where s is a complex variable. The periodic Green's function can be written in two parts by using the previous identity and splitting the path integration at the parameter $\mathtt{E}$ as

$$G_p(\mathbf{r}, \mathbf{r}_s) = G_{p1}(\mathbf{r}, \mathbf{r}_s) + G_{p2}(\mathbf{r}, \mathbf{r}_s) \tag{5.41}$$

where $G_{p1}(\mathbf{r}, \mathbf{r}_s)$ is given by

$$G_{p1}(\mathbf{r}, \mathbf{r}_s) = \frac{1}{4\pi} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-j(k_x m \mathtt{D_x} + k_y n \mathtt{D_y})} \times \frac{2}{\sqrt{\pi}} \int_0^{\mathtt{E}} e^{\left(-R_{mn}^2 s^2 + \frac{k_0^2}{4s^2}\right)} ds \tag{5.42}$$

and $G_{p2}(\mathbf{r}, \mathbf{r}_s)$ is given by

$$G_{p2}(\mathbf{r}, \mathbf{r}_s) = \frac{1}{4\pi} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-j(k_x m \mathtt{D_x} + k_y n \mathtt{D_y})} \times \frac{2}{\sqrt{\pi}} \int_{\mathtt{E}}^{\infty} e^{\left(-R_{mn}^2 s^2 + \frac{k_0^2}{4s^2}\right)} ds \tag{5.43}$$

with k_x, k_y and R_{mn} as the ones used in equation (5.37). For the integral in equation (5.43), Ewald transformation applies directly. More precisely, using the identity [72, Equation 7.4.34],

$$\frac{2}{\sqrt{\pi}} \int_{\mathtt{E}}^{\infty} e^{\left(-R_{mn}^2 s^2 + \frac{k_0^2}{4s^2}\right)} ds = \frac{1}{2R_{mn}} \left[e^{-jk_0 R_{mn}} \mathtt{erfc}\left(R_{mn}\mathtt{E} - \frac{jk}{2\mathtt{E}}\right) + e^{jk_0 R_{mn}} \mathtt{erfc}\left(R_{mn}\mathtt{E} + \frac{jk}{2\mathtt{E}}\right) \right] \tag{5.44}$$

where $\mathtt{erfc}$ is the complementary error function, the integral can be written as

$$G_{p2}(\mathbf{r}, \mathbf{r}_s) = \frac{1}{8\pi} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \frac{e^{-j(k_x m \mathtt{D_x} + k_y n \mathtt{D_y})}}{R_{mn}} \times \sum_{\pm} \left[e^{\pm jk_0 R_{mn}} \mathtt{erfc}\left(R_{mn}\mathtt{E} \pm \frac{jk}{2\mathtt{E}}\right) \right] \tag{5.45}$$

which is essentially a "modified" spatial-domain portion of the periodic Green's function. The summation over $\pm$ is a shorthand notation for the right hand side of equation (5.44) and it will be used along this chapter.