Ildar Ibragimov.

Five Decades in Probability and Statistics

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1. Linnik and creation of the Leningrad school of Probability and Statistics

Prehistory - Chebyshev, Markov, Lyapunov, Bernstein
Yury Vladimirovich Linnik (1915 -1972)
Linnik founded:

Laboratory of Statistical Methods in Leningrad Branch of Steklov Institute (headed by I.Ibragimov since 1973);

Chair of Probability Theory and Mathematical Statistics in Leningrad State University (headed by I.Ibragimov since 1997).

Linnik's heritage:
Use of powerful methods of analysis and especially theory of functions of complex variable.
Taste for book writing.
Warm and friendly relations with A.N.Kolmogorov and his Moscow school.

2. Linnik's students in Probability and Statistics

Some students in Probability and Statistics: I.Ibragimov, L. Klebanov, V.Petrov, A. Rukhin, A. Zinger, O.Shalaevsky, V.Skitovich, V.Statulyavichus, ...
...
Visiting researchers preparing second dissertation: I.Kubilius, A.Renyi
Also: A.Kagan, L.Khalfine, V.Sudakov, ...

3. 1951

First steps towards mathematical profession were not easy. In 1951 the young candidate was rejected by the university commission in charge of students' admission. Commission decided that he "was not good enough in mathematics" The true reasons, of course, had nothing to do with mathematical skills.

So I.Ibragimov spent his first student year in "Lesotechnicheskaya Academia" (i.e. Superior School of Forest Industry). With the help of Ibragimov's math professors, the affair was finally revised and arranged by University's rector mathematician A.D. Alexandrov. Eventually, student Ibragimov could enter the mathematical faculty from the second year studies.

He won a competition for students where Linnik was a president of jury. One of three books Linnik gave to the young winner was Kolmogorov's "Foundations of Probability Theory". This meeting was a start of their collaboration.

4. Unimodality of stable laws

Strong unimodality of probabilistic distributions

(first important result obtained by the student of 4-th year)

Definition The distribution function F is called *unimodal* if for some real a it is convex on $(-\infty, a]$ and concave on $[a, \infty)$.

Roughly speaking, the density of F is increasing, then, eventually, constant, and then decreasing.

Definition. (Ibragimov). The distribution is called *strongly unimodal* if its convolution with every unimodal distribution is unimodal.

The following result (1956) describes the class of strongly unimodal distributions.

Theorem. *The distribution is strongly unimodal iff and only if it has a logarithmically concave density.*

Example: normal distribution.

Important application to the following result:

Theorem. *Every stable law is unimodal.*

This was partially proved by Ibragimov and K.E.Chernin (1959) and in full generality by Yamazato (1978).

5. Classical limit theorems

Necessary and sufficient moment conditions for given convergence rate (1966)

Theorem. *Let X_j be i.i.d. and*

$$F_n(x) = P\left(\frac{1}{a_n}\sum_{j=1}^{n} X_j - b_n < x\right)$$

$$r_n = \inf_{a_n, b_n} \sup_x |F_n(x) - \Phi(x)|.$$

Then

$$r_n = O(n^{-1/2})$$

iff

$$EX^2\,\mathbf{1}_{\{|X|>z\}} = O(z^{-1}) \quad \text{and} \quad EX^3 1_{\{|X|<z\}} = O(1) \qquad (z \to \infty).$$

Similar conditions are given for the rates $n^{-p}, 0 < p < 1/2$.

Fast convergence rates in CLT (1966)

Ibragimov found necessary and sufficient conditions for unusually fast convergence rates in CLT $n^{-p},\ p > 1/2$, and proved, among others, the following remarkable result:

Proposition. *If for i.i.d. random variables X_j CLT holds with convergence rate $o(n^{-p})$ for all $p > 0$, then X_j are normal.*

Interestingly, for all non-normal stable laws the exponential convergence rate is possible.

Chebyshev-Cramér asymptotic expansions (1967)

Theorem. *Let X_j be i.i.d. random variables with zero mean and unit variance. Let*

$$F_n(x) = P\left(\frac{1}{n^{1/2}}\sum_{j=1}^{n} X_j < x\right).$$

Then

$$F_n(x) = \Phi(x) + \sum_{\nu=1}^{k-2} \frac{\exp(-x^2/2)}{\sqrt{2\pi}} \frac{Q_\nu(x)}{n^{\nu/2}} + o(n^{-(k-2)/2})$$

uniformly in x for $k \geq 3$; $Q_\nu(\cdot)$ - Chebyshev-Cramér polynomials iff the characteristic function satisfies

$$E\exp\{itX\} = \exp\left\{-t^2/2 + \sum_{\nu=3}^{k} \frac{(it)^\nu}{\nu!}\mu_\nu + o(|t|^k)\right\}, t \to 0.$$

The if part also needs Cramér (C) condition on ch.f.. Sufficient condition - essentially due to H.Cramér, necessary condition - to I.Ibragimov.

6. Weak dependence

Uniform mixing condition

Ibragimov introduced the following measure of weak dependence between the σ-fields $\mathcal{A}$ and $\mathcal{B}$.

$$\phi(\mathcal{A},\mathcal{B}) \;=\; \sup_{A\in\mathcal{A},B\in\mathcal{B}} |P(B|A)-P(B)| \;,$$

which is called uniform mixing coefficient or ϕ-mixing coefficient. It was important complement to other coefficients introduced earlier by M.Rosenblatt and Kolmogorov - Volkonskii.

Central limit theorem under mixing conditions

Ibragimov investigated when the Central Limit Theorem holds for stationary processes under various weak dependence conditions. His works stimulated many subsequent developments of the theory (e.g. by Davydov, Doukhan - Massart - Rio, ...). One result, still unsurpassed:

Theorem. (I. Ibragimov (1962)) *Let X_j be a centered stationary sequence of random variables with finite moments $E|X_j|^{2+\delta}$. Assume that*

$$D_n^2 = \mathbf{Var}\left(\sum_1^n X_j\right) \;\to\; \infty$$

and the uniform mixing holds, i.e.

$$\phi\left(\mathcal{F}(X_j\,, j\le 0), \mathcal{F}(X_j\,, j\ge n)\right) \;\to 0 \;.$$

Then CLT holds for sums $\frac{1}{D_n}\sum_1^n X_j$.

Famous *Ibragimov conjecture* claims that this statement remains true under weaker assumption that $E|X_j|^2$ is finite.

He also showed that the class of limit distributions in limit theorems for stationary processes under strong mixing condition is the same as for i.i.d. random variables, it coincides with the class of stable laws.

Continued fractions

Here is a natural example of sequence with uniform mixing property.

Theorem. *Let $x\in[0,1]$ be expanded in a continued fraction $x=[a_1(x),\ldots,a_n(x),\ldots]$, i.e.*

$$x = \frac{1}{a_1(x)+\frac{1}{a_2(x)+\ldots}} \qquad .$$

with (uniquely defined positive integers a_n. Consider the convergent fraction $p_n(x)/q_n(x)=[a_1(x),\ldots,a_n(x)]$. There exist the constants a and $\sigma>0$, such that

$$mes\left\{x : \frac{\ln q_n(x)-na}{\sigma\sqrt{n}} < z\right\} \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} \exp\Bigl(-\frac{u^2}{2}\Bigr)\,du.$$

This is a solution of an Erdös problem.

Recall that $a_n(\cdot)$ is a stationary uniformly mixing sequence on $\Bigl([0,1];\frac{dt}{\ln 2(1+t)}\Bigr)$.

This is only one example (finally proved in this form with important contribution of M.Gordin) where Ibragimov's general CLT for the functionals of processes with uniform mixing works for the endomorphisms from number theory (not only continued fractions but also dyadic expansions were handled this way)

Central limit theorem for martingale differences

Obtained independently by P.Billingsley and I.Ibragimov (1963).

Theorem. *Let (X_j) be a stationary martingale difference, i.e. $E(X_j|X_1 \dots X_{j-1}) = 0$, with unit variance. Then the distributions of normalized sums $\frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j$ converge to the standard normal law.*

Book:

I.A.Ibragimov, Yu.V.Linnik (1965). Independent and Stationary Sequences of Random Variables

Citation index (Institute for Scientific Information, USA): 263 references in research papers during 1993-2000 (very selective data).

7. Spectral theory of random processes

Toeplitz matrices

Consider a nonnegative function f on $[-\pi, \pi]$ (it will be later interpreted as the spectral density of a stationary process) and the Fourier coefficients

$$f_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-itk} \, dt.$$

Let $D_n(f)$ be the determinant of Toeplitz matrix

$$\begin{pmatrix} f_0 & f_1 & \cdots & f_n \\ f_{-1} & f_0 & \cdots & f_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ f_{-n} & f_{-n+1} & \cdots & f_0 \end{pmatrix}$$

G.Szegö established that

$$\lim_{n \to \infty} \frac{1}{n+1} \ln \left[D_n(f) \right] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \ln f(t) \, dt = D(f).$$

He also found sufficient conditions under which there exists a finite limit

$$d(f) = \lim_{n \to \infty} \left\{ \ln D_n(f) - (n+1) D(f) \right\},$$

i.e.

$$\ln D_n(f) = (n+1) D(f) + d(f) + o(1).$$

I.Ibragimov (1968) showed that this limit exists in any case and

$$d(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|\ln f(u) - \ln f(v)|^2}{(u-v)^2} \, dudv.$$

It is worthwhile to notice that the quantity $d(f)$ is equal to information about the future of Gaussian process with spectral density f which is contained in its "past".

Total regularity of stationary processes.

Let $H_{<0}$, $H^{>t}$ be the linear subspaces generated by the past $\{X_s, s < 0\}$ and the future $\{X_s, s > t\}$ of a stationary process X with spectral density f and the maximal covariance coefficient

$$\rho(t) = \sup_{\psi \in H_{<0}, \, \varphi \in H^{>t}} \frac{|E\psi\overline{\varphi}|}{\sqrt{E|\psi|^2 E|\varphi|^2}}.$$

We say that the process X is *totally regular*, if

$$\rho(t) \to 0 \qquad \text{for} \qquad t \to \infty.$$

(For Gaussian processes total regularity is equivalent to strong mixing condition.)

I.Ibragimov (1970) expressed in the language of the relative smoothness of the integral F of f necessary conditions and sufficient conditions for total regularity and investigated the rate of decreasing of $\rho(t)$ for real and vector-valued processes.

Theorem. *Let f be a positive integrable function on $[-\pi, \pi]$. Let $L(a, b)$ be a subspace in $L^2(f)$, generated by $\{ e^{i\lambda k} : a \le k \le b \}$. Let $\rho(n)$ be the cosine of the minimal angle between the spaces $L(-\infty, 0)$ and $L(n, \infty)$. Then*

$$\rho(n) \le n^{-\beta}, \qquad \beta = k + \alpha, \quad 0 < \alpha \le 1$$

if and only if

$$f(\lambda) = |P(\lambda)|^2 g(\lambda),$$

where P is a trigonometric polynomial, $g(\lambda) \ge m > 0$, and $g^{(k)}(\lambda)$ satisfies Hölder condition of order α.

...

The book

I.A.Ibragimov, Yu.A. Rozanov (1970). Gaussian Random Processes

`8. Functionals of random walk`

Consider i.i.d. steps X_j such that a properly normalized random walk

$$S_{kn} = \frac{1}{b_n} \sum_{j=1}^{k} X_j, \qquad k = tn(1 + o(1)),$$

converges weakly to an α-stable law with $\alpha > 1$.

There exists a wide class of functionals of random walks not covered by Donsker's invariance principle. For example, the time spent by random walk at zero, and other functionals whose distribution tends to that of the Brownian local time.

Ibragimov suggested original and very efficient method for investigation of additive functionals like

$$\eta_n = \sum_{k=1}^{n} f_n(S_{kn}),$$

where $f_n(x)$ is some sequence of functions. This method works under essentially weaker moment assumptions than earlier methods by Skorokhod and Slobodenyuk.

St-Flour lectures

I.A. Ibragimov (1985) Théorèmes limites pour marches aléatoires

The book

A.N. Borodin, I.A. Ibragimov (1994). Limit Theorems for Functionals Defined on Random Walks

`9. Smoothness of sample paths`

Main idea: embedding theorems from analysis yield Kolmogorov-type theorems providing (easy to verify) conditions for continuity, Hölder property etc of sample paths of random function. Here is only one simplest example (1973):

Theorem. *Let $p > 1$. Assume that a the random process $X(t), t \in [a, b]$ satisfies*

$$\mathbf{E}|X(t + h) - X(t)|^p \le K\, h^p.$$

Then the sample paths of X are absolutely continuous and $\mathbf{E}|X'(t)|^p \le K$.

10. Zeros of random polynomials

Let $P(x) = \sum_0^n \xi_k x^k$ be a polynomial with i.id. random coefficients. Let $N(P)$ denote the number of real zeros of P.

Theorem. (I.Ibragimov, N.B. Maslova (1971)) *If $\mathbf{E}\xi_j = 0$ and $\mathbf{E}\xi_j^2 < \infty$, then*

$$\mathbf{E}N(P) \sim \frac{2}{\pi} \ln n;$$

if $\mathbf{E}\xi_j \neq 0$ and $\mathbf{E}\xi_j^2 < \infty$, then

$$\mathbf{E}N(P) \sim \frac{1}{\pi} \ln n.$$

(Problem originates in the works of G.Polya, A.Offord and J.Littlewood. First results for normal and uniformly distributed variables were obtained by M.Kac; for symmetric Bernoulli variables - by P. Erdös and A.Offord.)

Later on, Ibragimov and O.Zeitouni (1997) made important work on the distribution of *complex* roots of random polynomials under fairly general conditions. Ibragimov and S.Podkorytov (1995) worked on random algebraic surfaces, which is a wide generalization of the root problem.

11. Statistical estimation

It was, probably, Linnik who, in early seventies, drew Ibragimov's attention to the asymptotic problems of Statistics (there was also an influence of A.Kagan's lectures on estimation theory). While basic results on summation of independent variables had been already obtained at that time, asymptotic Statistics was a research field with many open questions.

The earliest works of Ibragimov in Statistics (1962) concerned the estimation of the spectral measure for stationary, mainly Gaussian, processes.

The most part of the later work in Statistics had been done jointly by Ibragimov and R.Z. Khasminskii (since 1970; first joint publication dates back to 1972) and this collaboration yielded remarkable outcome.

In 1970-1980 they mainly considered the problems of parametric estimation. Among the results obtained, we can mention

-the minimal conditions of normality of maximal likelihood and Bayesian estimates (they essentially reduced the smoothness assumptions usually made in the basic model of independent sample from unknown distribution);

- comparison of sequential estimation and estimation with fixed number of observations (1974) (they showed that for smooth models sequential estimation with average number of observations N is not better (in minimax sense) than the estimation with N observations. Yet for non-smooth models (e.g. samples with discontinuous distribution density) sequential estimation do provide better results;

- study of asymptotic behavior of statistical models with distributions having infinite Fisher information.

In parametric estimation, they considered likelihood ratio as a random field depending on the parameter, which turned out to be extremely fruitful methodology.

One should especially mention the works on (parametric and nonparametric) estimation under Gaussian white noise. The research frame

$$dY(t) = S(t)dt + \epsilon\, dW(t), \quad \epsilon \to 0 \qquad (SW)$$

with functional parameter $S(t)$ to be estimated in Gaussian white noise $dW(t)$ became canonical in asymptotic Statistics.

All this material was included in the book

I.A. Ibragimov, R.Z. Khasminskii (1979) Statistical Estimation: Asymptotic Theory.

which remains, even 20 years later, a subject of numerous references and source of inspiration for many people.

Citation index (Institute for Scientific Information, USA): 256 references in research papers during 1993-2000 (very selective data).

In a series of later works (1984, 1986, 1991) Ibragimov and Khasminskii worked on estimation of a functional of signal in model (SW) (such as the value or the derivative of a signal at a fixed point, integral functionals, location of signal's maximum etc). These works also had much influence on the development of Statistics.

From 1977, Ibragimov and Khasminskii work more and more in nonparametric Statistics. Using Fano inequality from information theory, they obtained lower bounds for convergence rate of nonparametric estimates for signal in white noise, for distribution density, and regression. Many other people followed later this idea. Here is a typical result.

Assume *a priori* that unknown signal $S(t)$ belongs to the compact $\Sigma \subset L_p(0,1), 1 \le p \le \infty$. Let ℓ be a convex symmetric loss function. Then for every $\delta > 0$

$$\inf_T \sup_{S\in\Sigma} E_S^\epsilon \ell(||T-S||_p) \ge \ell(\delta)\left(1 - \frac{c(\epsilon,\Sigma)+\ln 2}{G_p(2\delta,\Sigma)-1}\right),$$

where the inf is taken over all estimates T based on the noisy observation Y from model (SW); $C(\epsilon,\Sigma)$ is the capacity of the channel (SW) under assumption $S \in \Sigma$ and $G_p(\delta,\Sigma)$ is the capacity of the set Σ. See also the survey (1990).

Ibragimov and Khasminskii also studied necessary and sufficient conditions for construction of consistent nonparametric estimates. Their remarkable result (analogous to that of A.N.Tikhonov in the theory of non-correct problems of mathematical physics) claims that it is impossible to construct such an estimate unless we have *a priori* information that the infinite-dimensional parameter under estimation belongs to a compact (1977, 1997).

12. Recent works

Almost sure limit theorems

This is a new research field which presents statistical point of view on classical weak limit theorems. The next result provides an interpretation of moments' convergence in CLT.

Theorem. (I.Ibragimov and M.Lifshits (1999)) *Let $S_k = \frac{1}{\sqrt{k}}\sum_1^k X_j$ be the normalized sum of i.i.d. centered random variables with unit variance. Let $f(\cdot)$ be a function which satisfies*

$$\int f(x)\ \exp\{-x^2/2\}\ dx < \infty$$

and some mild regularity conditions. Then, with probability one,

$$\lim_n \frac{1}{\log n}\sum_{k=1}^n \frac{f(S_k)}{k} = \frac{1}{\sqrt{2\pi}}\int f(x)\ \exp\{-x^2/2\}\ dx.$$

Estimation of SPDE coefficients

During last years, the Ibragimov and Khasminskii work much on statistical versions of inverse problems of mathematical physics (partial differential equations superposed with Gaussian white noise). Say, a deterministic equation

$$\frac{\partial}{\partial t}u(t,x) = (Lu)(t,x) + f(t,x)$$

with elliptic linear differential operator (in vector variable x)

$$(Lu)(t,x) := \sum_k a_k(t,x)(D_x^k u)(t,x)$$

becomes in stochastic setting

$$du(t) = Lu(t,x)\,dt + f(t,x)\,dt + \varepsilon\,dw(t).$$

One observes the sample path u of the solution of this stochastic PDE and tries to estimate PDE's parameter, for example, the functions a_k.

13. Ibragimov's students

The following is an incomplete list of Ibragimov's PhD-students in chronological order. In total, he was a supervisor of more than 25 defended dissertations.

Yu.Davydov, V.Solev, M.Gordin, Ya.Nikitin, O.Orevkova, T.Arak, N.Lyashenko, S.Gusev, T.Siraya, S.Vallander, B.Lifshits, I.Linnik, B.Tsyrelson, A.Tikhomirov, A.Borodin, M.Ermakov, M. Ginovyan,A. Zaitsev, N.Babayan, M. Radavichus, N. Bakirov, A. Teplyaev.

The brought important contributions in the theory of summation of independent variables, limit theorems under weak dependence, statistical estimation, efficiency of statistical criteria, Gaussian processes, Brownian motion, strong approximation ...

14. Titles and awards

Head of Laboratory of Statistical Methods in Leningrad Branch of Steklov Institute of Academy of Sciences since 1973; Head of Petersburg Branch of Steklov institute since 2000.

Head of Chair of Probability Theory and Mathematical Statistics in St-Petersburg State University since 1997.

Corresponding member of Academy of Sciences of USSR since 1990, Full member of Russian Academy of Sciences since 1997.

Fellow of Inst. of Math. Statist., USA, 1989.

St-Flour lecturer (1983), Wald lecturer (1989), Lecturer in the International World Mathematical Congress (1966).

Lenin Prize for the works in Probability Limit Theorems (with Yu.V. Linnik, Yu.V. Prohorov, Yu.A. Rozanov), 1970.

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