HW2

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1. **Random variables**: Results are given first:

function	random variable on $(\Omega, \mathcal{F}, \mathbb{P}_1)$	random variable on $(\Omega, \mathcal{P}(\Omega), \mathbb{P}_2)$
f	No	Yes
g	No	Yes
h	No	Yes
f^2	Yes	Yes
g^2	No	Yes
h^2	Yes	Yes

The key to judge a random variable is that the inverse image of every interval $X^-1[a, b] \in \mathcal{F}$, i.e. here it means if combinations of ω are subsets of $\mathcal{F}/\ \mathcal{P}(\Omega)$.

(a) For $(\Omega, \mathcal{P}(\Omega), \mathbb{P}_2)$, since $\mathcal{P}(\Omega)$ includes every possible subset, functions are always a random variable on it.

(b) $f : f(\omega) = -1$, the inverse of f: ω could be any combinations of apple, pear, or orange: not totally included in $\mathcal{F}$. As a result, f is not a random variable on $(\Omega, \mathcal{F}, \mathbb{P}_1)$.

(c) $g : g(\omega) = -1/1$. Also, the inverse could be any combinations of apple, pear or orange; not fully included in $\mathcal{F}$. g is not a random variable on $(\Omega, \mathcal{F}, \mathbb{P}_1)$.

(d) $h : h(\omega) = -1/1$. Also, the inverse could be any combinations of apple, pear or orange; not fully included in $\mathcal{F}$. h is not a random variable on $(\Omega, \mathcal{F}, \mathbb{P}_1)$.

(e) $f^2 : f^2(\omega) = 1$, while $f(\omega) = -1$ for $apple/pear/orange$. As a result, corresponding set have to be $\{\varnothing\}$, a subset of $\mathcal{F}$. As a result, f^2 is a random variable on $(\Omega, \mathcal{F}, \mathbb{P}_1)$.

(f) $g^2 : g^2(\omega) = 1$, so the corresponding set could be any combination of $pear/orange$. Set $\{pear, orange\}$ is not a subset of $\mathcal{F}$. As a result, g^2 is not a random variable on $(\Omega, \mathcal{F}, \mathbb{P}_1)$.

(g) $h^2 : h^2(\omega) = 1$. The only possible ω can be $pear$, or it takes $\{\varnothing\}$, both are subsets of $\mathcal{F}$. As a result, h^2 is a random variable on $(\Omega, \mathcal{F}, \mathbb{P}_1)$.

2. **Independence**:

(a) X_1 and X_2: $P(X_1 = 0) = P(X_1 = 1) = P(X_2 = 0) = P(X_2 = 1) = 1/2$. As a result, $P(X_1 = 0){*}P(X_2 = 0) = 1/4$. When $P(X_1 = 0) = 0, \omega \in \{a, b\}$, which automatically means $X_2 = 0$ as well, further indicating $P(X_1 \cap X_2) = 1/2$. So, $P(X_1 \cap X_2) \neq P(X_1 = 0) * P(X_2 = 0)$, X_1 and X_2 are not independent.

(b) X_1 and X_3: $P(X_3 = 0) = 1$. $P(X_1 = 0) * P(X_3 = 0) = 1/2 = P(X_1 \cap X_3) = P(\omega \in \{a, b\})$. Similarly, $P(X_1 = 1) * P(X_3 = 0) = 1/2 = P(X_1 \cap X_3) = P(\omega \in \{c, d\})$: X_1 and X_3 are independent.

(c) X_1 and X_4: $P(X_4 = 0) = P(X_4 = 1) = 1/2. P(X_1 = 0) * P(X_4 = 0) = P(X_1 \cap X_4 = \{0\}) = P(\omega \in \{a\}) = 1/4, P(X_1 = 1) * P(X_4 = 1) = P(X_1 \cap X_4 = \{1\}) = P(\omega \in \{d\}) = 1/4 => X_1$ and X_4 are independent.

(d) X_1 and X_5: $P(X_5 = 0) = P(X_5 = 1) = 1/2. P(X_1 \cap X_5 = \{0\}) = 0$, which is not equal to $P(X_1 = 0) * P(X_5 = 0)$ $=> X_1$ and X_5 are not independent.

(e) X_1 and X_6: $P(X_1 \cap X_6 = \{0\}) = 0 \neq P(X_1 = 0) * P(X_6 = 0) = 1/8$. $P(X_1 \cap X_6 = \{1\}) = P(\omega \in \{d\}) = 1/4 \neq P(X_1 = 1) * P(X_6 = 1) = 3/8 => X_1$ and X_6 are not independent.

(f) X_1 and X_7: $P(X_7 = 0) = P(X_7 = 1) = 1/2. P(X_1 = 0) * P(X_7 = 0) = P(X_1 \cap X_7 = \{0\}) = P(\omega \in \{a\}) = 1/4, P(X_1 = 1) * P(X_7 = 1) = P(X_1 \cap X_7 = \{1\}) = P(\omega \in \{c\}) = 1/4 => X_1$ and X_7 are independent.

(g) Similar to (b). $P(X_2 = 0) * P(X_3 = 0) = 1/2 = P(X_2 \cap X_3) = P(\omega \in \{a, b\})$. Similarly, $P(X_2 = 1) * P(X_3 = 0) = 1/2 = P(X_2 \cap X_3) = P(\omega \in \{c, d\})$: X_2 and X_3 are independent.

(h) X_1, X_2 and X_3: $P(X_1 \cap X_2 \cap X_3 = \{0\}) = P(\omega \in \{a, b\}) = 1/2$, while $P(X_1 = 0) * P(X_2 = 0) * P(X_3 = 0) = 1/2 * 1/2 * 1 = 1/4 => P(X_1 \cap X_2 \cap X_3) \neq P(X_1 = 0) * P(X_2 = 0) * P(X_3 = 0)$, so X_1, X_2 and X_3 are not independent.

(i) Similar to (b) and (g). It can be seen that X_3 does not have any impact on other variables. $P(X_3 \cap X_7) = P(X_3) * P(X_7)$, so X_3 and X_7 are independent.

(j) X_1, X_3 and X_7: $P(X_1 \cap X_3 \cap X_7 = \{0\}) = P(\omega \in \{a\}) = 1/4$. $P(X_1 = 0) * P(X_3 = 0) * P(X_7 = 0) = 1/2 * 1 * 1/2 = 1/4 => P(X_1 \cap X_2 \cap X_3) = P(X_1 = 0) * P(X_3 = 0) * P(X_7 = 0)$. When $P(X_3 = 1)$ is involved, the products are always $0 =>$ X_1, X_3 and X_7 are independent.

3. **Bernoulli's utility function**:

(a) I assume $W_1(H)$ and $W_1(T)$ means when the coin is tossed and outcomes are H/T. As a result, p and $(1-p)$ are not involved because outcomes are given.

For a unit of wealth, $f * V_H$ will be the terminal wealth of the toss with $(1-f)$ left. So $W_1(H) = \omega_0[fV_H + (1-f)]$. Similarly, $W_1(T) = \omega_0[fV_T + (1-f)]$.

(b) Just calculate using previous formulas:

p	V_H	V_T	ω_0	f	$W_1(H)$	$W_1(T)$
0.99	0.97	1000	1000	0.0	1000	1000
0.99	0.97	1000	1000	0.25	992.5	250750
0.99	0.97	1000	1000	1.00	970	1000000

(c) $W_1 = pW_1(H) + (1-p)W_1(T)$, where $W_1(H) = \omega_0[fV_H + (1-f)]$ and $W_1(T) = \omega_0[fV_T + (1-f)]$. So, $W_1 = \omega_0 * (p[fV_H + (1-f)] + (1-p)[fV_T + (1-f)]) = \omega_0 * ((1-f) + f[pV_H + (1-p)V_T])$.

Utility function $E[log_2(W_1)] = log_2(\omega_0) + log_2[(1-f) + f(pV_H + (1-p)V_T)]$.

(d) Using the above formula:

p	V_H	V_T	ω_0	f	expected value $E[W_1]$	utility $E[log_2(W_1)]$
0.99	0.97	1000	1000	0.0	1000	9.97
0.99	0.97	1000	1000	0.25	3490.075	11.77
0.99	0.97	1000	1000	1.00	10960.3	13.42

(e) The f that maximizes the utility will maximize W_1. ω_0 does not affect the result as a constant, i.e. to maximize f in function $g(f) = (1-f) + (pV_H + (1-p)V_T)f$. Take the derivative of g, $g' = (pV_H + (1-p)V_T) - 1$ on range $f \in [0, 1]$. So,

- if $(pV_H + (1-p)V_T) > 1$, $g(f)$ is an increasing function on $f \in [0, 1]$, then $f = 1$ will maximize utility;
- if $(pV_H + (1-p)V_T) < 1$, $g(f)$ is a decreasing function on $f \in [0, 1]$, then $f = 0$ will maximize utility;
- if $(pV_H + (1-p)V_T) = 1$, $g(f)$ stays constant on $f \in [0, 1]$.