HW5

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November 29, 2020

1. **Nearest Neighbor and Metric Spaces**:
   (a) Yes, from the symmetry of Nearest Neighbor: $d(X_1, X_2) = d(X_2, X_1)$, if X_1 is the nearest neighbor of X_2, then X_2 is the nearest neighbor of X_1.
   (b) $d(X_2, X_3) = d(X_2, X_1) + d(X_1, X_3) \geq d(X_2, X_1) + 0 => d(X_2, X_1) \leq d(X_2, X_3)$
   (c) From Triangle inequality, it is known that $d(X_2, X_3) \leq d(X_2, X_1) + d(X_1, X_3)$. Equation 1 says $d(X_2, X_3) = d(X_2, X_1) + d(X_1, X_3)$, so X_1 is in $[X_2, X_3]$ and could be anywhere in between.
      i. Possible, e.g. $[X_2, X_1, X_3, X_4]$
      ii. Possible, e.g. $[X_4, X_2, X_1, X_3]$ and X_4 is very close to X_2.
      iii. Impossible, given $d(X_2, X_1) \leq d(X_2, X_3)$, when X_1 is not X_3, X_1 is the 2nd nearest; when X_1 is X_3, X_1 is still the 2nd nearest.
   (d) following the above deduction, when $d(X_1, X_4) = d(X_1, X_3) + d(X_3, X_4)$, X_3 is in $[X_1, X_4]$. Also, it is known that X_1 in $[X_2, X_3]$ so that the whole picture is like $[X_2, X_1, X_3, X_4]$.
      i. Could be
      ii. Couldn't be
      iii. Couldn't be

2. **Lipschitz functions**:
   (a) $f(x) = x^{\frac{1}{3}}$: $f'(x) = 1/(3x^{-2/3})$: with x increases, $f'(x)$ decreases. When $x -> 0$, $f'(x) -> \infty$, so:
      i. $f(x)$ is not Lipschitz on $[0, \infty)$ because $f'(x) -> \infty$ when $x -> 0$.
      ii. $f(x)$ is not Lipschitz on $[0, 1]$ because $f'(x) -> \infty$ when $x -> 0$.
      iii. $f(x)$ is Lipschitz on $[2, \infty)$. $f'(x)_{max} = f'(2) = \frac{1}{3\sqrt[3]{4}}$ so the smallest constant is $\frac{1}{3\sqrt[3]{4}}$.
      iv. $f(x)$ is Lipschitz on $[1, 2]$. $f'(x)_{max} = f'(1) = 1/3 => L = 1/3$.
   (b) $f(x) = x^{\frac{4}{3}}$: $f'(x) = \frac{4}{3}x^{\frac{1}{3}}$.
      i. $[0, \infty), [2, \infty)$: when $x -> \infty$, $f(x) -> \infty$ so $f(x)$ is not Lipschitz.
      ii. $[0, 1]$: $f(x)$ is Lipschitz. $f'(x)_{max} = f'(1) = \frac{4}{3} => L = \frac{4}{3}$
      iii. $[1, 2]$: $f(x)$ is Lipschitz. $f'(x)_{max} = f'(2) = \frac{4}{3}2^{1/3} => L = \frac{4}{3}2^{1/3}$
   (c)
   $$f(x) = \begin{cases} 0, & \text{if } x \leq 1 \\ \text{2(x-1)}, & \text{if } x \in [1, 2) \\ 1, & \text{if } x \geq 2 \end{cases} \tag{1}$$
   One thing to notice is that when $x -> 2$ from the left to when $x = 2$, there is a plunge from 2 to 1, which means the derivative from the left to where $x = 2$ is ∞. No other places would find a $f'(x) -> \infty$. As a result, intervals that include $x = 2$ from left side are not Lipschitz. Other places are Lipschitz.
      i. $[0, \infty)$: not Lipschitz
      ii. $[0, 1]$: $f(x) = 0$: Lipschitz, constant is 0.
      iii. $[2, \infty)$: $f(x) = 1$: Lipschitz, constant is 0.
      iv. $[1, 2]$: plunge included; not Lipschitz
   (d)
   $$f(x) = \begin{cases} 0, & \text{if } x \leq 1 \\ \text{2(x-1)}, & \text{if } x \in [1, 2) \\ 2, & \text{if } x \geq 2 \end{cases} \tag{2}$$
   The only difference from (c) is that $f(x)$ is continuous this time at $x = 2$. $f(x)$ is also bounded by $[0, 2]$. As a result, $f(x)$ is Lipschitz anywhere. Specifically, Lipschitz constant is 2 on $[0, \infty)$ and $[1, 2]$, but 0 on $[0, 1]$ and $[2, \infty)$ because $f(x)$ is constant.

3. **Hölder functions**:

(a) For $f(0) = f(1) = f(2) = 0$ case, if $f(x) = 0$ always holds, this function satisfies all following Hölder class. As a result, the 1st column are all checked.

(b) For case $f(0) = f(1) = 0$ and $f(2) = 1$: the smoothest function would be

$$f(x) = \begin{cases} 0, & x \in [0, 1] \\ \text{x-1}, & x \in [1, 2] \end{cases} \tag{3}$$

In that case, $|f(x) - f(y)| = |x - y|$ if both $x, y \in [1, 2]$. When one variable is 0, i.e. $x = 0$ or $y = 0$, then $f(x) = 0$ or $f(y) = 0$.

i. $\mathcal{F}^{(0,\frac{1}{2})}$ $<=>$ if $|f(x) - f(y)| \leq \frac{1}{2}|x - y|$: Impossible, because the smoothest would be $|x - y|$, which is larger than $\frac{1}{2}|x - y|$.

ii. $\mathcal{F}^{(0,1)}$: Possible, the given function above is exactly $|x - y|$.

iii. $\mathcal{F}^{(0,2)}$: Possible, because $|x - y| \leq 2|x - y|$.

iv. $\mathcal{F}^{(1,\frac{1}{2})}$: For $f'(x) - f'(y)$, it is either 0 or 1. When $f'(x) - f'(y) = 0$ x and y are both in $[0, 1]$ or $[1, 2]$. Given the left hand side is 0, $f'(x) - f'(y) = 0 \leq \frac{1}{2}|x - y|$. When one variable in $[0, 1]$ and the other in $[1, 2]$, e.g. y in $[0, 1]$, then $f(y) = 0$. Left hand side $= f'(x) - 0 = 1$, right hand side $= \frac{1}{2}|x - y|$. When $x = 2, y = 0$, right hand side $= 1$, so this is possible.

v. $\mathcal{F}^{(1,1)}$: $<=>$ $f'(x) - f'(y) \leq |x - y|$. Same analysis as above: When $y = 0$ and right hand side is $|x|$, when $x \in [1, 2]$, this always holds so it is possible.

vi. $\mathcal{F}^{(1,2)}$: $f'(x) - f'(y) \leq |x - y|$ is possible, so $f'(x) - f'(y) \leq 2|x - y|$ is also possible.

(c) For case $f(0) = f(2) = 0$ and $f(1) = 1$.
For $\mathcal{F}^{(0,x)}$, consider the function

$$f(x) = \begin{cases} x, & x \in [0, 1] \\ \text{2-x}, & x \in [1, 2] \end{cases} \tag{4}$$

i. $\mathcal{F}^{(0,\frac{1}{2})}$: Impossible, $f(x) - f(y) = |x - y|$.

ii. $\mathcal{F}^{(0,1)}$: Possible: $f(x) - f(y) = |x - y|$.

iii. $\mathcal{F}^{(0,2)}$: Possible: $f(x) - f(y) = |x - y| \leq 2|x - y|$.

For $\mathcal{F}^{(1,x)}$, consider the function $f(x) = 1 - (x - 1)^2$, then $|f'(x) - f'(y)| = |2(1 - x) - 2(1 - y)| = 2|x - y|$. Then:

i. $\mathcal{F}^{(1,2)}$: Possible.

ii. $\mathcal{F}^{(1,1)}$: Impossible.

iii. $\mathcal{F}^{(1,0)}$: Impossible.

As a result, the table would be

$f(0) =$	0	0	0
$f(1) =$	0	0	1
$f(2) =$	0	1	0
$\mathcal{F}^{(0,\frac{1}{2})}$	✓		
$\mathcal{F}^{(0,1)}$	✓	✓	✓
$\mathcal{F}^{(0,2)}$	✓	✓	✓
$\mathcal{F}^{(1,\frac{1}{2})}$	✓	✓	
$\mathcal{F}^{(1,1)}$	✓	✓	
$\mathcal{F}^{(1,2)}$	✓	✓	✓

(d) If $f \in \mathcal{F}^{(p,C)}$, i.e. $|f^{(p)}(x) - f^{(p)}(y)| \leq C|x - y|$, then $|g^{(p)}(x) - g^{(p)}(y)| = |(1 - f(x))^{(p)} - (1 - f(y))^{(p)}| = |f^{(p)}(x) - f^{(p)}(y)| \leq C|x - y| => g(x)$ is always $\mathcal{F}^{(p,C)}$. The key is that constant addition/subtraction does not affect derivative operations.

(e) $g(x) = f(1 - x)$ is not always $\mathcal{F}^{(p,C)}$ because $f(x)$ can take many different forms of functions, in which case it cannot be guaranteed only $|f^{(p)}(x) - f^{(p)}(y)|$ is enough.