Re master equation

We redefine H_0 and H_I as:

$$H_0 = \frac{\omega\sigma_z}{2} + \Omega_0 b^\dagger b$$

$$H_I = g\sigma_z(b+b^\dagger) + (b+b^\dagger)\sum_k h_k(b_k + b_k^\dagger)$$

$$H_B = \sum_k \Omega_k b_k^\dagger b_k$$

then H_I is writen as

$$H_I = \sum_\alpha A_\alpha \otimes B_\alpha \quad \text{where}$$

$$A_1 = g\sigma(b+b^\dagger) \qquad B_1 = \mathbb{1}$$
$$A_2 = (b+b^\dagger) \qquad B_2 = \sum_k h_k(b_k+b_k^\dagger)$$

For the M.E. we need the interaction operators in the interactional picture

$$A(t) = e^{iH_St} A\, e^{-iH_St}$$

For this we will use the Hausdorf Formula, so we will evalute this commutators

$$[iH_St, A], \quad [iH_St, [iH_St, A]], \ \ldots$$

For A_1

$$A_1(t) = e^{iH_st}\left(g\sigma_z(b+b^\dagger)\right)e^{-iH_st}$$

$$[iH_st, g\sigma_z(b+b^\dagger)] = it\left[\frac{\omega\sigma_z}{2} + \Omega_0 b^\dagger b, g\sigma_z(b+b^\dagger)\right]$$

$$= it\Omega_0 g\sigma_z[b^\dagger b, b+b^\dagger] = it\Omega_0 g\sigma_z(-b+b^\dagger)$$

$$[iH_st,[iH_st, g\sigma_z(b+b^\dagger)]] = it\left[\frac{\omega\sigma_z}{2} + \Omega_0 b^\dagger b, it\Omega_0 g\sigma_z(-b+b^\dagger)\right]$$

$$= (it\Omega_0)^2 g\sigma_z[b^\dagger b, -b+b^\dagger] = (it\Omega_0)^2 g\sigma_z(b+b^\dagger)$$

High order commutator will give $b+b^\dagger$ or $-b+b^\dagger$
so, we write the seres as

$$A_1(t) = g\sigma_z b\sum_{n=0}^{\infty}\frac{(-1)^n(it\Omega_0)^n}{n!} + g\sigma_z b^\dagger\sum_{n=0}^{\infty}\frac{(it\Omega_0)^n}{n!}$$

$$= g\sigma_z b\, e^{-i\Omega_0 t} + g\sigma_z b\, e^{i\Omega_0 t}$$

From this calculation we also get

$$A_2(t) = e^{iH_st}(b+b^\dagger)e^{-iH_st}$$

$$= b\, e^{-i\Omega_0 t} + b^\dagger e^{i\Omega_0 t}$$

Now we work the redfield eq. (from Gernots notes) in the interactional picture

$$\dot{\rho}_S = -\int_0^\infty \sum_{\alpha\beta} \left\{ C_{\alpha\beta}(\tau) \left[A_\alpha(t), A_\beta(t-\tau) \rho_S(t) \right] + h.c. \right\} d\tau$$

For this we calcule $C_{\alpha\beta}$ with $\alpha,\beta = 1,2$

$$C_{\alpha\beta}(t_1,t_2) = Tr\{ B_\alpha(t_1) B_\beta(t_2) \bar{\rho}_0 \}$$

this are in the I.P.

in this case we use a thermal state

$$\bar{\rho}_B = \frac{e^{-\beta \sum \Omega_k b^\dagger b}}{Z}$$

and therefore we have $[H_B, \bar{\rho}_B] = 0$, then the correlation fuction can be writen ase

$$C_{\alpha\beta}(t) = Tr\{ e^{iH_B t} B_\alpha e^{-iH_B t} B_\beta \bar{\rho}_B \}$$

then

$$C_{11}(\tau) = Tr\{ e^{iH_B\tau} \mathbb{1} e^{-iH_B\tau} \mathbb{1} \bar{\rho}_B \} = 1$$

$$C_{12}(\tau) = Tr\{ e^{iH_B} \mathbb{1} e^{-iH_B\tau} \sum_k h_k (b_k + b_k^\dagger) \bar{\rho}_B \} =$$

$$= \sum_k h_k \langle b_k \rangle + h_k \langle b_k^\dagger \rangle = 0$$

$C_{21}(\tau) = 0$ same argument above

$$C_{22}(\tau) = \mathrm{Tr}\left\{ e^{iH_B\tau} \sum_k h_k (b_k + b_k^\dagger) e^{-iH_B\tau} \sum_{k'} h_{k'} (b_{k'} + b_{k'}^\dagger) \bar{\rho}_B \right\}$$

$$= \frac{1}{2\pi} \int J^{(1)}(\omega) \left[1 + n_B(\omega)\right] e^{-i\omega\tau} d\omega$$

We write the specific elements of the Redfield equation

$$\dot{\rho} = -\int_0^\infty \sum_{\alpha\beta} \left\{ C_{\alpha\beta}(\tau) \left[A_\alpha(t), A_\beta(t-\tau)\rho_S(t)\right] + h.c \right\} d\tau$$

$$= -\int_0^\infty \left\{ \left[g\sigma_z \left(b e^{-i\Omega t} + b^\dagger e^{i\Omega t}\right), g\sigma_z \left(b e^{-i\Omega(t-\tau)} + b^\dagger e^{i\Omega(t-\tau)}\right) \rho_S(t) \right] \right.$$
$$\left. + h.c \right\} d\tau$$

$$- \int_0^\infty \left\{ C_{22}(\tau) \left[b e^{-i\Omega\tau} + b^\dagger e^{-i\Omega\tau}, \left(b e^{-i\Omega(t-\tau)} + b^\dagger e^{i\Omega(t-\tau)}\right) \rho_S(t) \right] \right.$$
$$\left. + h.c. \right\} d\tau$$

the second terme, we know will lead the the M.E. making the R.W.A (ie. neglect $e^{-2i\Omega}$ terms)

$$\gamma\left(b\rho b^\dagger - \frac{1}{2}\{b^\dagger b, \rho\}\right) + \bar{\gamma}\left(b^\dagger \rho b - \frac{1}{2}\{b b^\dagger, \rho\}\right)$$

Now we will work the 1 term

$$-\int_0^\infty \operatorname{Tr}_B\left[g\sigma_z\left(b e^{-i\Omega t}+b^\dagger e^{i\Omega t}\right), g\sigma_z\left(b e^{-i\Omega(t-\tau)}+b^\dagger e^{i\Omega(t-\tau)}\right)\rho_S(t)\right]$$

$$+h.c \quad d\tau$$

$$= -\int g^2\sigma_z^2\Big[e^{-i\Omega(2t-\tau)}[b, b\rho] + e^{-i\Omega\tau}[b, b^\dagger\rho]$$

$$+ e^{i\Omega\tau}[b^\dagger, b\rho] + e^{i\Omega(2t-\tau)}[b^\dagger, b^\dagger\rho]$$

$$+h.c\Big]\,d\tau$$

We apply the R.W.A, not taking into account the term with $e^{\pm i\Omega 2t}$, and we use the facta $\sigma_z^2 = \mathbb{1}$

$$= -g^2\left[\int_0^\infty e^{-i\Omega\tau}[b, b^\dagger\rho] - \int_0^\infty e^{i\Omega\tau}[b^\dagger, b\rho] + h.c\right]$$

We perform

$$\int_0^\infty e^{\pm i\Omega\tau} = \frac{\pm i}{\Omega}$$

$$= \frac{g^2 i}{\Omega}[b, b^\dagger\rho] + \frac{g^2 i}{\Omega}[b^\dagger, b\rho] + h.c$$

Opening the comutator and the h.c.

$$= \frac{g^2 i}{\Omega}\left(b b^\dagger\rho + b^\dagger b\rho - \rho b b^\dagger - \rho b^\dagger b\right]$$

$$= \frac{g^2 i}{\Omega}\left([b b^\dagger, \rho] + [b^\dagger b, \rho]\right)$$

then The total ME in the interaction picture is

$$\dot{\rho}_s = \gamma\left(b\rho_s b^\dagger - \frac{1}{2}\{b^\dagger b, \rho_s\}\right) + \bar{\gamma}\left(b^\dagger\rho_s b - \frac{1}{2}\{bb^\dagger, \rho_s\}\right)$$

$$+ \underbrace{\frac{g^2}{2}\left(i[bb^\dagger, \rho_s] + i[b^\dagger b, \rho_s]\right)}_{\text{Unitary change}}$$