1. Arbitrage Pricing Theory

In this exercise, you will be given the data to investigate a particular example of the general class of models discussed in class, known as Arbitrage Pricing Theory (APT) models.

The model is of the form

$$R_{t,t+1} = X_t f_{t,t+1} + \epsilon_{t,t+1}$$

where R and X are given in the accompanying files, and ϵ is assumed to follow a normal distribution $\epsilon \sim N(0, D)$ where $D = \operatorname{diag}(\sigma_1^2, \ldots, \sigma_n^2)$. This is in the same form as the linear APT models discussed in lecture.

There is one file per day; each file contains certain columns which basically give X_t. Each file also has a column containing R sampled over the interval $[t, t+1)$, and a few other useful columns. The columns are as follows:

(1) ID: a unique stock identifier that doesn't change through time
(2) NAME: the name of the corporation, for example EXXON MOBIL CORP
(3) BETA, MOMENTUM, SIZE, VOL: style factor loadings, which becomes columns of X
(4) GICCODE: an industry classification, which you can treat as a categorical factor, whose unique levels become columns of X
(5) XDATE: the date on which the factor loadings are sampled, usually one trading day before T
(6) T: the date t in $R_{t,t+1}$
(7) RDATE: the date $t+1$ in $R_{t,t+1}$
(8) R: the numerical value of $R_{t,t+1}$
(9) RESVOL: the value of σ_i for this stock, estimated using data knowable before t
(10) ADVP: average daily volume times price, in dollars
(11) tir, td: alpha factors, sampled before t, which may or may not help predict R

You can then regress the R against the X in the same file with no lookahead or other data problems. The matrix X_t will be $n \times k$ where n is the number of stocks, and k is the number of factors.

Problem 1. (a) *For each t, compute the OLS factor returns*

$$\hat{f}_{t+1} = (X_t' X_t)^{-1} X_t' R_{t,t+1}$$

and also the weighted-least-squares (WLS) versions, with weights $1/\sigma_i^2$. (Apply some outlier trimming to large values of $1/\sigma_i^2$, or equivalently, small values of σ_i^2.)

(b) *For each of the alpha factors, plot the cumulative-sum time series of the corresponding element of $\hat{f}$ with both methods (OLS and WLS) shown together for comparison purposes. Are they significantly different, statistically?*

(c) *Estimate the mean factor returns by the sample mean of the OLS and WLS estimates:*

$$\hat{f} := \frac{1}{T} \sum_{t=1}^{T} \hat{f}_{t+1}$$

For brevity, only print the two components of $\hat{f}$ corresponding to the alpha factors, but do this for both OLS and WLS.

Problem 2. *For each date t in the sample, define a composite alpha, denoted by the Greek letter α, as the simple arithmetic sum of TD and TIR.*

If F is the factor-return covariance matrix, let

$$\Sigma = XFX' + D$$

where to get D, use your EWMA estimates from a previous problem, and estimate F using an expanding-window standard sample covariance of $\hat{f}$.

The unconstrained Markowitz portfolio is defined as the solution

$$h^* = \underset{h}{\operatorname{argmax}}(h'\alpha - \frac{\kappa}{2} h'\Sigma h) \tag{1}$$

(a) *Write a function that can compute h^* without ever forming an $n \times n$ matrix in computer memory (all of the matrices you use must be $n \times k$ where k is the number of factors, or $k \times k$, or diagonal like D). So for example you never actually form the full Σ.*

(b) *Find κ so that, with constant κ, the average gross market value (GMV = long + short) of h^* is 500 million USD.*

(c) *With this choice of κ, using the Markowitz-optimization function you wrote previously, plot the time-series of cumulative P/L before costs, ie. the time-series*

$$\pi_{t+1} := \sum_{\tau \le t} (\Sigma_\tau^{-1} \alpha_\tau)' R_{\tau,\tau+1}$$

(d) *Also compute the cost of trading the above sequence, assuming that a trade of size $advp_i/100$ in stock i, (which is to say, one percent of average daily volume), moves the price by 10 basis points in market impact, and the impact function is linear in price. Therefore a trade of size δ dollars costs*

$$10bps \times \frac{\delta^2}{0.01 advp_i}$$

Compute the daily time series of total cost of the Markowitz trades, and plot it cumulatively alongside π_{t+1}.

(e) *Referring to the lecture on risk decompositions, compute and plot (as daily time series) the percent of the variance in these portfolios due to alpha factors, non-alpha factors, and idiosyncratic (non-factor) variance.*

2. Mixed-effects Models

Problem 3. (a) *Estimate the parameters of the factor-return process using a linear mixed-effects (LME) model as discussed by Prof. Benveniste. In particular, assume a model of the form*

$$r_s = X_s\beta + X_s b_s + \epsilon_s \tag{2}$$

$$b_s \sim N(0, \mathbf{\Theta}) \tag{3}$$

$$\epsilon_s \sim N(0, \sigma^2 I) \tag{4}$$

Write a function that can estimate $\hat{\beta}$, $\mathbf{\Theta}$, and σ^2 using the EM algorithm. Apply it to this data, but do not print out all of $\hat{\beta}$; for brevity, only print the two components of $\hat{\beta}$ corresponding to the alpha factors.

(b) *Are your estimates for $\hat{\beta}$ for the alpha factors different from the estimates $\hat{f}$ derived previously? If so, why do you think this is the case?*

Recall that under the model,

$$r_s \sim N(X_s\beta, X_s\mathbf{\Theta}X_s' + \sigma^2 I) \tag{5}$$

Hence a reasonable thing to use for α in a portfolio optimization might be $X_s\hat{\beta}$.

Define $\tilde{\beta}$ to be $\hat{\beta}$ with the non-alpha-factor components set to zero. Define

$$\tilde{\alpha} = X_s \tilde{\beta}$$

In words, this amounts to taking the subset of $X_s\hat{\beta}$ coming from the alpha factors; do not use industry, style etc. in forming the composite α.

(c) *With these tools, re-do the previous exercise in which you simulate the Markowitz portfolio, but use covariance matrix as in* (5) *and use the alpha just described and use $\tilde{\alpha}$ in place of α. Be sure to use rolling estimates of the LME parameters $\hat{\beta}, \hat{\mathbf{\Theta}}, \hat{\sigma}^2$ in a lookahead-free way. Plot the associated π_{t+1}^{LME} on the same graph alongside the π_{t+1} defined above.*

(d) *Finally, generate the same comparison of π_{t+1}^{LME} alongside the π_{t+1} where the optimizations are done with a long-only constraint, i.e.*

$$h^* = \underset{h}{\operatorname{argmax}}(h'\alpha - \frac{\kappa}{2}h'\Sigma h) \text{ subject to } h_i \geq 0 \ \forall \ i = 1, \dots, n$$

3. Reinforcement Learning

We consider a simple simulation of a market for a single security with trading frictions. This market admits arbitrage, and the security price is not a martingale.

Assume a tradable security with a strictly positive price process $p_t > 0$. Further suppose that there is some "equilibrium price" p_e such that $x_t = \log(p_t/p_e)$ has dynamics

$$dx_t = -\lambda x_t + \sigma\,\xi_t \tag{6}$$

where $\xi_t \sim N(0,1)$ and ξ_t, ξ_s are independent when $t \neq s$. This means that p_t tends to revert to its long-run equilibrium level p_e with mean-reversion rate λ, and is a standard discretization of the Ornstein-Uhlenbeck process. Take parameters $\lambda = \log(2)/H;$, where $H = 5$ is the half-life, $\sigma = 0.1$, and the equilibrium price is $p_e = 50$.

The trade size δn_t in a single interval is limited to at most K round lots, where a "round lot" is usually 100 shares (most institutional equity trades are in integer multiples of round lots). Also we assume a maximum position size of M round lots. Consequently, the space of possible trades, and also the action space, is

$$\mathcal{A} = \text{LotSize} \cdot \{-K, -K+1, \dots, K\}$$

The action space has cardinality $|\mathcal{A}| = 2K+1$. Letting $\mathcal{H}$ denote the possible values for the holding n_t, then similarly $\mathcal{H} = \{-M, -M+1, \dots, M\}$ with cardinality $|\mathcal{H}| = 2M+1$.

Take $K = 5$ and $M = 10$ and TickSize $= 0.1$. With the parameters as above, the probability that the price path ever exits the region $[0.1, 100]$ is small enough that no aspect of the problem depends on these bounds. Concretely, the space of possible prices is:

$$\mathcal{P} = \text{TickSize} \cdot \{1, 2, \dots, 1000\} \subset \mathbb{R}_+$$

We charge a spread cost of one tick size for any trade. If the bid-offer spread were equal to two ticks, then this fixed cost would correspond to the slippage incurred by an aggressive fill which crosses the spread to execute. Hence

$$\text{SpreadCost}(\delta n) = \text{TickSize} \cdot |\delta n| \tag{7}$$

We also assume that there is permanent price impact which has a linear functional form: each round lot traded is assumed to move the price one tick, hence leading to a dollar cost $|\delta n_t| \times \text{TickSize}/\text{LotSize}$ per share traded, for a total dollar cost for all shares

$$\text{ImpactCost}(\delta n) = (\delta n)^2 \times \text{TickSize}/\text{LotSize}. \tag{8}$$

The total cost is the sum of (7) and (8).

The state of the environment $s_t = (p_t, n_{t-1})$ will contain the security prices p_t, and the agent's position, in shares, coming into the period: n_{t-1}. Therefore the *state space* is the Cartesian product $\mathcal{S} = \mathcal{H} \times \mathcal{P}$. The agent then chooses an action $a_t = \delta n_t \in \mathcal{A}$ which changes the position to $n_t = n_{t-1} + \delta n_t$ and observes a profit/loss equal to $\delta v_t = n_t(p_{t+1} - p_t) - c_t$, and a reward

$$R_{t+1} = \delta v_{t+1} - 0.5\kappa\,(\delta v_{t+1})^2$$

Various parameters control the learning rate, discount rate, risk-aversion, etc:

$$\kappa = 10^{-4}, \gamma = 0.999, \alpha = 0.001, \epsilon = 0.1.$$

Problem 4. *Design and implement simple Q-learning system which learns to exploit the arbitrage in this system. Train it on simulated data, generated by simulating the process* (6)*. Simulate using the trained system to trade for 500,000 periods and estimate its annualized Sharpe ratio if each of the periods were a single day. The annualized Sharpe ratio of daily returns* r_t *is* $N \times mean(r_t)/\sqrt{N} var(r_t)$*, where* N *is the number of trading days per year, about 252 in the US equity market.*