HW3

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1. **Probability measures**:

   For $\mathbb{P}_2$, $p(x) = x$ when $x = 2^{-i}$ for $i \in \mathbb{N}$.

   (a) $\{0\}$: $0 \notin 2^{-i}$ for any $i \in \mathbb{N}$, $P(\{0\}) = 0$.
   (b) $\{\frac{1}{2}\}$: $i = 1, x = 2^{-1}, p(\frac{1}{2}) = 1/2$
   (c) $\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$: $\frac{1}{4}, \frac{1}{2}$ are in set $\{2^{-i}\}, i \in \mathbb{N}, p = \frac{1}{4} + \frac{1}{2} = 3/4$.
   (d) All 2^{-2i} are in set $\{2^{-i}\}$. $\sum_{n=1}^{\infty} 2^{-2i} = \sum_{n=1}^{\infty} 4^{-i} = \frac{\frac{1}{4}(1-\frac{1}{4}^n)}{1-\frac{1}{4}} = \frac{1}{3}$
   (e) $[0, \frac{1}{2}]$: In fact, only $\frac{1}{2}$ inside the range has a point mass so $p([0, \frac{1}{2}]) = p(\frac{1}{2}) = \frac{1}{2}$.
   (f) $p([\frac{1}{4}, \frac{1}{2}]) = p(\{\frac{1}{4}, \frac{1}{2}\}) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$.
   (g) $p([\frac{1}{4}, \frac{1}{2}] \cup [\frac{3}{4}, 1]) = p([\frac{1}{4}, \frac{1}{2}]) = p(\{\frac{1}{4}, \frac{1}{2}\}) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$.

   For $\mathbb{P}_1$, the cumulative distribution is $P(x) = 1 - e^{-x}, x > 0$ and $0, otherwise$ given the density.

   (a) $\{0\}, \{\frac{1}{2}\}, \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$: For a continuous function with a density, each point corresponds to a probability 0. As a result, all probabilities are 0.
   (b) 2^{-2i} for i in $\mathbb{N}$: This set includes infinite number of points, on an infinite range $[0, \infty]$. $P([0, \infty]) = 1$. $P(\{2^{-i}\}) = \sum_{n=1}^{\infty} 4^{-i} = \frac{1}{3}$. For infinite number of points, probabilities would be the same whether P has a density or a point mass function.
   (c) For ranges $[0, \frac{1}{2}], [\frac{1}{4}, \frac{1}{2}], [\frac{1}{4}, \frac{1}{2}] \cup [\frac{3}{4}, 1]$, use cdf $P(x) = 1 - e^{-x}, x > 0$:
      i. $P(\frac{1}{2}) = 1 - e^{-\frac{1}{2}}$
      ii. $P(\frac{1}{2}) - P(\frac{1}{4}) = e^{-\frac{1}{4}} - e^{-\frac{1}{2}}$
      iii. $(P(\frac{1}{2}) - P(\frac{1}{4})) + (P(1) - P(\frac{3}{4})) = e^{-\frac{1}{4}} + e^{-\frac{3}{4}} - e^{-\frac{1}{2}} - e^{-1}$

	$\mathbb{P}_1$	$\mathbb{P}_2$
$\{0\}$	0	0
$\{\frac{1}{2}\}$	0	1/2
$\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$	0	3/4
2^{-2i} for i in $\mathbb{N}$	1/3	1/3
$[0, \frac{1}{2}]$	$1 - e^{-\frac{1}{2}}$	1/2
$[\frac{1}{4}, \frac{1}{2}]$	$e^{-\frac{1}{4}} - e^{-\frac{1}{2}}$	3/4
$[\frac{1}{4}, \frac{1}{2}] \cup [\frac{3}{4}, 1]$	$e^{-\frac{1}{4}} + e^{-\frac{3}{4}} - e^{-\frac{1}{2}} - e^{-1}$	3/4

2. **Cumulative distribution functions**:

   (a) $F_{p,\theta}(x) = p + (1-p)(1 - exp(-\theta x)) => P(S_t = x) = F' = \theta(1-p)e^{-\theta x}$ for $x \geq 0 => P(S_1 = 0) = \theta(1-p)$.
   To simplify F, $F_{p,\theta}(x) = p + (1-p)(1 - e^{-\theta x}) = p + (1-p) - (1-p)e^{-\theta x} = 1 + (p-1)e^{-\theta x}$. Also, I assume $p \neq 1$ because when $p = 1$, F, P would both be 0 and this whole problem would be meaningless.
   For range distribution in (b),(c) and (d), it is asking for cumulative distributions. I can just use $F_{p,\theta}(x)$ to calculate the probability.
   (b) For b), $F = 0$ when $-\frac{1}{2} \leq S_1 < 0$; when $S_1 = 0, F(0) = 1 + p - 1 = p => \mathbb{P}(-\frac{1}{2} \leq S_1 \leq 0) = p$
   (c) $\mathbb{P}(-\frac{1}{2} \leq S_1 \leq \frac{1}{2}) = \mathbb{P}(0 \leq S_1 \leq \frac{1}{2}) = 1 + (p-1)e^{-\frac{\theta}{2}}$
   (d) $\mathbb{P}(0 \leq S_1 < \infty) = 1$
   (e) When $x > 0$, x can take on any real value so it is not a finite set, so this probability distribution has no point mass function.
   (f) The premise for a function f to be a density function is that f needs to be continuous. $P(0) = \theta(1-p)$ and $p \neq 1$ from (a) so it is not 0, but $\lim_{x \to 0^-} f(x) = 0$ so P is not continuous. As a result, this probability distribution have no density.

(g) From above, it's known this probability distribution has neither point mass function nor density so it cannot be a mixture of both.

(h) It is singular because it has no point mass or density, and it's not a mixture.

(i)
$$E[S_1] = \int_{-\infty}^{\infty} (x-K)^+ \theta(1-p)e^{-\theta x}dx = \int_K^{\infty} \theta x(1-p)e^{-\theta x}dx - \int_K^{\infty} \theta(1-p)Ke^{-\theta x}dx$$
$$= \frac{1-p}{\theta}\int_{\theta K}^{\infty} te^{-t}dt - \theta(1-p)K\int_K^{\infty} e^{-\theta x}dx = (1-p)(\frac{1}{\theta}(1+\theta K)e^{-\theta K} - Ke^{-\theta K}) = \frac{1-p}{\theta}e^{-\theta K}$$
.

3. **Law of Large Numbers**:

(a)

$$S_t = exp((\mu - \frac{\sigma^2}{2})t + \sigma B_t) => log(S_t) = (\mu - \frac{\sigma^2}{2})t + \sigma B_t => log(\frac{S_{t_n}}{S_{t_{n-1}}}) = (\mu - \frac{\sigma^2}{2})(t_n - t_{n-1}) + \sigma(B_{t_n} - B_{t_{n-1}})$$

Since $B_{t_2} - B_{t_1}, B_{t_3} - B_{t_2}, ..., B_{t_n} - B_{t_n-1}$ are independent for any $t_1 < t_2 < t_3 < ... < t_{n-1} < t_n; t_2 - t_1, ..., t_n - t_{n-1}$ are also independent. As a result, the two terms are both independent, which makes $log(\frac{S_{t_n}}{S_{t_{n-1}}})$ independent and $\frac{S_{t_n}}{S_{t_{n-1}}}$ as well.

(b)
$$E[log(\frac{S_{t+1}}{S_t})] = (\mu - \frac{\sigma^2}{2}) + \sigma E[B_{t+1} - B_t] = \mu - \frac{\sigma^2}{2}$$

(c)
$$\lim_{n\to\infty}\sum_{i=1}^{n} log(\frac{S_{i+1}}{S_i}) = n(\mu - \frac{1}{2}\sigma^2) + \sigma\sum_{i=1}^{n}(B_{i+1} - B_i)$$
$$=> \frac{\lim_{n\to\infty}\sum_{i=1}^{n} log(\frac{S_{i+1}}{S_i})}{n} = (\mu - \frac{1}{2}\sigma^2) + \lim_{n\to\infty}\sigma\frac{\sum_{i=1}^{n}(B_{i+1} - B_i)}{n}.$$

$(\mu - \frac{1}{2}\sigma^2)$ is a constant. If the above formula converges,

$$\lim_{n\to\infty}\sigma\frac{\sum_{i=1}^{n}(B_{i+1} - B_i)}{n}$$

must converge. According to the Law of Large Numbers, for this to converge, each factor in $\sum_{i=1}^{n}(B_{i+1} - B_i)$ must be IID. If they are IID, they can be added up as $n(B_2 - B_1)$, or written as $(B_{n+1} - B_n) + (B_n - B_{n-1}) + ... + (B_2 - B_1) = B_{n+1} - B_1 . B_1 \sim N(0,1)$, then all I need is to make the term with larger variance, $B_{n+1} \sim N(0, n+1)$ converge. The cumulative distribution function $F_{B_{n+1}}(x) = \Phi(\frac{x}{n+1})$, for $n -> \infty, F(x) = \Phi(0) = 0.5$, so the probability is 0.5, which indicates the original

$$\lim_{n\to\infty}\sum_{i=1}^{n} log(\frac{S_{i+1}}{S_i})$$

converges with a probability 0.5.
For each i from 1 to n, $E[B_{i+1} - B_i] = 0$. According to the Law of Large Numbers,

$$P(\lim_{n\to\infty}\sigma\frac{\sum_{i=1}^{n}(B_{i+1} - B_i)}{n} = E[B_{i+1} - B_i] = 0) = 1$$

, so
$$\frac{\lim_{n\to\infty}\sum_{i=1}^{n} log(\frac{S_{i+1}}{S_i})}{n} = (\mu - \frac{1}{2}\sigma^2).$$