The Mathematics of Machine Learning
Homework Assignment 2

September 30, 2020

This assignment is due on October 8th. Please make sure to show all work so that you can get partial credit. Also, please list any parts of definitions or properties required for the problems. The assignment was designed to be done without programming. There are 3 problems, each with several parts. Each problem lists the number of points out of 100 on it. This assignment will count as 10% of your grade.

1. **Random variables:** (36 points total) Let $\Omega = \{apple, pear, orange\}$ and $\mathcal{F} = \{\varnothing, \{apple, orange\}, \{pear\}, \{apple, pear, orange\}\}$. Note that $\mathcal{F}$ is an algebra. Let $\mathbb{P}_1 : \mathcal{F} \to [0,1]$ and $\mathbb{P}_2 : \mathcal{P}(\Omega) \to [0,1]$ be probability measures. We define the functions $f, g, h : \Omega \to \mathbb{R}$ as follows:

ω	$f(\omega)$	$g(\omega)$	$h(\omega)$
apple	-1	-1	-1
pear	-1	1	1
orange	-1	1	-1

We write f^2 for the function obtained by first applying f and then squaring the result, that is, $f^2(x) = f(x)^2$ and similarly for the other functions. Please fill out the following table indicating whether each function is a random variable on the listed probability space (3 points each):

function	random variable on $(\Omega, \mathcal{F}, \mathbb{P}_1)$?	random variable on $(\Omega, \mathcal{P}(\Omega), \mathbb{P}_2)$?
f		
g		
h		
f^2		
g^2		
h^2		

f: 1. 都是-1,倒过来apo都有; F: 不是rm; P:是
g: {-1, 1}, 倒过来也是apo都有; F:不是rm; P: 是
h: 同 <— 这就是我问S，B的定义是什么的原因

f^2: 都是1， 倒过来apo都没有; F: 是rm， 因为有空集; P: 更是
g^2: 倒过来就是po，o： F没有， 所以不是； P是
h^2: 倒过来只有p， 所以F -> 是rm, P更是rm

上面这样做的前提是S是集合 {1, -1}

f, h, g^2 are rm: 说明如果取值， reverse image对应的是
这个值对应的所有可能的选项组成的set:
f: -1 => {apple, pear, orange}
h: -1 => {apple, orange}; 1 => {pear}

但是g^2? 1 => {pear, orange} ???

2. **Independence:** (24 points total) Let $\Omega = \{a, b, c, d\}$ and consider the probability space $(\Omega, \mathcal{P}(\Omega), \mathbb{P})$ where $\mathbb{P}$ is the uniform distribution. We define 3 random variables X_1, X_2, X_3 and X_4 on this probability space by the following table:

ω	$X_1(\omega)$	$X_2(\omega)$	$X_3(\omega)$	$X_4(\omega)$	$X_5(\omega)$	$X_6(\omega)$	$X_7(\omega)$
a	0	0	0	0	1	1	0
b	0	0	0	1	1	1	1
c	1	1	0	0	0	0	1
d	1	1	0	1	0	1	0

(a) (2 points) Are X_1 and X_2 independent? no

(b) (2 points) Are X_1 and X_3 independent? yes

(c) (2 points) Are X_1 and X_4 independent? yes

(d) (2 points) Are X_1 and X_5 independent? no

(e) (2 points) Are X_1 and X_6 independent? no

(f) (2 points) Are X_1 and X_7 independent? yes

(g) (2 points) Are X_2 and X_3 independent? yes

(h) (4 points) Are X_1, X_2 and X_3 independent? no

(i) (2 points) Are X_3 and X_7 independent? yes

(j) (4 points) Are X_1, X_3 and X_7 independent? yes

3. **Bernoulli's utility function:**(40 points total) Consider the flip of an unfair coin, that is, the probability space $(C, \mathcal{P}(C), \mathbb{P})$ where $C = \{H, T\}$ and $\mathbb{P}$ is defined by $\mathbb{P}(\{H\}) = p$ and, as required, $\mathbb{P}(\{T\}) = 1 - p$. Recall that Bernoulli suggested the use of $E[\log_2(W_1)]$ where W_1 is the terminal wealth after some bet or investment, as a method of evaluating the utility of the bet or investment. You have the option of placing money in a bet which yields, for each unit of currencey invested, V_H if heads are tossed and V_T if tails are tossed. Assume that you start with wealth w_0 units of currency and invest some fraction f of your wealth into this bet.

a,b: 确定了是H/T;
c开始: 没有确定是H/T

(a) (8 points) What are the formulas for $W_1(H)$ and $W_1(T)$, that is, the terminal wealth after making the above bet?

(b) (6 points) Calculate $W_1(H)$ and $W_1(T)$ for the following scenarios using this formula:

p	V_H	V_T	w_0	f	$W_1(H)$	$W_1(T)$
0.99	0.97	1000	1000	0.0	1000	1000
0.99	0.97	1000	1000	0.25	992.5	250750
0.99	0.97	1000	1000	1.0	970	1000000

(c) (8 points) What is the formula for the utility of the bet?

(d) (6 points) Calculate the expected value and utility for the following scenarios using this formula:

p	V_H	V_T	w_0	f	expected value $E[W_1]$	utility $E[\log 2(W_1)]$
0.99	0.97	1000	1000	0.0	1000	9.97
0.99	0.97	1000	1000	0.25	3490.075	11.77
0.99	0.97	1000	1000	1.0	10960.3	13.42

(e) (12 points) What is the formula for the optimal value of f, that is, the value which maximizes the utility?