1 Basic equations

Basic equations to start from: MHD with non-adiabatic effects added:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot (\rho \boldsymbol{v}) \\
\rho \frac{\partial \boldsymbol{v}}{\partial t} &= -\nabla p - \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \rho \boldsymbol{g} \\
\rho \frac{\partial T}{\partial t} &= -\rho \boldsymbol{v} \cdot \nabla T - (\gamma - 1) p \nabla \cdot \boldsymbol{v} - (\gamma - 1)\rho \mathscr{L} + (\gamma - 1)\nabla \cdot (\boldsymbol{\kappa} \cdot \nabla T) \\
\frac{\partial B}{\partial t} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{B})
\end{aligned} \tag{1}$$

Linearize them with flow, so $\boldsymbol{v}_0 \neq 0$. Also use a vector potential $\boldsymbol{B}_1 = \nabla \times \boldsymbol{A}_1$ for the perturbed magnetic field and replace p_1 by $T_0\rho_1 + \rho_0 T_1$ (from the linearized ideal gas law):

$$\begin{aligned}
\frac{\partial \rho_1}{\partial t} &= -\nabla \cdot (\rho_0 \boldsymbol{v}_1) - \nabla \cdot (\rho_1 \boldsymbol{v}_0) \\
\rho_0 \frac{\partial \boldsymbol{v}_1}{\partial t} &= -\rho_0 \boldsymbol{v}_0 \cdot \nabla \boldsymbol{v}_1 - \rho_0 \boldsymbol{v}_1 \cdot \nabla \boldsymbol{v}_0 - \rho_1 \boldsymbol{v}_0 \cdot \nabla \boldsymbol{v}_0 - \nabla(T_0\rho_1 + \rho_0 T_1) + (\nabla \times \boldsymbol{B}_0) \times (\nabla \times \boldsymbol{A}_1) \\
&\quad + [\nabla \times (\nabla \times \boldsymbol{A}_1)] \times \boldsymbol{B}_0 + \rho_1 \boldsymbol{g} \\
\rho_0 \frac{\partial T_1}{\partial t} &= -\rho_0 \boldsymbol{v}_1 \cdot \nabla T_0 - \rho_0 \boldsymbol{v}_0 \cdot \nabla T_1 - (\gamma - 1)\rho_0 T_0 \nabla \cdot \boldsymbol{v}_1 - (\gamma - 1)\rho_1 \mathscr{L}_0 - (\gamma - 1)\rho_0 (\mathscr{L}_T T_1 + \mathscr{L}_\rho \rho_1) \\
&\quad + (\gamma - 1)\nabla \cdot (\boldsymbol{\kappa}_0 \cdot \nabla T_1) + (\gamma - 1)\nabla \cdot (\boldsymbol{\kappa}_1 \cdot \nabla T_0) \\
\frac{\partial \boldsymbol{A}_1}{\partial t} &= \boldsymbol{v}_1 \times \boldsymbol{B}_0 + \boldsymbol{v}_0 \times (\nabla \times \boldsymbol{A}_1) \\
\frac{p_1}{p_0} &= \frac{T_1}{T_0} + \frac{\rho_1}{\rho_0}
\end{aligned} \tag{2}$$

Denote coordinate system by (u_1, u_2, u_3), which is (x, y, z) in Cartesian and (r, θ, z) in cylindrical coordinates. Equilibrium profiles are only dependent on u_1, taken to be

$$\begin{aligned}
\rho_0 &= \rho_0(u_1) \\
p_0 &= p_0(u_1) \\
T_0 &= T_0(u_1) \\
\boldsymbol{v}_0 &= v_{02}(u_1)\hat{\boldsymbol{e}}_2 + v_{03}(u_1)\hat{\boldsymbol{e}}_3 \\
\boldsymbol{B}_0 &= B_{02}(u_1)\hat{\boldsymbol{e}}_2 + B_{03}(u_1)\hat{\boldsymbol{e}}_3
\end{aligned} \tag{3}$$

so $\nabla \cdot \boldsymbol{v}_0 = 0$ and obviously $\nabla \cdot \boldsymbol{B}_0 = 0$ and $\nabla \cdot \boldsymbol{B}_1 = 0$, take gravity as $\boldsymbol{g} = -g\hat{\boldsymbol{e}}_1$. Vectors $\hat{\boldsymbol{e}}_1$, $\hat{\boldsymbol{e}}_2$ and $\hat{\boldsymbol{e}}_3$ are orthonormal basis vectors representing directions along (u_1, u_2, u_3). Apply Fourier transformation using

$$f_1(u_1, u_2, u_3, t) = \widetilde{f}_1(u_1)\mathrm{e}^{-i\omega t + ik_2u_2 + ik_3u_3} \tag{4}$$

1.1 Slab geometry

Below all tildes are dropped for convenience.

$$
\begin{aligned}
\omega\rho_1 &= -iv_1\rho_0' - i\rho_0(v_1' + ik_2v_2 + ik_3v_3) + (v_{02}k_2 + v_{03}k_3)\rho_1 \\
\omega\rho_0 v_1 &= \rho_0(k_2v_{02} + k_3v_{03})v_1 - i(\rho_0T_1 + \rho_1T_0)' + iB_{02}'(a_3' - ik_3a_1) - iB_{03}'(a_2' - ik_2a_1) \\
&\quad + iB_{03}\left[k_3^2a_2 - k_2k_3a_3 - (a_2' - ik_2a_1)'\right] - iB_{02}\left[k_2^2a_3 - k_2k_3a_2 + (ik_3a_1 - a_3')'\right] - ig\rho_1 \\
\omega\rho_0 v_2 &= k_2(T_0\rho_1 + \rho_0T_1) + \rho_0(k_2v_{02} + k_3v_{03})v_2 - i\rho_0v_{02}'v_1 - B_{02}'(k_2a_3 - k_3a_2) \\
&\quad - iB_{03}\left[(k_2^2 + k_3^3)a_1 + i(k_2a_2 + k_3a_3)'\right] \\
\omega\rho_0 v_3 &= k_3(T_0\rho_1 + \rho_0T_1) + \rho_0(k_2v_{02} + k_3v_{03})v_3 - i\rho_0v_{03}'v_1 - B_{03}'(k_2a_3 - k_3a_2) \\
&\quad + iB_{02}\left[(k_2^2 + k_3^2)a_1 + i(k_2a_2 + k_3a_3)'\right] \\
\omega\rho_0 T_1 &= -i\rho_0v_1T_0' + \rho_0(k_2v_{02} - k_3v_{03})T_1 - i(\gamma-1)\rho_0T_0(v_1' + ik_2v_2 + ik_3v_3) - i(\gamma-1)\mathscr{L}_0\rho_1 \\
&\quad - i(\gamma-1)\rho_0\mathscr{L}_T T_1 - i(\gamma-1)\rho_0\mathscr{L}_\rho\rho_1 - i(\gamma-1)(\kappa_\parallel - \kappa_\perp)\frac{1}{B^2}(k_2B_{02} + k_3B_{03})^2T_1 \\
&\quad + i(\gamma-1)(\kappa_\perp T_1')' - i\kappa_\perp(\gamma-1)(k_2^2 + k_3^2)T_1 + i(\gamma-1)(\kappa_{1,\perp}T_0')' \\
&\quad + i(\gamma-1)(\kappa_\parallel - \kappa_\perp)\frac{1}{B^2}\left[(k_2k_3B_{02} + k_3^2B_{03})a_2 - (k_2k_3B_{03} + k_2^2B_{02})a_3\right]T_0' \\
\omega a_1 &= iB_{03}v_2 - iB_{02}v_3 + iv_{02}a_2' + iv_{03}a_3' + (k_2v_{02} + k_3v_{03})a_1 \\
\omega a_2 &= -iB_{03}v_1 - v_{03}(k_2a_3 - k_3a_2) \\
\omega a_3 &= iB_{02}v_1 - v_{02}(k_3a_2 - k_2a_3)
\end{aligned}
\tag{5}
$$

where the prime denotes ∂_{u_1}.

1.2 Cylindrical geometry

Derivations in cylindrical coordinates, explicitly done to see how the scale factor enters into the equations. For consistency with the slab case the same Fourier transformation is used (usually $k_2 = m$ and $k_3 = k$).

$$
\begin{aligned}
\omega\rho_1 &= -iv_1\rho_0' - i\rho_0\left(v_1' + \frac{v_1}{r} + \frac{1}{r}ik_2v_2 + ik_3v_3\right) + \left(\frac{1}{r}v_{02}k_2 + v_{03}k_3\right)\rho_1 \\
\omega\rho_0v_1 &= -i\left(T_0\rho_1 + T_1\rho_0\right)' - i\frac{B_{03}'}{r}\left[(ra_2)' - ik_2a_1\right] + i\frac{(rB_{02})'}{r}\left(a_3' - ik_3a_1\right) \\
&\quad + iB_{03}\left\{k_3^2a_2 - \frac{1}{r}k_2k_3a_3 - \left[\frac{1}{r}\left((ra_2)' - ik_2a_1\right)\right]'\right\} + \left(\frac{v_{02}k_2}{r} + v_{03}k_3\right)\rho_0v_1 \\
&\quad - iB_{02}\left\{\frac{1}{r^2}k_2^2a_3 - \frac{1}{r}k_2k_3a_2 + \frac{1}{r}\left[r\left(ik_3a_1 - a_3'\right)\right]'\right\} + i\frac{2}{r}\rho_0v_{02}v_2 + i\frac{1}{r}v_{02}^2\rho_1 - ig\rho_1 \\
\omega\rho_0v_2 &= \frac{1}{r}(T_0\rho_1 + \rho_0T_1)k_2 + \rho_0\left(\frac{1}{r}v_{02}k_2 + v_{03}k_3\right)v_2 + \frac{1}{r^2}(rB_{02})'(rk_3a_2 - k_2a_3) \\
&\quad + B_{03}\left[-ik_3^2a_1 + k_3a_3' + \frac{1}{r^2}k_2(ra_2)' - \frac{i}{r^2}k_2^2a_1\right] - i\rho_0\frac{1}{r}v_{02}v_1 - i\rho_0v_{02}'v_1 \\
\omega\rho_0v_3 &= (T_0\rho_1 + \rho_0T_1)k_3 - B_{03}'\left(\frac{1}{r}k_2a_3 - k_3a_2\right) + iB_{02}\left[\frac{1}{r^2}ik_2(ra_2)' + \frac{1}{r^2}k_2^2a_1 + k_3^2a_1 + ik_3a_3'\right] \\
&\quad + \rho_0\left(\frac{1}{r}k_2v_{02} + k_3v_{03}\right)v_3 - i\rho_0v_{03}'v_1 \\
\omega\rho_0T_1 &= -i\rho_0v_1T_0' + \rho_0\left(\frac{1}{r}k_2v_{02} + k_3v_{03}\right)T_1 - i(\gamma-1)\rho_0T_0\left[\frac{1}{r}(rv_1)' + \frac{1}{r}ik_2v_2 + ik_3v_3\right] \\
&\quad - i(\gamma-1)(\kappa_\parallel - \kappa_\perp)\frac{1}{B^2}\left(\frac{1}{r}k_2B_{02} + k_3B_{03}\right)^2T_1 + i(\gamma-1)\frac{1}{r}(r\kappa_\perp T_1')' \\
&\quad - i(\gamma-1)\kappa_\perp\left(\frac{1}{r^2}k_2^2 + k_3^2\right)T_1 + i(\gamma-1)\frac{1}{r}(r\kappa_{\perp,1}T_0')' \\
&\quad + i(\gamma-1)(\kappa_\perp - \kappa_\parallel)\frac{1}{B^2}\left[(B_{03}k_3^2 + B_{02}k_2k_3)a_2 - \frac{1}{r}(B_{02}k_2^2 + B_{03}k_2k_3)a_3\right] \\
&\quad - i(\gamma-1)\rho_1\mathscr{L}_0 - i(\gamma-1)\rho_0(\mathscr{L}_TT_1 + \mathscr{L}_\rho\rho_1) \\
\omega a_1 &= iB_{03}v_2 - iB_{02}v_3 + iv_{02}a_2' + iv_{03}a_3' + \left(\frac{1}{r}k_2v_{02} + k_3v_{03}\right)a_1 + i\frac{1}{r}v_{02}a_2 \\
\omega a_2 &= -iB_{03}v_1 - \left(\frac{1}{r}k_2a_3 - k_3a_2\right)v_{03} \\
\omega a_3 &= iB_{02}v_1 - \left(k_3a_2 - \frac{1}{r}k_2a_3\right)v_{02}
\end{aligned}
\tag{6}
$$

1.3 Introducing the scale factor

Now define the scale factor ε, where $\varepsilon = 1$ and $\varepsilon' = 0$ in Cartesian coordinates. In cylindrical coordinates, $\varepsilon = r$ and $\varepsilon' = 1$. Apply the following transformation to the perturbed quantities:

$$\begin{array}{cccc} \varepsilon\rho_1 = \widetilde{\rho_1} & i\varepsilon v_1 = \widetilde{v_1} & v_2 = \widetilde{v_2} & \varepsilon v_3 = \widetilde{v_3} \\ \varepsilon T_1 = \widetilde{T_1} & a_1 = \widetilde{a_1} & i\varepsilon a_2 = \widetilde{a_2} & i a_3 = \widetilde{a_3} \end{array}$$

Tildes are dropped below for convenience.

$$\begin{aligned}
\omega\frac{\rho_1}{\varepsilon} &= -\frac{1}{\varepsilon}\rho_0' v_1 - \frac{1}{\varepsilon}\rho_0\left(v_1' - k_2 v_2 - k_3 v_3\right) + \frac{1}{\varepsilon}\left(\frac{1}{\varepsilon}v_{02}k_2 + v_{03}k_3\right)\rho_1 \\
\omega\rho_0\frac{v_1}{\varepsilon} &= \left(\frac{T_0\rho_1 + \rho_0 T_1}{\varepsilon}\right)' + \frac{1}{\varepsilon}B_{03}'\left(a_2' - k_2 a_1\right) - \frac{(\varepsilon B_{02})'}{\varepsilon}\left(a_3' - k_3 a_1\right) \\
&\quad - B_{03}\left\{\frac{k_3^2}{\varepsilon}a_2 - \frac{k_2 k_3}{\varepsilon}a_3 - \left[\frac{1}{\varepsilon}\left(a_2' - k_2 a_1\right)\right]'\right\} + \frac{1}{\varepsilon}\left(\frac{v_{02}k_2}{\varepsilon} + v_{03}k_3\right)\rho_0 v_1 \\
&\quad + B_{02}\left\{\frac{k_2^2}{\varepsilon^2}a_3 - \frac{k_2 k_3}{\varepsilon^2}a_2 - \frac{1}{\varepsilon}\left[\varepsilon\left(a_3' - k_3 a_1\right)\right]'\right\} - 2\frac{\varepsilon'}{\varepsilon}\rho_0 v_{02} v_2 - \frac{\varepsilon'}{\varepsilon^2}v_{02}^2\rho_1 + \frac{1}{\varepsilon}g\rho_1 \\
\omega\rho_0 v_2 &= \frac{1}{\varepsilon^2}(T_0\rho_1 + \rho_0 T_1)k_2 + \rho_0\left(\frac{1}{\varepsilon}k_2 v_{02} + k_3 v_{03}\right)v_2 + \frac{1}{\varepsilon^2}(\varepsilon B_{02})'\left(k_3 a_2 - k_2 a_3\right) \\
&\quad + B_{03}\left[-\left(k_3^2 + \frac{k_2^2}{\varepsilon^2}\right)a_1 + \frac{k_2}{\varepsilon^2}a_2' + k_3 a_3'\right] - \frac{1}{\varepsilon^2}\rho_0(\varepsilon v_{02})' v_1 \\
\omega\rho_0\frac{v_3}{\varepsilon} &= \frac{1}{\varepsilon}(T_0\rho_1 + \rho_0 T_1)k_3 + \frac{1}{\varepsilon}\rho_0\left(\frac{1}{\varepsilon}k_2 v_{02} + k_3 v_{03}\right)v_3 + \frac{1}{\varepsilon}B_{03}'\left(k_3 a_2 - k_2 a_3\right) \\
&\quad - B_{02}\left[-\left(k_3^2 + \frac{k_2^2}{\varepsilon^2}\right)a_1 + \frac{k_2}{\varepsilon^2}a_2' + k_3 a_3'\right] - \frac{1}{\varepsilon}\rho_0 v_{03}' v_1 \\
\omega\rho_0\frac{T_1}{\varepsilon} &= -\frac{1}{\varepsilon}\rho_0 T_0' v_1 + \frac{1}{\varepsilon}\rho_0\left(\frac{k_2}{\varepsilon}v_{02} + k_3 v_{03}\right)T_1 - (\gamma - 1)\frac{1}{\varepsilon}\rho_0 T_0\left(v_1' - k_2 v_2 - k_3 v_3\right) \\
&\quad - i(\gamma - 1)\frac{(\kappa_\parallel - \kappa_\perp)}{\varepsilon}\frac{1}{B^2}\left(\frac{k_2}{\varepsilon}B_{02} + k_3 B_{03}\right)T_1 + i(\gamma - 1)\frac{1}{\varepsilon}\left[\varepsilon\kappa_\perp\left(\frac{T_1}{\varepsilon}\right)'\right]' \\
&\quad - i(\gamma - 1)\frac{\kappa_\perp}{\varepsilon}\left(\frac{k_2^2}{\varepsilon^2} + k_3^2\right)T_1 + i(\gamma - 1)\frac{1}{\varepsilon}\left(\varepsilon\kappa_{\perp,1}T_0'\right)' \\
&\quad + i(\gamma - 1)\frac{(\kappa_\perp - \kappa_\parallel)}{\varepsilon}\frac{1}{B^2}\left[\left(k_2 k_3 B_{02} + k_3^2 B_{03}\right)a_2 - \left(k_2^2 B_{02} + k_2 k_3 B_{03}\right)a_3\right] \\
&\quad - i(\gamma - 1)\mathscr{L}_0\frac{1}{\varepsilon}\rho_1 - i(\gamma - 1)\frac{1}{\varepsilon}\rho_0\left(\mathscr{L}_T T_1 + \mathscr{L}_\rho\rho_1\right) \\
\omega a_1 &= -B_{03}v_2 + \frac{1}{\varepsilon}B_{02}v_3 - \frac{1}{\varepsilon}v_{02}a_2' - v_{03}a_3' + \left(\frac{k_2}{\varepsilon}v_{02} + k_3 v_{03}\right)a_1 \\
\omega\frac{a_2}{\varepsilon} &= -\frac{1}{\varepsilon}B_{03}v_1 - \frac{1}{\varepsilon}\left(k_2 a_3 - k_3 a_2\right)v_{03} \\
\omega a_3 &= \frac{1}{\varepsilon}B_{02}v_1 + \frac{1}{\varepsilon}\left(k_2 a_3 - k_3 a_2\right)v_{02}
\end{aligned} \tag{7}$$

1.3.1 Change in thermal conduction tensor

The linearized part of the thermal conduction tensor in cylindrical coordinates is given by

$$\kappa_{\perp,1} = \frac{\partial \kappa_\perp}{\partial T} T_1 + \frac{\partial \kappa_\perp}{\partial \rho} \rho_1 + 2 \left(B_{02} B_\theta + B_{03} B_z \right) \frac{\partial \kappa_\perp}{\partial (B^2)} \tag{8}$$

Applying the transformations for the perturbed variables and introducing the vector potential for B_1, this can be rewritten as

$$\varepsilon \kappa_{\perp,1} = \frac{\partial \kappa_\perp}{\partial T} T_1 + \frac{\partial \kappa_\perp}{\partial \rho} \rho_1 - 2\varepsilon B_{02} \left(a_3' - k_3 a_1 \right) \frac{\partial \kappa_\perp}{\partial (B^2)} + 2 B_{03} \left(a_2' - k_2 a_1 \right) \frac{\partial \kappa_\perp}{\partial (B^2)} \tag{9}$$

which can be substituted in the above equations.

2 Finite Elements

Define the state vector of the system as $\boldsymbol{w} = (\rho_1, v_1, v_2, v_3, T, a_1, a_2, a_3)$.