The Mathematics of Machine Learning
Homework Assignment 4

November 4, 2020

This assignment is due on November 5th. Please make sure to show all work so that you can get partial credit. Also, please list any parts of definition, properties or theorems required for the problems. The assignment was designed to be done without programming though a calculator or spreadsheet might help with the calculations There are 3 problems, each with several parts. Each problem lists the number of points out of 100 on it. This assignment will count as 10% of your grade.

1. **The Central Limit Theorem:** (5 points each for 40 points total) We are trading a daily rebalanced portfolio w, that is, a portfolio which is rebalanced daily to match proportion w_i in the *ith* stock. The daily returns of this portfolio are modeled as IID and having the following properties:

$$E\left[\ln\left(1+w^T R\right)\right] = \mu = 0.0004$$

$$E\left[\left(\ln\left(1+w^T R\right)-\mu\right)^2\right] = \sigma = 0.01$$

这个少了个sqrt; sigma还是0.01

We wish to if the portfolio seems to be performing worse than the model would predict (people don't often seem to check if their portfolio is performing better than their model would predict). Calculate the asymptotic Central Limit Theorem approximation of the probabilities that the portfolio has the values given at the times given in the following tables. You can assume that there are 252 business days in a year.

Year 0	Year 1 Value	Probability	Year 2 Value	Probability
\$1,000,000	$\leq$ \$1,000,000		$\leq$ \$1,000,000	
\$1,000,000	$\leq$ \$900,000		$\leq$ \$900,000	

Year 0	Year 5 Value	Probability	Year 10 Value	Probability
\$1,000,000	$\leq$ \$1,000,000		$\leq$ \$1,000,000	
\$1,000,000	$\leq$ \$900,000		$\leq$ \$900,000	

1 mi对应的return = (1mi - 1mi)/(1mi) = 0
900 000对应return = -100000/1mi = -0.1

Jensen's inequality: f(t1X1+t2X2) <= t1f(X1) + t2f(X2), f: convex
Probability: f convex, f(E[X]) <= E[f(X)]

2. **Supermartingales and conditional expectation:** (8 points each for 24 points total) Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be a sequence of σ-algebras and $X_1, X_2, \ldots$ a sequence of random variables such that X_i is $\mathcal{F}_i$ measureable. Suppose X is a supermartingale, that is[1]:

$$E[X_{n+1}|\mathcal{F}_n] \leq X_n$$

Please solve the following and make sure to show what properties are used in each subproblem.

(a) Is $Y_1, Y_2, Y_3, \ldots$ a supermartingale with respect to $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \ldots$ where $Y_i = X_{2i}$ and $\mathcal{G}_i = \mathcal{F}_{2i}$?

(b) Is $\log(X_1), \log(X_2), \log(X_3), \ldots$ a supermartingale with respect to $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots$?

need only show that the conditions of Theorem 4.1.1, which are listed in the statement of the theorem, hold

(c) Is $-X_1^2, -X_2^2, -X_3^2, \ldots$ a supermartingale with respect to $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots$?

3. **Simplified universality for nearest neighbor:** (6 points each for 36 points total) Which of the following models does the simplified universality result in the book apply to?

f: should be continuous and bounded

unbounded (a) $Y_i = \alpha X_i + \epsilon_i$ where α is a constant, and X_i and ϵ_i are IID (independent of everything else).

bounded (b) $Y_i = \frac{1}{1+\exp(-\alpha X_i)} + \epsilon_i$ where α is a constant, and X_i and ϵ_i are IID (independent of everything else).

epsilon changes: f uncontinuous (c) $Y_i = \frac{1}{1+\exp(-\alpha X_i) - \epsilon_i}$ where α is a constant, and X_i and ϵ_i are IID (independent of everything else).

uncontinuous (d) $Y_i = \mathbb{1}_{X_i \geq 0}(\alpha X_i) + \epsilon_i$ where α is a constant, and X_i and ϵ_i are IID (independent of everything else).

Y_{i-1}: const: same as b (e) $Y_i = \frac{1}{1+\exp(-\alpha X_i - \beta Y_{i-1})} + \epsilon_i$ where α and β are constants, and X_i and ϵ_i are IID (independent of everything else).

epsilon changes: f uncontinuous (f) $Y_i = \frac{1}{1+\exp(-\alpha X_i)} + X_i \epsilon_i$ where α is a constant, and X_i and ϵ_i are IID (independent of everything else).

[1] Note that there was an error in the book in the definition of a supermartingale; it had had an equality rather than an inequality