HOFEM-AIRBUS Transmission/Reflection

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GREMA
RADIOFREQUENCY, ELECTROMAGNETICS, MICROWAVES & ANTENNAS

Outline

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We start considering different alternatives to implement the TX/RX conditions in the context of the present FEM formulation coded in HOFEM

Outline

FEM Formulation

- Formulation based on double curl vector wave equation (use of **E** or **H**).

$$\boldsymbol{\nabla} \times \left(f_r^{-1} \boldsymbol{\nabla} \times \mathbf{V}\right) - k_0^2 g_r \mathbf{V} = -jk_0 H_0 \mathbf{P} + \nabla \times f_r^{-1} \mathbf{L}$$

Table: Formulation magnitudes and parameters

	$\mathbf{V}$	$\mathbf{V}^d$	$\bar{\bar{f}}_r$	$\bar{\bar{g}}_r$	h	$\mathbf{P}$	$\mathbf{L}$
Form. **E**	$\mathbf{E}$	$\mathbf{H}$	$\bar{\bar{\mu}}_r$	$\bar{\bar{\epsilon}}_r$	η	$\mathbf{J}$	$\mathbf{M}$
Form. **H**	$\mathbf{H}$	$\mathbf{E}$	$\bar{\bar{\epsilon}}_r$	$\bar{\bar{\mu}}_r$	$-\frac{1}{\eta}$	$\mathbf{M}$	$-\mathbf{J}$

FEM Formulation (cont.)

- Use of $\mathbf{H}(\text{curl})$ spaces:

$$\mathbf{H}(\text{curl})_0 = \{\mathbf{W} \in \mathbf{H}(\text{curl}),\ \hat{\mathbf{n}} \times \mathbf{W} = 0 \ \text{ on } \ \Gamma_\text{D}\} \tag{1}$$

$$\mathbf{H}(\text{curl}) = \{\mathbf{W} \in L^2,\ \boldsymbol{\nabla} \times \mathbf{W} \in L^2\} \tag{2}$$

- and Galerkin method leads to (with respect to the double-curl term only) to:

$$\int_\Omega (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \left(\bar{\bar{f}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{V}\right) d\Omega + \underbrace{\int_\Gamma \mathbf{F} \cdot (\hat{\mathbf{n}} \times \underbrace{(\bar{\bar{f}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{V})}_{-jk_0 h_0 \mathbf{V}^d}) d\Gamma}_{\text{“natural” b.c.}}$$

- More precisely
  - Form. $\mathbf{E}$: $\bar{\bar{f}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{V} = -jk_0\eta_0 \, \mathbf{H}$
  - Form. $\mathbf{H}$: $\bar{\bar{f}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{V} = +j\dfrac{k_0}{\eta_0} \, \mathbf{E}$

FEM Formulation (cont.)

The integral boundary term $\int_\Gamma \mathbf{F} \cdot (\hat{\mathbf{n}} \times (\bar{\bar{f}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{V})) d\Gamma$

- Can be used to weakly impose some boundary conditions of the problem
- For instance, Neumann boundary condition

$$\hat{\mathbf{n}} \times \left(\bar{\bar{f}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{V} \right) = \boldsymbol{\Psi}_\mathrm{N}$$

$$\longrightarrow \quad \int_\Gamma \mathbf{F} \cdot (\hat{\mathbf{n}} \times (\bar{\bar{f}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{V})) d\Gamma = \int_\Gamma \boldsymbol{\Psi}_\mathrm{N} \, d\Gamma$$

- For instance, ABC boundary condition

$$\hat{\mathbf{n}} \times \left(\bar{\bar{f}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{V} \right) + \gamma \, \hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{V} = \boldsymbol{\Psi}_\mathrm{C}$$

$$\longrightarrow \quad \int_\Gamma \mathbf{F} \cdot (\hat{\mathbf{n}} \times (\bar{\bar{f}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{V})) \, d\Gamma = \int_\Gamma \boldsymbol{\Psi}_\mathrm{C} d\Gamma + \gamma \int_\Gamma (\hat{\mathbf{n}} \times \mathbf{F}) \cdot (\hat{\mathbf{n}} \times \mathbf{V}) \, d\Gamma_\mathrm{C}$$

FEM Formulation (cont.)

- Note that the implementation of above boundary conditions is straightforward because
  - Either, the integrand of the boundary term was a known function
  - or, it can be expressed in terms of the primal unknown $\mathbf{V}$
- At the end of the day, the contribution of the boundary term is translated into algebra as
  - extra contributions for the values of the matrix coefficients (related to the existing g_i associated to the corresponding boundary).
  - I.e., no extra degrees of freedom g_i are needed

What does it happen when that is not the case?

FEM Continuity Conditions

- Assuming $\mathbf{V} = \sum_i g_i \mathbf{N}_i \leftrightarrow \{g\}$

Between Neighbour Elements 1 and 2

- Two continuity conditions:
  - Strong continuity of $\hat{\mathbf{n}} \times \mathbf{V}$

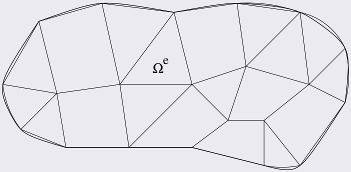
$$\{g\}_1 = \{g\}_2$$

  - Weak continuity of $\hat{\mathbf{n}} \times \mathbf{V}^d$

$$\int_\Gamma \mathbf{F} \cdot (\hat{\mathbf{n}}_1 \times (\bar{\bar{f}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{V}_1)) d\Gamma$$

$$+ \int_\Gamma \mathbf{F} \cdot (\hat{\mathbf{n}}_2 \times (\bar{\bar{f}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{V}_2)) d\Gamma = 0$$

We simply omit the term



Note that $\hat{\mathbf{n}} \times \mathbf{F}_1 = \hat{\mathbf{n}} \times \mathbf{F}_2$ and that
$\mathbf{F} \cdot (\hat{\mathbf{n}} \times (\bar{\bar{f}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{V}) = (\hat{\mathbf{n}} \times \mathbf{F}) \cdot (\hat{\mathbf{n}} \times (\bar{\bar{f}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{V})$

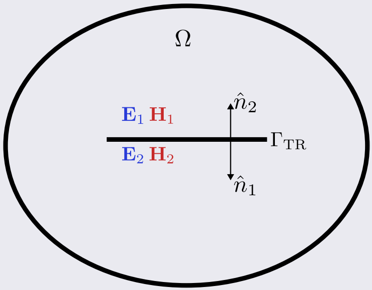
FEM Continuity Conditions (cont.)

Between Elements Simulating a Thin Sheet

- In general, we need to break continuity on
  - $\hat{\mathbf{n}} \times \mathbf{V} \Rightarrow$ Duplication of degrees of freedom: $\{g\}_1, \{g\}_2$
  - $\hat{\mathbf{n}} \times \mathbf{V}^d \Rightarrow$ The integral boundary term can not be omitted
- Two linearly independent equations involving four magnitudes $\mathbf{E}_1, \mathbf{E}_2, \mathbf{H}_1, \mathbf{H}_2$:

$$[\ ]_{2x2}\,\{X\}_{2x1} = \{B\}_{2x1}$$

- Different possibilities depending on the magnitudes chosen as sources (right hand side B) and responses X
- Clear analogy with two-port network parameters of linear circuit analysis: $\mathbf{E} \leftrightarrow V \qquad \mathbf{H} \leftrightarrow I$



The Immittance Approach

The Simplest (Least Invasive) Approach

- We express $\hat{\mathbf{n}} \times \mathbf{V}^d$ for region 1 and region 2 in terms of $\hat{\mathbf{n}} \times \mathbf{V}$ of region 1 and region 2, i.e.,

$$\begin{Bmatrix} \hat{\mathbf{n}} \times \mathbf{V}_1^d \\ \hat{\mathbf{n}} \times \mathbf{V}_2^d \end{Bmatrix} = \begin{bmatrix} YZ_{11} & YZ_{12} \\ YZ_{21} & YZ_{22} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{n}} \times \mathbf{V}_1 \\ \hat{\mathbf{n}} \times \mathbf{V}_2 \end{Bmatrix}$$

  - This is equivalent to an immitance characterization of the equivalent two-port network connecting upper and lower regions separated by the sheet.
    - ⋆ Form. **E**: $\mathbf{V} = \mathbf{E}$, $\mathbf{V}^d = \mathbf{H}$, immittance $\equiv$ admitance
    - ⋆ Form. **E**: $\mathbf{V} = \mathbf{E}$, $\mathbf{V}^d = \mathbf{H}$, immittance $\equiv$ impedance
- We substitute $\hat{\mathbf{n}} \times \mathbf{V}^d$ on the integral boundary terms corresponding to region 1 and region 2 by its linear combination of $\hat{\mathbf{n}} \times \mathbf{V}_1$ and $\hat{\mathbf{n}} \times \mathbf{V}_2$.

The Immittance Approach (cont.)

The Simplest (Least Invasive) Approach

- For instance, with **E**-formulation we have

$$\int_\Gamma \mathbf{F}_1 \cdot (\hat{\mathbf{n}} \times (\bar{\bar{\mu}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{E}_1)) d\Gamma = y_{11} \int_\Gamma \mathbf{F}_1 \cdot (\hat{\mathbf{n}} \times \mathbf{E}_1)) d\Gamma + y_{12} \int_\Gamma \mathbf{F}_1 \cdot (\hat{\mathbf{n}} \times \mathbf{E}_2)) d\Gamma$$

$$\int_\Gamma \mathbf{F}_2 \cdot (\hat{\mathbf{n}} \times (\bar{\bar{\mu}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{E}_2)) d\Gamma = y_{21} \int_\Gamma \mathbf{F}_2 \cdot (\hat{\mathbf{n}} \times \mathbf{E}_1)) d\Gamma + y_{22} \int_\Gamma \mathbf{F}_2 \cdot (\hat{\mathbf{n}} \times \mathbf{E}_2)) d\Gamma$$

where $\hat{\mathbf{n}} \times \mathbf{F}_1 = \hat{\mathbf{n}} \times \mathbf{F}_2$ (conformal mesh above-below the sheet).

The Immittance Approach (cont.)

The Simplest (Least Invasive) Approach

- Thus, the TX/RX continuity conditions imposed by the sheet are translated into algebra simply as extra contributions for the values of the matrix coefficients (related to the duplicated degrees of freedom $\{g\}_1$, $\{g\}_2$ associated to the sheet).

Advantages

- Simple (non code invasing)
  - No new variational unknowns
  - No new equations but simply additional terms to the existing ones
  - Only requires replication of uknonws on sheet: $\{g\}_1 \neq \{g\}_2$
- Reciprocity of the material ($y_{12} = y_{21}$) $\Rightarrow$ symmetry of FEM matrix
- Short circuit (PEC) and open circuit (PMC) conditions naturally reproducible

The Immittance Approach (cont.)

The Simplest (Least Invasive) Approach

Disadvantages

- Total transmission (sheet transparency) not reproducible
  - Singular system of equations
  - This situation must be either avoided or taken into account explicitly

Short Circuit (PEC) Sheet

- We have $y_{12} = y_{21} = 0$ and $y_{11} = y_{22} = \infty$
- Thus, the equation associated to $\{g\}_1$ has a dominant term

$$\ldots + y_{11} \int_{\Gamma} \mathbf{F}_1 \cdot (\hat{\mathbf{n}} \times \mathbf{E}_1)) d\Gamma + y_{12} \int_{\Gamma} \mathbf{F}_1 \cdot (\hat{\mathbf{n}} \times \mathbf{E}_2)) d\Gamma \approx \infty \int_{\Gamma} \mathbf{F}_1 \cdot (\hat{\mathbf{n}} \times \mathbf{E}_1)) d\Gamma$$

  that corresponds to impose $\hat{\mathbf{n}} \times \mathbf{E}_1 = 0$, i.e., algebraically $\{g\}_1 = 0$
- Analogously, with equation associated to $\{g\}_2$

$$\ldots + y_{21} \int_{\Gamma} \mathbf{F}_2 \cdot (\hat{\mathbf{n}} \times \mathbf{E}_1)) d\Gamma + y_{22} \int_{\Gamma} \mathbf{F}_2 \cdot (\hat{\mathbf{n}} \times \mathbf{E}_2)) d\Gamma \approx \infty \int_{\Gamma} \mathbf{F}_2 \cdot (\hat{\mathbf{n}} \times \mathbf{E}_2)) d\Gamma$$

  that corresponds to impose $\hat{\mathbf{n}} \times \mathbf{E}_2 = 0$, i.e., algebraically $\{g\}_2 = 0$

Open Circuit (PMC) Sheet

- We have $y_{12} = y_{21} = 0$ and $y_{11} = y_{22} = 0$
- Thus, the equation associated to $\{g\}_1$ is equivalent to have

$$\int_\Gamma \mathbf{F}_1 \cdot (\hat{\mathbf{n}} \times (\bar{\bar{\mu}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{E}_1)) d\Gamma = 0$$

  that corresponds to weakly impose $\hat{\mathbf{n}} \times \bar{\bar{\mu}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{E}_1 = 0$ (Neumann b.c.) i.e., $\hat{\mathbf{n}} \times \mathbf{H}_1 = 0$.
- Analogously, with equation associated to $\{g\}_2$

$$\int_\Gamma \mathbf{F}_2 \cdot (\hat{\mathbf{n}} \times (\bar{\bar{\mu}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{E}_2)) d\Gamma = 0$$

  that corresponds to weakly impose $\hat{\mathbf{n}} \times \bar{\bar{\mu}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{E}_2 = 0$ (Neumann b.c.) i.e., $\hat{\mathbf{n}} \times \mathbf{H}_2 = 0$.

Total Transmission (sheet transparency)

- It is clear that from

$$\begin{Bmatrix} \hat{\mathbf{n}} \times \mathbf{V}_1^d \\ \hat{\mathbf{n}} \times \mathbf{V}_2^d \end{Bmatrix} = \begin{bmatrix} YZ_{11} & YZ_{12} \\ YZ_{21} & YZ_{22} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{n}} \times \mathbf{V}_1 \\ \hat{\mathbf{n}} \times \mathbf{V}_2 \end{Bmatrix}$$

  is not possible to impose full continuity of $\hat{\mathbf{n}} \times \mathbf{V}$, $\hat{\mathbf{n}} \times \mathbf{V}^d$, i.e.,

$$\begin{Bmatrix} \hat{\mathbf{n}} \times \mathbf{V}_1 \\ \hat{\mathbf{n}} \times \mathbf{V}_1^d \end{Bmatrix} = \begin{Bmatrix} \hat{\mathbf{n}} \times \mathbf{V}_2 \\ \hat{\mathbf{n}} \times \mathbf{V}_2^d \end{Bmatrix}$$

- Technically, we have $y_{12} = y_{21} = \infty$ and $y_{11} = y_{22} = \infty$

The Immittance Approach (cont.)

Special Cases

Total Transmission (sheet transparency)

- Thus, the equation associated to $\{g\}_1$ has two dominant terms

$$\ldots + \infty \int_\Gamma \mathbf{F}_1 \cdot (\hat{\mathbf{n}} \times \mathbf{E}_1))d\Gamma + \infty \int_\Gamma \mathbf{F}_1 \cdot (\hat{\mathbf{n}} \times \mathbf{E}_2))d\Gamma$$

- Analogously, with equation associated to $\{g\}_2$

$$\ldots + \infty \int_\Gamma \mathbf{F}_2 \cdot (\hat{\mathbf{n}} \times \mathbf{E}_1))d\Gamma + \infty \int_\Gamma \mathbf{F}_2 \cdot (\hat{\mathbf{n}} \times \mathbf{E}_2))d\Gamma$$

Total Transmission (sheet transparency)

- Both equations tends to be the same equation

  $\Rightarrow$ Singular FEM matrix

  (I mush still check signs for y_{ij}) Nevertheless, the above resulting equations can never impose continuity of $\hat{\mathbf{n}} \times \mathbf{V}$

The Immittance Approach (cont.)

Special Cases

Workaround

- We must identify the case, i.e., when y_{ij} is "large enough"
- Then, we have two main possibilities:
  - Ignore TR/RX boundary conditions for that sheet, i.e., do nothing
    - Do not replicate degres of freedom $\{g\}$
    - Do not alter FEM equations
  - Recover full continuity, i.e., continuity of $\hat{\mathbf{n}} \times \mathbf{V}$ and $\hat{\mathbf{n}} \times \mathbf{V}^d$ once replication of $\{g\}$ has been perfomed
    - Several possibilities

The Immittance Approach (cont.)

Special Cases

How to Recover Full Continuity?

- We keep the integral boundary term $\int_\Gamma \mathbf{F} \cdot (\hat{\mathbf{n}} \times (\bar{\bar{f}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{V}))d\Gamma$ for region 1 and region 2
- We add two equations to recover continuity of $\hat{\mathbf{n}} \times \mathbf{V}$ and $\hat{\mathbf{n}} \times \mathbf{V}^d$

Continuity of $\hat{\mathbf{n}} \times \mathbf{V}$

- We add one strong condition equation

$$\{g\}_1 = \{g\}_2$$

- Perfomed at the algebraic level, i.e., operating in the matrix by fixing all coefficients on the corresponding row to zero except the two being related (paired)
- The above operation is simple, trivial I would say.
- However, it introduces asymmetry in the FEM matrix

The Immittance Approach (cont.)

Special Cases

Continuity of $\hat{\mathbf{n}} \times \mathbf{V}^d$, i.e., of $\hat{\mathbf{n}} \times (\bar{\bar{f}}_r^{-1} \nabla \times \mathbf{V})$

- We add a equation to weakly force continuity of the dual variable

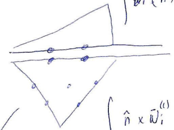
$$\int_{\Gamma} \mathbf{F}_1 \cdot (\hat{\mathbf{n}} \times (\bar{\bar{\mu}}_r^{-1} \nabla \times \mathbf{E}_1)) d\Gamma = \int_{\Gamma} \mathbf{F}_2 \cdot (\hat{\mathbf{n}} \times (\bar{\bar{\mu}}_r^{-1} \nabla \times \mathbf{E}_2)) d\Gamma \quad (3)$$

- Note that the above equation after discretization introduces asymmetry in the matrix
  - Although, $\hat{\mathbf{n}} \times \mathbf{F}_1 = \hat{\mathbf{n}} \times \mathbf{F}_2$ due to conformity of the mesh
  - the discretization of $\hat{\mathbf{n}} \times \nabla \times \mathbf{E}$ involves all basis functions of the finite element, i.e., not only basis functions on the common boundary (sheet)

The Immittance Approach (cont.)

Special Cases

$$\int \hat{n}\times \bar{W}_i^{(1)} \cdot \left(\hat{n}\times \mu_r^{-1}\left(\nabla\times N_j^{(1)}\right)\right) d\Gamma$$



$$\int \hat{n}\times \bar{W}_i^{(1)} \cdot \left(n\times \mu_r^{-1}\, \nabla\times N_j^{(2)}\right) d\Gamma$$

$\nabla\times E$ depends on all g_i of the element

The Immittance Approach (cont.)

Special Cases

- The above procedure of recovering full continuity can also be approached by defining a new variational unknown for the dual field, i.e., for $\hat{\mathbf{n}} \times \mathbf{V}^d$; basically,
  - A **J** for **E**-formulation
  - A **M** for **H**-formulation
- Once being involved with new variational uknowns it is much worthier to consider other approaces (described later in this presentation??)

How to Recover Full Continuity? (EQUIVALENT METHOD)
need to think twice if is really equivalent

- We use ABCs in each region, i.e., we substitute in the integral boundary term of region 1 and region 2 as follows $\int_\Gamma \mathbf{F} \cdot (\hat{\mathbf{n}} \times (\bar{\bar{f}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{V}))d\Gamma$ with

$$\int_\Gamma \mathbf{F}_1 \cdot (\hat{\mathbf{n}} \times (\bar{\bar{f}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{V}_1))\, d\Gamma = jk_0 \int_\Gamma (\hat{\mathbf{n}} \times \mathbf{F}_1) \cdot (\hat{\mathbf{n}} \times \mathbf{V}_1)\, d\Gamma_C$$
$$\int_\Gamma \mathbf{F}_2 \cdot (\hat{\mathbf{n}} \times (\bar{\bar{f}}_r^{-1} \boldsymbol{\nabla} \times \mathbf{V}_2))\, d\Gamma = jk_0 \int_\Gamma (\hat{\mathbf{n}} \times \mathbf{F}_2) \cdot (\hat{\mathbf{n}} \times \mathbf{V}_2)\, d\Gamma_C$$

  This is trivial
- We add one strong condition equation

$$\{g\}_1 = \{g\}_2$$

The $[g]$, $[h]$, $[ABCD]$ Approaches

$[g]$ Parameters

$$\begin{Bmatrix} I_1 \\ V_2 \end{Bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{Bmatrix} V_1 \\ I_2 \end{Bmatrix}$$

$[h]$ Parameters

$$\begin{Bmatrix} V_1 \\ I_2 \end{Bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{Bmatrix} I_1 \\ V_2 \end{Bmatrix}$$

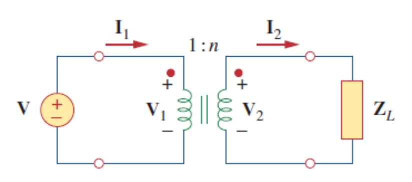
$[ABCD]$ Parameters

$$\begin{Bmatrix} V_1 \\ I_1 \end{Bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{Bmatrix} V_2 \\ I_2 \end{Bmatrix}$$

The [g], [h], [$ABCD$] Approaches

Total Transmission (sheet transparency)

Analogy: Ideal Transformer



$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & n \end{bmatrix}$$

$$\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{n} \\ -\frac{1}{n} & 0 \end{bmatrix}$$

The $[g]$, $[h]$, $[ABCD]$ Approaches
FEM Implementation

- Duplication of degrees of freedom: $\{g\}_1, \{g\}_2$
- Keeping the integral boundary terms in region 1 and region 2
- Adding two equations to weakly enforce the corresponding tranmsision conditions (given by $[g]$, $[h]$, $[ABCD]$, etc)
  - Optionally, definition of a new variational unknown for the dual field, i.e., for $\hat{\mathbf{n}} \times \mathbf{V}^d$; basically,
    - A $\mathbf{J}$ for $\mathbf{E}$-formulation
    - A $\mathbf{M}$ for $\mathbf{H}$-formulation

    for helping to set the equations

Elaborar sobre la muy buena idea de Sergio de ¡utilizar relaciones lineales entre combinaciones de V e I como sucede en la carateterización con los parámetros S

Outline

TITULO

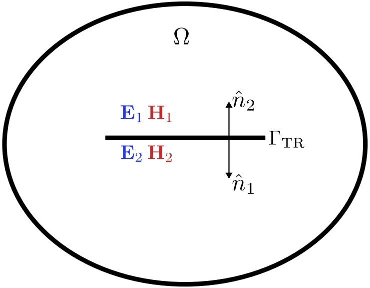
Hola

Outline

We describe the implementation of the TX/RX conditions uisng the characterization of the material sheet in terms of its immittance (impedance/admittance) matrix

Outline

Formulation



$$\hat{n}_1 \times (\mu_r^{-1} \nabla \times \mathbf{E}_1) - \frac{jk_0}{\eta} y_{11} \hat{n}_1 \times (\hat{n}_1 \times \mathbf{E}_1) - \\ - \frac{jk_0}{\eta} y_{12} \hat{n}_2 \times (\hat{n}_2 \times \mathbf{E}_2) = 0,$$

$$\hat{n}_2 \times (\mu_r^{-1} \nabla \times \mathbf{E}_2) - \frac{jk_0}{\eta} y_{21} \hat{n}_1 \times (\hat{n}_1 \times \mathbf{E}_1) - \\ - \frac{jk_0}{\eta} y_{22} \hat{n}_2 \times (\hat{n}_2 \times \mathbf{E}_2) = 0,$$

Note that y_{xx} are relative to the vacuum admittance.

Formulation (cont.)

Find $\mathbf{E} \in \mathbf{H}_0(\text{curl}, \Omega)$ such that

$$\left(\nabla \times \mathbf{w}, \mu_r^{-1} \nabla \times \mathbf{E}\right)_\Omega - k_0^2 \left(\mathbf{w}, \varepsilon_r \mathbf{E}\right)_\Omega + jk_0 \left\langle \hat{n} \times \mathbf{w}, \hat{n} \times \mathbf{w} \right\rangle_{\Gamma_\text{C}} =$$
$$\left(\mathbf{w}, \mathbf{F}\right)_\Omega - \left\langle \hat{n} \times (\mathbf{w} \times \hat{n}), \mathbf{\Psi}_\text{N} \right\rangle_{\Gamma_\text{N}} - \left\langle \hat{n} \times (\mathbf{w} \times \hat{n}), \mathbf{\Psi}_\text{C} \right\rangle_{\Gamma_\text{C}} \quad \forall\, \mathbf{w} \in \mathbf{H}_0(\text{curl}, \Omega).$$

with

$$\left(\mathbf{w}, \mathbf{v}\right)_\Omega = \int_\Omega \mathbf{w}^* \cdot \mathbf{v} d\Omega,$$
$$\left\langle \mathbf{w}, \mathbf{v} \right\rangle_\Gamma = \int_\Gamma \mathbf{w}^* \cdot \mathbf{v} d\Gamma.$$

Formulation (cont.)

For *upper* elements on Γ_{TR} (side 1), we have

$$\begin{aligned}
&\mathrm{LHS}_1 \\
&\quad + j\frac{k_0}{\eta}\left\langle \hat{n}\times(\mathbf{w}_1\times\hat{n}), y_{11}\hat{n}\times(\mathbf{w}_1\times\hat{n})\right\rangle_{\Gamma_{\mathrm{TR}}} + j\frac{k_0}{\eta}\left\langle \hat{n}\times(\mathbf{w}_1\times\hat{n}), y_{12}\hat{n}\times(\mathbf{w}_2\times\hat{n})\right\rangle_{\Gamma_{\mathrm{TR}}} = \\
&\quad \mathrm{RHS}_1,
\end{aligned}$$

whereas for *lower* elements (side 2), we get

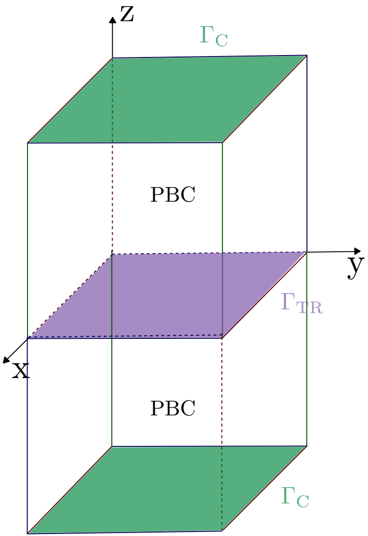
$$\begin{aligned}
&\mathrm{LHS}_2 \\
&\quad + j\frac{k_0}{\eta}\left\langle \hat{n}\times(\mathbf{w}_2\times\hat{n}), y_{21}\hat{n}\times(\mathbf{w}_1\times\hat{n})\right\rangle_{\Gamma_{\mathrm{TR}}} + j\frac{k_0}{\eta}\left\langle \hat{n}\times(\mathbf{w}_2\times\hat{n}), y_{22}\hat{n}\times(\mathbf{w}_2\times\hat{n})\right\rangle_{\Gamma_{\mathrm{TR}}} = \\
&\quad \mathrm{RHS}_2,
\end{aligned}$$

FEM implementation

- The DOFs will be doubled for the faces and the interior edges.
- The exterior edges of Γ_{TR} are not doubled.
  - Identified by code: the edges associated to two faces are interior.
  - If the boundaries of the sheet belong to PBC, the edges of Γ_{TR} are also doubled.

Outline

Problem to be solved



Simulation of an infinite medium with transmission/reflection sheet that divides the space into two halves.

- Γ_{TR}: Transmission/reflection sheet defined with

$$\mathbf{Y} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}.$$

- Γ_C: ABC with excitation with polarization E_y
- The vertical faces are set to PBC

Testbench

- $\mathbf{Y} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$: sanity check, we should get same result as the two halves with a PMC.
- $\mathbf{Y} = \mathbb{I}$: sanity check, we should get same result as the two halves with an ABC.
- Change lower Γ_C by PEC and solve analytic problem with four media: final test.
  - Obtain parameters for $\mathbf{Y}$ of the equivalent problem.
  - Get same solutions for the electric field.
  - Transparent? Puede ser que aproximar con $1e6$. Quizás con ABCD.

Outline

HOFEM implementation

- New boundary condition: `TRBC`.
  - We define a normal, $\hat{n}_{\mathrm{TRBC}}$ to detect lower and upper side. Upper side is the closer to $\hat{n}_{\mathrm{TRBC}}$.
  - Definition of y_{11}, y_{12}, y_{21}, and y_{22} as relative values with respect to vacuum admittance.
- Two options for implementation
  - Integers defined in `tetrahedra_element`.
  - Allocatable array of $1 \times N_{\mathrm{elem,TR}}$ where the two positions (stored in boundary conditions module, accessible from `mesh_reordering_module` and `elementary_terms_3D`):
    1. $10\times$ Neighbor element identifier (to couple $\mathbf{w}_2$ and $\mathbf{w}_1$).
    2. Integer 1,2 (side) (to extract the values of y_{11},y_{12},y_{21}, and y_{22}).
- Significant methods involved:
  - Postprocessing over `reordering_DOF_algorithm_3D`.
  - `calc_boundary_3D_nxNi_nxNi_term_of_this_element`.
  - Construction of the MUMPS-related matrix: different number of non-zeros per element, assembly of coupled elements (now single-element assembly).