The Mathematics of Machine Learning
Homework Assignment 4

November 18, 2020

This assignment is due on November 30th. Please make sure to show all work so that you can get partial credit. Also, please list any parts of definitions, properties or theorems required for the problems. This assignment will count as 10% of your grade.

1. **Nearest Neighbor and Metric Spaces:**(24 points total) Let (S, d) be a metric space. We observe random variables $(X_1, Y_1), (X_2, Y_2), \ldots, (X_4, Y_4)$ where X_i takes values in S. Demonstrate or provide a counterexample for each of the following questions:

   (a) (3 points) If X_1 is the nearest neighbor of X_2, is X_2 always the nearest neighbor of X_1?

   a). yes; 2.symmetry:
   d(X1, X2) = d(X2, X1)

   (b) (3 points) Suppose that the following holds:

   $$d(X_2, X_3) = d(X_2, X_1) + d(X_1, X_3) \tag{1}$$

   Is it always true that $d(X_2, X_1) \leq d(X_2, X_3)$?

   b). d(X2, X3) = d(X2, X1) + d(X1, X3)
   >= d(X2, X1) + 0 = d(X2, X1).
   => d(X2, X1) <= d(X2, X3),
   equals when X1 is X3

   (c) (9 points; 3 points each) If Equation (1) holds, which of the following are possible:

   i. X_1 is the nearest neighbor to X_2
   ii. X_1 is the second nearest neighbor to X_2
   iii. X_1 is the third nearest neighbor to X_2

   c) 3.Triangle inequality:
   d(X2, X3) <= d(X2, X1) + d(X1, X3),
   (1) d(X2, X3) = d(X2, X1) + d(X1, X3)
   => X1 in line[X2, X3], could be anywhere
   => i. could be: e.g. [X2, X1, X3, X4]
   ii. could be: e.g. [X4->X2, X1, X3]
   iii. couldn't be: d(X2, X1) <= d(X2, X3):
   when X1 is not X3: X1: 2nd nearest;
   when X1 is X3: still 2nd nearest

   (d) (9 points; 3 points each) If in addition to Equation (1), the following also holds:

   $$d(X_1, X_4) = d(X_1, X_3) + d(X_3, X_4)$$

   which of the following are possible:

   i. X_1 is the nearest neighbor to X_2
   ii. X_1 is the second nearest neighbor to X_2
   iii. X_1 is the third nearest neighbor to X_2

   d): X1 in [X2, X3] & X3 in [X1, X4]
   => format: [X2, X1, X3, X4]

   i. could be
   ii. can't be
   iii. can't be

2. **Lipschitz functions:**(32 points total; 2 points each) For each of the functions listed on the rows, on the intervals listed on the columns, show whether the function is Lipschitz and, if so, find the smallest Lipschitz constant:

	$[0, \infty)$	$[0, 1]$	$[2, \infty)$	$[1, 2]$
$f(x) = x^{\frac{1}{3}}$	N	Y	Y	Y
$f(x) = x^{\frac{4}{3}}$	N	Y	N	Y
$f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ 2(x-1) & \text{if } x \in [1, 2) \\ 1 & \text{if } x \geq 2 \end{cases}$	N	Y	Y	N
$f(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ 2(x-1) & \text{if } x \in [1, 2) \\ 2 & \text{if } x \geq 2 \end{cases}$	Y	Y	Y	Y

x^1/3:
[0, inf), [0,1]: x -> 0: f'(x): infinity; not Lipschitz;
[1,2]: polynomial on bounded interval: Lipschitz
[2, inf): f'(x): bounded: Lipschitz

x^(4/3):
[0, inf), [2, inf): x -> inf: f'(x): inf: not Lipschitz
[0, 1], [1,2]: bounded: Lipschitz

f:
[0, inf): not Lipschitz: <= not continuous =>
x1->2, 2<-x2: L: inf;
[1,2]: not Lipschitz: same reason, except 2=x2;
[0,1], [2, inf): constant: Lipschitz

f: all Lipschitz: bounded and continuous;

3. **Hölder functions:**(44 points total) We consider functions $f : \mathbb{R} \to \mathbb{R}$ such that f is (p, C)-Hölder, that is, $f \in \mathcal{F}^{(p,C)}$.

(a) (36 points; 2 points each) For each of the sets of function values listed in the first 3 rows, indicate whether it is possible for a function in the listed Hölder class listed in the subsequent rows to have those values.

$f(0) =$	0	0	0
$f(1) =$	0	0	1
$f(2) =$	0	1	0
$\mathcal{F}^{(0,\frac{1}{2})}$	✓	N	N
$\mathcal{F}^{(0,1)}$	✓	✓	✓
$\mathcal{F}^{(0,2)}$	✓	✓	✓
$\mathcal{F}^{(1,\frac{1}{2})}$	✓	✓	
$\mathcal{F}^{(1,1)}$	✓	✓	
$\mathcal{F}^{(1,2)}$	✓	✓	✓

i.e. | f(x)-f(y) | <= 1/2(x-y)

e.g. 分段函数: f(x)=0, x-1: 如果有一个是0，那么右边也是0

同

p=1:

010: f(x)=1-(x-1)^2; F(1,2)刚好=> F(1,1/2), F(1,1)都不行

(b) (4 points) If a function $f \in \mathcal{F}^{(p,C)}$, is the function $g(x) = 1 - f(x)$ also always in $\mathcal{F}^{(p,C)}$?

(c) (4 points) If a function $f \in \mathcal{F}^{(p,C)}$, is the function $g(x) = f(1 - x)$ also always in $\mathcal{F}^{(p,C)}$?