.5194/gmd-9-2007-2016, 2016.

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