Limit Theorems Which Establish a Keeping Intensity Relationship Between Poisson and Ornstein-Uhlenbeck Processes

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A number of functional limit theorems which establish a relationships between the Poisson type and Ornstein-Uhlenbeck type processes are given and discussed. Applications to statistics and financial modeling also discussed. We describe a construction for passages to the limit such that the linear time scaling property for the Poisson process directly transits to the same property for the Ornstein-Uhlenbeck process, i.e. $\lambda = \beta$.

As the Ornstein-Uhlenbeck process we understand the stationary, markovian, gaussian process $U(t), t \geq 0$. In the case of U is a centered and normalized process we call it as a standard one. From the definition above it follows that $\mathrm{cov}(U(t_1), U(t_2)) = \exp\{-\beta|t_2 - t_1|\}$, where $\beta > 0$ is a viscosity coefficient.

The Basic Limit Theorem

Let $\{\Pi_i(s)\},\ i = 1, 2, \ldots,\ s \geq 0$, be a sequence of independent identically distributed Poisson processes with the joint intensity $\lambda > 0$.

Let $\{\varepsilon_i^{(j)}\},\ \ i = 1, 2, \ldots,\ j = 1, 2, \ldots$, be a two dimensional array of mutually independent identically distributed random variables (iid rv's), $\mathbf{E}\varepsilon_1^{(1)} = 0,\ \mathbf{D}\varepsilon_1^{(1)} = 1$. As a sampling distribution we consider the simple binomial one: $\varepsilon_1^{(1)} = \pm 1,\ 1/2$. We suppose that the sequence $\{\Pi_i\}$ is independent of the family $\{\varepsilon_i^{(j)}\}$.

In all text below we suppose that N is a natural number tending to infinity.

We examine limit behavior of sums of the rv's $\{\varepsilon_i^{(j)}\}$ when the upper index $j \equiv j(i)$ is a random variable with values determined by values of the

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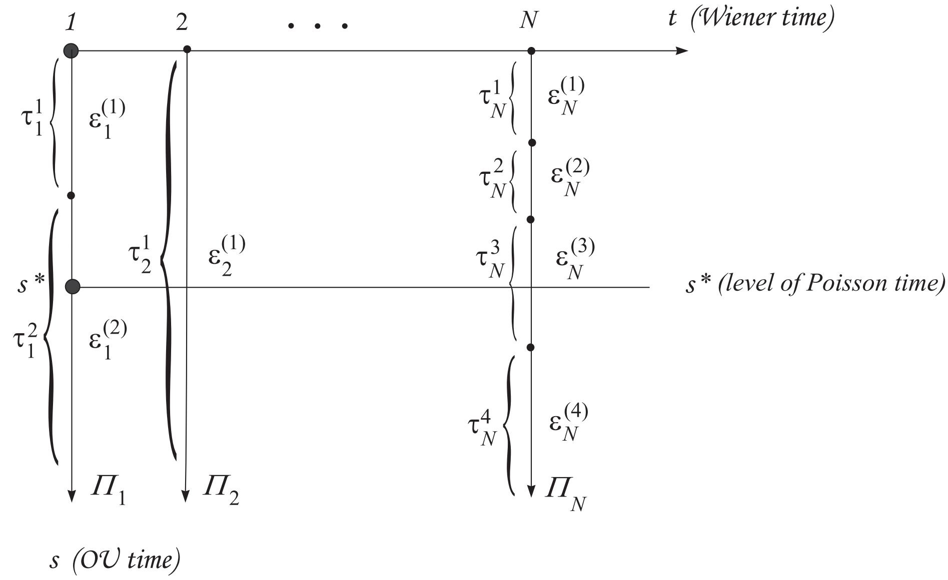
Poisson process Π_i (with the number i). Such kind of constructions has the following interpretation.

It is well known that a time moment for the k-th jump of the Poisson process equals to the sum $\sum_{j=1}^{k} \tau^j$, where $\{\tau^j\}$, $j = 1, 2, \ldots,$ — is the sequence of iid rv's having the joint exponential distribution law $\mathcal{E}xp(\lambda)$, $\lambda > 0$. We denote the interval between the two consecutive time moments of the jumps by

$$\Theta^j \triangleq [\tau^{j-1}, \tau^{j-1} + \tau^j)$$

and name it as j-th spacing, $j = 1, 2, \ldots$.

Let us denote $\{\Theta_i^j\}$, $i = 1, 2, \ldots,$ $j = 1, 2, \ldots$ the array of spacings corresponding to the processes $\{\Pi_i\}$. So, the enumeration of the upper index j in $\varepsilon_i^{(j)}$ by values of Poisson processes means that we weight the spacing Θ_i^j by the corresponding random weight $\varepsilon_i^{(j)}$.



Pic.1. An Example of the Sequence of the Poisson Processes for a Limit of the Ornstein-Uhlenbeck Type

Theorem 1. Let $\left\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}, 1\right\}$ be the nodes of the uniform partition of the interval $[0, 1] \ni s$. In the nodes we define the $\sqrt{N}$-normalized cumulative sums over the Poisson processes

$$S_N(s) \triangleq \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \varepsilon_i^{(\Pi_i(s)+1)}, \tag{0}$$

and in the remaining points $s \in [0,1]$ we define the values of $S_N(s)$ by the linear interpolation.

The following weak convergence is valid in the space of continuous function $\mathbf{C}_{[0,1]}$, as $N \to \infty$,

$$S_N(s) \implies U(s)\,,$$

where $U(s)$ is the [standard] Ornstein-Uhlenbeck process with the viscosity coefficient λ coinciding with the intensity coefficient of every Poisson process from the sequence $\{\Pi_i(t)\}$.

Next we consider the generalization of $S_N(s)$, defined by (0), to random broken planes. Following theorem 2 from section below allows us the fact of convergency the random broken planes to a field that we name as the Wiener-Ornstein-Uhlenbeck field (WOU field).

Family of the Wiener-Ornstein-Uhlenbeck Fields

Let $W(t,s)$, $t \geq 0$, $s \geq 0$ be a standard Brownian sheet, i.e. a centered gaussian function with the covariance

$$\operatorname{cov}(W(t_1,s_1), W(t_2,s_2)) = \min(t_1,t_2)\min(s_1,s_2)\,.$$

We define WOU field by the Lamperti transform applying to the time scale s,

$$Z(t,s) \equiv Z_\beta(t,s) = \exp\{\beta s\} W(t,\, e^{-2\beta s})\,, \tag{1}$$

where $\beta > 0$ is the viscosity coefficient.

Model Comment 1. We interpret the time t as an extrinsic (observed) time and we name the time s by the intrinsic time. Model interpretation of the intrinsic time is as follows. Let we observe the Brownian motion (Bm) W (log-prices of some financial asset, for instance) and we have a delay with the fixing of the small increments $\mathrm{d}W$. We suppose that during this delay the increments $\mathrm{d}W$ can replace with their independent copies. And the time between the replacements is driven by the Poisson process Π, i.e. the replacements occur in the points of jumps Π. We assume that for every t the delays of observations $\mathrm{d}W(t)$ are independent and identically distributed. Thus we will observe another Bm. So, in terms of WOU field the prime Bm $W(t)$ is the $Z(t,0)$ and the observed after delays Bm is $Z(t,s^*)$, where s^* is the time of the delay (see Pic.1) and β in Z is the intensity of the driving Poisson processes.

The field $Z(t,s)$ is a tensor product of the Wiener process and the Ornstein-Uhlenbeck process, i.e. the WOU field is a continuous gaussian centered function,

$$\operatorname{cov}(Z(t_1,s_1),Z(t_2,s_2)) = \exp\{-\beta|s_2-s_1|\}\min(t_1,t_2)\,.$$

Remark that this form of the covariance guarantees the existence of the WOU field.

We consider the diagonal process $Z(t) \stackrel{\triangle}{=} Z(t,t)$ of the following peculiar interest. The fact is that the diagonal process $Z(t)$ is a zero mean a gaussian markovian process with the linearly increasing variance $\mathbf{D}Z(t) = t$.

Model Comment 2. The process $Z(t)$ can play a role of an alternative to the Brownian motion. The covariance of the diagonal process coincide with covariance of the product independent Ornstein-Uhlenbeck and Wiener processes $\operatorname{cov}(Z(t_1),Z(t_2)) = \exp\{-\beta|t_1-t_2|\}\min(t_1,t_2)$. Such type of covariance directly corresponds to the stochastic volatility models in finance. An effect is that applying the diagonal process we work with the normal distribution for log-prices whereas in the stochastic volatility models the log-prices follows to a complicate distribution with practically very sensitive and unstable parameters.

The essential defect (single, probably) of the diagonal process is that it is not a homogeneous one. The conditional moments of $Z(t)$ are as follows

$$\mathbf{E}\left\{Z(t)|Z(u)=z\right\} = ze^{-\beta(t-u)}\,,$$

$$\mathbf{D}\left\{Z(t)|Z(u)=z\right\} = t - ue^{-2\beta(t-u)}\,, \tag{2}$$

for $u \le t$.

The simplest way to characterize the diagonal process is by the following Lamperti type transform for the standard Bm W,

$$Z(t) \overset{distr}{=} \exp\{\beta t\} W(te^{-2\beta t}) \,. \tag{3}$$

Proposition 1 below characterizes the diagonal process in the moving average form and in the form of the Langevin stochastic differential equation.

Proposition 1. There exists the Bm $B(u)$ such that

$$Z(t) = e^{-\beta t} \int_0^t e^{\beta u} \sqrt{1 + 2\beta u}\, \mathrm{d}B(u) \,, \tag{4}$$

and

$$\mathrm{d}Z(t) = -\beta Z(t)\mathrm{d}t \,+ \sqrt{1 + 2\beta t}\, \mathrm{d}B(t) \tag{5}$$

with the starting point $Z(0) = 0$.

We consider the following generalization of the WOU field to the non-homogeneous case named the n-hWOU field. Let $f(s)$, $s \geq 0$, $f(0) = 1$ be the positive, strictly monotone, increasing, and differential function. We define the n-hWOU field by the following generalization of the transform of Lamperti type

$$Z_f(t, s) = f(s) W\left(t, \frac{1}{f^2(s)}\right) \,. \tag{6}$$

It is easy to calculate the covariance for the n-hWOU field,

$$\mathrm{cov}(Z_f(t_1, s_1), Z_f(t_2, s_2)) = \frac{f(s_1) f(s_2)}{f^2(\min(s_1, s_2))} \min(t_1, t_2) \,. \tag{7}$$

Proposition 2. There exists the Bm $B(u)$ such that the corresponding diagonal process $Z_f(t) \overset{\triangle}{=} Z_f(t, t)$ follows to the equality

$$Z_f(t) = \frac{1}{f(t)} \left(\int_0^t f(u) \sqrt{1 \,+ 2u\, (\ln f(u))'}\, \mathrm{d}B(u) \right) \,. \tag{8}$$

Proposition 3 establishes the relationships between the diagonal processes for fields $Z_f(s, t)$ and $Z(s, t)$.

Proposition 3. From the form of the covariance of the $Z_f(s, t)$ it is directly follows,

$$Z_f(t, t) \overset{distr}{=} Z(\ln f(t), t) \,. \tag{9}$$

Model Comment 3. Proposition 3 includes that for investigation of distribution properties of the processes $Z(t, g(t))$, where $g(t)$ is a positive increasing function, it is sufficient to examine the diagonal process $Z_f(t)$ of n-hWOU field with $f = e^g$. In model sense for the intrinsic time the function $g(t)$ regularizes non-homogeneity of the delays in observations of $\mathrm{d}W(t)$.

The equation (9) obviously implies that $\mathbf{D}Z_f(t,t) = t$.

The following limit theorem 2 allows us the fact of convergency random broken planes to the [homogeneous] WOU field.

Theorem 2. Let us split the rectangle $[0,T]\times[0,1] \ni (t,s),\ T<\infty$ into rectangular parts by the nodes $\left\{(\frac{iT}{N}, \frac{j}{N})\right\};\ \ i,j \in \{0,1,\dots,N\}$. We define the random broken rectangles as follows

$$S_N(t,s) \equiv \frac{1}{\sqrt{N}} \sum_{i=1}^{[Nt]} \varepsilon_i^{\left(\Pi_i\left(\frac{[Ns]}{N}\right)+1\right)} ,$$

where $[Nt]$ and $[Ns]$ denote the integer parts. Then the following weak convergence is valid in the space of step functions $\mathbf{D}^{\mathbf{Q}}_{[0,T]\times[0,1]}$ equipping with the uniform metric and having jumps in a countable fixed set, as $N \to \infty$,

$$S_N(t,s) \implies Z(t,s) ,$$

where $Z(t,s),\ \ (t,s) \in [0,T]\times[0,1]$, is the WOU field.

The following theorem 3 states a convergency of non-homogeneous Poisson processes (with spacings weighted by the iid rv's $\{\varepsilon\}$) to the non-homogeneous Ornstein-Uhlenbeck (n-hOU) process.

Non-homogeneous Ornstein-Uhlenbeck Process as a Limit of Weighted Non-homogeneous Poisson Processes

We define the n-hOU process by generalization of Lamperti transform,

$$U_f(s) = f(s) W\left(\frac{1}{f^2(s)}\right) , \tag{10}$$

where $f(s),\ s \geq 0,\ f(0) = 1$ is the positive, strictly monotone, increasing, and differential function. Obviously, the n-hOU process defined by (10) is a truncation of the n-hWOU field defined by (6).

In particular case, $f(s) = \exp\{-\beta s\}$ we have in (10) the standard Ornstein-Uhlenbeck process with the viscosity coefficient $\beta > 0$. The fact is that the process $U_f(s)$ is a stationary one iff the function f is the exponential.

The equivalent definition of the n-hOU process we give by the following generalization of the Langevin stochastic differential equation (SDE),

$$\mathrm{d}X(t) = -H(t)X(t)\mathrm{d}t + \sqrt{2H(t)}\mathrm{d}B(t)\,, \tag{11}$$

where the viscosity parameter is as follows

$$H(t) = \frac{\mathrm{d}\ln f(t)}{\mathrm{d}t}\,, \qquad t \geq 0\,,$$

and the initial value $X(0) \in \mathcal{N}(0,1)$ does not depends on the Bm $B(t)$.

Let $\{\Pi_i^f(s)\},\ i = 1, 2, \ldots\ s \geq 0$, be a sequence of independent identically distributed non-homogeneous Poisson processes (Cox processes) with the joint intensity function $\Lambda(s) = \ln f(s)$, where $f(s)$, $s \geq 0$, $f(0) = 1$ is the positive, strictly monotone, increasing, and differential function.

Theorem 3. Let $\left\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}, 1\right\}$ be nodes of the uniform partition of the interval $[0,1] \ni s$. In the nodes we define the $\sqrt{N}$-normalized cumulative sums of the variables $\{\varepsilon\}$ indexed by the non-homogeneous Poisson processes

$$S_N^f(s) \equiv \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \varepsilon_i^{(\Pi_i^f(s)+1)}\,,$$

and in the remaining points $s \in [0,1]$ we define the values of $S_N^f(s)$ by the linear interpolation.

The following weak convergence is valid in the space of continuous function $\mathbf{C}_{[0,1]}$, as $N \to \infty$,

$$S_N^f(s) \implies U_f(s)\,,$$

where $U_f(s)$ is the non-homogeneous Ornstein-Uhlenbeck process satisfied (10) with the viscosity function

$$H(s) = \frac{\mathrm{d}\Lambda(s)}{\mathrm{d}s}\,.$$