行列式的混合运算

- $|\mathbf{A}+\mathbf{B}| = ?$
- $|\mathbf{A}^T| = |\mathbf{A}|^T$
- $|k\mathbf{A}| = k^n|\mathbf{A}|$
- $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}|$

例

$$\begin{aligned}\begin{vmatrix} a+x & b+y \\ c+w & d+z \end{vmatrix} &= \begin{vmatrix} a & b \\ c+w & d+z \end{vmatrix} + \begin{vmatrix} x & y \\ c+w & d+z \end{vmatrix} \\ &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ w & z \end{vmatrix} + \begin{vmatrix} x & y \\ c & d \end{vmatrix} + \begin{vmatrix} x & y \\ w & z \end{vmatrix}\end{aligned}$$

示意

$$\begin{vmatrix} 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \\ a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \\ 0 & a_{12} & -a_{11}b_{11} & -a_{11}b_{12} \\ 0 & a_{22} & -a_{21}b_{11} & -a_{21}b_{12} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & b_{11} & b_{12} \\ 0 & 1 & b_{21} & b_{22} \\ 0 & 0 & -a_{11}b_{11} - a_{12}b_{21} & -a_{11}b_{12} - a_{12}b_{22} \\ 0 & 0 & -a_{21}b_{11} - a_{22}b_{21} & -a_{21}b_{12} - a_{22}b_{22} \end{vmatrix}$$

$$= \begin{vmatrix} \mathbf{E} & \mathbf{B} \\ \mathbf{O} & -\mathbf{AB} \end{vmatrix} = \begin{vmatrix} \mathbf{E} & \mathbf{B} \\ \mathbf{A} & \mathbf{O} \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

A_{ij}是其代数余子式，则

$$x_1A_{13} + x_2A_{23} + x_3A_{33} + x_4A_{43} = \begin{vmatrix} a_1 & b_1 & x_1 & d_1 \\ a_2 & b_2 & x_2 & d_2 \\ a_3 & b_3 & x_3 & d_3 \\ a_4 & b_4 & x_4 & d_4 \end{vmatrix}$$

于是有

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = a_{1i}A_{1j} + a_{2i}A_{2j} + \cdots + a_{ni}A_{nj} = 0, \quad i \neq j$$

例 (1.4.19)

$$|\mathbf{A}| = \begin{vmatrix} 1 & 1 & 0 & 2 \\ -1 & -1 & 1 & 1 \\ 2 & 5 & -1 & -1 \\ 3 & 2 & -2 & 0 \end{vmatrix}, \quad \text{求} M_{12} + 2M_{22} + 3M_{32} + 4M_{42}$$

解：

$$M_{12} + 2M_{22} + 3M_{32} + 4M_{42} = -A_{12} + 2A_{22} - 3A_{32} + 4A_{42}$$

$$= \begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 2 & 1 & 1 \\ 2 & -3 & -1 & -1 \\ 3 & 4 & -2 & 0 \end{vmatrix} = -18$$

注：

如果不是某一行或某一列的代数余子式的组合，该如何计算？

定义方阵$\mathbf{A}=[a_{ij}]$的伴随矩阵为$\mathbf{A}^*=[A_{ij}]^T$，其满足

$$\mathbf{AA}^*=\mathbf{A}^*\mathbf{A}=|\mathbf{A}|\mathbf{E}=\begin{bmatrix}|\mathbf{A}| & & & \\ & |\mathbf{A}| & & \\ & & \ddots & \\ & & & |\mathbf{A}|\end{bmatrix}$$

示意

$$\begin{bmatrix}a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33}\end{bmatrix}=\begin{bmatrix}|\mathbf{A}| & & \\ & |\mathbf{A}| & \\ & & |\mathbf{A}|\end{bmatrix}$$

n阶行列式计算

- 观察行列式数字分布规律是否可利用
- 可以尝试利用行列式性质寻找递推关系

例 (1.4.14)

$$|\mathbf{A}| = \begin{vmatrix} a_0 & b_1 & b_2 & \cdots & b_n \\ c_1 & a_1 & 0 & \cdots & 0 \\ c_2 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & 0 & 0 & \cdots & a_n \end{vmatrix} = \begin{vmatrix} a_0 - \frac{b_1 c_1}{a_1} & b_1 & b_2 & \cdots & b_n \\ 0 & a_1 & 0 & \cdots & 0 \\ c_2 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_n & 0 & 0 & \cdots & a_n \end{vmatrix}$$

$$= \begin{vmatrix} a_0 - \sum_{k=1}^{n} \frac{b_k c_k}{a_k} & b_1 & b_2 & \cdots & b_n \\ 0 & a_1 & 0 & \cdots & 0 \\ 0 & 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix} = \left(a_0 - \sum_{k=1}^{n} \frac{b_k c_k}{a_k}\right) \prod_{i=1}^{n} a_i$$

例 (1.4.14)

$$D_n = \begin{vmatrix} a_0 & b_1 & b_2 & \cdots & b_n \\ c_1 & a_1 & & & \\ c_2 & & a_2 & & \\ \vdots & & & \ddots & \\ c_n & & & & a_n \end{vmatrix} = a_n D_{n-1} + (-1)^{n+1+1} c_n \begin{vmatrix} b_1 & b_2 & \cdots & b_{n-1} & b_n \\ a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & 0 \end{vmatrix}$$

$$= a_n D_{n-1} + (-1)^n c_n (-1)^{1+n} b_n \begin{vmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_{n-1} \end{vmatrix} = a_n D_{n-1} - \frac{b_n c_n}{a_n} \prod_{i=1}^{n} a_i$$

例 (范德蒙行列式)

$$
\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^2(x_1-x_4) & x_2^2(x_2-x_4) & x_3^2(x_3-x_4) & 0 \end{vmatrix}
$$

$$
= \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1-x_4 & x_2-x_4 & x_3-x_4 & 0 \\ x_1(x_1-x_4) & x_2(x_2-x_4) & x_3(x_3-x_4) & 0 \\ x_1^2(x_1-x_4) & x_2^2(x_2-x_4) & x_3^2(x_3-x_4) & 0 \end{vmatrix}
$$

$$
=(-1)^{1+4} \begin{vmatrix} x_1-x_4 & x_2-x_4 & x_3-x_4 \\ x_1(x_1-x_4) & x_2(x_2-x_4) & x_3(x_3-x_4) \\ x_1^2(x_1-x_4) & x_2^2(x_2-x_4) & x_3^2(x_3-x_4) \end{vmatrix}
$$

$$
=(x_4-x_1)(x_4-x_2)(x_4-x_3) \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix}
$$

例 (范德蒙行列式)

$$D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}$$

由

$$D_n = x_n^{n-1} A_{nn} + \cdots + x_n A_{2n} + A_{1n}$$

以及A_{in}中不包含x_n可知，$D_n = f(x_n)$关于x_n是一个最高$n-1$次多项式。又因为当$x_n = x_i, i = 1, \ldots, n-1$时，行列式两列相同，所以有因式分解

$$f(x_n) = c(x_n - x_1)(x_n - x_2)\cdots(x_n - x_{n-1})$$

比较$n-1$次项系数得$c = A_{nn}$，即

$$D_n = (x_n - x_1)(x_n - x_2)\cdots(x_n - x_{n-1}) D_{n-1}$$

作业

第38页练习1.4:

A4(6,7), B2(2,4), B3