

# SB - RC - Master equation (Dephasing case)

After applying the RC-map to the spin-boson Hamiltonian we find.

$$H = H_S + H_E + H_I$$

$$H_S = \frac{\omega}{2} \sigma_z + \Delta_0 b^\dagger b + g \sigma_z (b^\dagger + b) + \frac{g^2}{\Delta_0} \sigma_z^2 + \Delta_0 \Delta (b^\dagger b)^2$$

$$H_E = \sum_n \Delta_n b_n^\dagger b_n$$

$$H_I = (b + b^\dagger) \sum_n h_n (b_n + b_n^\dagger)$$

State time evolution - von Neumann

$$\dot{\rho}(t) = -i [H_S + H_E + H_I, \rho(t)] \quad (1)$$

Move to the interaction picture

$$\rho_I(t) = \exp(i(H_S + H_E)t) \rho(t) \exp(-i(H_S + H_E)t)$$

$$\stackrel{(1)}{\rightarrow} \dot{\rho}_I(t) = -i [H_I(t), \rho_I(t)]. \quad (2)$$

with

$$H_I(t) = \exp(i(H_S + H_E)t) H_I \exp(-i(H_S + H_E)t)$$

Formal integration of (2) yields

$$\rho_I(t) = \rho_I(0) - i \int_0^t dt' [H_I(t'), \rho_I(t')], \quad (3)$$

which after substituting in (2) leads to

$$\dot{\rho}_I(t) = -i [H_I(t), \rho_I(0)] - \int_0^t dt' [H_I(t), [H_I(t'), \rho_I(t')]] \quad (4)$$

Define now the reduced state of the system as

$$\bar{\rho}_S(t) = \text{Tr}_E(\rho_I(t)),$$

and assume that  $\rho(0) = \bar{\rho}_S \otimes \rho_E$ , with

$$\text{Tr}_E(H_I(t) \rho_I(0)) = 0. \quad (\text{which can be shown to hold for the conditions in our problem})$$

Trace over the environment on both sides of (4)

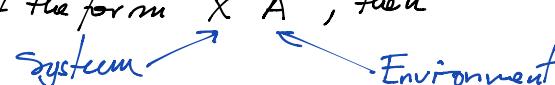
$$\stackrel{(4)}{\Rightarrow} \dot{\bar{\rho}}_I(t) = - \int_0^t dt' \text{Tr}_E ([H_I(t), [H_I(t'), \bar{\rho}_I(t')]]) \quad (5)$$

\* Approximation: At this point we take

$$\rho_I(t) = \bar{\rho}_I(t) \otimes \rho_E.$$

- i)  $H_I \ll H_S, H_B$
- ii) The state of the bath doesn't change (No memory).
- iii) The state of the system does change.
- iv) A weaker condition could have been imposed:

For interactions of the form  $X A$ , then



$$\text{Tr}_E ([A_I(t), [A(t'), \bar{\rho}_I(t')]]) \approx$$

$$\bar{\rho}_I(t) \otimes \underbrace{\text{Tr}_E ([A_I(t), [A(t'), \rho_E]])}_{\text{Bath correlations.}}$$

(Assume they don't change).

$$\stackrel{(5)}{\Rightarrow} \dot{\bar{\rho}}_I(t) = - \int_0^t dt' \text{Tr}_E ([H_I(t), [H_I(t'), \bar{\rho}_I(t') \otimes \rho_E]]) \quad (6)$$

Markov Approximation.

We assume the bath correlation time to be small.

- i)  $\bar{\rho}_I(t') \rightarrow \bar{\rho}_I(t)$  (Markov approx: Yields a first order differential equation for  $\bar{\rho}_I$ ).
- ii) We let the upper limit of the integral to go to infinity.

$$\begin{aligned} \stackrel{(6)}{\Rightarrow} \dot{\bar{\rho}}_I(t) &= - \int_0^{+\infty} dt' \text{Tr}_E ([H_I(t), [H_I(t'), \bar{\rho}_I(t) \otimes \rho_E]]) ; \quad \tau = t - t' \\ &= \int_0^{+\infty} d\tau \text{Tr}_E ([H_I(t), [H_I(t-\tau), \bar{\rho}_I(t) \otimes \rho_E]]). \end{aligned} \quad (7)$$

\* We can choose appropriate forms for  $H_S$  and  $H_I$  and find different master equations.

## Redfield equation

To recover the redfield equation, let us consider  $H_I(t)$  in detail.

$$H_I(t) = \exp(i(H_S + H_E)t)(b + b^\dagger) \sum_k h_k (b_k + b_k^\dagger) \exp(-i(H_S + H_E)t).$$

Using  $e^{iH_E t} b_k e^{-iH_E t} = b_k e^{-i\omega_k t}$

$$H_I(t) = \exp(iH_S t)(b + b^\dagger) \exp(-iH_S t) \sum_k h_k (b_k e^{-i\omega_k t} + b_k^\dagger e^{i\omega_k t}) \quad (8)$$

$\Gamma(t)$

Equation (7) on the Schrödinger equation can now be rewritten as

$$\begin{aligned} \dot{\rho}(t) &= -i[H_S, \rho(t)] - \int_0^{+\infty} d\tau \text{Tr}_E ([H_I, [H_I(t-\tau), \bar{\rho}(t) \otimes \rho_E]]) \\ &= -i[H_S, \bar{\rho}(t)] - \int_0^{+\infty} d\tau [\text{Tr}_E (\Pi(0) \Pi(\tau) \rho_E) [(b + b^\dagger), \bar{e}^{iH_S \tau} (b + b^\dagger) \bar{e}^{iH_S \tau}] \bar{\rho}(t)] \\ &\quad + \text{Tr}_E (\Pi(\tau) \Pi(0) \rho_E) [\bar{e}^{iH_S \tau} (b + b^\dagger) \bar{e}^{iH_S \tau} \bar{\rho}(t), (b + b^\dagger)] \\ &= -i[H_S, \bar{\rho}(t)] - \int_0^{+\infty} d\tau C(\tau) [(b + b^\dagger), \bar{e}^{iH_S \tau} (b + b^\dagger) \bar{e}^{iH_S \tau} \bar{\rho}(t)] \\ &\quad - \int_0^{+\infty} d\tau C(-\tau) [\bar{e}^{iH_S \tau} (b + b^\dagger) \bar{e}^{iH_S \tau} \bar{\rho}(t), (b + b^\dagger)] \quad (9) \end{aligned}$$

This is the Redfield equation reported by G. Shuller.

With the correlation function of the residual bath.

$$C(\tau) = \text{Tr}_E (\Pi(0) \Pi(\tau) \rho_E) = \text{Tr}_E (\Pi(-\tau) \Pi(0) \rho_E)$$

which we write also as.

$$C(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \delta(\omega) e^{-i\omega\tau} d\tau = C^*(-\tau)$$

To proceed further we use the eigen basis of  $H_{\text{sys}}$  ( $H_{\text{sys}}|a\rangle = E_a|a\rangle$ ), to write

$$\begin{aligned}\dot{\bar{\rho}}(t) &= -i[H_s, \bar{\rho}(t)] - \sum_{ab} \langle a | (b + b^\dagger) | b \rangle \\ &\quad \times \left\{ - \int_0^{+\infty} d\tau C(\tau) e^{i(E_b - E_a)\tau} [ (b + b^\dagger), |a\rangle \langle b| \bar{\rho}(t)] \right. \\ &\quad \left. - \int_0^{+\infty} d\tau C(-\tau) e^{i(E_b - E_a)\tau} [ |a\rangle \langle b| \bar{\rho}(t), (b + b^\dagger) ] \right\}.\end{aligned}$$

Let us now consider

$$\begin{aligned}\int_0^{+\infty} d\tau C(\tau) e^{i(E_b - E_a)\tau} &= \int_{-\infty}^{+\infty} d\tau \Theta(\tau) C(\tau) e^{i(E_b - E_a)\tau} \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dw \gamma(w) \int_{-\infty}^{+\infty} d\tau e^{i(E_b - E_a - w)\tau} \frac{1}{2} (1 + \text{sng}(\tau)) \\ &= \frac{\gamma(E_b - E_a)}{2} + \frac{i}{2\pi} P \int_{-\infty}^{+\infty} dw \frac{\gamma(w)}{E_b - E_a - w} \approx \frac{\gamma(E_b - E_a)}{2}.\end{aligned}$$

The equation of motion becomes

$$\dot{\bar{\rho}}(t) = -i[H_s, \bar{\rho}] + \sum_{ab} \langle a | (b + b^\dagger) | b \rangle \left\{ -\gamma_{ab} [ (b + b^\dagger), |a\rangle \langle b| \bar{\rho}(t)] \right. \\ \left. - \gamma_{ba} [ |a\rangle \langle b| \bar{\rho}(t), (b + b^\dagger) ] \right\} \quad (10)$$

## The Markovian ME.

We continue with the calculation in the interaction picture starting with equation (7).

$$\dot{\bar{\rho}}_I(t) = - \int_0^{+\infty} d\tau \text{Tr}_E \left( [H_I(t), [H_I(t-\tau), \bar{\rho}_I(t) \otimes \rho_E]] \right). \quad (7)$$

We now use Eq.(8) to write.

$$H_I(t) = \exp(iH_0 t)(b + b^\dagger) \exp(-iH_0 t) \Gamma(t),$$

and we evaluate it using, and correcting Liu's results:

$$e^{-iH_0 t} (b + b^\dagger) e^{iH_0 t} = \frac{i \sin(x\sqrt{y})}{\sqrt{y}} (-b + b^\dagger) + \cos(x\sqrt{y})(b + b^\dagger) \\ + \cos(x\sqrt{y}) \frac{2y}{\delta\omega} \sigma_z - \frac{2y}{\delta\omega y} \sigma_z$$

with  $x = \Omega t$  and  $y = (1+4\Delta)$ .

For the moment we restrict to the simpler case with  $\Delta=0 \Rightarrow y=1$ .

$$e^{-iH_0 t} (b + b^\dagger) e^{iH_0 t} = \underbrace{(b + \frac{g}{\Omega} \sigma_z)}_{X} e^{-i\Omega t} + \underbrace{(b^\dagger + \frac{g}{\Omega} \sigma_z)}_{X^\dagger} e^{i\Omega t} - \underbrace{\frac{2g}{\Omega\omega} \sigma_z}_{Y} \\ = X e^{-i\Omega t} + X^\dagger e^{i\Omega t} - Y \\ = X(t) + X^\dagger(t) - Y.$$

Equation (7) can now be rewritten as

$$\dot{\bar{\rho}}_I(t) = - \int_0^{+\infty} d\tau \text{Tr}_E \left( [ (X(t) + X^\dagger(t) - Y) \Gamma(t), \right. \\ \left. [(X(t-\tau) + X^\dagger(t-\tau) - Y) \Gamma(t-\tau), \bar{\rho}_I(t) \otimes \rho_E]] \right)$$

and after some algebra

$$\dot{\bar{\rho}}_I(t) = - \int_0^{+\infty} d\tau \{ C(\tau) [ (X(t) + X^\dagger(t) - Y), (X(t-\tau) + X^\dagger(t-\tau) - Y) \bar{\rho}_I(t) ] \\ + C(-\tau) [ \bar{\rho}_I(t) (X(t-\tau) + X^\dagger(t-\tau) - Y), (X(t) + X^\dagger(t) - Y) ] \}$$

There are several terms that are obtained from expanding the commutators.  
Consider a typical one

$$(X(t) + X^\dagger(t) - Y)(X(t-\tau) + X^\dagger(t-\tau) - Y) \\ = \cancel{X^2} e^{-i\Omega(2t-\tau)} + X^\dagger X e^{i\Omega\tau} - Y X \cancel{e^{-i\Omega(t-\tau)}} + X X^\dagger e^{-i\Omega\tau} \\ + \cancel{X^\dagger X^\dagger} e^{i\Omega(2t-\tau)} - Y X^\dagger \cancel{e^{i\Omega(t-\tau)}} - X Y \cancel{i\Omega t} - X^\dagger Y \cancel{e^{i\Omega t}} + Y^2$$

## Rotating wave approximation.

In the above expression we neglect the rapidly rotating terms with  $t$ , as they will give negligible contributions.

$$(X(t) + X^t(t) - Y)(X(t-\tau) + X^t(t-\tau) - Y) \approx X^t X e^{i\omega\tau} + X X^t e^{-i\omega\tau} + Y^2$$

Equation (7) now reduces to

$$\begin{aligned}\dot{\bar{\rho}}_I(t) = & - \int_0^\infty d\tau c(\tau) \{ (X^t X e^{i\omega\tau} + X X^t e^{-i\omega\tau} + Y^2) \bar{\rho}_I(t) \\ & - (X \bar{\rho}_I(t) X^t e^{i\omega\tau} + X^t \bar{\rho}_I(t) X e^{-i\omega\tau} + Y \bar{\rho}_I(t) Y) \} \\ & - \int_0^\infty d\tau c(-\tau) \{ \bar{\rho}_I(t) (X^t X e^{-i\omega\tau} + X X^t e^{i\omega\tau} + Y^2) \\ & - (X \bar{\rho}_I(t) X^t e^{-i\omega\tau} + X^t \bar{\rho}_I(t) X e^{i\omega\tau} + Y \bar{\rho}_I(t) Y) \}\end{aligned}$$

We now consider

$$\int_0^\infty d\tau c(\tau) e^{i\omega\tau} = \Gamma(\omega) \quad c(-\tau) = c^*(\tau).$$

$$\int_0^\infty d\tau c(-\tau) e^{i\omega\tau} = \int_0^\infty d\tau c^*(\tau) e^{i\omega\tau} = \left( \int_0^\infty d\tau c(\tau) e^{-i\omega\tau} \right)^* = \Gamma^*(-\omega)$$

$$\begin{aligned}\dot{\bar{\rho}}_I = & -\Gamma(\omega) (X^t X \bar{\rho}_I - X \bar{\rho}_I X^t) - \Gamma(-\omega) (X X^t \bar{\rho}_I - X^t \bar{\rho}_I X) \\ & - \Gamma^*(\omega) (\bar{\rho}_I X^t X - X \bar{\rho}_I X^t) - \Gamma^*(-\omega) (X X^t \bar{\rho}_I - X^t \bar{\rho}_I X) \\ & - \Gamma(0) (Y^2 \bar{\rho}_I - Y \bar{\rho}_I Y) - \Gamma^*(0) (\bar{\rho}_I Y^2 - Y \bar{\rho}_I Y)\end{aligned}$$

We now use

$$\Gamma(\omega) = \frac{\delta(\omega)}{2} + \frac{i}{2} \sigma(\omega) \quad \Gamma(-\omega) = \frac{\delta(\omega)}{2} + \frac{i}{2} \bar{\sigma}(\omega)$$

$$Y(\omega) = J(\omega) (\bar{n}(\omega) + 1) \quad \bar{J}(\omega) = J(\omega) \bar{n}(\omega)$$

$$\bar{n}(\omega) = (e^{\beta\omega} - 1)^{-1}$$

and arrive to the Lindblad master equation. (We neglect the Lamb shift)

$$\begin{aligned}\dot{\bar{\rho}}_I = & \delta(\omega) (X \bar{\rho}_I X^t - \frac{1}{2} \{ X^t X, \bar{\rho}_I \}) + \bar{J}(\omega) (X^t \bar{\rho}_I X - \frac{1}{2} \{ X X^t, \bar{\rho}_I \}) \quad (11) \\ & + \delta(0) (Y \bar{\rho}_I Y - \frac{1}{2} \{ Y^2, \bar{\rho}_I \})\end{aligned}$$

Our lovely master equation! (in the interaction picture)

Remember:  $X = b + \frac{g}{\Omega} J_z$ . This non-trivial change is what is going to make the difference!

$$Y = \frac{g}{\Omega} J_z$$

Can we find a left unitary that separates this?

Going to the Schrödinger picture.

$$\begin{aligned}\dot{\rho}(t) = & -i[H_S, \bar{\rho}(t)] + \gamma(\omega_0)(X\bar{\rho}(t)X^+ - \frac{1}{2}\{X^+X, \bar{\rho}(t)\}) \\ & + \bar{\gamma}(\omega_0)(X^+\bar{\rho}(t)X - \frac{1}{2}\{X^+X^+, \bar{\rho}(t)\}) \\ & + \gamma(0)(Y\bar{\rho}(t)Y - \frac{1}{2}\{Y^2, \bar{\rho}(t)\})\end{aligned}\quad (12)$$

master equation! (in the Schrödinger picture).

## Some notes

(This notes may be useful for the generalization when  $\Delta \neq 0$ .)

$$e^{iHst} (b + b^*) e^{-iHst} = e^{i\bar{\Omega}t} X^+ + e^{-i\bar{\Omega}t} X - Y.$$

with  $X = \frac{1}{2} \left[ b \left( 1 - \frac{1}{\Delta} \right) + b^* \left( 1 + \frac{1}{\Delta} \right) + \bar{g} \sigma_z \right] ; Y = \bar{g} \sigma_z.$

$$X^+ = \frac{1}{2} \left[ b^* \left( 1 - \frac{1}{\Delta} \right) + b \left( 1 + \frac{1}{\Delta} \right) + \bar{g} \sigma_z \right]$$

$$\bar{g} = \frac{2g}{\sqrt{\Delta}} = \frac{2g}{\Delta(1+4\Delta)} \quad \text{with} \quad \bar{\Delta} = \sqrt{1+4\Delta}$$

$$\bar{\Delta} = \Delta \sqrt{1+4\Delta} = \Delta \bar{\Delta}$$

Notice that:

$$[H_S, X] = -\bar{\Delta} X \quad \text{and} \quad [H_S, X^+] = \bar{\Delta} X^+ \quad \text{and} \quad [H_S, Y] = 0.$$

Notice that we may write again

$$X = \frac{1}{2} \left[ \underbrace{(b + b^*)}_x - \frac{1}{\Delta} \underbrace{(b - b^*)}_y + \bar{g} \sigma_z \right] \quad \text{@ The field quadratures.}$$

The general equation of motion will have the same form of eq. (12) but with the new operators and parameters.

$$\begin{aligned} \bar{\rho}(t) = & -i[H_S, \bar{\rho}(t)] + \gamma(\bar{\omega}) (X \bar{\rho}(t) X^+ - \frac{1}{2} \{X^+ X, \bar{\rho}(t)\}) \\ & + \bar{\gamma}(\bar{\omega}) (X^+ \bar{\rho}(t) X - \frac{1}{2} \{X X^+, \bar{\rho}(t)\}) \\ & + \gamma(\bar{\omega}) (Y \bar{\rho}(t) Y - \frac{1}{2} \{Y^2, \bar{\rho}(t)\}) \end{aligned}$$

**master equation!** (in the Schrödinger picture). (12)