Valleyless Sequences

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Abstract

Valleyless sequences of finite length n and maximum entry k occur in tree enumeration problems and provide an interesting correspondence between permutations and compositions. In this paper we introduce the notion of *valleyless* sequences, explore the correspondence and enumerate them using the method of generating functions.

1 Introduction

Throughout this article we use the notation $\mathcal{S}_n$ to denote finite sequences of length n. A length n sequence $s = s_1, s_2, \ldots, s_n$ of positive integers is a permutation whenever each s_i is a distinct member of the set $\{1, 2, \ldots, n\}$. In this case we write the sequence s as a word $\pi = \pi_1 \pi_2 \ldots \pi_n$ and denote the set of all permutations of length n as $\mathcal{P}_n$. In the general case, the sequence of positive integers s can be associated with a composition of the integer $s_1 + s_2 + \ldots + s_n$. The composition has n parts where s_i is the size of the ith part. We let $\mathcal{C}_n$ be the set of all compositions of the positive integer n.

A sequence is defined herein to be *valleyless* provided $1 \leq i < j < k \leq n$ implies $s_j \geq \min\{s_i, s_k\}$. A graph of the pairs (i, s_i) reveals that a valleyless sequence never has a valley. As a sequence, it is either nondecreasing, nonincreasing, or is nondecreasing to a point and thereafter nonincreasing. We let $\mathcal{V}_n$ be the set of all length n valleyless sequences of positive integers and $\mathcal{V}_{n,k}$ the set of all length n valleyless sequences of positive integers with maximum part k.

Valleyless sequences occur in tree enumeration problems [3] and provide an interesting correspondence between permutations and compositions. We explore this correspondence first and then enumerate valleyless sequences in three different ways.

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2 Valleyless permutations

A permutation $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathcal{P}_n$ is uniquely determined by its inversion table $I(\pi) = (a_1, a_2, \ldots, a_n)$. The entry a_k is the number of symbols in the word π to the left of k that are greater than k. Thus, for all k, $0 \le a_k \le n-k$. The total number of inversions of π is $i(\pi) = a_1 + a_2 + \ldots + a_n$ and

$$\sum_{\pi \in \mathcal{S}_n} q^{i(\pi)} = (1+q)(1+q+q^2) \cdots (1+q+q^2+\ldots+q^{n-1}).$$

As an example we have $\pi = 2731546$, $a_1 = 3$, $I(\pi) = (3,0,1,2,1,1,0)$, and $i(\pi) = 8$.

A finite sequence $s = s_1, s_2, \ldots, s_n$ viewed as a composition $s_1 + s_2 + \ldots + s_n = N$ can be encoded as a unique subset of $\{1, 2, \ldots, N-1\}$ by $\Theta(s) = \{s_1, s_1 + s_2, \ldots, s_1 + s_2 + \ldots + s_{n-1}\}$. For example, the sequence $s = 2, 3, 2, 1, 1$ is the composition $2+3+2+1+1 = 9$ with $\Theta(s) = \{2, 5, 7, 8\}$. The process is reversible so that there are 2^{N-1} different compositions of N.

Next, we present a result relating *valleyless* permutations and compositions.

Theorem 1. *The subset of valleyless permutations in $\mathcal{P}_n$ is in one-to-one correspondence with the compositions of n. That is, $|\mathcal{V}_n \cap \mathcal{P}_n| = |\mathcal{C}_n|$.*

Proof. Let $\pi = \pi_1 \pi_2 \ldots \pi_n$ be valleyless, then either $\pi_1 = 1$ or $\pi_n = 1$. Add one to every entry of π and then append 1 to the beginning or end. The new permutation is valleyless of length $n + 1$. The process is reversible so there are twice as many valleyless permutations of length $n+1$ as there are of length n. Since $|\mathcal{P}_1 \cap \mathcal{S}_1| = 1$ we have $|\mathcal{V}_n \cap \mathcal{P}_n| = 2^{n-1} = |\mathcal{C}_n|$ and we are done. $\qquad\square$

The statement of the theorem implies a stronger correspondence than the cardinality result. There is indeed such a correspondence which we present now. Our goal in presenting the above proof was to introduce the reader to the technique of altering a sequence by adding one to every entry and then appending a 1 at the beginning or end. We will use this technique in the future. The stronger result is obtained from the following.

Theorem 2. *A permutation $\pi = \pi_1 \pi_2 \ldots \pi_n$ with inversion table $I(\pi) = (a_1, a_2, \ldots, a_n)$ is valleyless if and only if $a_k = 0$ or $n - k$, $k = 1, 2, \ldots, n$.*

Proof. Suppose that a permutation $\pi = \pi_1 \pi_2 \ldots \pi_n$ with inversion table $(a_1, a_2, \ldots, a_n)$ is valleyless. If $a_k > 0$ for some k, $1 \le k \le n$, then there is at least one symbol in π to the left of k that is greater than k. If there is

any symbol in π greater than k to the right of k, then k would be a valley contradicting our assumption. Hence all numbers greater than k must be to the left of k and $a_k = n - k$. Conversely, suppose that $I(\pi) = (a_1, a_2, \ldots, a_n)$ and $a_k = 0$ or $n - k$, $k = 1, 2, \ldots, n$. We want to show that π belongs to $\mathcal{V}_n \cap \mathcal{P}_n$. Suppose not, then there is at least one symbol π_k in π such that

$$\pi_k < \pi_i, \text{ for some } i < k \text{ and } \pi_k < \pi_j, \text{ for some } j > k.$$

Then $a_k \neq 0$ and $a_k < n - k$ contradicting our assumption. Hence π belongs to $\mathcal{V}_n \cap \mathcal{P}_n$ and we are done. □

Note that there are two choices for each a_k except for $k = n$ in which case there is only one. Thus, there are 2^{n-1} valleyless permutations of length n. If we again use q to record the number of inversions in a permutation, then we have

Corollary 3.

$$\sum_{\pi \in \mathcal{V}_n \cap \mathcal{P}_n} q^{i(\pi)} = (1 + q)(1 + q^2) \cdots (1 + q^{n-1}).$$

3 Permutations with a valley

One of the fundamental statistics associated with a permutation $\pi = \pi_1 \cdots \pi_n$ is the descent set $D(\pi) = \{i | \pi_i > \pi_{i+1}\}$. See [4] for the enumeration of permutations by their descent set. We say that a length n sequence $s = s_1, s_2, \cdots, s_n$ has exactly k valleys if

$$|\{j : s_j < min\{s_{j-1}, s_{j+1}\}\}| = k.$$

In this section we obtain a recursive formula for the generating function of permutations with exactly k valleys which we denote by $g_k(x)$ and use the formula to reproduce the table of permutations with peaks in [1].

We have shown in the previous section that $g_0(x)$ satisfies the relation

$$g_0(x) = \frac{x}{1 - 2x}. \tag{1}$$

One can easily see that length $n + 1$ permutations with exactly $k + 1$ valleys can be obtained from:

1. length n permutations with exactly $k+1$ valleys by adding 1 to every entry and then appending a 1 either at the beginning, end, to the left or to the right of the $k+1$ valleys.

2. length n permutations with exactly k valleys by adding 1 to every entry and then inserting a 1 in one of the $(n-1) - 2k = n - (2k+1)$ middle positions.

Hence

$$g_{k+1}(x) = (2(k+1)+2)xg_{k+1}(x) + x^{2k+3}D_x\left(\frac{g_k(x)}{x^{2k+1}}\right), \text{ for } k \geq 0. \quad (2)$$

Substituting $f_k(x) = \frac{g_k(x)}{x^{2k+1}}$ in (2) we obtain a system of linear differential equations

$$f_{k+1}(x) = \frac{1}{1 - (2(k+1)+2)x}D_x(f_k(x)) \quad (3)$$

which can also be written in matrix form as

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ \vdots \end{bmatrix} = \left[\frac{1}{1-2x}, \frac{1}{1-4x}, \frac{1}{1-6x}, \frac{1}{1-8x}, \ldots\right] \cdot D_x \begin{bmatrix} x \\ f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \end{bmatrix}$$

Even though (3) is an elegant recurrence relation just like many known simple relations for the number of partitions of an integer, we do not know how to solve it at this time.

k	$g_k(x)$
0	$\frac{x}{1-2x}$
1	$\frac{2x^3}{(1-2x)^2(1-4x)}$
2	$\frac{16x^5(1-3x)}{(1-2x)^3(1-4x)^2(1-6x)}$
3	$\frac{16x^7(17-184x+636x^2-720x^3)}{(1-2x)^4(1-4x)^3(1-6x)^2(1-8x)}$
4	$\frac{256x^9(31-788x+8096x^2-43132x^3+126072x^4-192672x^5+120960x^6)}{(1-2x)^5(1-4x)^4(1-6x)^3(1-8x)^2(1-10x)}$

Table 1: Generating functions of permutations with valleys

The formal Taylor series expansion of the functions in in Table 1 reproduces the table of permutations with peaks in [1].

k/n	1	2	3	4	5	6	7	8	9	10
0	1	2	4	8	16	32	64	128	256	512
1			2	16	88	416	1824	7680	31616	128512
2					16	272	2880	24576	185856	1304832
3							272	7936	137216	1841152
4									7936	353792

Table 2: The number of permutations of length n with exactly k valleys

If we denote the number of permutations of n numbers with k valleys by $P(n,k)$, the functional recursion (2) can also be written as

$$P(n,k) = 2(k+1)P(n-1,k) + (n-2k)P(n-1,k-1) \qquad (4)$$

and notice the remarkable similarity of the recurrence relation (4) with that of Eulerian numbers [2, 6]

$$E(n,k) = (k+1)E(n-1,k) + (n-k)E(n-1,k-1).$$

4 Valleyless sequences

In this section we enumerate valleyless sequences of length n and maximum entry k which we have denoted by $\mathcal{V}_{n,k}$ using the method of generating functions. Table 3 shows the cardinalities of $\mathcal{V}_{n,k}$ for $1 \leq n \leq 6$ and $1 \leq k \leq 5$.

n/k	1	2	3	4	5
1	1	1	1	1	1
2	1	3	5	7	9
3	1	6	15	28	45
4	1	10	35	84	165
5	1	15	70	210	495
6	1	21	126	462	1287

Table 3: Valleyless Sequences of Length n with maximum entry k

Let $V(x,y)$ be a generating function which enumerates valleyless sequences of length n and maximum entry k. Then

$$V(x,y) = \sum_{n,k} V_{n,k}x^n y^k = 1 + xy + xy^2 + \dots,$$

where $V_{n,k} = |\mathcal{V}_{n,k}|$. As pointed out in the proof of Theorem 1, any valleyless sequence of length n and maximum entry k can be obtained from the base case:

$$1, 11, 111, 1111, \ldots$$

by adding one to every entry and then appending a 1 at the beginning or end. Hence $V(x,y)$ satisfies the recursion

$$V(x,y) = \underbrace{\{x + x^2 y + x^3 y + \ldots\}}_{\text{base case}} + \overbrace{\{yV(x,y)\}}^{\text{adding 1}} \times \underbrace{\{1 + 2x + 3x^2 + \ldots\}}_{\text{appending a 1}}$$

$$= \frac{xy}{1-x} + yV(x,y)\frac{1}{(1-x)^2}.$$

Therefore,

$$V(x,y)\left(1 - \frac{y}{(1-x)^2}\right) = \frac{xy}{1-x} \text{ and } V(x,y) = \frac{xy}{1 - x - \frac{y}{1-x}}. \qquad (5)$$

On the other hand, it is well known [5] that

$$\sum_{k=0}^{\infty} \binom{n-1+k}{k} y^k = \frac{1}{(1-y)^n}.$$

Using this closed form and the 'snake oil' [7] method, we observe that

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \binom{n-1+2(k-1)}{2(k-1)} y^k\right) x^n \qquad (6)$$

$$= \sum_{n=1}^{\infty} \frac{y}{2}\left(\frac{1}{(1-\sqrt{y})^n} + \frac{1}{(1+\sqrt{y})^n}\right) x^n$$

$$= \frac{y}{2}\sum_{n=1}^{\infty}\left(\frac{x}{1-\sqrt{y}}\right)^n + \frac{y}{2}\sum_{n=1}^{\infty}\left(\frac{x}{1+\sqrt{y}}\right)^n$$

$$= \frac{y}{2}\left(\frac{x}{1-\sqrt{y}-x} + \frac{x}{1+\sqrt{y}-x}\right)$$

$$= \frac{xy}{(1-x) - \frac{y}{1-x}}. \qquad (7)$$

Hence from (5) and (6) one can see that

Theorem 4. *The number of valleyless sequences of length n with maximum entry k is*

$$V_{n,k} = \binom{n-1+2(k-1)}{2(k-1)}.$$

5 q-Analog

If we use q to record the sum of the entries of members of $\mathcal{V}_{n,k}$ and let $V(x, q, y)$ its generating function, then

$$V(x, q, y) = \sum_{n,p,k} V_{n,p,k} x^n q^p y^k.$$

Using the same argument as in the previous case, we see that $V(x, q, y)$ satisfies the functional recursion

$$
\begin{aligned}
V(x, q, y) &= \frac{xqy}{1 - xq} + yV(xq, q, y)\frac{1}{(1 - xq)^2} \\
\Leftrightarrow (1 - xq)^2 V(x, q, y) &= xqy - x^2 q^2 y + yV(xq, q, y). \quad (8)
\end{aligned}
$$

Equation (7) is a linear $q-$difference equation and we seek series solution of the form

$$V(x, q, y) = \sum_{n=0}^{\infty} b_n(x, q) y^n. \quad (9)$$

Substituting (8) into (7) and comparing coefficients of y^n we obtain:

$$b_n(x, q) = \frac{b_{n-1}(xq, q)}{(1 - xq)^2} \text{ for } n > 1, \quad (10)$$

where $b_0 = 0$ and $b_1 = \frac{xq}{1-xq}$.
Repeated application of equation (9) gives the explicit solution

$$b_n = \frac{xq^n(1 - xq^n)}{(xq)_n^2}, \quad (11)$$

where the common $q-$symbol

$$(xq)_n = \prod_{i=1}^{n}(1 - xq^i).$$

Therefore,

$$
\begin{aligned}
V(x, q, y) &= \frac{xq}{1 - xq} y + \sum_{n=2}^{\infty} \frac{xq^n(1 - xq^n)}{(xq)_n^2} y^n \\
&= \sum_{n=1}^{\infty} \frac{xq^n(1 - xq^n)}{(xq)_n^2} y^n \quad (12)
\end{aligned}
$$

An implementation of (11) using Maple[1] shows that the number of valleyless sequences of length 10, sum of entries 20, and maximum part 5, for instance, is 325.

6 A nonlinear relation

Expanding $V(x, q, y)$ in x as $V(x, q, y) = \sum a_n(q, y)x^n$, substituting it into (5), and comparing coefficients of x^n, we obtain a three term recurrence relation

$$a_n = \frac{2qa_{n-1} - q^2 a_{n-2}}{1 - yq^n} \text{ for } n > 2 \tag{13}$$

where $a_0 = 0, a_1 = \frac{yq}{1-yq}$, and $a_2 = \frac{q^2 y + q^3 y^2}{(yq;q)_2}$.

This nonlinear recurrence relation provides yet another way of enumerating valleyless sequences. Indeed, one can obtain valleyless sequences of length n from sequences of length $n - 1$ by adding a 1 at the beginning or end, and then appending 1s to every entry. This process contributes $\frac{2q}{1-yq^n}a_{n-1}(q, y)$ to $a_n(q, y)$. However, sequences with the same first and last entry, for example 23522, are counted twice in this algorithm and we need to subtract $\frac{q^2}{1-yq^n}a_{n-2}(q, y)$, the number of valleyless sequences of length n with the same first and last entry. Hence,

$$a_n(q, y) = \frac{2q}{1 - yq^n}a_{n-1}(q, y) - \frac{q^2}{1 - yq^n}a_{n-2}(q, y).$$

Unlike (9) equation (12) is nonlinear and we do not know its explicit solution.

7 Conclusion

We came across valleyless sequences while enumerating ordered trees by their number of leaves, total path length and number of vertices. Given a positive integer n, consider its composition or ordered partition and draw a composition tree corresponding to n. Then try to construct all possible ordered trees of total path length n by identifying non-terminal nodes or vertices of the composition tree.

A sequence obtained from an ordered partition of a positive integer n by taking the minimum of two consecutive entries gives a sequence without

[1] Send e-mail to **zelekem@wpunj.edu** to obtain the Maple Package **bxq** accompanying this paper.

isolated valley and we hope that the ideas developed in this article will provide an alternative means to solve the ordered tree enumeration problem.

8   Acknowledgements

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