1 Basic equations

Basic equations to start from: MHD with non-adiabatic effects added:

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot (\rho \boldsymbol{v}) \\
\rho \frac{\partial \boldsymbol{v}}{\partial t} &= -\nabla p - \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \rho \boldsymbol{g} \\
\rho \frac{\partial T}{\partial t} &= -\rho \boldsymbol{v} \cdot \nabla T - (\gamma - 1) p \nabla \cdot \boldsymbol{v} - (\gamma - 1) \rho \mathscr{L} + (\gamma - 1) \nabla \cdot (\boldsymbol{\kappa} \cdot \nabla T) \\
\frac{\partial B}{\partial t} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{B})
\end{aligned}
\tag{1}
$$

Linearize them with flow, so $\boldsymbol{v}_0 \neq 0$. Also use a vector potential $\boldsymbol{B}_1 = \nabla \times \boldsymbol{A}_1$ for the perturbed magnetic field and replace p_1 by $T_0\rho_1 + \rho_0 T_1$ (from the linearized ideal gas law):

$$
\begin{aligned}
\frac{\partial \rho_1}{\partial t} &= -\nabla \cdot (\rho_0 \boldsymbol{v}_1) - \nabla \cdot (\rho_1 \boldsymbol{v}_0) \\
\rho_0 \frac{\partial \boldsymbol{v}_1}{\partial t} &= -\rho_0 \boldsymbol{v}_0 \cdot \nabla \boldsymbol{v}_1 - \rho_0 \boldsymbol{v}_1 \cdot \nabla \boldsymbol{v}_0 - \rho_1 \boldsymbol{v}_0 \cdot \nabla \boldsymbol{v}_0 - \nabla (T_0 \rho_1 + \rho_0 T_1) + (\nabla \times \boldsymbol{B}_0) \times (\nabla \times \boldsymbol{A}_1) \\
&\quad + [\nabla \times (\nabla \times \boldsymbol{A}_1)] \times \boldsymbol{B}_0 + \rho_1 \boldsymbol{g} \\
\rho_0 \frac{\partial T_1}{\partial t} &= -\rho_0 \boldsymbol{v}_1 \cdot \nabla T_0 - \rho_0 \boldsymbol{v}_0 \cdot \nabla T_1 - (\gamma - 1) \rho_0 T_0 \nabla \cdot \boldsymbol{v}_1 - (\gamma - 1) \rho_1 \mathscr{L}_0 - (\gamma - 1) \rho_0 (\mathscr{L}_T T_1 + \mathscr{L}_\rho \rho_1) \\
&\quad + (\gamma - 1) \nabla \cdot (\boldsymbol{\kappa}_0 \cdot \nabla T_1) + (\gamma - 1) \nabla \cdot (\boldsymbol{\kappa}_1 \cdot \nabla T_0) \\
\frac{\partial \boldsymbol{A}_1}{\partial t} &= \boldsymbol{v}_1 \times \boldsymbol{B}_0 + \boldsymbol{v}_0 \times (\nabla \times \boldsymbol{A}_1) \\
\frac{p_1}{p_0} &= \frac{T_1}{T_0} + \frac{\rho_1}{\rho_0}
\end{aligned}
\tag{2}
$$

Denote coordinate system by (u_1, u_2, u_3), which is (x, y, z) in Cartesian and (r, θ, z) in cylindrical coordinates. Equilibrium profiles are only dependent on u_1, taken to be

$$
\begin{aligned}
\rho_0 &= \rho_0(u_1) \\
p_0 &= p_0(u_1) \\
T_0 &= T_0(u_1) \\
\boldsymbol{v}_0 &= v_{02}(u_1)\hat{\boldsymbol{e}}_2 + v_{03}(u_1)\hat{\boldsymbol{e}}_3 \\
\boldsymbol{B}_0 &= B_{02}(u_1)\hat{\boldsymbol{e}}_2 + B_{03}(u_1)\hat{\boldsymbol{e}}_3
\end{aligned}
\tag{3}
$$

so $\nabla \cdot \boldsymbol{v}_0 = 0$ and obviously $\nabla \cdot \boldsymbol{B}_0 = 0$ and $\nabla \cdot \boldsymbol{B}_1 = 0$, take gravity as $\boldsymbol{g} = -g\hat{\boldsymbol{e}}_1$. Vectors $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$ are orthonormal basis vectors representing directions along (u_1, u_2, u_3). Apply Fourier transformation using

$$
f_1(u_1, u_2, u_3, t) = \widetilde{f}_1(u_1) \mathrm{e}^{-i\omega t + ik_2 u_2 + ik_3 u_3}
\tag{4}
$$

1.1 Slab geometry

Below all tildes are dropped for convenience.

$$
\begin{aligned}
\omega\rho_1 &= -iv_1\rho_0' - i\rho_0(v_1' + ik_2v_2 + ik_3v_3) + (v_{02}k_2 + v_{03}k_3)\rho_1 \\
\omega\rho_0 v_1 &= \rho_0(k_2v_{02} + k_3v_{03})v_1 - i(\rho_0T_1 + \rho_1T_0)' + iB_{02}'(a_3' - ik_3a_1) - iB_{03}'(a_2' - ik_2a_1) \\
&\quad + iB_{03}\left[k_3^2a_2 - k_2k_3a_3 - (a_2' - ik_2a_1)'\right] - iB_{02}\left[k_2^2a_3 - k_2k_3a_2 + (ik_3a_1 - a_3')'\right] - ig\rho_1 \\
\omega\rho_0 v_2 &= k_2(T_0\rho_1 + \rho_0T_1) + \rho_0(k_2v_{02} + k_3v_{03})v_2 - i\rho_0v_{02}'v_1 - B_{02}'(k_2a_3 - k_3a_2) \\
&\quad - iB_{03}\left[(k_2^2 + k_3^3)a_1 + i(k_2a_2 + k_3a_3)'\right] \\
\omega\rho_0 v_3 &= k_3(T_0\rho_1 + \rho_0T_1) + \rho_0(k_2v_{02} + k_3v_{03})v_3 - i\rho_0v_{03}'v_1 - B_{03}'(k_2a_3 - k_3a_2) \\
&\quad + iB_{02}\left[(k_2^2 + k_3^2)a_1 + i(k_2a_2 + k_3a_3)'\right] \\
\omega\rho_0 T_1 &= -i\rho_0v_1T_0' + \rho_0(k_2v_{02} - k_3v_{03})T_1 - i(\gamma-1)\rho_0T_0(v_1' + ik_2v_2 + ik_3v_3) - i(\gamma-1)\mathscr{L}_0\rho_1 \\
&\quad - i(\gamma-1)\rho_0\mathscr{L}_TT_1 - i(\gamma-1)\rho_0\mathscr{L}_\rho\rho_1 - i(\gamma-1)(\kappa_\parallel - \kappa_\perp)\frac{1}{B^2}(k_2B_{02} + k_3B_{03})^2T_1 \\
&\quad + i(\gamma-1)(\kappa_\perp T_1')' - i\kappa_\perp(\gamma-1)(k_2^2 + k_3^2)T_1 + i(\gamma-1)(\kappa_{1,\perp}T_0')' \\
&\quad + i(\gamma-1)(\kappa_\parallel - \kappa_\perp)\frac{1}{B^2}\left[(k_2k_3B_{02} + k_3^2B_{03})a_2 - (k_2k_3B_{03} + k_2^2B_{02})a_3\right]T_0' \\
\omega a_1 &= iB_{03}v_2 - iB_{02}v_3 + iv_{02}a_2' + iv_{03}a_3' + (k_2v_{02} + k_3v_{03})a_1 \\
\omega a_2 &= -iB_{03}v_1 - v_{03}(k_2a_3 - k_3a_2) \\
\omega a_3 &= iB_{02}v_1 - v_{02}(k_3a_2 - k_2a_3)
\end{aligned}
\tag{5}
$$

where the prime denotes ∂_{u_1}.

1.2 Cylindrical geometry

Derivations in cylindrical coordinates, explicitly done to see how the scale factor enters into the equations. For consistency with the slab case the same Fourier transformation is used (usually $k_2 = m$ and $k_3 = k$).

$$
\begin{aligned}
\omega\rho_1 &= -iv_1\rho_0' - i\rho_0\left(v_1' + \frac{v_1}{r} + \frac{1}{r}ik_2v_2 + ik_3v_3\right) + \left(\frac{1}{r}v_{02}k_2 + v_{03}k_3\right)\rho_1 \\
\omega\rho_0 v_1 &= -i\left(T_0\rho_1 + T_1\rho_0\right)' - i\frac{B_{03}'}{r}\left[(ra_2)' - ik_2a_1\right] + i\frac{(rB_{02})'}{r}\left(a_3' - ik_3a_1\right) \\
&\quad + iB_{03}\left\{k_3^2a_2 - \frac{1}{r}k_2k_3a_3 - \left[\frac{1}{r}\left((ra_2)' - ik_2a_1\right)\right]'\right\} + \left(\frac{v_{02}k_2}{r} + v_{03}k_3\right)\rho_0v_1 \\
&\quad - iB_{02}\left\{\frac{1}{r^2}k_2^2a_3 - k_2k_3a_2 + \frac{1}{r}\left[r\left(ik_3a_1 - a_3'\right)\right]'\right\} + i\frac{2}{r}\rho_0v_{02}v_2 + i\frac{1}{r}v_{02}^2\rho_1 - ig\rho_1 \\
\omega\rho_0 v_2 &= \frac{1}{r}(T_0\rho_1 + \rho_0T_1)k_2 + \rho_0\left(\frac{1}{r}v_{02}k_2 + v_{03}k_3\right)v_2 + \frac{1}{r^2}(rB_{02})'(rk_3a_2 - k_2a_3) \\
&\quad + B_{03}\left[-ik_3^2a_1 + k_3a_3' + \frac{1}{r^2}k_2(ra_2)' - \frac{i}{r^2}k_2^2a_1\right] - i\rho_0\frac{1}{r}v_{02}v_1 - i\rho_0v_{02}'v_1 \\
\omega\rho_0 v_3 &= (T_0\rho_1 + \rho_0T_1)k_3 - B_{03}'\left(\frac{1}{r}k_2a_3 - k_3a_3\right) + iB_{02}\left[\frac{1}{r^2}ik_2(ra_2)' - \frac{1}{r^2}ik_2a_1 + k_3^2a_1 + ik_3a_3'\right] \\
&\quad + \rho_0\left(\frac{1}{r}k_2v_{02} + k_3v_{03}\right) - i\rho_0v_{03}'v_1 \\
\omega\rho_0 T_1 &= -i\rho_0v_1T_0' + \rho_0\left(\frac{1}{r}k_2v_{02} + k_3v_{03}\right)T_1 - i(\gamma-1)\rho_0T_0\left[\frac{1}{r}(rv_1)' + \frac{1}{r}ik_2v_2 + ik_3v_3\right] \\
&\quad - i(\gamma-1)(\kappa_\parallel - \kappa_\perp)\frac{1}{B^2}\left(\frac{1}{r}k_2B_{02} + k_3B_{03}\right)^2 T_1 + i(\gamma-1)\frac{1}{r}(\kappa_\perp T_1')' \\
&\quad - i(\gamma-1)\kappa_\perp\left(\frac{1}{r^2}k_2^2 + k_3^2\right)T_1 + i(\gamma-1)\frac{1}{r}(r\kappa_{\perp,1}T_0')' \\
&\quad + i(\gamma-1)(\kappa_\perp - \kappa_\parallel)\frac{1}{B^2}\left[(B_{03}k_3^2 + B_{02}k_2k_3)a_2 - \frac{1}{r}(B_{02}k_2^2 + B_{03}k_2k_3)a_3\right] \\
&\quad - i(\gamma-1)\rho_1\mathscr{L}_0 - i(\gamma-1)\rho_0(\mathscr{L}_T T_1 + \mathscr{L}_\rho\rho_1) \\
\omega a_1 &= iB_{03}v_2 - iB_{02}v_3 + iv_{02}a_2' + iv_{03}a_3' + \left(\frac{1}{r}k_2v_{02} + k_3v_{03}\right)a_1 + \frac{1}{r}v_{02}a_2 \\
\omega a_2 &= -iB_{03}v_1 - \left(\frac{1}{r}k_2a_3 - k_3a_2\right)v_{03} \\
\omega a_3 &= iB_{02}v_1 - \left(k_3a_2 - \frac{1}{r}k_2a_3\right)v_{02}
\end{aligned}
\tag{6}
$$