The Mathematics of Machine Learning
Homework Assignment 3

October 15, 2020

This assignment is due on October 21st. In the week following this date, the instructor and grader will meet with you online to go over your assignment and possibly also prior assignments, in place of a midterm exam.

Please make sure to show all work so that you can get partial credit. Also, please list any parts of definition, properties or theorems required for the problems. The assignment was designed to be done without programming. There are 3 problems, each with several parts. Each problem lists the number of points out of 100 on it. This assignment will count as 10% of your grade and the one on one meeting will count as another 10% of your grade.

1. **Probability measures:** (28 points total) Let $\Omega = \mathbb{R}$ and consider the following probability measures on the Borel σ-algebra $\mathcal{B}$:

   (a) $\mathbb{P}_1$ has density $f : \mathbb{R} \to [0, \infty)$ defined by:

$$f(x) = \begin{cases} \exp(-x) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

P(x) = 1 - e^(-x), x > 0

   (b) $\mathbb{P}_2$ has point mass function $p : \mathbb{R} \to [0, 1]$ defined by:

$$p(x) = \begin{cases} 2^{-i} & \text{if } x = 2^{-i} \text{ for some } i \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

   For 2 points each, please fill out each cell of the following table with the probability of the set listed in the header of the row corresponding to the cell under the measure listed in the header of the column corresponding to the cell.

这里是有限到无穷的交界 —>

	$\mathbb{P}_1$	$\mathbb{P}_2$
$\{0\}$	0	0
$\{\frac{1}{2}\}$	0	1/2
$\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$	0	1/4+1/2=3/4
$\{2^{-2i} : \text{for } i \text{ in } \mathbb{N}\}$	1/3	sigma—>1/3
$[0, \frac{1}{2}]$	1 - e^(-1/2)	1/2
$[\frac{1}{4}, \frac{1}{2}]$	e^(-1/4) - e^(-1/2)	3/4
$[\frac{1}{4}, \frac{1}{2}] \cup [\frac{3}{4}, 1]$	e^(-1/4)+e^(-3/4) - e^(-1/2)-e^(-1)	3/4

P1: {2^(-2i)}: 相比于无穷多个点，
还是有限个所以是0
Countable number of points

P2 = 1 for [0, 1/2]:
因为2^(-i)，i从1到无穷累加得到1

这个可以和下面的[0, 1/2]的答案对比看出区别

2. **Cumulative distribution functions:** (36 points total) Let S_t be a random variable corresponding to the price of a financial asset at time t and let $\mathbb{P}_{S_t}$ be the distribution of the price after some time has passed. The cumulative distribution for $\mathbb{P}_{S_1}$, denoted $F_{p,\theta} : \mathbb{R} \to [0,1]$ for some parameters $p \in [0,1]$ and $\theta \in (0,\infty)$, is defined as follows:

(1-p)(1-exp(-t x))

$$F_{p,\theta}(x) = \begin{cases} p + 1 - \exp(-\theta x) & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

和上一题最大不同点就是x=0时候，
和从负轴趋近0的时候，F和P都是跳跃的

jump: => p-0 = p: jump的时候
并不连续，所以并不能求导
得到结果，而是直接相减两点的差距

Please answer the following questions for 4 points each:

(a) What is $\mathbb{P}(S_1 = 0)$? Note that this is the probability that the stock goes bankrupt.

P(x) = F' = theta*(1-p) * e^(-theta * x), x>=0
0, x<0;
P(0) = theta(p-1)

(b) What is $\mathbb{P}\left(-\frac{1}{2} \leq S_1 \leq 0\right)$? = F(0) = 0+(1-p)(1-e^0) = 0.

(c) What is $\mathbb{P}\left(-\frac{1}{2} \leq S_1 \leq \frac{1}{2}\right)$? = P(0 <= S1 <= 1/2) = 1+ (p-1)*exp(-1/2 t)

(d) What is $\mathbb{P}(0 \leq S_1 < \infty)$? = 1

e) No, f) No:
p:是x=0处的point mass;
1-e^(-t x):是一个density
所以没有纯粹的point mass/density;
但是是个mixture

(e) Does this probability distribution have a point mass function?

(f) Does this probability distribution have a density?

(g) Is this probability distribution a mixture of a distribution with a point mass function and one with a density? <—— yes

(h) Is this probability distribution singular?

(i) A European option on S_1 with strike K has value $(S_1 - K)^+$. What is the expected value of the option?

e) P(x) = theta * e^(-theta x), x所在的集合Omega并不是一个finite set，所以没有point mass function
f) 没有density，因为pdf的前提条件是f是连续的，这里的P充当了f的作用，但是P在x=0处不连续
g) 并不是mixture，没有point mass function, 也没有density
h) 是singular：既没有point mass也没有density也不是这两者的mixture
i) E(S1) = (1-p/theta) * e^(-k*theta)

3. **Law of Large Numbers:** (36 points total) In the Black-Scholes world, a stock price, S_t, behaves according to a geometric Browian motion with mean μ and variance σ^2, that is:

$$S_t = \exp\left(\sigma B_t + \left(\mu - \frac{\sigma^2}{2}\right) t\right)$$

for a Brownian motion B_t with cumulative distribution function:

$$F_{B_t}(x) = \Phi\left(\frac{x}{\sqrt{t}}\right)$$

where $\Phi : \mathbb{R} \to [0, \infty)$ is the cumulative distribution function of the normal distribution. Note that B_t has independent increments, that is, $B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, \ldots, B_{t_n} - B_{t_{n-1}}$ are independent for any $t_1 < t_2 < t_3 < \ldots < t_{n-1} < t_n$. Please answer the following questions for 12 points each:

(a) Are $\frac{S_{t_2}}{S_{t_1}}, \frac{S_{t_2}}{S_{t_1}}, \ldots, \frac{S_{t_2}}{S_{t_1}}$ independent for every $t_1 < t_2 < t_3 < \ldots < t_{n-1} < t_n$? Please indicate why you believe this to be the case.

(b) What is the geometric return, that is, $E\left[\log\left(\frac{S_{t+1}}{S_t}\right)\right]$?

(c) What is the probability that the sample geometric return, that is:

$$\frac{1}{n} \lim_{n\to\infty} \sum_{i=1}^{n} \log\left(\frac{S_{i+1}}{S_i}\right)$$

converges? When this converges, what value or values does it converge to?