HW4

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1. **Central Limit Theorem**:

   The first thing to notice is that starting with $\$1mi$, the return would be 0 if ending wealth is also $1mi$ and, return would be $(900000 - 1mi)/1mi = -0.1$ if ending with \$900000.

   Year 1 corresponds to 252 days, Year 2 to 504 days, Year 5 to 1260 days and Year 10 to 2520 days.

   To use Central limit theorem, I need to convert this variable to another variable z $N(0,1)$ which conforms to Standard Normal distribution, e.g. for Year1, $X = 0$, $P(W < \$1000000) = P(return < 0) = P(z < \frac{0-0.0004}{0.01/\sqrt{252}}) = \Phi(\frac{0-0.0004}{0.01/\sqrt{252}}) = \Phi(-0.635)$. Use the same logic for the remaining seven blanks. In total:

   (a) Year 1, $P(return \le 0) = P(z \le \frac{-0.0004}{0.01/\sqrt{252}}) = \Phi(-0.635) = 0.2627$

   (b) Year 1, $P(return \le -0.1) = P(z \le \frac{-0.1004}{0.01/\sqrt{252}}) = \Phi(-159.38) = 0$

   (c) Year 2, $P(return \le 0) = P(z \le \frac{-0.0004}{0.01/\sqrt{504}}) = \Phi(-0.898) = 0.1846$

   (d) Year 2, $P(return \le -0.1) = P(z \le \frac{-0.1004}{0.01/\sqrt{504}}) = \Phi(-225.39) = 0$

   (e) Year 5, $P(return \le 0) = P(z \le \frac{-0.0004}{0.01/\sqrt{1260}}) = \Phi(-1.42) = 0.0778$

   (f) Year 5, $P(return \le -0.1) = P(z \le \frac{-0.1004}{0.01/\sqrt{1260}}) = \Phi(-356.38) = 0$

   (g) Year 10, $P(return \le 0) = P(z \le \frac{-0.0004}{0.01/\sqrt{2520}}) = \Phi(-2.01) = 0.0222$

   (h) Year 10, $P(return \le -0.1) = P(z \le \frac{-0.1004}{0.01/\sqrt{2520}}) = \Phi(-504) = 0$

2. **Super martingales and conditional expectation**:

   (a) $E[E[Y_{i+1}|\mathcal{G}_i]] = E[E[X_{2i+2}|\mathcal{F}_{2i}]] \le E[X_{2i+1}] = E[X_{2i+1}|\mathcal{F}_{2i}] \le X_{2i} = Y_i$. As a result, $Y_1, Y_2 ...$ is a supermartingale w.r.t. $\mathcal{G}_1, \mathcal{G}_2 ...$

   (b) $log(X)$ is a concave function. Given Jensen's inequality, $E[\Phi(x)] \le \Phi(E[x])$ holds when $\Phi(x)$ is concave $=>$ $E[log(X)] \le log(E[X]) => E[log(X_{n+1})|\mathcal{F}_n] \le log(E[log(X_{n+1}|\mathcal{F}_n]) = log(X_n)$. As a result, $log(X_1), log(X_2)...$ is a supermartingale.

   (c) $-X^2$ is also a concave function. Given Jensen's inequality, $E[-X_{n+1}^2|\mathcal{F}_n] \le (E[-X_{n+1}|\mathcal{F}_n])^2 = X_n^2$. As a result, $-X_1^2, -X_2^2...$ is a supermartingale. concave,但是并不是monotonic,所以不能用Jensen，不是supermartingale

3. **Simplified universality for nearest neighbors**:

   To apply the simplified universality, models need to be formed like $Y_i = f(X_i) + \epsilon_i$ and f should be continuous and bounded.

   (a) $Y_i = \alpha X_i + \epsilon_i$: αX_i is obviously not bounded: e.g. for α positive, when X_i goes to infinity, αX_i goes to infinity too. Thus simplified universality does not apply to this one.

   (b) $Y_i = \frac{1}{1+exp(-\alpha X_i)} + \epsilon_i$: Yes. Since $exp(-\alpha X_i) \in (0, \infty)$, function $f(X_i) = \frac{1}{1+exp(-\alpha X_i)}$ is bounded by $(0,1)$. It's continuous as well. Thus simplified universality can be applied.

   (c) $Y_i = \frac{1}{1+exp(-\alpha X_i)-\epsilon_i} = [\frac{1}{1+exp(-\alpha X_i)-\epsilon_i} - \epsilon_i] + \epsilon_i => f(X_i) = \frac{1}{1+exp(-\alpha X_i)-\epsilon_i} - \epsilon_i$. Given each ϵ_i could be different, $f(X_i)$ would jump upside down at each X_i, making it not continuous. Thus simplified universality does not apply.

   (d) $Y_i = 1_{X_i \ge 0}(\alpha X_i) + \epsilon_i$: function is not continuous when X_i goes from very small negative value to 0: goes from 0 abruptly to 1. Thus simplified universality cannot be applied.

   (e) $Y_i = \frac{1}{1+exp(-\alpha X_i)*exp(-\beta Y_{i-1})} + \epsilon_i$: Given Y_{i-1}, $e^{\beta Y_{i-1}}$ is a constant. As a result, $f(X_i)$ can be seen as $\frac{1}{1+c*e^{(}-\alpha X_i)}$: not very different from (b): continuous and bounded by $(0,1)$. Thus simplified universality can be applied.

   (f) $Y_i = \frac{1}{1+exp(-\alpha X_i)} + X_i\epsilon_i = [\frac{1}{1+exp(-\alpha X_i)} + (X_i - 1)\epsilon_i] + \epsilon_i$, so $f(X_i) = \frac{1}{1+exp(-\alpha X_i)} + (X_i - 1)\epsilon_i$. Since ϵ_i is inside $f(X_i)$, it's the same as (c): ϵ_i would make $f(X_i)$ not continuous everywhere. $f(X_i)$ not continuous, making simplified universality cannot be applied.

e) Can't put into Y = f(X) + e形式; Yj-1 is a parameter, can't be applied