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# The Rainbow Connection Number of a Flower $(C_m, K_n)$ Graph and a Flower $(C_3, F_n)$ Graph

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**Abstract**

An edge-colored graph $G$ is rainbow connected, if any two vertices are connected by a path whose edges have distinct colors. Such a path is called a rainbow path. The smallest number of colors needed in order to make $G$ rainbow connected is called the rainbow connection number of $G$, denoted by $rc(G)$. In this paper, we determine the rainbow connection number of a flower $(C_m, K_n)$ and a flower $(C_3, F_n)$ graph.



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*Keywords:* rainbow connection, rainbow path, cycle, complete, fan.

*2000 MSC:* 05C15, 05C76, 05C40

## 1. Introduction

All graphs considered in this paper are simple, finite, and connected. The concept of rainbow coloring was introduced by Chartrand *et al.* 2009[1] . Let $G = (V(G), E(G))$ be a nontrivial connected graph. For some positive integers $k$, we define a coloring $c : E(G) \to \{1, 2, ..., k\}$ of the edges of $G$ such that the adjacent edges can be colored the same. A path $P$ in $G$ is called a *rainbow path*, if no two edges of $P$ are colored the same. A rainbow path connecting two vertices $x$ and $y$ in $G$ is called *rainbow $x-y$ path*. A graph $G$ is said *rainbow-connected*, if for every two vertices $x$ and $y$ of $G$, there exists a rainbow $x-y$ path. In this case, the coloring $c$ is called a *rainbow $k$-coloring* of $G$. The minimum $k$ such that $G$ has a rainbow $k$-coloring is called the *rainbow connection number* of $G$. Clearly that $diam(G) \leq rc(G)$, where $diam(G)$ denotes the diameter of $G$.

Chartrand *et al.* 2009[1] determined the rainbow connection number for some classes of graphs. They showed that $rc(G) = 1$ if only if $G$ is a complete graph, and $rc(G) = |E(G)|$ if only if $G$ is a tree. They also determined the rainbow connection number of cycles and wheels. In 2013[5] , Syarizal *et al.* determined the rainbow connection number of fans and suns. Li and Sun 2009[2] obtained some results on bounds for rainbow connection number of product graphs, including Cartesian product, composition, and join of graphs. An overview about rainbow connection number can be

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found in a book of Li and Sun 2012[4] and a survey by Li *et al.* 20013[3] . In this paper, we determine the rainbow connection number of a flower $(C_m, K_n)$ and a flower $(C_3, F_n)$ graph. For simplifying, define $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$.

## 2. Main Results

**Definition 1.** *Let $C_m$ be a cycle on m vertices and $K_n$ be a complete graph on n vertices. A flower $(C_m, K_n)$ graph is a graph formed by taking one copy of $C_m$ and m copies of $K_n$ and grafting the i−th copy of $K_n$ at the i−th edge of $C_m$. A flower $(C_m, K_n)$ graph is denoted by $F_{C_m,K_n}$.*

Let $m$ and $n$ be two positive integers with $m \geq 3$ and $n \geq 3$. We define the vertex set and the edge set of $F_{C_m,K_n}$ as follows.

$$V(F_{C_m,K_n}) = \{u_i \mid i \in [1, m]\} \cup \{u_j^{i,i+1} \mid i \in [1, m], j \in [1, n-2], u_j^{m,m+1} = u_j^{m,1}\}.$$

$$\begin{aligned} E(F_{C_m,K_n}) = &\{u_iu_{i+1} \mid i \in [1, m], u_{m+1} = u_1\} \cup \{u_iu_j^{i,i+1} \mid i \in [1, m], j \in [1, n-2], u_j^{m,m+1} = u_j^{m,1}\} \cup \\ &\{u_{i+1}u_j^{i,i+1} \mid i \in [1, m], j \in [1, n-2], u_{m+1} = u_1, u_j^{m,m+1} = u_j^{m,1}\} \cup \\ &\{u_j^{i,i+1}u_s^{i,i+1} \mid i \in [1, m], j \in [1, n-2], s \in [1, n-2], j \neq s, u_j^{m,m+1} = u_j^{m,1}, u_s^{m,m+1} = u_s^{m,1}\}. \end{aligned}$$

For illustration, an $F_{C_8,K_5}$ graph is given in Figure 1.(*a*).

**Theorem 2.** *Let m and n be two positive integers with $m \geq 3$ and $n \geq 3$. Let $C_m$ be a cycle on m vertices and $K_n$ be a complete graph on n vertices. Then rainbow connection number of $F_{C_m,K_n}$ is*

$$rc(F_{C_m,K_n}) = \lfloor \tfrac{m}{2} \rfloor + 1.$$

*Proof.* We construct $c : E(F_{C_m,K_n}) \longrightarrow [1, \lfloor \frac{m}{2} \rfloor + 1]$. Since $diam(F_{C_m,K_n}) = \lfloor \frac{m}{2} \rfloor + 1$, clearly $rc(F_{C_m,K_n}) \geq \lfloor \frac{m}{2} \rfloor + 1$. Next, it remains to show that $rc(F_{C_m,K_n}) \leq \lfloor \frac{m}{2} \rfloor + 1$. We define a coloring by using $\lfloor \frac{m}{2} \rfloor + 1$ colors as follows. Let $i \in [1, \lfloor \frac{m}{2} \rfloor + 1]$, $j \in [1, n-2]$, $k \in [\lceil \frac{m}{2} \rceil + 2, m]$, $u_{m,m+1} = u_{m,1}$, and $u_{m+1}u_j^{m,m_1} = u_1u_j^{m,1}$.

$$(c(e)) = \begin{cases} i, & \text{if } e = u_iu_j^{i,i+1} \text{ or } e = u_{i+1}u_j^{i,i+1} \text{ or } e = u_iu_{i+1}; \\ \lceil \frac{m}{2} \rceil, & \text{if } e = u_mu_1 \text{ or } e = u_1u_j^{m,1}, \text{ and } m \text{ is odd}; \\ k - \lceil \frac{m}{2} \rceil, & \text{if } e = u_{k-1}u_k \text{ or } e = u_ku_j^{k-1,k}, k \geq 5, \text{ and } m \text{ is odd}; \\ k - \frac{m}{2}, & \text{if } e = u_ku_{k+1} \text{ or } e = u_{k+1}u_j^{k,k+1}, k \geq 5, \text{ and } m \text{ is even}; \\ 1, & \text{elsewhere.} \end{cases}$$

We consider any two vertices $x, y \in V(F_{C_m,K_n})$. It is obvious that there exists a rainbow $x - y$ path if $x$ is adjacent to $y$. We shall evaluate for $x$ that is not adjacent to $y$ in three following cases. Let $i \in [1, m]$, $l \in [1, m]$, $i < l, u_{m+1} = u_1, u_0 = u_m$, $j \in [1, n-2]$, and $k \in [1, n-2]$.

**Case 1.** $x = u_i$ and $y = u_l$ with $d(u_iu_l) = \lfloor \frac{m}{2} \rfloor$

A rainbow $x - y$ path is $u_i, u_{i+1}, \ldots, u_{l-1}, u_l$.

**Case 2.** $x = u_j^{i,i+1}$ and $y = u_k^{l,l+1}$ with $l - i \leq \lfloor \frac{m}{2} \rfloor + 1$

A rainbow $x - y$ path is $u_j^{i,i+1}, u_{i+1}, u_{i+2}, \ldots, u_l, u_j^{l,l+1}$.

**Case 3.** $x = u_j^{i,i+1}$ and $y = u_k^{l,l+1}$ with $l - i > \lfloor \frac{m}{2} \rfloor + 1$

A rainbow $x - y$ path is $u_j^{i,i+1}, u_i, u_{i-1}, \ldots, u_m, u_{m-1}, \ldots, u_l, u_j^{l,l+1}$.

For any two vertices $x, y \in V(F_{C_m,K_n})$, we can find a rainbow $x - y$ path. So, $c$ is a rainbow coloring. Therefore, $rc(F_{C_m,K_n}) = \lfloor \frac{m}{2} \rfloor + 1$, where $m \geq 3$ and $n \geq 3$. □

For illustration, a rainbow colored $F_{C_m,K_n}$ is given in Figure 1.(*b*).

**Definition 3.** *Let $C_3$ be a cycle on 3 vertices. Consider $F_n = P_n + \{x\}$, where $P_n = v_1, v_2, \ldots, v_n$ is a path on n vertices, and x is adjacent to $v_i$ for every $i \in [1, n]$. A flower $(C_3, F_n)$ is a graph formed by taking one copy of $C_3$ and 3 copies of $F_n$ and grafting the $(v_1, x) \in E(F_n)$ in the each edge of $C_3$. A flower $(C_3, F_n)$ is denoted by $F_{C_3,F_n}$.*

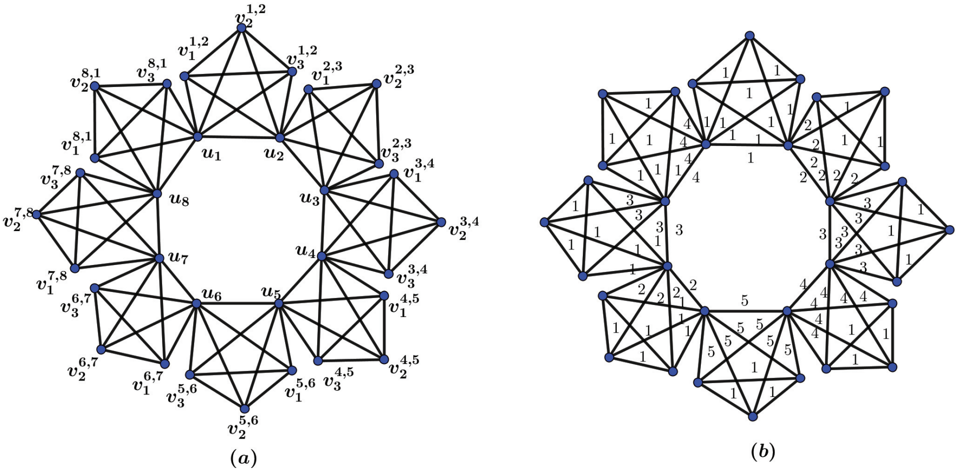


Fig. 1. ($a$). $F_{C_8,K_5}$ and (b). A rainbow 5−coloring on $F_{C_8,K_5}$

Let $n$ be a positive integer with $n \geq 3$. We define the vertex set and the edge set of $F_{C_3,F_n}$ as follows.

$$V(F_{C_3,F_n}) = \{u_i \mid i \in [1,3]\} \cup \{u_j^{i,i+1} \mid i \in [1,3],\, j \in [1,n-1],\, u_j^{3,4} = u_j^{3,1}\}.$$

$$E(F_{C_3,F_n}) = \{u_iu_{i+1} \mid i \in [1,3], u_4 = u_1\} \cup \{u_iu_j^{i,i+1} \mid i \in [1,3],\, j \in [1,n-1]\} \cup$$
$$\{u_j^{i,i+1}u_{j+1}^{i,i+1} \mid i \in [1,3],\, j \in [1,n-2],\, u_j^{3,4} = u_j^{3,1}\} \cup \{u_iu_1^{i,i+1} \mid i \in [1,3],\, u_j^{3,4} = u_j^{3,1}\}.$$

There are two non isomorphic graphs of $F_{C_3,F_n}$. For illustration, we refer to see Figure 2.

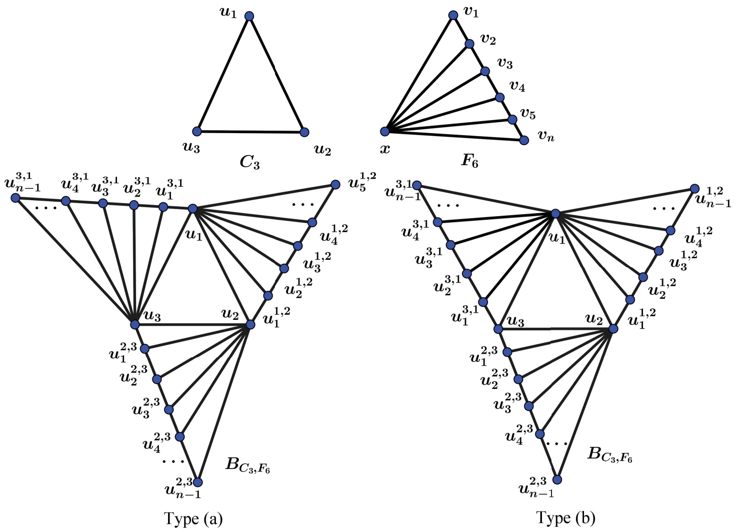


Fig. 2. Type ($a$) and Type ($b$) are two types of $F_{C_3,F_n}$ which are not isomorphic to each other.

**Theorem 4.** *Let $n$ be a positive integer with $n \geq 3$. Let $C_3$ be a cycle on 3 vertices and $F_n$ be a fan on $n+1$ vertices. Then rainbow connection number of $F_{C_3,F_n}$ is*

$$rc(F_{C_3,F_n}) = \begin{cases} 3, \text{ for } n \leq 6; \\ 4, \text{ for } n \geq 7. \end{cases}$$

*Proof.* Observe Figure 2. Since the prove for two types of $F_{C_3,F_n}$ graphs are similar, we only prove for Type (*a*). We shall consider two cases.

**Case 1.** $n \le 6$

Since $diam(F_{C_3,F_n}) = 3$, it follows that $rc(F_{C_3,F_n}) \ge 3$. Now we prove the lower bound that is $rc(F_{C_3,F_n}) \le 3$. Let $a$ and $p$ be two integers, to simplify in writing we define $ap \bmod a = a$. We construct a coloring by using 3 colors as follows.

$$(c(e)) = \begin{cases} 1, & \text{if } e = u_1u_2 \text{ or } e = u_3u_1^{2,3} \text{ or } e = u_3u_j^{3,1} \text{ for } j \in [1, n-1];\\ 2, & \text{if } e = u_2u_3 \text{ or } e = u_1u_1^{3,1} \text{ or } e = u_1u_j^{1,2} \text{ for } j \in [1, n-1];\\ 3, & \text{if } e = u_3u_1 \text{ or } e = u_2u_1^{1,2} \text{ or } e = u_2u_j^{2,3} \text{ for } j \in [1, n-1];\\ k \bmod 3, & \text{if } e = u_k^{i,i+1}u_{k+1}^{i,i+1} \text{ and } i \in [1,3], k \in [1, n-2], \text{ where } u_k^{3,4} = u_k^{3,1}. \end{cases}$$

We consider any two vertices $x, y \in V(F_{C_3,F_n})$. It is obvious that there exists a rainbow $x - y$ path if $x$ is adjacent to $y$. We shall evaluate for $x$ that is not adjacent to $y$ as follows. Let $i \in [1,3]$, $s \in [1,3]$, $i \ne s$, $j \in [1, n-1]$, $u_j^{3,4} = u_j^{3,1}$, $k \in [1, n-1]$, and $u_k^{3,4} = u_k^{3,1}$.

**Subcase 2.1.** $n \le 6$, $x = u_j^{i,i+1}$, and $y = u_k^{s,s+1}$ with $i \ne s$ and $j \le k$

There exists a rainbow $x - y$ path, namely $u_j^{i,i+1}, u_i, u_s, u_k^{s,s+1}$.

**Subcase 2.2.** $n \le 5$, $x = u_j^{i,i+1}$, and $y = u_k^{i,i+1}$ with $j < k$

There exists a rainbow $x - y$ path, namely $u_j^{i,i+1}, u_{j+1}^{i,i+1}, \ldots, u_k^{i,i+1}$.

**Subcase 2.3.** $n = 6$, $x = u_j^{i,i+1}$, and $y = u_k^{i,i+1}$ with $j < k$

There exists a rainbow $x - y$ path, namely $u_1^{i,i+1}, u_{i+1}, u_i, u_5^{i,i+1}$ for $j \ne 1$ and $k \ne 5$, and $u_j^{i,i+1}, u_{j+1}^{i,i+1}, \ldots, u_k^{i,i+1}$ for other $j$ and $k$.

So, $c$ is a rainbow coloring. Therefore, $rc(F_{C_3,F_n}) = 3$ where $n \ge 6$.

**Case 2.** $n \ge 7$

In order to show that $rc(F_{C_3,F_n}) \le 4$, we construct a coloring $c : E(F_{C_3,F_n}) \longrightarrow [1, 4]$ defined as follows.

$$(c(e)) = \begin{cases} 1, \text{ if } e = u_iu_j^{i,i+1} \text{ for } i \in [1,3], j \in [1, n-1], \text{ and } j = 1 \bmod 2;\\ 2, \text{ if } e = u_iu_j^{i,i+1} \; i \in [1,3], j \in [2, n-1], \text{ and } j = 0 \bmod 2;\\ 3, \text{ if } e = u_iu_{i+1};\\ 4, \text{ elsewhere.} \end{cases}$$

We consider any two vertices $x, y \in V(F_{C_3,F_n})$. It is obvious that there exists a rainbow $x - y$ path if $x$ is adjacent to $y$. We shall evaluate for $x$ that is not adjacent to $y$ in two following subcases. Let $i \in [1,3]$, $s \in [1,3]$, $i \ne s$, $j \in [1, n-1]$, $u_j^{3,4} = u_j^{3,1}$, $k \in [1, n-1]$, $j \le k$, $u_k^{3,4} = u_k^{3,1}$, and $u_0^{s,s+1} = u_n^{s,s+1}$.

**Subcase 2.1.** $x = u_j^{i,i+1}$ and $y = u_k^{i,i+1}$

There exists a rainbow $x - y$ path, namely $u_j^{i,i+1}, u_i, u_k^{i,i+1}$ if $j$ and $k$ have a same parity, and $u_j^{i,i+1}, u_i, u_{k-1}^{i,i+1}, u_k^{i,i+1}$ if $j$ and $k$ have different parity.

**Subcase 2.2.** $x = u_j^{i,i+1}$ and $y = u_k^{s,s+1}$

There exists a rainbow $x - y$ path, namely $u_j^{i,i+1}, u_i, u_s, u_{k-1}^{s,s+1}, u_k^{s,s+1}$ if $j$ and $k$ have a same parity, and $u_j^{i,i+1}, u_i, u_s, u_k^{s,s+1}$ if $j$ and $k$ have different parity.

So, $c$ is a rainbow coloring. Thus, $rc(F_{C_3,F_n}) \le 4$ for $n \ge 7$.

Next, it remains to show that $rc(F_{C_3,F_n}) \ge 4$. Assume to the contrary $rc(F_{C_3,F_n}) \le 3$. Let $c'$ be a rainbow 3−coloring on $F_{C_3,F_n}$. Without loss of generality, assume that $c'(u_1, u_3^{1,2}) = 1$. Since $u_3^{1,2}, u_1, u_2, u_j^{2,3}$ is the only $u_3^{1,2} - u_j^{2,3}$ path of length three in $F_{C_3,F_n}$ for every $j \in [2, 7]$, it forces $c'(u_1, u_2) = 2$ and $c'(u_2, u_j^{2,3}) = 3$ or vice versa. But there is no rainbow $u_2^{2,3} - u_7^{2,3}$ path. We get a contradiction. Therefore, $rc(F_{C_3,F_n}) \ge 4$. Hence,

$$rc(F_{C_3,F_n}) = \begin{cases} 3, \text{ for } n \le 6;\\ 4, \text{ for } n \ge 7. \end{cases}$$

□

For illustration, rainbow 3−colorings on $F_{C_3,F_5}$ and on $F_{C_3,F_6}$, and a rainbow 4−coloring on $F_{C_3,F_8}$ are given in Figure 3 and Figure 4, respectively.

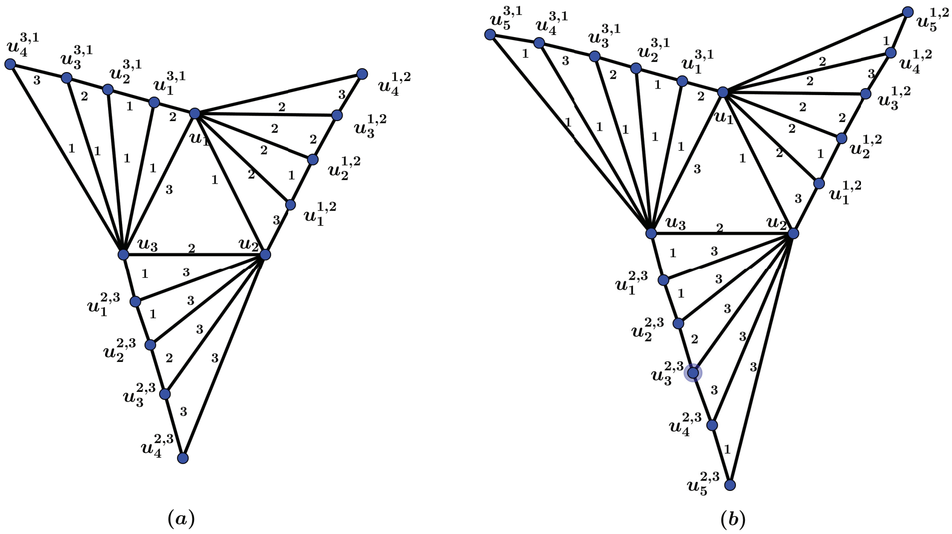


Fig. 3. (*a*). A rainbow 3−coloring on $F_{C_3,F_5}$ and (*b*). A rainbow 3−coloring on $F_{C_3,F_6}$

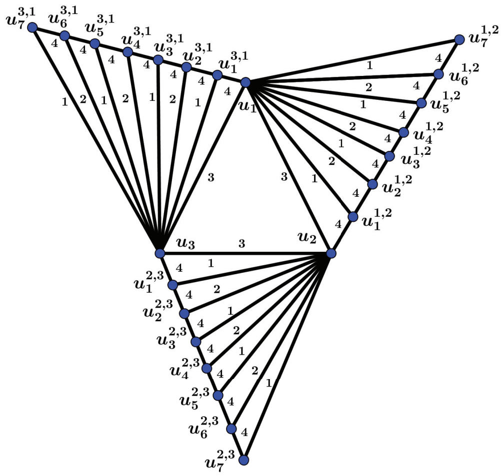


Fig. 4. A rainbow 4−coloring on $F_{C_3,F_8}$

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