SB – RC – Master equation (Dephasing case).

After applying the RC-map to the spin-boson Hamiltonian we find.

$$H = H_S + H_E + H_I$$

$$H_S = \frac{\omega}{2}\sigma_z + \Omega_0 b^+ b + g\,\sigma_z\,(b^+ + b) + \frac{g^2}{\Omega_0}\sigma_z^2 + \Omega_0 \Delta (b + b^+)^2$$

$$H_E = \sum_k \Omega_k b_k^+ b_k$$

$$H_I = (b + b^+) \sum_k h_k (b_k + b_k^+)$$

State time evolution – von Neumann

$$\dot{\rho}(t) = -i\,[H_S + H_E + H_I, \rho(t)] \qquad (1)$$

Move to the interaction picture

$$\rho_I(t) = \exp(i(H_S + H_E)t)\,\rho(t)\,\exp(-i(H_0 + H_{10})t)$$

$$\xrightarrow{(1)} \dot{\rho}_I(t) = -i\,[H_I(t), \rho_I(t)]. \qquad (2)$$

with

$$H_I(t) = \exp(i(H_S + H_E)t)\,H_I\,\exp(-i(H_S + H_E)t)$$

Formal integration of (2) yields

$$\rho_I(t) = \rho_I(0) - i\int_0^t dt'\,[H_I(t'), \rho_I(t')], \qquad (3)$$

which after substituting in (2) leads to

$$\dot{\rho}_I(t) = -i\,[H_I(t), \rho_I(0)] - \int_0^t dt'\,[H_I(t), [H_I(t'), \rho_I(t')]] \qquad (4)$$

Define now the reduce state of the system as

$$\bar{\rho}_I(t) = \mathrm{Tr}_E(\rho_I(t)),$$

and assume that $\rho(0) = \bar{\rho}_S \otimes \rho_E$, with

$$\mathrm{Tr}_E(H_I(t)\,\rho_I(0)) = 0.$$ (Which can be show to hold for the conditions in our problem)

Trace over the environment on both sides of (4)

$$\overset{(4)}{\Longrightarrow} \quad \dot{\bar{\rho}}_I(t) = -\int_0^t dt' \, Tr_E\left([H_I(t), [H_I(t'), \rho_I(t')]]\right) \qquad (5)$$

⊛ Approximation: At this point we take

$\rho_I(t) = \bar{\rho}_I(t) \otimes \rho_E$.

i) $H_I \ll H_S, H_B$

ii) The state of the bath doesn't change (No memory).

iii) The state of the system does change.

iv) A weaker condition could have been imposed:

For interactions of the form $\underset{\text{System}}{X} \; \underset{\text{Environment}}{A}$, then

$$Tr_E\left([A_I(t), [A(t'), \rho_I(t')]]\right) \approx$$

$$\bar{\rho}_I(t) \otimes \underbrace{Tr_E\left([A_I(t), [A(t'), \rho_E]]\right)}_{\text{Bath correlations.}}$$

(Assume they don't change).

$$\overset{(5)}{\Longrightarrow} \quad \dot{\bar{\rho}}_I(t) = -\int_0^t dt' \, Tr_E\left([H_I(t), [H_I(t'), \bar{\rho}_I(t') \otimes \rho_E]]\right) \quad (6)$$

Marcov Approximation.

We assume the bath correlation time to be small.

i) $\bar{\rho}_I(t') \longrightarrow \bar{\rho}_I(t)$ (Markov approx: Yields a first order differential equation for $\bar{\rho}_I$).

ii) We let the upper limit of the integral to go to infinity.

$$\overset{(6)}{\Longrightarrow} \quad \dot{\bar{\rho}}_I(t) = -\int_0^{+\infty} dt' \, Tr_E\left([H_I(t), [H_I(t'), \bar{\rho}_I(t) \otimes \rho_E]]\right) \; ; \quad \tau = t - t'$$

$$= -\int_0^{+\infty} d\tau \, Tr_E\left([H_I(t), [H_I(t-\tau), \bar{\rho}_I(t) \otimes \rho_E]]\right). \quad (7)$$

⊛ We can choose appropriate forms for H_S and H_S and find different master equations.

Redfield equation

To recover the redfield equation, let us consider $H_I(t)$ in detail.

$$H_I(t) = \exp(i(H_S+H_E)t)(b+b^\dagger)\sum_k h_k(b_k+b_k^\dagger)\exp(-i(H_S+H_E)t).$$

Using $e^{iH_E t}\, b_k\, e^{-iH_E t} = b_k e^{-i\Omega_k t}$

$$H_I(t) = \exp(iH_S t)(b+b^\dagger)\exp(-iH_S t)\underbrace{\sum_k h_k\left(b_k e^{-i\Omega_k t} + b_k^\dagger e^{i\Omega_k t}\right)}_{\Gamma(t).} \quad (8)$$

Equation (7) in the Schrödinger equation can now be written as

$$\dot{\bar\rho}(t) = -i[H_S,\bar\rho(t)] - \int_0^{+\infty} d\tau\, Tr_E\left([H_I,[H_I(t-\tau),\bar\rho(t)\otimes\rho_E]]\right).$$

$$= -i[H_S,\bar\rho(t)] - \int_0^{+\infty} d\tau \Big\{ Tr_E\left(\Gamma(0)\Gamma(\tau)\rho_E\right)\left[(b+b^\dagger),\, e^{-iH_S\tau}(b+b^\dagger)e^{+iH_S\tau}\bar\rho(t)\right]$$

$$+ Tr_E\left(\Gamma(\tau)\Gamma(0)\rho_E\right)\left[e^{-iH_S\tau}(b+b^\dagger)e^{+iH_S\tau}\bar\rho(t),(b+b^\dagger)\right]\Big\}$$

$$= -i[H_S,\bar\rho(t)] - \int_0^{+\infty} d\tau\, C(\tau)\left[(b+b^\dagger),\, e^{-iH_S\tau}(b+b^\dagger)e^{+iH_S\tau}\bar\rho(t)\right]$$

$$- \int_0^{+\infty} d\tau\, C(-\tau)\left[e^{-iH_S\tau}(b+b^\dagger)e^{+iH_S\tau}\bar\rho(t),(b+b^\dagger)\right]. \quad (9)$$

This is the Redfield equation reported by G. Shaller.

With the correlation function of the residual bath.

$$C(\tau) = Tr_E\left(\Gamma(0)\Gamma(\tau)\rho_E\right) = Tr_E\left(\Gamma(-\tau)\Gamma(0)\rho_E\right)$$

which we write also as.

$$C(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\gamma(\omega)\, e^{-i\omega\tau}\, d\tau = C^*(-\tau)$$

To proceed further we use the eigen basis of H_{sys} $\left(H_S|a\rangle = E_a|a\rangle\right)$, to write

$$\dot{\bar{\rho}}(t) = -i[H_S, \bar{\rho}(t)] - \sum_{ab} \langle a|(b+b^+)|b\rangle$$
$$\times \Big\{ -\int_0^{+\infty} d\tau\, C(\tau)\, e^{i(E_b-E_a)\tau} \left[(b+b^+), |a\rangle\langle b| \bar{\rho}(t) \right]$$
$$- \int_0^{+\infty} d\tau\, C(-\tau)\, e^{i(E_b-E_a)\tau} \left[|a\rangle\langle b| \bar{\rho}(t), (b+b^+) \right] \Big\}.$$

Let us now consider

$$\int_0^{+\infty} d\tau\, C(\tau)\, e^{i(E_b-E_a)\tau} = \int_{-\infty}^{+\infty} d\tau\, \Theta(\tau)\, C(\tau)\, e^{i(E_b-E_a)\tau}$$
$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega\, \gamma(\omega) \int_{-\infty}^{+\infty} d\tau\, e^{i(E_b-E_a-\omega)\tau}\, \frac{1}{2}\left(1 + \mathrm{Sng}(\tau)\right)$$
$$= \frac{\gamma(E_b-E_a)}{2} + \frac{i}{2\pi} \mathcal{P} \int_{-\infty} d\omega\, \frac{\gamma(\omega)}{E_b-E_a-\omega} \approx \frac{\gamma(E_b-E_a)}{2}.$$

The equation of motion becomes

$$\dot{\bar{\rho}}(t) = -i[H_S, \rho] + \sum_{ab} \langle a|b+b^+|b\rangle \Big\{ -\gamma_{ab} \left[(b+b^+), |a\rangle\langle b| \bar{\rho}(t) \right]$$
$$- \gamma_{ba} \left[|a\rangle\langle b| \bar{\rho}(t), (b+b^+) \right] \Big\} \qquad (10)$$

The Markovian ME.

We continue with the calculation in the interaction picture starting with equation (7).

$$\dot{\bar{\rho}}_I(t) = -\int_0^{+\infty} d\tau \, Tr_E \left([H_I(t), [H_I(t-\tau), \bar{\rho}_I(t) \otimes \rho_E]] \right). \quad (7)$$

We now use Eq. (8) to write.

$$H_I(t) = \exp(iH_s t)(b+b^+)\exp(-iH_s t)\,\Gamma(t),$$

and we evaluate it using, and correcting dwis results:

$$e^{-iH_s t}(b+b^+)e^{iH_s t} = i\frac{\sin(x\sqrt{y})}{\sqrt{y}}(-b+b^+) + \cos(x\sqrt{y})(b+b^+)$$
$$+ \cos(x\sqrt{y})\frac{2g}{\Omega_0 y}\sigma_z - \frac{2g}{\Omega_0 y}\sigma_z$$

with $x = \Omega t$ and $y = (1+4\Delta)$.

For the moment we restrict to the simpler case with $\Delta = 0 \Rightarrow y = 1$.

$$e^{-iH_s\tau}(b+b^+)e^{iH_s t} = \underbrace{\left(b+\frac{g}{\Omega}\sigma_z\right)}e^{-i\Omega t} + \underbrace{\left(b^+ + \frac{g}{\Omega}\sigma_z\right)}e^{i\Omega t} - \underbrace{\frac{2g}{\Omega_0}\sigma_z}$$
$$= X e^{-i\Omega t} + X^+ e^{i\Omega t} - Y.$$
$$= X(t) + X^+(t) - Y.$$

Equation (7) can now be rewritten as

$$\dot{\bar{\rho}}_I(t) = -\int^{+\infty} d\tau \, Tr_E \Big([(X(t)+X^+(t)-Y)\Gamma(t),$$
$$[(X(t-\tau)+X^+(t-\tau)-Y)\Gamma(t-\tau), \rho_I(t)\otimes\rho_E]] \Big)$$

and after some algebra

$$\dot{\bar{\rho}}_I(t) = -\int_0^{+\infty} d\tau \Big\{ C(\tau)[(X(t)+X^+(t)-Y), (X(t-\tau)+X^+(t-\tau)-Y)\bar{\rho}_I(t)]$$
$$+ C(-\tau)[\bar{\rho}_I(t)(X(t-\tau)+X^+(t-\tau)-Y), (X(t)+X^+(t)-Y)]\Big\}$$

There are several term that are obtained from expanding the commutators. Consider a typical one

$$(X(t)+X^+(t)-Y)(X(t-\tau)+X^+(t-\tau)-Y)$$
$$= \cancel{X^2 e^{-i\Omega(2t-\tau)}} + X^+X e^{i\Omega\tau} - \cancel{YX e^{-i\Omega(t-\tau)}} + XX^+ e^{-i\Omega\tau}$$
$$+ \cancel{X^+X^+ e^{i\Omega(2t-\tau)}} - \cancel{YX^+ e^{i\Omega(t-\tau)}} - \cancel{XY e^{i\Omega t}} - \cancel{X^+Y e^{i\Omega t}} + Y^2$$

Rotating wave approximation.

In the above expression we neglect the rapidly rotating terms with t, as they will give negligible contributions.

$$(X(t)+X^\dagger(t)-Y)(X(t-\tau)+X^\dagger(t-\tau)-Y) \approx X^\dagger X e^{i\Omega\tau} + XX^\dagger e^{-i\Omega\tau} + Y^2$$

Equation (7) now reduces to

$$\begin{aligned}
\dot{\bar{\rho}}_I(t) = &-\int_0^{+\infty} d\tau\, c(\tau)\{(X^\dagger X e^{i\Omega\tau} + XX^\dagger e^{-i\Omega\tau} + Y^2)\bar{\rho}_I(t) \\
&\qquad - (X\bar{\rho}_I(t)X^\dagger e^{i\Omega\tau} + X^\dagger\bar{\rho}_I(t)X e^{-i\Omega\tau} + Y\bar{\rho}_I(t)Y)\} \\
&-\int_0^{+\infty} d\tau\, c(-\tau)\{\bar{\rho}_I(t)(X^\dagger X e^{-i\Omega\tau} + XX^\dagger e^{i\Omega\tau} + Y^2) \\
&\qquad - (X\bar{\rho}_I(t)X^\dagger e^{-i\Omega\tau} + X^\dagger\bar{\rho}_I(t)X e^{i\Omega\tau} + Y\bar{\rho}_I(t)Y)\}
\end{aligned}$$

We now consider

$$\int_0^{+\infty} d\tau\, c(\tau) e^{i\Omega\tau} = \Gamma(\Omega) \qquad\qquad c(-\tau) = c^*(\tau).$$

$$\int_0^{+\infty} d\tau\, c(-\tau) e^{i\Omega\tau} = \int_0^{+\infty} d\tau\, c^*(\tau) e^{i\Omega\tau} = \left(\int_0^{+\infty} d\tau\, c(\tau) e^{-i\Omega t}\right)^* = \Gamma^*(-\Omega)$$

$$\begin{aligned}
\Rightarrow \dot{\bar{\rho}}_I = &-\Gamma(\Omega)(X^\dagger X\bar{\rho}_I - X\bar{\rho}_I X^\dagger) - \Gamma(-\Omega)(XX^\dagger\bar{\rho}_I - X^\dagger\bar{\rho}_I X) \\
&-\Gamma^*(\Omega)(\bar{\rho}_I X^\dagger X - X\bar{\rho}_I X^\dagger) - \Gamma^*(-\Omega)(XX^\dagger\bar{\rho}_I - X^\dagger\bar{\rho}_I X) \\
&-\Gamma(0)(Y^2\bar{\rho}_I - Y\bar{\rho}_I Y) - \Gamma^*(0)(\bar{\rho}_I Y^2 - Y\bar{\rho}_I Y)
\end{aligned}$$

We now use

$$\Gamma(\Omega) = \frac{\gamma(\Omega)}{2} + \frac{i}{2}\sigma(\Omega) \qquad\qquad \Gamma(-\Omega) = \frac{\bar{\gamma}(\Omega)}{2} + \frac{i}{2}\bar{\sigma}(\Omega)$$

$$\gamma(\Omega) = J(\Omega)(\bar{n}(\Omega)+1) \qquad\qquad \bar{\gamma}(\Omega) = J(\Omega)\bar{n}(\Omega)$$

$$\bar{n}(\Omega) = (e^{\beta\Omega}-1)^{-1}$$

and arrive to the Lindblad master equation. (We neglect the Lamb shift)

$$\begin{aligned}
\dot{\bar{\rho}}_I = &\ \gamma(\Omega)(X\bar{\rho}_I X^\dagger - \tfrac{1}{2}\{X^\dagger X, \bar{\rho}_I\}) + \bar{\gamma}(\Omega)(X^\dagger\bar{\rho}_I X - \tfrac{1}{2}\{XX^\dagger, \bar{\rho}_I\}) \qquad (11) \\
&+\gamma(0)(Y\bar{\rho}_I Y - \tfrac{1}{2}\{Y^2, \bar{\rho}_I\})
\end{aligned}$$

Our lovely master equation! (in the interaction picture)

Remember: $\boxed{X = b + \frac{g}{\Omega}\sigma_z.}$ This non-trivial change is what is going to make the difference!

$$\boxed{Y = \frac{2g}{\Omega_b} J_z}$$

⊛ Can we find a left unitary that separates this?

Going to the Schrödinger picture.

$$\begin{aligned}\dot{\bar{\rho}}(t) = -i[H_S, \bar{\rho}(t)] &+ \gamma(\Omega)\left(X\bar{\rho}(t)X^\dagger - \tfrac{1}{2}\{X^\dagger X, \bar{\rho}(t)\}\right)\\ &+ \bar{\gamma}(\Omega)\left(X^\dagger\bar{\rho}(t)X - \tfrac{1}{2}\{XX^\dagger, \bar{\rho}(t)\}\right)\\ &+ \gamma(0)\left(Y\bar{\rho}(t)Y - \tfrac{1}{2}\{Y^2, \bar{\rho}(t)\}\right)\end{aligned} \qquad (12)$$

master equation! (in the Schrödinger picture).

Some notes

(This notes may be useful for the generalization when $\Delta \neq 0$)

$$e^{iH_s t}(b+b^\dagger)e^{-iH_s t} = e^{i\bar{\Omega}t}X^\dagger + e^{-i\bar{\Omega}t}X - Y .$$

with $X = \frac{1}{2}\left[b\left(1-\frac{1}{\bar{\Delta}}\right) + b^\dagger\left(1+\frac{1}{\bar{\Delta}}\right) + \bar{g}\,\sigma_z \right]$; $Y = \bar{g}\,\sigma_z$.

$$X^\dagger = \frac{1}{2}\left[b^\dagger\left(1-\frac{1}{\bar{\Delta}}\right) + b\left(1+\frac{1}{\bar{\Delta}}\right) + \bar{g}\,\sigma_z \right]$$

$$\bar{g} = \frac{2g}{\bar{\Omega}\bar{\Delta}} = \frac{2g}{\Omega(1+4\Delta)} \quad \text{with} \quad \bar{\Delta} = \sqrt{1+4\Delta}$$

$$\bar{\Omega} = \Omega\sqrt{1+4\Delta} = \Omega\bar{\Delta}$$

Notice that:

$$[H_s, X] = -\bar{\Omega}X \quad \text{and} \quad [H_s, X^\dagger] = \bar{\Omega}X^\dagger \quad \text{and} \quad [H_s, Y] = 0.$$

Notice that we may write again

$$X = \frac{1}{2}\left[\underbrace{(b+b^\dagger)}_{x} - \frac{1}{\bar{\Delta}}\underbrace{(b-b^\dagger)}_{y} + \bar{g}\,\sigma_z \right]$$

(*) The field quadratures.

The general equation of motion will have the same form of eq. (12) but with the new operators and parameters.

$$\begin{aligned}\dot{\bar{\rho}}(t) = {} & -i[H_s, \bar{\rho}(t)] + \gamma(\bar{\Omega})\left(X\bar{\rho}(t)X^\dagger - \tfrac{1}{2}\{X^\dagger X, \bar{\rho}(t)\}\right) \\ & + \bar{\gamma}(\bar{\Omega})\left(X^\dagger\bar{\rho}(t)X - \tfrac{1}{2}\{XX^\dagger, \bar{\rho}(t)\}\right) \\ & + \gamma(\bar{0})\left(Y\bar{\rho}(t)Y - \tfrac{1}{2}\{Y^2, \bar{\rho}(t)\}\right)\end{aligned} \qquad (13)$$

master equation! (in the Schrödinger picture).