1. $$\lim_{n\to\infty} \mathbb{P}\left(V_{n+1} \le \exp(n\mu + \sqrt{n}\sigma x)\right) = \Phi(x)$$

$$\mathbb{P}(V_{n+1} \le y)$$

$$y = \exp(n\mu + \sqrt{n}\sigma x)$$

$$x = \frac{\ln y - n\mu}{\sigma\sqrt{n}}$$

$$\approx \Phi\left(\frac{\ln y - n\mu}{\sigma\sqrt{n}}\right)$$

Year 0	Year 1 Value	Prob	Year 2 Value	Prob
\$1m	≤ \$1m	26.3%	≤ \$1m	18.5%
\$1m	≤ \$900k	9.7%	≤ \$900k	8.6%

Year 0	Year 5 Value	Prob	Year 10 Value	Prob
\$1m	≤ \$1m	7.8%	≤ \$1m	2.2%
\$1m	≤ \$900k	4.3%	≤ \$900k	1.3%

NOTE: ~~#~~ significant digits above are fine because of an ambiguity in the problem

2.

a. $E[Y_{n+1} | \mathcal{G}_n] = E[X_{2n+2} | \tilde{\mathcal{F}}_{2n}]$

$= E[E[X_{2n+2} | \mathcal{F}_{2n+1}] | \tilde{\mathcal{F}}_{2n}]$

by the law of iterated expectation

$\leq E[X_{2n+1} | \tilde{\mathcal{F}}_{2n}]$

by the supermartingale property and monotonicity of expectation

$\leq X_{2n}$ by the supermartingale property

Yes

b. $E[\log(X_{n+1}) | \tilde{\mathcal{F}}_n]$

$\leq \log(E[X_{n+1} | \tilde{\mathcal{F}}_n])$

by Jensen's inequality

$\leq \log(X_n)$

by the supermartingale property and monotonicity of log

Yes

2 c The answer is no because $f(x) = -x^2$ is not monotonically increasing

A slightly better answer is to give a counterexample of which there are many

Here is one:

The (deterministic) sequence $X_n = 2^{-n}$ is a supermartingale since:

$$E[X_{n+1} | \tilde{\mathcal{F}}_n] = E[2^{-n-1} | \tilde{\mathcal{F}}_n]$$
$$= 2^{-n-1} < 2^{-n} = X_n$$

However, $Y_n = -X_n^2 = -2^{-2n}$ is not:

$$E[Y_{n+1} | \tilde{\mathcal{F}}_n] = E[-2^{-2n-2} | \tilde{\mathcal{F}}_n] = -2^{-2n-2}$$
$$> -2^{-2n}$$

3. a. No. Y_i are not bounded except in trivial case

b. Yes.

c. No. The noise is not additive

d. No. $f(X_i)$ is not continuous. The statement of this problem has a bug so I would be generous here.

e. No. f has Y_{i-1} as a parameter.

f. No. The noise is not identically distributed.