The master equation

We redefine H_0 and H_I as:

$$H_0 = \frac{\omega \sigma_z}{2} + \Omega_0 b^\dagger b$$

$$H_I = g\sigma_z(b+b^\dagger) + (b+b^\dagger)\sum_k h_k(b_k + b_k^\dagger)$$

$$H_B = \sum_k \Omega_k b_k^\dagger b_k$$

then H_I is writen as

$$H_I = \sum_\alpha A_\alpha \otimes B_\alpha \quad \text{where}$$

$$A_1 = g\sigma(b+b^\dagger) \qquad B_1 = \mathbb{1}$$

$$A_2 = (b+b^\dagger) \qquad B_2 = \sum_k h_k(b_k + b_k^\dagger)$$

For the M.E. we need the interaction operators in the interactional picture

$$A(t) = e^{iH_S t} A e^{-iH_S t}$$

For this we will use the Hausdorf Formula, so we will evalute this commutators

$$[iH_S t, A], \quad [iH_S t, [iH_S t, A]], \quad \dots$$

For A_1

$$A_1(t) = e^{iH_s t}\left(g\sigma_z(b+b^\dagger)\right) e^{-iH_s t}$$

$$[iH_s t, g\sigma_z(b+b^\dagger)] = it\left[\frac{w\sigma_z}{2} + \Omega_0 b^\dagger b, g\sigma_z(b+b^\dagger)\right]$$

$$= it\Omega_0 g\sigma_z[b^\dagger b, b+b^\dagger] = it\Omega_0 g\sigma_z(-b+b^\dagger)$$

$$[iH_s t,[iH_s t, g\sigma_z(b+b^\dagger)]] = it\left[\frac{w\sigma_z}{2} + \Omega_0 b^\dagger b, it\Omega_0 g\sigma_z(-b+b^\dagger)\right]$$

$$= (it\Omega_0)^2 g\sigma_z[b^\dagger b, -b+b^\dagger] = (it\Omega_0)^2 g\sigma_z(b+b^\dagger)$$

High order commutator will give $b+b^\dagger$ or $-b+b^\dagger$
so, we write the seres as

$$A_1(t) = g\sigma_z b\sum_{n=0}^{\infty}\frac{(-1)^n(it\Omega_0)^n}{n!} + g\sigma_z b^\dagger\sum_{n=0}^{\infty}\frac{(it\Omega_0)^n}{n!}$$

$$= g\sigma_z b e^{-i\Omega_0 t} + g\sigma_z b e^{i\Omega_0 t}$$

From this calculation we also get

$$A_2(t) = e^{iH_s t}(b+b^\dagger) e^{-iH_s t}$$

$$= b e^{-i\Omega_0 t} + b^\dagger e^{i\Omega_0 t}$$

the evolution of the full system

$$\dot{\rho} = -i\,[H_S + H_F + H_B, \rho]$$

we move tho the interaction picture

$$\dot{\rho}(t) = -i\,[H_I(t), \rho(t)] \quad (1)$$

We integrate

$$\rho(t) - \rho(0) = -i\int_0^t [H_I(t'), \rho(t')]\,dt' \quad (2)$$

and reinserted in (1)

$$\dot{\rho}(t) = -i\,[H_I(t), \rho_0] - \int_0^t [H_I(t), [H_I(t'), \rho(t')]]\,dt$$

and take the parcial trace over the bath

$$Tr_B\{\dot{\rho}(t)\} = \dot{\rho}_S(t) = -i\,Tr_B\{[H_I(t), \rho_0]\}$$
$$- \int_0^t Tr_B\{[H_I(t), [H_I(t'), \rho(t')]]\}\,dt'$$

Now we perform the Born aproximation

$$\rho(t) = \rho_S(t) \otimes \overline{\rho}_B$$

$$\dot{\rho}_s(t) = -i\sum_\alpha \Big(A_\alpha(t)\,\rho_s^0\, Tr\{B_\alpha(t)\bar{\rho}_B\} - \rho_s^0 A_\alpha(t)\, Tr\{\bar{\rho}_B B_\alpha(t)\}$$

$$- \int_0^t Tr_B\{[H_I(t), [H_I(t'), \rho(t')]]\}\, dt' \quad (3)$$

Before going further, we derive and Identity, take Tr_B over (2) we have

$$Tr_B\{\rho(t)\} = Tr_B\{\rho(0)\} - \int_0^t Tr_B\{[H(t'), \rho(t')]\}\, dt'$$

$$\rho_s(t) = \rho_s^0 - i\sum_\alpha \int_0^t Tr_B\{[A_\alpha(t')\otimes B_\alpha(t'), \rho_s(t')\otimes\bar{\rho}_B]\}\, dt'$$

$$= \rho_s^0 - \sum_\alpha \int_0^t A_\alpha(t')\rho_s(t')\, Tr\{B_\alpha(t')\bar{\rho}_B\}$$

$$- \rho_s(t') A_\alpha(t')\, Tr\{\bar{\rho}_B B_\alpha(t')\}$$

$$= \rho_0 - i\sum_\alpha \int_0^t [A_\alpha(t'), \rho_s(t')]\, Tr\{B_\alpha(t')\bar{\rho}_B\}$$

We have $\langle B_1\rangle = 1$ and $\langle B_2\rangle = 0$

then

$$\rho_s(t) = \rho_0 - i\int_0^t [A_1(t'), \rho_s(t')]\, dt'$$

then we calculate $[A_1(t), \rho_s(t)]$ (we suppose $A_1(t)$ is well behave and has no singularities)

$$-i\left[A_1(t), \rho_s(t)\right] = -i\left[A_1(t), \rho_0\right] - \int_0^t \left[A_1(t), \left[A_1(t'), \rho_s(t')\right]\right] dt' \qquad (4)$$

Now we continue with eq (3)

$$\dot{\rho}_s(t) = -i\sum_\alpha \left[A_\alpha(t), \rho_s^0\right] \mathrm{Tr}\left\{B_\alpha(t)\bar{\rho}_B\right\} - \int_0^t \mathrm{Tr}_B\left[H_I(t), \left[H_I(t'), \rho(t')\right]\right] dt'$$

taking into account that $\langle B_1 \rangle = 1$ $\langle B_2 \rangle = 0$

$$\dot{\rho}_s = -i\left[A_1(t), \rho_s^0\right] - \int_0^t \mathrm{Tr}_B\left[H_I(t), \left[H_I(t'), \rho(t')\right]\right] dt'$$

We can write the second term in term of the correlation functions

$$\dot{\rho}_s = -i\left[A_1(t), \rho_s^0\right] - \sum_{\alpha\beta}\left\{\int_0^t C_{\alpha\beta}(t-t')\left[A_\alpha(t), A_\beta(t')\rho_s(t')\right] + C_{\beta\alpha}(t'-t)\left[\rho_s(t')A_\beta(t'), A_\alpha(t)\right]\right\}dt'$$

where $C_{\alpha\beta}(t_1, t_2) = Tr\{B_\alpha(t_1) B_\beta(t_2) \bar{\rho}_B\}$

if $[H_B, \bar{\rho}_B] = 0$, that is our case

$$C_{\alpha\beta}(t_1 - t_2) = Tr\{e^{iH_B(t_1-t_2)} B_\alpha e^{-iH_B(t_1-t_2)} B_\beta \bar{\rho}_B\}$$

Then we calculat the $C_{\alpha\beta}$

$$C_{11}(t_1 - t_2) = 1$$
$$C_{12}(t_1 - t_2) = 0$$
$$C_{21}(t_1 - t_2) = 0$$
$$C_{22}(t_1 - t_2) = \frac{1}{2\pi} \int J^{(1)}(\omega) [1 + n_B(\omega)] e^{-i\omega(t_1 - t_2)} d\omega$$

Then

$$\dot{\rho}_s = -i[A_1(t), \rho_s^0]$$
$$- \int_0^t [A_1(t), A_1(t') \rho_s(t')] + [\rho_s(t') A_1(t'), A_1(t)] dt'$$
$$- \int_0^t C_{22}(t-t') [A_2(t), A_2(t') \rho_s(t')] + C_{22}(t'-t) [\rho_s(t') A_2(t'), A_2(t)] d$$

we notice that the second term can be writen as

$$[A_1(t), [A_1(t'), \rho_s(t')]]$$

and we can use the identity (4)

$$\dot{\rho}_s = -i\left[A(t), \rho_s(t)\right]$$

$$- \int_0^t C_{zz}(t-t')\left[A_z(t), A_z(t')\rho_s(t')\right] + C_{zz}(t'-t)\left[\rho_s(t') A_z(t'), A_z(t)\right]$$

In this point we perform the Markov approximation

$$\rho_s(t') \longrightarrow \rho_s(t)$$

$$\tau = t - t'$$

$$\text{limit} \rightarrow \infty$$

$$\dot{\rho}_s(t) = -i\left[A(t), \rho_s(t)\right]$$

$$- \int_0^\infty C_{zz}(\tau)\left[A_z(t), A_z(t-\tau)\rho_s(t)\right] + C_{zz}(-\tau)\left[\rho_s(t) A_z(t-\tau), A_z(t)\right] d\tau$$

this will lead us to the well known M.E. in the I.P.

$$\dot{\rho}_s(t) = -i\left[A_1(t), \rho_s(t)\right]$$

$$+ \gamma\left(b\rho_s(t)b^\dagger - \frac{1}{2}\left\{b^\dagger b, \rho_s(t)\right\}\right)$$

$$+ \bar{\gamma}\left(b^\dagger\rho_s(t) b - \frac{1}{2}\left\{bb^\dagger, \rho_s(t)\right\}\right)$$

$$A_1 = g\sigma_z(b+b^\dagger)$$