

Hedge P&L Analysis

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Overview

Hedge P&L
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P&L of
Hedged
Options

Effect of
Different
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Strategies

Effect of
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- 1 P&L of Hedged Options
- 2 Effect of Different Hedging Strategies
- 3 Effect of Discrete Hedging
- 4 Effect of Transaction Cost

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We consider the P&L of a hedged European call option. Let

- $0 = t_0 < \dots < t_n = T$ be equally spaced time intervals where T is the expiry of a derivative, $\delta t = t_i - t_{i-1}$
- S_i : stock price at time t_i
- $C_i = C(t_i, S_i)$: price of call option at time t_i when the stock price is S_i
- $\Delta_i = \Delta(t_i, S_i)$: hedge ratio for time t_i when stock price S_i . It is the number of shares of stock we short.

Note that Δ_i is an arbitrary hedging strategy. It does not have to be the BSM hedge ratio.

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Suppose we start with a call option. At time t_0 , we short Δ_0 shares of stocks and deposit the cash amount $\Delta_0 S_0$ into a bank account. Our portfolio at time t_0 consists of

- 1 A call option
- 2 $-\Delta_0$ shares of stock
- 3 $\Delta_0 S_0$ cash

The total value is C_0 .

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At time t_1 , before we re-hedge our positions. The value of assets in our portfolio is

1 Call option: C_1

2 Stock: $-\Delta_0 S_1$

3 Cash: $\Delta_0 S_0 e^{r\delta t}$

The total value is $C_1 - \Delta_0 S_1 + \Delta_0 S_0 e^{r\delta t}$.

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At time t_1 , we short another $\Delta_1 - \Delta_0$ stocks to re-hedge. Our portfolio is

1 A call option

2 Stock: $-\Delta_1$

3 Cash: $\Delta_0 S_0 e^{r\delta t} + (\Delta_1 - \Delta_0) S_1$

The total value is $C_1 - \Delta_1 S_1 + \Delta_0 S_0 e^{r\delta t} + (\Delta_1 - \Delta_0) S_1$.

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At time t_2 , before we re-hedge, the value of our assets is

1 Call option: C_2

2 Stock: $-\Delta_1 S_2$

3 Cash: $\Delta_0 S_0 e^{2r\delta t} + (\Delta_1 - \Delta_0) S_1 e^{r\delta t}$

The total value is $C_2 - \Delta_1 S_2 + \Delta_0 S_0 e^{2r\delta t} + (\Delta_1 - \Delta_0) S_1 e^{r\delta t}$.

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At time t_2 , we short $\Delta_2 - \Delta_1$ to re-hedge, our portfolio is

1 A call option

2 Stock: $-\Delta_2$

3 Cash: $\Delta_0 S_0 e^{2r\delta t} + (\Delta_1 - \Delta_0) S_1 e^{r\delta t} + (\Delta_2 - \Delta_1) S_2$

The total value is

$$C_2 - \Delta_2 S_2 + \Delta_0 S_0 e^{2r\delta t} + (\Delta_1 - \Delta_0) S_1 e^{r\delta t} + (\Delta_2 - \Delta_1) S_2.$$

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Continue this process, at time t_n , after re-hedge our portfolio is

1 A call option

2 Stock: $-\Delta_n$

3 Cash:

$$\Delta_0 S_0 e^{nr\delta t} + (\Delta_1 - \Delta_0) S_1 e^{(n-1)r\delta t} + \dots + (\Delta_n - \Delta_{n-1}) S_n$$

The total value is

$$C_n - \Delta_n S_n + \Delta_0 S_0 e^{nr\delta t} + (\Delta_1 - \Delta_0) S_1 e^{(n-1)r\delta t} + \dots + (\Delta_n - \Delta_{n-1}) S_n$$

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In the limit when $\delta t \rightarrow 0$, we can replace the summation with integral

$$C_T - \Delta_T S_T + \Delta_0 S_0 e^{rT} + \int_0^T e^{r(T-t)} S(t) [d\Delta(t)]_b$$

Note that for an integrand

$$\int_0^T \alpha(t, \omega) [d\Delta(t)]_b = \lim \sum_i \alpha(t_{i+1}, \omega) (\Delta(t_{i+1}) - \Delta(t_i))$$

is the backward Ito's Integral.

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Denote

$$V = e^{-rT} \left(C_T - \Delta_T S_T + \Delta_0 S_0 e^{rT} + \int_0^T e^{r(T-t)} S(t) [d\Delta(t)]_b \right)$$

- V is the value of a hedged option at time 0
- In general V is random and path-dependent
- In BSM model, if we choose hedge ratio $\Delta(t)$ to be BSM delta, then V is deterministic and equal to the price of call option at time $t = 0$.

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From

$$\begin{aligned} & d(e^{r(T-t)}S(t)\Delta(t)) \\ = & -re^{r(T-t)}S(t)\Delta(t)dt + e^{r(T-t)}\Delta(t)dS(t) \\ & + e^{r(T-t)}S(t)[d\Delta(t)]_b \end{aligned}$$

we can solve

$$\begin{aligned} \int_0^T e^{r(T-t)}S(t)[d\Delta(t)]_b &= S_T\Delta_T - e^{rT}S_0\Delta_0 \\ &- \int_0^T e^{r(T-t)}\Delta(t)(dS(t) - rS(t)dt) \end{aligned}$$

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Substituting this into V we have

$$V = e^{-rT} C_T - \int_0^T e^{-rt} \Delta(t) (dS(t) - rS(t)dt)$$

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If we work with BSM model

$$dS(t) - rS(t)dt = \sigma S(t)dW(t)$$

In this case

$$V = e^{-rT} C_T - \int_0^T e^{-rt} \Delta(t) \sigma S(t) dW(t)$$

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- Note that

$$\mathbb{E}[V] = e^{-rt} \mathbb{E}[C_T]$$

which is exactly risk neutral pricing formula.

- If the stock price follows Geometric Brownian Motion with drift r , the expectation is irrelevant to hedging strategy $\Delta(t)$. Even if we don't hedge, the expected value is still the same.

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Assume

- The stock price follows

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma_r dW(t)$$

in the real world measure.

- Market is pricing option with σ_i
- We predict the realized volatility σ_r will be greater than σ_i .

How do we make a profit from this misprice in the market?

⇒ We buy the option from market and delta hedge it according to BSM hedge ratio $\Delta(\sigma_r)$.

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Suppose our prediction is correct and the realized volatility turns out to be σ_r . We would expect to see P&L

$$V^r(0) - V^i(0)$$

- $V^r(0)$ is the price of option based on σ_r . That what the fair value we think should be.
- $V^i(0)$ is the price of option based on σ_i . That is what market is pricing.
- We use superscript i and r to denote the values based on realized volatility σ_r and implied volatility σ_i respectively.

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Is our expectation justified?

If it is true, how is this future known profit realized over time?

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Suppose at time t , our portfolio consists of

- Long an option whose market value is $V^i(t)$
- Short $\Delta^r(t)$ shares of stock
- Cash $\Delta^r(t)S - V^i(t)$

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P&L from t to $t + dt$ is

$$\begin{aligned} & dP\&L(t) \\ = & dV^i(t) - \Delta^r(t)dS + r(\Delta^r(t)S - V^i(t))dt \end{aligned}$$

The total value of portfolio is $dP\&L(t)$ and its PV is $e^{-rt}dP\&L(t)$.

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After re-hedging at $t + dt$, our portfolio

- Long a option whose market value is $V^i(t + dt)$
- Short $\Delta^r(t + dt)$ shares of stock
- Cash $\Delta^r(t + dt)S - V^i(t + dt) + dP\&L(t)$

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If we exclude the cash amount $dP\&L(t)$ from our portfolio, the value of our portfolio will be zero at time $t + dt$.

For this reduced portfolio, P&L from $t + dt$ to $t + 2dt$ is

$$\begin{aligned} & dP\&L(t + dt) \\ = & dV^i(t + dt) - \Delta^r(t + dt)dS \\ & + r(\Delta^r(t + dt)S - V^i(t + dt))dt \end{aligned}$$

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The total P&L from $t + dt$ to $t + 2dt$ is

$$dP\&L(t + dt) + dP\&L(t)(e^{rdt} - 1)$$

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The 2-period P&L from t to $t + 2dt$ is

$$dP\&L(t + dt) + dP\&L(t)e^{rdt}$$

Its PV is

$$e^{-r(t+dt)}dP\&L(t + dt) + e^{-rt}dP\&L(t)$$

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Hence the PV of P&L from 0 to T is

$$P\&L = \int_0^T e^{-rt} dP\&L(t)$$

where

$$dP\&L(t) = dV^i(t) - \Delta^r(t)dS + r(\Delta^r(t)S - V^i(t))dt$$

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From the derivation of BSM equation, we know that

$$dV^r(t) - \Delta^r(t)dS = r(V^r(t) - \Delta^r(t)S)dt$$

Using this to simplify $dP\&L(t)$, we have

$$\begin{aligned} dP\&L(t) &= dV^i(t) - dV^r(t) - r(V^i(t) - V^r(t))dt \\ &= e^{rt}d[e^{-rt}(V^i(t) - V^r(t))] \end{aligned}$$

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The P&L formula becomes

$$\begin{aligned} P\&L &= \int_0^T e^{-rt} dP\&L(t) \\ &= \int_0^T d[e^{-rt} (V^i(t) - V^r(t))] \\ &= [V^i(t) - V^r(t)]_{t=0}^T \\ &= V^r(0) - V^i(0) \end{aligned}$$

since $V^r(T) = V^i(T)$.

This is what we expected to see.

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If we know the (future) realized volatility and hedge continuously, the final P&L at the expiry is deterministic and equal to the difference between the value of option based on realized volatility and implied volatility.

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How does the P&L vary over the whole hedging process?

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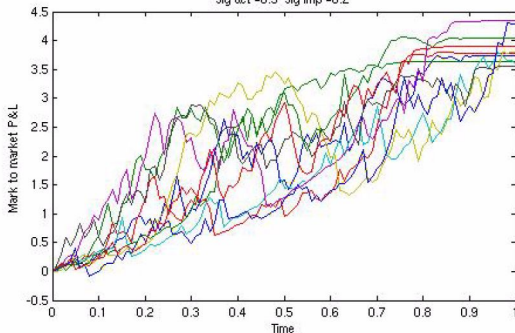
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P&L for a delta-hedged option marked to market and hedged at realized volatility
theoretical P&L = 3.7807
implied call rice = 10.4506 actual call price = 14.2313
nstep=100 npath=10
stock price = 100 time to exp = 1 strike = 100
 $r = 0.05$ $\mu = 0.1$ $D = 0$
sig act = 0.3 sig imp = 0.2



Source: Derman (2008)

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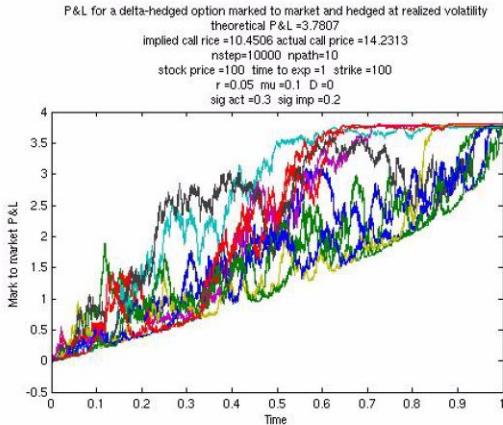
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Source: Derman (2008)

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It appears that

- P&L shows some degree of randomness.
- P&L converges eventually as time approaches to expiry
- For sufficiently large number of simulation steps, P&L converges to theoretical value $V^r(0) - V^i(0)$.
- P&L seems to have a time-dependent lower bound.

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To see that P&L is random, we notice that the P&L from time 0 to t is

$$(V^i(t) - V^r(t)) - (V^i(0) - V^r(0))$$

The first term $V^i(t) - V^r(t)$ depends on the stock price $S(t)$ which is random.

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To better understand how P&L changes, let's have a closer look at

$$dP\&L(t) = dV^i - \Delta^r dS + r (\Delta^r S - V^i) dt$$

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Applying Ito's Lemma to $V^i(t)$, we have

$$dV^i(t) = \Theta^i dt + \Delta^i dS + \frac{1}{2} \Gamma^i S^2 \sigma_r^2 dt$$

where Θ^i , Δ^i and Γ^i are BSM theta, delta and gamma respectively based on implied volatility.

Note that the volatility in last term is σ_r since that is the real volatility for the return of stock price.

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The P&L from t to $t + dt$ is

$$\begin{aligned}dP\&L(t) &= dV^i - \Delta^r dS + r(\Delta^r S - V^i) dt \\&= \left[\Theta^i dt + \Delta^i dS + \frac{1}{2} \Gamma^i S^2 \sigma_r^2 dt \right] \\&\quad - \Delta^r dS + r(\Delta^r S - V^i) dt \\&= \left[\Theta^i + \frac{1}{2} \Gamma^i S^2 \sigma_r^2 \right] dt + (\Delta^i - \Delta^r) dS \\&\quad + r(\Delta^r S - V^i) dt\end{aligned}$$

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From BSM equation, we have

$$\Theta^i + \Delta^i rS + \frac{1}{2} \Gamma^i S^2 \sigma_i^2 = rV^i$$

Hence

$$\begin{aligned} dP\&L(t) &= \frac{1}{2} \Gamma^i S^2 (\sigma_r^2 - \sigma_i^2) dt \\ &+ (\Delta^i - \Delta^r) [(\mu - r)Sdt + \sigma_r SdW] \end{aligned}$$

Note that

- $dP\&L(t)$ is random even in the infinitesimal time interval $(t, t + dt)$ unless $\Delta^i = \Delta^r$.
- When $\Delta^i = \Delta^r$, we get the P&L equation we derived in last class.

Hedging with Implied Volatility

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Realized volatility is not known, in practice, traders usually hedge with implied volatility. How does P&L look like in this case?

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If we hedge with implied volatility, i.e., use hedge ratio Δ_i ,
P&L from t to $t + dt$ is

$$dP\&L(t) = \frac{1}{2} \Gamma^i S^2 (\sigma_r^2 - \sigma_i^2) dt$$

\Rightarrow Non-random in the infinitesimal interval.

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PV of P&L from time 0 to T is

$$P\&L = \frac{1}{2} \int_0^T e^{-rt} \Gamma^i S^2 (\sigma_r^2 - \sigma_i^2) dt$$

- Γ^i is random as it depends on stock price.
- In contrast to delta hedging with realized volatility, P&L is not deterministic and highly path-dependent!
- When option is deep ITM or OTM, Γ^i is small, in this case P&L is insensitive to volatility.

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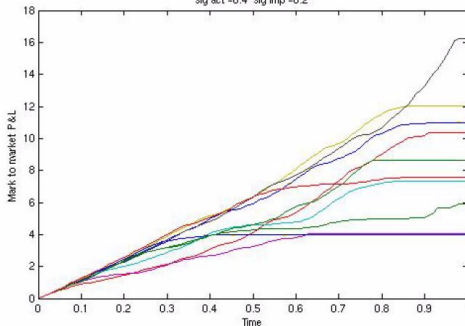
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P&L for a delta-hedged option marked to market and hedged at implied volatility
theoretical price = 14.0043
nstep=100 npath=10
stock price = 100 time to exp = 1 strike = 110
r = 0.05 mu = 0.1 D = 0
sig act = 0.4 sig imp = 0.2



Source: Derman (2008)

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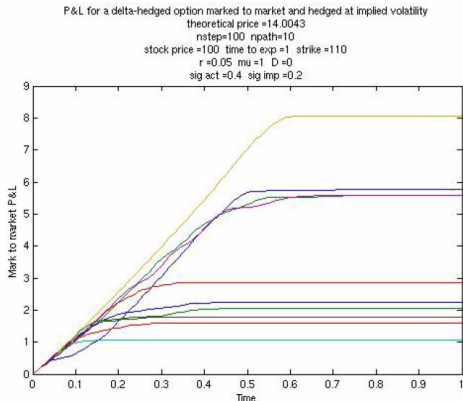
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Source: Derman (2008)

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Our previous analysis is based on continuous hedging. In reality traders hedge

- at equally spaced time intervals
- when change in risks exceeds a threshold

How does hedging frequency impact P&L?

Monte Carlo Simulations

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We assume

- BSM model
- Hedge at regular time intervals.
- Hedge with implied volatility σ_i

Monte Carlo Simulations

Case 1: $\sigma_i = \sigma_r$, # of steps = 21

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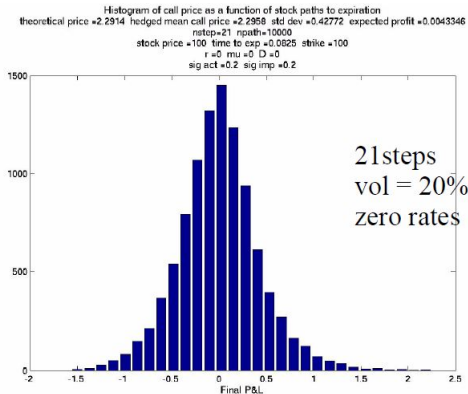
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Source: Derman (2008)

Monte Carlo Simulations

Hedge P&L
Analysis

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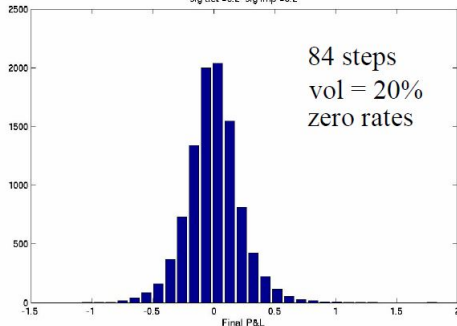
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Case 2 : $\sigma_i = \sigma_r$, # of steps = 84

Histogram of call price as a function of stock paths to expiration
theoretical price =2.2914 hedged mean call price =2.2918 std dev =0.21644 expected profit =0.00035751
nstep=84 npath=10000
stock price =100 time to exp =0.0625 strike =100
r =0 mu =0 D =0
sig act =0.2 sig imp =0.2



Source: Derman (2008)

Monte Carlo Simulations

Case 3: $\sigma_i = 40\%$, $\sigma_r = 20\%$, # of steps = 21

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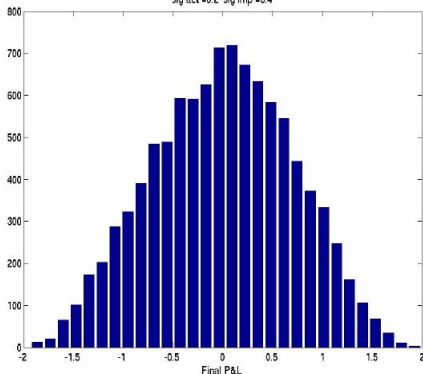
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Histogram of call price as a function of stock paths to expiration
theoretical price = 2.2914 hedged mean call price = 2.2973 std dev = 0.70614 expected profit = 0.0056942
nstep=21 npath=10000
stock price = 100 time to exp = 0.0825 strike = 100
r = 0 mu = 0 D = 0
sig act = 0.2 sig imp = 0.4



Source: Derman (2008)

Monte Carlo Simulations

Case 4 : $\sigma_i = 40\%$, $\sigma_r = 20\%$, # of steps = 84

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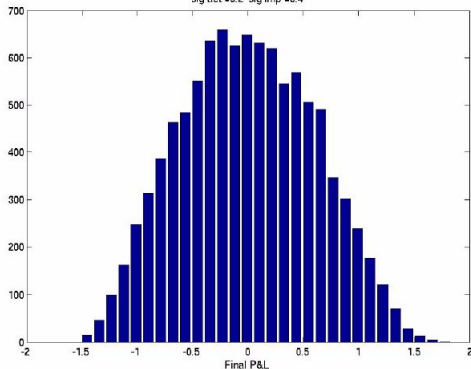
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Histogram of call price as a function of stock paths to expiration
theoretical price =2.2914 hedged mean call price =2.2942 std dev =0.60714 expected profit =0.0028089
nstep=84 npath=10000
stock price =100 time to exp =0.0825 strike =100
r =0 mu =0 D =0
sig act =0.2 sig imp =0.4



Source: Derman (2008)

Monte Carlo Simulations

Case 5: $\mu = 20\%$, $r = 0\%$, $\sigma_i = \sigma_r = 20\%$, # of steps = 21

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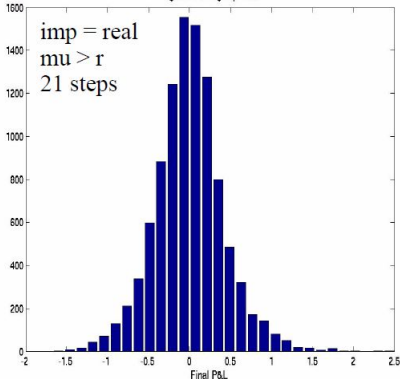
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Histogram of call price as a function of stock paths to expiration
theoretical price = 2.2914 hedged mean call price = 2.2957 std dev = 0.42195 expected profit = 0.0042507
nstep=21 npath=10000
stock price = 100 time to exp = 0.0825 strike = 100
r = 0 mu = 0.2 D = 0
sig act = 0.2 sig imp = 0.2



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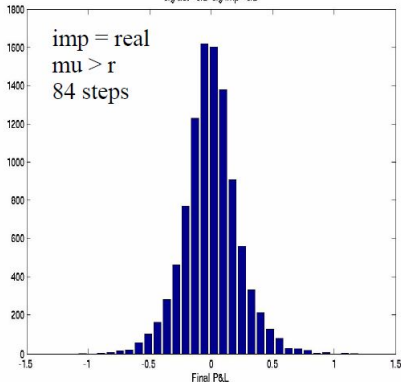
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Case 6 : $\mu = 20\%$, $r = 0\%$, $\sigma_i = \sigma_r = 20\%$, # of steps = 84

Histogram of call price as a function of stock paths to expiration
theoretical price = 2.2914 hedged mean call price = 2.2698 std dev = 0.21892 expected profit = 0.0016632
nstep=84 npaths=10000
stock price = 100 time to exp = 0.0825 strike = 100
r = 0 mu = 0.2 D = 0
sig act = 0.2 sig imp = 0.2



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Observations

- When $\sigma_i = \sigma_r$, quadruple the number of hedging \Rightarrow halve the standard deviation.
- When $\sigma_i \neq \sigma_r$, increasing the number of hedging does not seem to reduce the standard deviation much.

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Hedging portfolio

- Long one call option
- Short $\frac{\partial C}{\partial S}$ shares of stock

Portfolio value

$$\pi = C - \frac{\partial C}{\partial S} S$$

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Between t to $t + \Delta t$, if fully hedged, P&L will be

$$r\pi\Delta t$$

Discrete hedged P&L using realized volatility is

$$C(t + \Delta t, S + \Delta S) - \frac{\partial C}{\partial S} \Delta S - C$$

Hedging error is

$$\begin{aligned} HE &= C(t + \Delta t, S + \Delta S) - \frac{\partial C}{\partial S} \Delta S - C - r\pi\Delta t \\ &= C(t + \Delta t, S + \Delta S) - \frac{\partial C}{\partial S} \Delta S - C - r \left(C - \frac{\partial C}{\partial S} S \right) \Delta t \end{aligned}$$

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Using Taylor's expansion, we have

$$\begin{aligned} HE &= C(t + \Delta t, S + \Delta S) - \frac{\partial C}{\partial S} \Delta S - C - r\pi \Delta t \\ &\approx \left[C + \frac{\partial C}{\partial t} \Delta t + \frac{\partial C}{\partial S} \Delta S + \frac{1}{2} \Gamma (\Delta S)^2 \right] \\ &\quad - \frac{\partial C}{\partial S} \Delta S - C - r \left(C - \frac{\partial C}{\partial S} S \right) \Delta t \\ &= \left[\frac{\partial C}{\partial t} - r \left(C - \frac{\partial C}{\partial S} S \right) \right] \Delta t + \frac{1}{2} \Gamma (\Delta S)^2 \\ &\approx \left[\frac{\partial C}{\partial t} - r \left(C - \frac{\partial C}{\partial S} S \right) \right] \Delta t + \frac{1}{2} \Gamma \sigma^2 S^2 (\Delta W(t))^2 \end{aligned}$$

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From BSM equation,

$$\frac{\partial C}{\partial t} - r \left(C - \frac{\partial C}{\partial S} S \right) = -\frac{1}{2} \Gamma \sigma^2 S^2$$

hence

$$\begin{aligned} HE &= \frac{1}{2} \Gamma \sigma^2 S^2 ((\Delta W(t))^2 - \Delta t) \\ &= \frac{1}{2} \Gamma \sigma^2 S^2 (Z^2 - 1) \Delta t \end{aligned}$$

where

$$Z = \frac{\Delta W(t)}{\sqrt{\Delta t}}$$

is a standard normal distribution and independent of Γ and S .

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Let $t_0 < t_1 < \dots < t_n$ be equally spaced time intervals for hedging, total hedging error is

$$HE = \frac{1}{2} \frac{\sigma^2 T}{n} \sum_i \Gamma_i S_i^2 (Z_i^2 - 1)$$

where Z_i are i.i.d. standard normal and independent of $\Gamma_i S_i^2$.

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Note that

$$\begin{aligned}\mathbb{E}[HE] &= \frac{1}{2} \frac{\sigma^2 T}{n} \sum_i \mathbb{E} [\Gamma_i S_i^2 (Z_i^2 - 1)] \\ &= \frac{1}{2} \frac{\sigma^2 T}{n} \sum_i \mathbb{E} [\Gamma_i S_i^2] \mathbb{E} [(Z_i^2 - 1)] \\ &= 0\end{aligned}$$

The variance of hedging error is

$$\begin{aligned}\sigma_{HE}^2 &= \mathbb{E}[HE^2] \\ &= \frac{1}{4} \frac{\sigma^4 T^2}{n^2} \sum_i \mathbb{E} (\Gamma_i S_i^2)^2 \mathbb{E} (Z_i^2 - 1)^2 \\ &= \frac{1}{2} \frac{\sigma^4 T^2}{n^2} \sum_i \mathbb{E} (\Gamma_i S_i^2)^2\end{aligned}$$

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We can prove that (HW)

$$\mathbb{E} (\Gamma_i S_i^2)^2 \approx \Gamma_0^2 S_0^4 \sqrt{\frac{T^2}{T^2 - t_i^2}}$$

Hence, the variance of hedging error is

$$\sigma_{HE}^2 \approx \frac{1}{2} \frac{\sigma^4 T^2}{n^2} \Gamma_0^2 S_0^4 \sum_i \sqrt{\frac{T^2}{T^2 - t_i^2}}$$

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The variance of hedging error is

$$\begin{aligned}\sigma_{HE}^2 &\approx \frac{1}{2} \frac{\sigma^4 T^2}{n^2} \Gamma_0^2 S_0^4 \sum_i \sqrt{\frac{T^2}{T^2 - t_i^2}} \\ &= \frac{1}{2} \frac{\sigma^4 T}{n} \Gamma_0^2 S_0^4 \sum_i \sqrt{\frac{T^2}{T^2 - t_i^2}} \Delta t_i \\ &\approx \frac{1}{2} \frac{\sigma^4 T}{n} \Gamma_0^2 S_0^4 \int_0^T \sqrt{\frac{T^2}{T^2 - t^2}} dt \\ &= \frac{\pi}{4} \frac{\sigma^4 T^2}{n} \Gamma_0^2 S_0^4\end{aligned}$$

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The standard deviation of hedging error is

$$\sigma_{HE} \approx \sqrt{\frac{\pi}{4}} \frac{\sigma^2 T}{\sqrt{n}} \Gamma_0 S_0^2$$

Using the fact that

$$\Gamma_0 S_0^2 = \frac{1}{\sigma T} \frac{\partial C}{\partial \sigma}$$

we have

$$\sigma_{HE} \approx \sqrt{\frac{\pi}{4}} \frac{\sigma}{\sqrt{n}} \frac{\partial C}{\partial \sigma}$$

Quadruple the number of steps \Rightarrow halve the standard deviation

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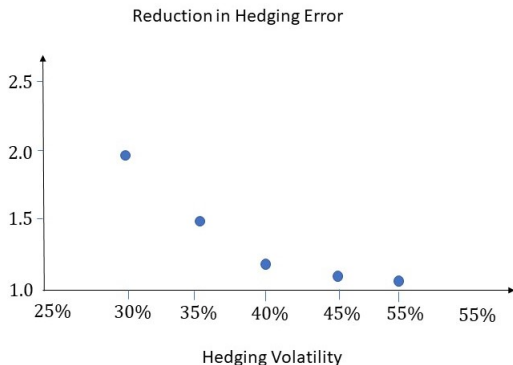
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In the analysis we assume hedging with realized volatility.
What if hedging volatility is not equal to realized volatility?

Discrete Hedging Error

Assume realized volatility is 30%, the following graph shows the reduction of hedging error when we increase the number of hedging from 100 to 400 using MC.



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Assuming BSM model, we have

- If we estimate future realized volatility correctly and hedge continuously, the P&L will be the value of option.
- If we hedge discretely at realized volatility, P&L is random (but centered around BSM value). The standard deviation is proportional to $\frac{1}{\sqrt{n}}$.
- If the implied volatility is not equal to realized volatility and we hedge continuously at implied volatility, P&L is path-dependent and unpredictable.
- If we hedge discretely at implied volatility, P&L is path-dependent and unpredictable. There is also a random component due to it.

Modeling Transaction Cost

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Assume the transaction cost is a fraction k of share prices.

If we long (or short) N shares of stock, the transaction cost will be $kS|N|$.

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Suppose we hedge with BSM hedge ratio $\Delta(t, S(t))$. From t to $t + dt$, the additional shares of stock we need to buy (or sell) is

$$\Delta(t + dt, S(t + dt)) - \Delta(t, S(t))$$

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$$\begin{aligned} & \Delta(t + dt, S(t + dt)) - \Delta(t, S(t)) \\ &= \frac{\partial \Delta}{\partial S} dS + O(dt) \\ &= \frac{\partial \Delta}{\partial S} \sigma S dW(t) + O(dt) \\ &= \frac{\partial \Delta}{\partial S} \sigma S \sqrt{dt} Z + O(dt) \\ &\approx \frac{\partial^2 V}{\partial S^2} \sigma S \sqrt{dt} Z \end{aligned}$$

where

$$Z = \frac{dW(t)}{\sqrt{dt}}$$

is a standard normal distribution.

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The transaction cost from t to $t + dt$ with delta hedging is

$$k\sigma S^2\sqrt{dt} \left| \frac{\partial^2 V}{\partial S^2} \right| |Z|$$

The (conditional) expectation of transaction cost is

$$k\sigma S^2\sqrt{dt} \left| \frac{\partial^2 V}{\partial S^2} \right| \mathbb{E}[|Z| \mid \mathcal{F}_t] = k\sigma S^2\sqrt{dt} \left| \frac{\partial^2 V}{\partial S^2} \right| \sqrt{\frac{2}{\pi}}$$

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The transaction cost at an infinitesimal time interval is $O(\sqrt{dt})$.

If we choose $dt = \frac{T}{n}$ where n is the number of intervals, the total transaction cost is

$$n \times O(\sqrt{dt}) = O(\sqrt{n}) \rightarrow \infty \quad \text{as } n \rightarrow \infty$$

\Rightarrow Continuous hedging implies infinite transaction cost.

Pricing with Transaction Cost

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Let V be the price of a derivative, consider delta hedged portfolio π

- long V
- short Δ shares of stocks

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The change of value of the portfolio between t to $t + dt$ including transaction cost is

$$\begin{aligned}d\pi &= dV - \Delta dS - \text{transaction cost} \\&= dV - \Delta dS - k\sigma S^2 \sqrt{dt} \left| \frac{\partial^2 V}{\partial S^2} \right| |Z| \\&= \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt - k\sigma S^2 \sqrt{dt} \left| \frac{\partial^2 V}{\partial S^2} \right| |Z|\end{aligned}$$

This is not a perfect hedge as the last term is not deterministic!

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The (conditional) expectation of return $\mathbb{E}[d\pi|\mathcal{F}_t]$ is

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt - k \sigma S^2 \sqrt{dt} \left| \frac{\partial^2 V}{\partial S^2} \right| \sqrt{\frac{2}{\pi}}$$

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We assume to earn riskless rate of return r

$$\mathbb{E}[d\pi|\mathcal{F}_t] = r(V - \Delta S) dt$$

\implies

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt - k\sigma S^2 \sqrt{dt} \left|\frac{\partial^2 V}{\partial S^2}\right| \sqrt{\frac{2}{\pi}} = r(V - \Delta S) dt$$

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Canceling dt we can derive a pricing PDE similar to BSM equation

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - k\sigma S^2 \left| \frac{\partial^2 V}{\partial S^2} \right| \sqrt{\frac{2}{\pi dt}} = rV$$

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For call/put option, gamma is positive, we have

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} + \frac{1}{2}\hat{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV$$

where

$$\hat{\sigma}^2 = \sigma^2 - 2k\sigma\sqrt{\frac{2}{\pi dt}}$$

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- For a long position in call/put option, the effect of including transaction cost is equivalent to reducing volatility. This makes option cheaper.
- dt is the hedging interval.
- Increasing hedging frequency \Leftrightarrow higher transaction cost \Leftrightarrow lower $\hat{\sigma}$.

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For a short position in call/put option,

$$d\pi = -dV + \Delta dS - \text{transaction cost}$$

We can show

$$\hat{\sigma}^2 = \sigma^2 + 2k\sigma\sqrt{\frac{2}{\pi dt}}$$

⇒ This increases the price of call/put option we short

⇒ Reduce the value of our portfolio.

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Enamuel Derman (2008)

Dynamic Replication: Realities and Myths of Option Pricing, *Lecture 2 (Spring 2008)*

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Thank you!