The master equation

We redefine H_0 and H_I as:

$$H_0 = \frac{\omega\sigma_z}{2} + \Omega_0 b^\dagger b$$

$$H_I = g\sigma_z(b+b^\dagger) + (b+b^\dagger)\sum_k h_k(b_k + b_k^\dagger)$$

$$H_B = \sum_k \Omega_k b_k^\dagger b_k$$

then H_I is writen as

$$H_I = \sum_\alpha A_\alpha \otimes B_\alpha \quad \text{where}$$

$$A_1 = g\sigma(b+b^\dagger) \qquad B_1 = \mathbb{1}$$
$$A_2 = (b+b^\dagger) \qquad B_2 = \sum_k h_k(b_k+b_k^\dagger)$$

For the M.E. we need the interaction operators in the interactional picture

$$A(t) = e^{iH_S t} A e^{-iH_S t}$$

For this we will use the Hausdorf Formula, so we will evalute this commutators

$$[iH_S t, A], \quad [iH_S t, [iH_S t, A]], \ \ldots$$

For A_1

$$A_1(t) = e^{iH_s t}\left(g\sigma_z(b+b^\dagger)\right) e^{-iH_s t}$$

$$[iH_s t, g\sigma_z(b+b^\dagger)] = it\left[\frac{\omega\sigma_z}{2} + \Omega_0 b^\dagger b, g\sigma_z(b+b^\dagger)\right]$$

$$= it\Omega_0 g\sigma_z[b^\dagger b, b+b^\dagger] = it\Omega_0 g\sigma_z(-b+b^\dagger)$$

$$[iH_s t, [iH_s t, g\sigma_z(b+b^\dagger)]] = it\left[\frac{\omega\sigma_z}{2} + \Omega_0 b^\dagger b, it\Omega_0 g\sigma_z(-b+b^\dagger)\right]$$

$$= (it\Omega_0)^2 g\sigma_z[b^\dagger b, -b+b^\dagger] = (it\Omega_0)^2 g\sigma_z(b+b^\dagger)$$

High order commutator will give $b+b^\dagger$ or $-b+b^\dagger$

so, we write the seres as

$$A_1(t) = g\sigma_z b\sum_{n=0}^{\infty}\frac{(-1)^n(it\Omega_0)^n}{n!} + g\sigma_z b^\dagger\sum_{n=0}^{\infty}\frac{(it\Omega_0)^n}{n!}$$

$$= g\sigma_z b\, e^{-i\Omega_0 t} + g\sigma_z b\, e^{i\Omega_0 t}$$

From this calculation we also get

$$A_2(t) = e^{iH_s t}(b+b^\dagger)e^{-iH_s t}$$

$$= b\, e^{-i\Omega_0 t} + b^\dagger e^{i\Omega_0 t}$$

We now start the calculation for the master equation with ($\rho_0 = \rho_S^0 \otimes \bar{\rho}_B$)

$$\dot{\rho}_S = -i\,Tr\{[H_I(t), \rho_0]\}$$
$$- \int_0^\infty Tr\{[H_I(t),[H_I(t'), \rho_S(t') \otimes \bar{\rho}_B]]\} dt'$$

After performing the Born and Markov approximations we have

$$\dot{\rho}_S = -i\,Tr\{[H_I(t), \rho_0]\}$$
$$- \int_0^\infty \sum_{\alpha\beta} \{ C_{\alpha\beta}(\tau) [A_\alpha(t), A_\beta(t-\tau)\rho_S(t)] + h.c\} d\tau$$

We now works the second term

$$-\int_0^\infty \sum_{\alpha\beta} \{ C_{\alpha\beta}(\tau) [A_\alpha(t), A_\beta(t-\tau)\rho_S(t)] + h.c\} d\tau$$

with $\alpha, \beta = 1,2$, For now we will calculate only the term 2,2

$$-\int_0^\infty C_{22}(\tau)[A_2(t), A_2(t-\tau)\rho_S(t)] + h.c\} d\tau$$

For this we calculate C_{22}

$$C_{\alpha\beta}(t_1,t_2) = \mathrm{Tr}\{B_\alpha(t_1) B_\beta(t_2) \bar{\rho}_B\}$$

this are in the I.P.

in this case we use a thermal state

$$\bar{\rho}_B = \frac{e^{-\beta\sum \Omega_k b^\dagger b}}{Z}$$

and therefore we have $[H_B, \bar{\rho}_B] = 0$, then the correlation fuction can be writen ase

$$C_{\alpha\beta}(t) = \mathrm{Tr}\{e^{iH_B t} B_\alpha e^{-iH_B t} B_\beta \bar{\rho}_B\}$$

$$C_{22}(\tau) = \mathrm{Tr}\{e^{iH_B\tau}\sum_k h_k(b_k+b_k^\dagger) e^{-iH_B\tau}\sum_{k'} h_{k'}(b_{k'}+b_{k'}^\dagger)\bar{\rho}_B\}$$

$$= \frac{1}{2\pi}\int J^{(1)}(\omega)[1+n_B(\omega)] e^{-i\omega\tau} d\omega$$

Then

$$-\int_0^\infty C_{22}(\tau)[A_2(t), A_2(t-\tau)\rho_S(t)] + h.c.\} d\tau =$$

$$-\int_0^\infty \{C_{22}(\tau)[b e^{-i\Omega\tau} + b^\dagger e^{-i\Omega\tau}, (b e^{-i\Omega(t-\tau)} + b^\dagger e^{i\Omega(t-\tau)})\rho_S(t)]$$

$$+ h.c.\} d\tau$$

this term , we know will lead the the M.E. making the R.W.A (ie. neglect $e^{-2i\Omega}$ terms)

$$\gamma\left(b\rho b^{+}-\frac{1}{2}\{b^{+}b,\rho\}\right)+\bar{\gamma}\left(b^{+}\rho b-\frac{1}{2}\{bb^{+},\rho\}\right)$$

In resume we have at the point

$$\dot{\rho}_s = -i\,\mathrm{Tr}\{[H_I(t),\rho_0]\}$$

$$-\int_0^{\infty}\sum_{\substack{\alpha,\beta=\\ \{1,1;\,1,2;\,2,1\}}}\{C_{\alpha\beta}(\tau)[A_\alpha(\tau),A_\beta(t-\tau)\rho_s(t)]+h.c\}\,d\tau$$

$$+\gamma\left(b\rho_s(t)b^{+}-\frac{1}{2}\{b^{+}b,\rho_s(t)\}\right)+\bar{\gamma}\left(b^{+}\rho_s(t)b-\frac{1}{2}\{bb^{+},\rho_s(t)\}\right)$$

where we remember that an operator with time depends is in the interaction picture

$$O(t) = e^{iH_st}\,O\,e^{-iH_st}$$

Now we will perform a transformation in the Hamiltonian system and B_1 operator in order to make the 1'st term equal to

$$H_s \to H_s + \langle B_1\rangle A_1$$

$$B_1 \to B_1 - \langle B_1\rangle \mathbb{1}$$

Having that $B_1 = \mathbb{1}$, then $\langle B_1 \rangle = 1$
the new H_S and B_1 goes

$$H_S = \frac{\omega \sigma_z}{2} + \Omega_0 b^{\dagger} b + g \sigma_z (b + b^{\dagger})$$

$$B_1 = \mathbb{1} - \mathbb{1} = \mathbb{O}$$

With this tranformation

$$-i \mathrm{Tr}_B \{ [H_I(t), \rho_0] \}$$

$$= -i \sum_{\alpha} [A_\alpha(t) \rho_S^0 \mathrm{Tr} \{ B_\alpha(t) \bar{\rho}_B \} - \rho_S^0 A_\alpha(t) \mathrm{Tr} \{ \bar{\rho}_B B_\alpha(t) \}]$$

$$\mathrm{Tr} \{ B_1(t) \bar{\rho}_B \} = 0$$

$$\mathrm{Tr} \{ B_2(t) \bar{\rho}_B \} = 0$$

then

$$-i \mathrm{Tr}_B \{ [H_I(t), \rho_0] \} = \mathbb{O}$$

The only mising terms are the ones of the sumatory, we evalute $C_{\alpha\beta}(\tau)$ with the tranformed $B_1 = \mathbb{O}$

then

$$C_{11}(\tau) = \mathrm{Tr} \{ e^{iH_B\tau} O \, e^{-iH_B\tau} O \bar{\rho}_B \} = 0$$

$$C_{12}(t) = \mathrm{Tr} \{ e^{iH_B} O \, e^{-iH_B\tau} \sum_K h_K (b_K + b_K^{\dagger}) \bar{\rho}_B \} = 0$$

$$C_{21}(\tau) = 0$$

then all this elements are 0

$$\dot{\rho}_s = -i\,Tr\{\cancel{[H_I(t), \rho_0]}^{\,0}\}$$

$$- \int_0^\infty \sum_{\substack{\alpha,\beta= \\ \{1,1;\,1,2;\,2,1\}}} \{ \cancel{C_{\alpha\beta}(\tau)[A_\alpha(\tau), A_\beta(t-\tau)\rho_s(t)] + h.c}^{\,0} \} d\tau$$

$$+\gamma\left(b\rho_s(t)b^+ - \frac{1}{2}\{b^+b, \rho_s(t)\}\right) + \bar{\gamma}\left(b^+\rho_s(t)b - \frac{1}{2}\{bb^+, \rho_s(t)\}\right),$$

in the end the M.E in the interaction picture is

$$\dot{\rho}_s(t) = \gamma\left(b\rho_s(t)b^+ - \frac{1}{2}\{b^+b, \rho_s(t)\}\right) + \bar{\gamma}\left(b^+\rho_s(t)b - \frac{1}{2}\{bb^+, \rho_s(t)\}\right)$$

where $H_s = \frac{\omega\sigma_z}{2} + \Omega_0 b^+b + g\,\sigma_z(b+b^+)$