Bayesian Statistics and Hierarchical Bayesian Modeling for Psychological Science

Lecture 11

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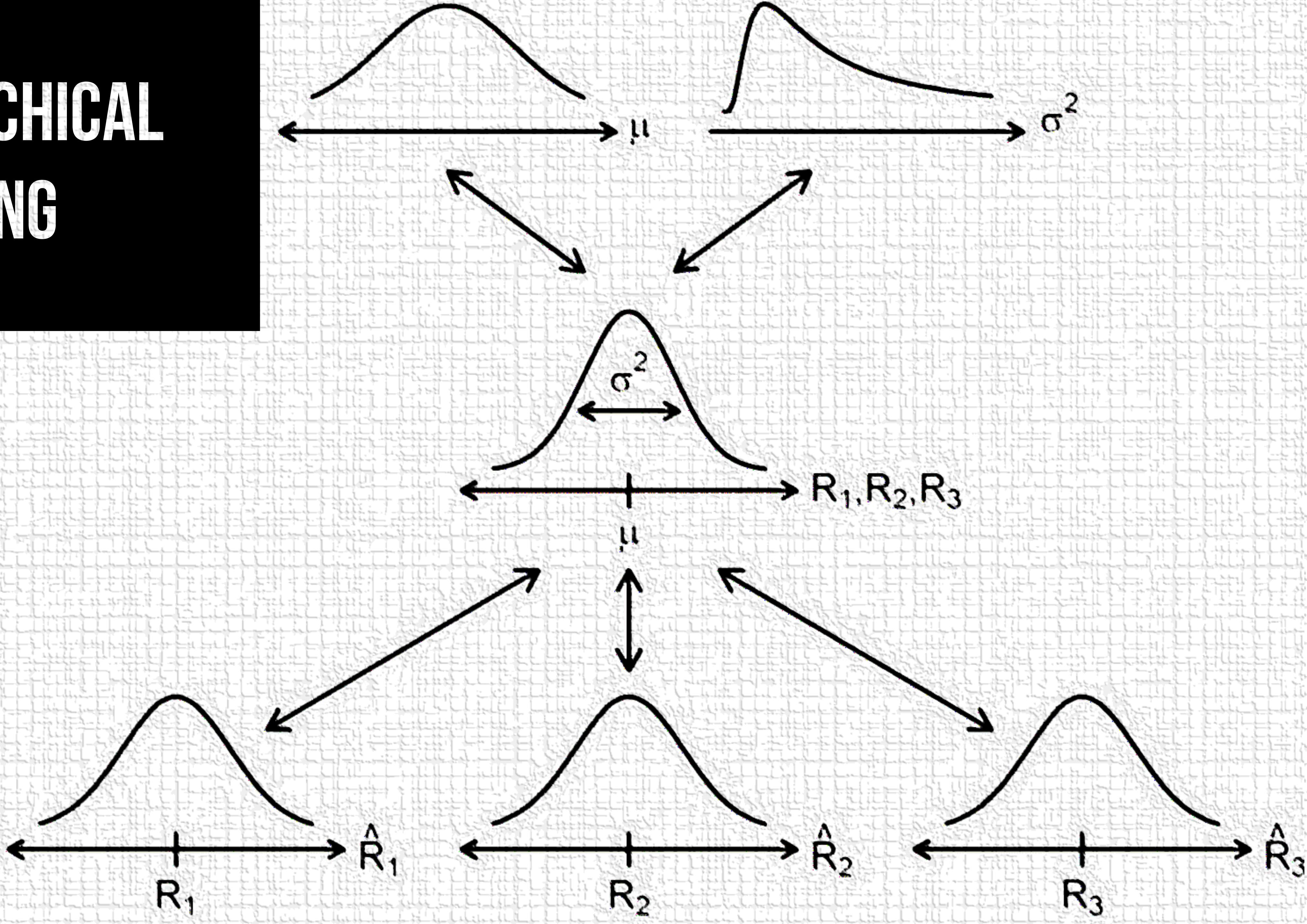
https://github.com/lei-zhang/BayesCog_Wien

universität wien

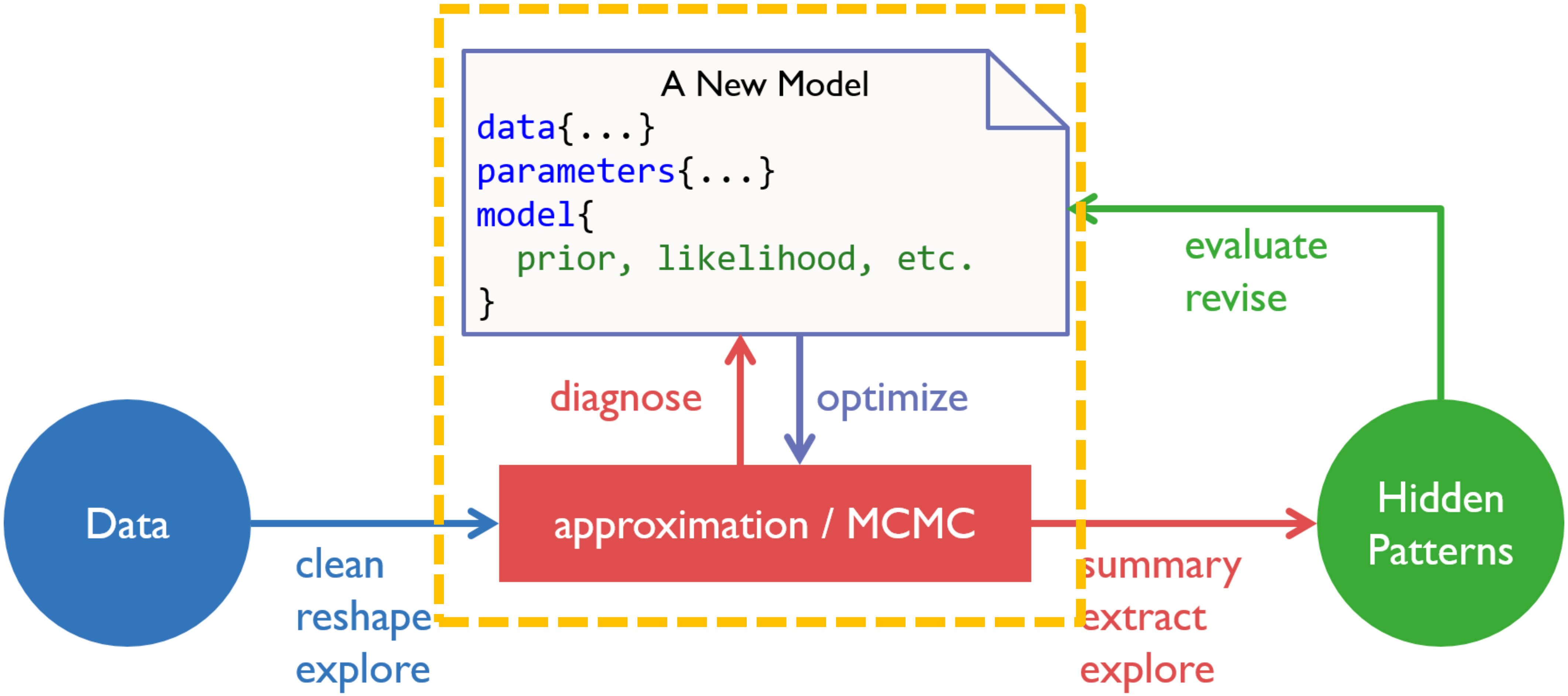
Fakultät für Psychologie

Bayesian warm-up?

HIERARCHICAL MODELING



cognitive model
statistics
computing



Why Hierarchical Bayesian Cognitive Modeling?

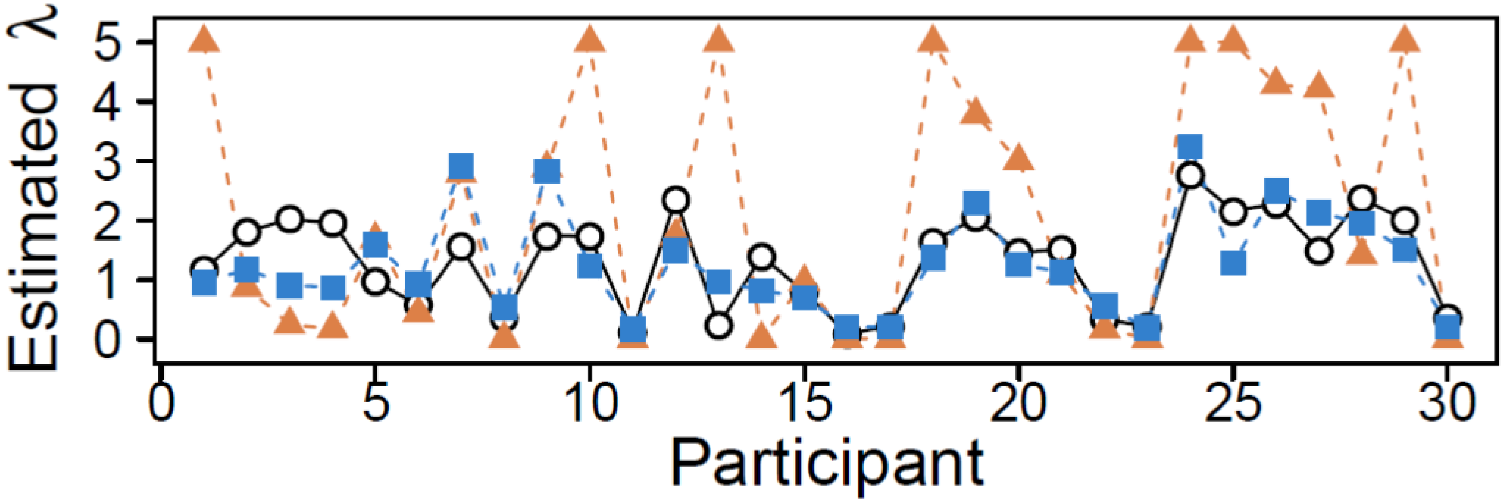
cognitive model
statistics
computing

Simulation study

Hierarchical Bayesian

Maximum likelihood

Actual values



Ahn et al. (2011)

Why Hierarchical Bayesian Cognitive Modeling?

Fixed effects

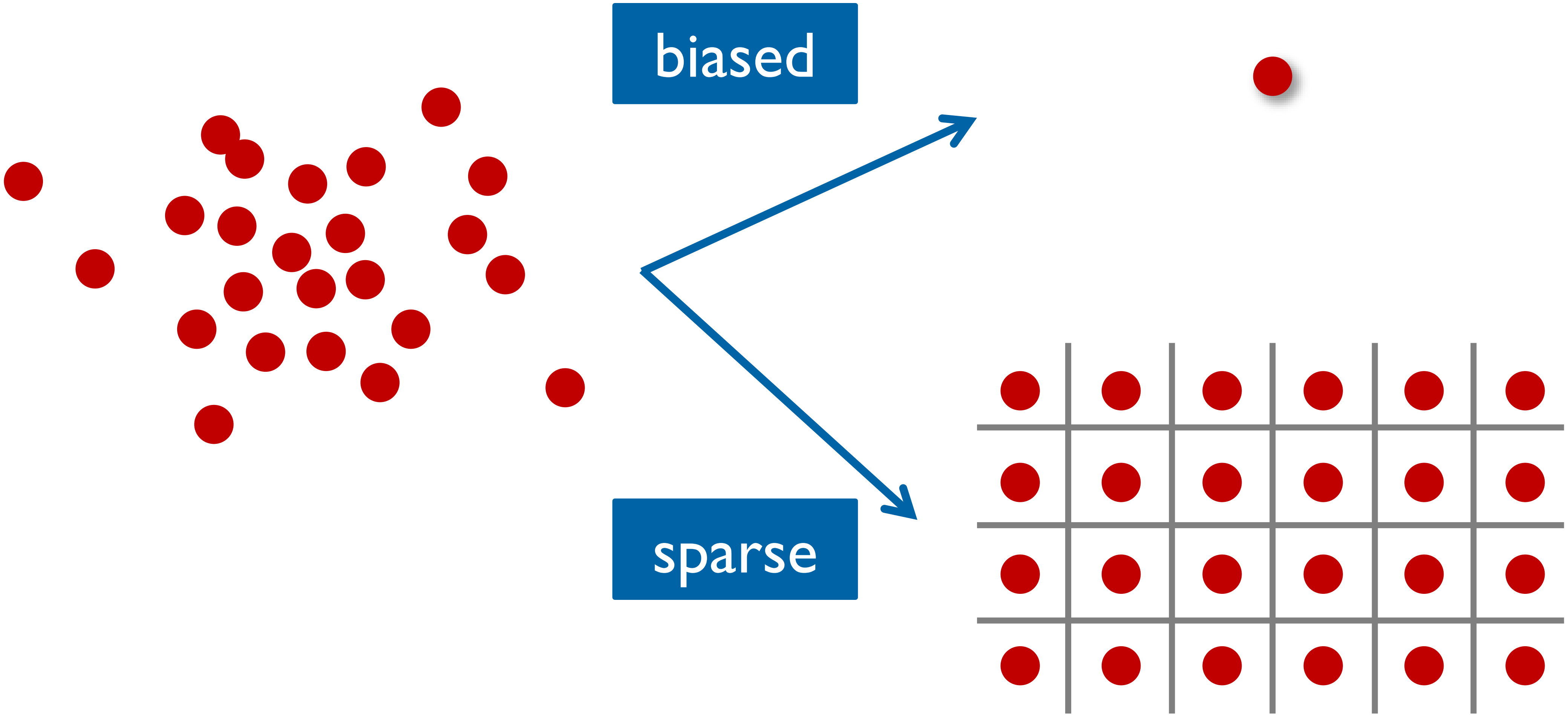
- all subjects are fitted with the same set of parameters
- worse model fit than "random effects"

Random effects

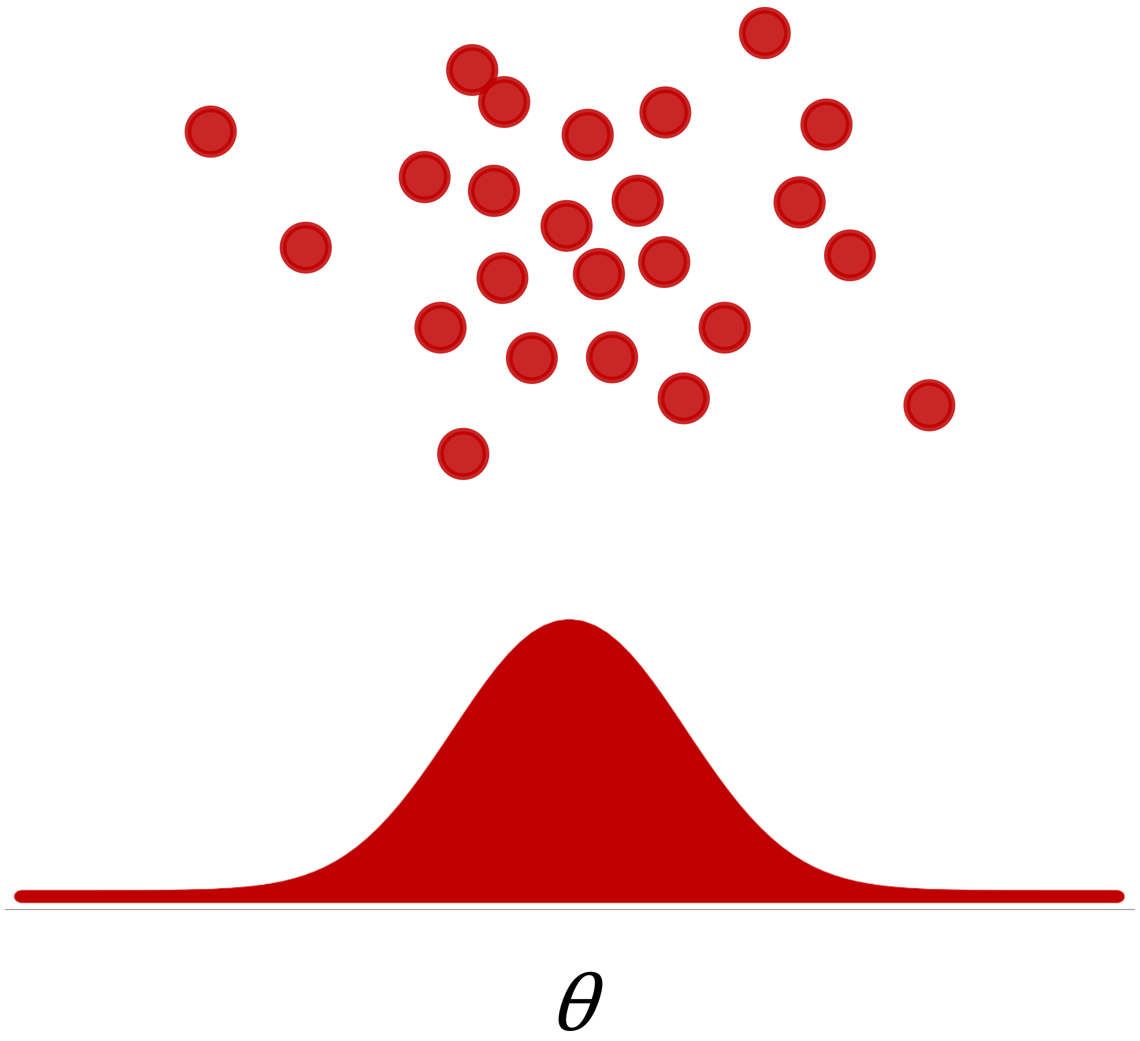
- each subject is fitted independently of the others
- best model fit for each subject
- parameter estimates can be noisy

Adapted from Jan Gläscher's workshop

Fitting Multiple Participants



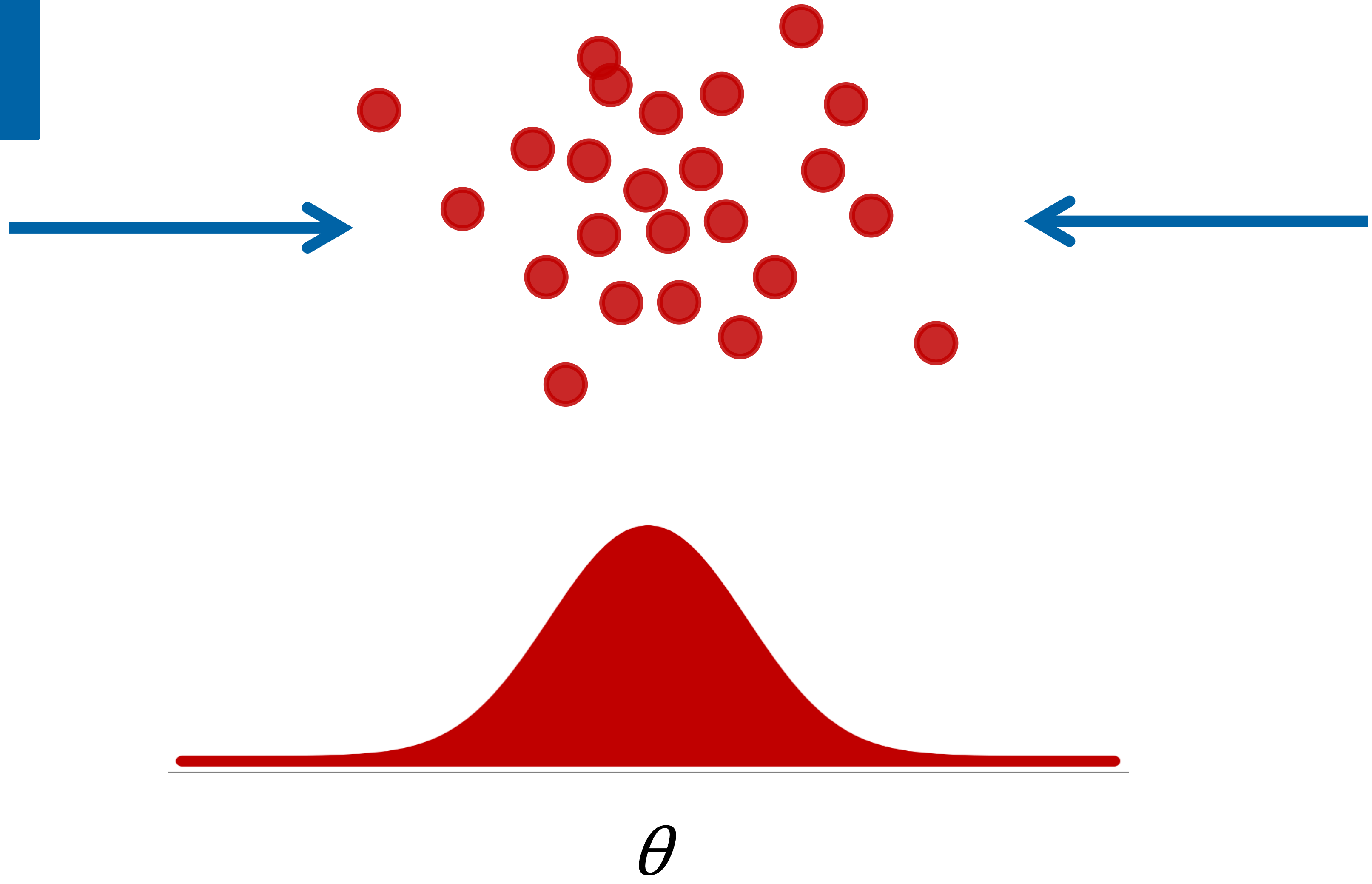
Fitting Multiple Participants



Fitting Multiple Participants

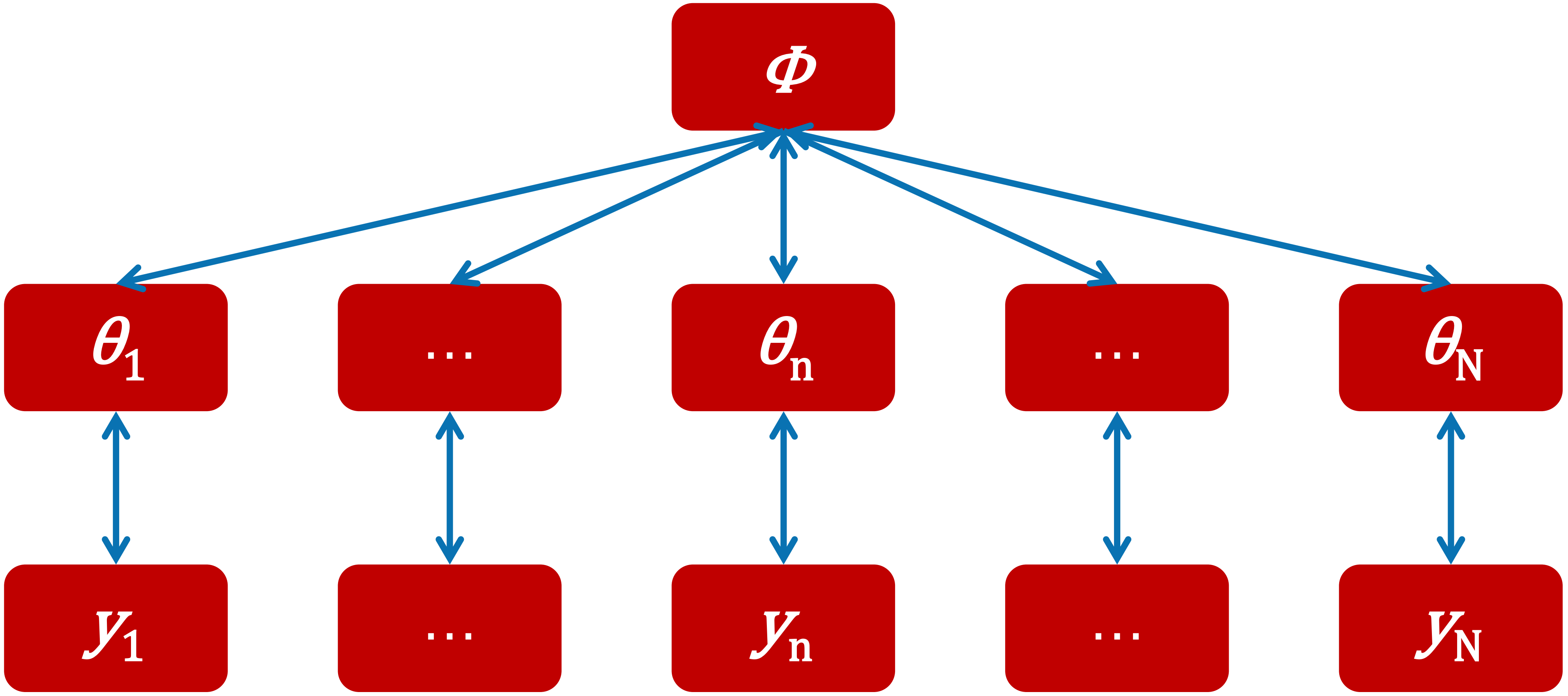
cognitive model
statistics
computing

shrinkage effect



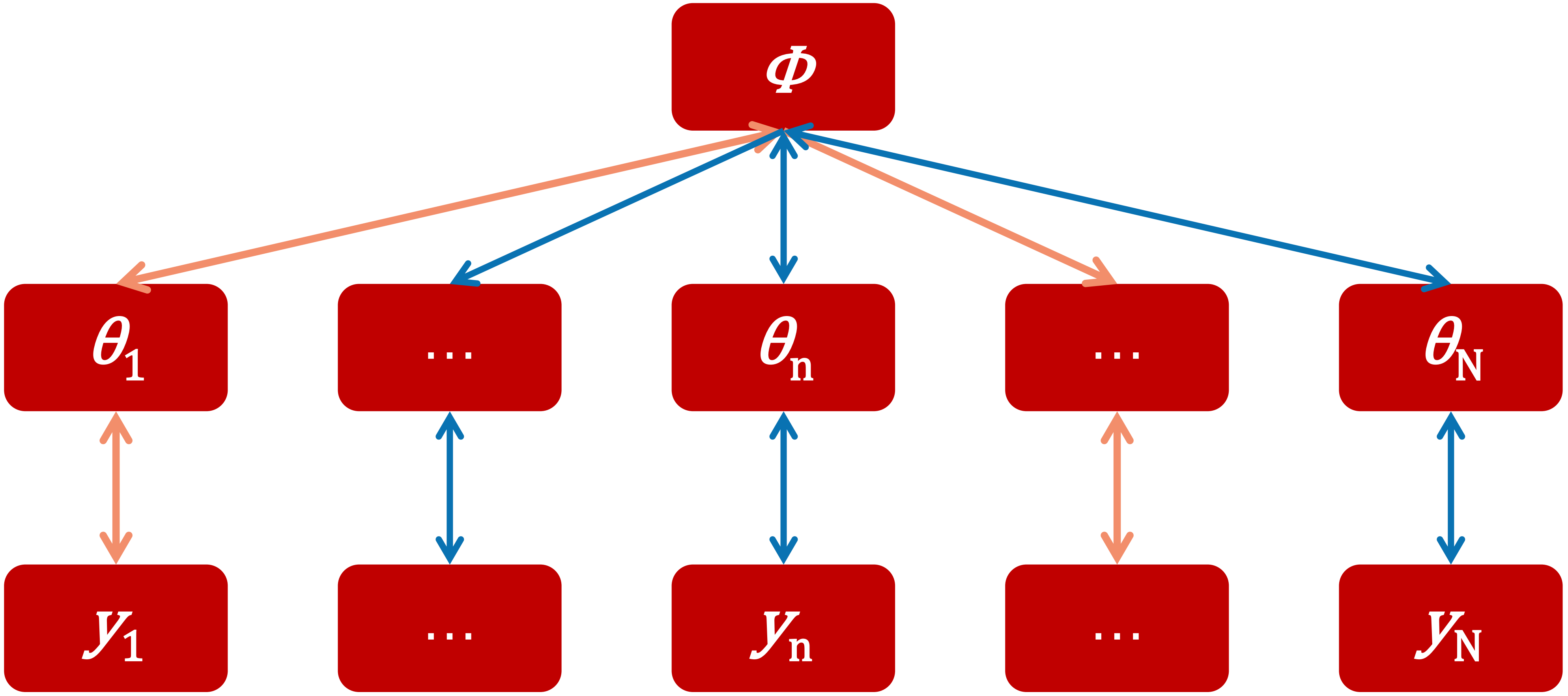
Hierarchical Structure

cognitive model
statistics
computing



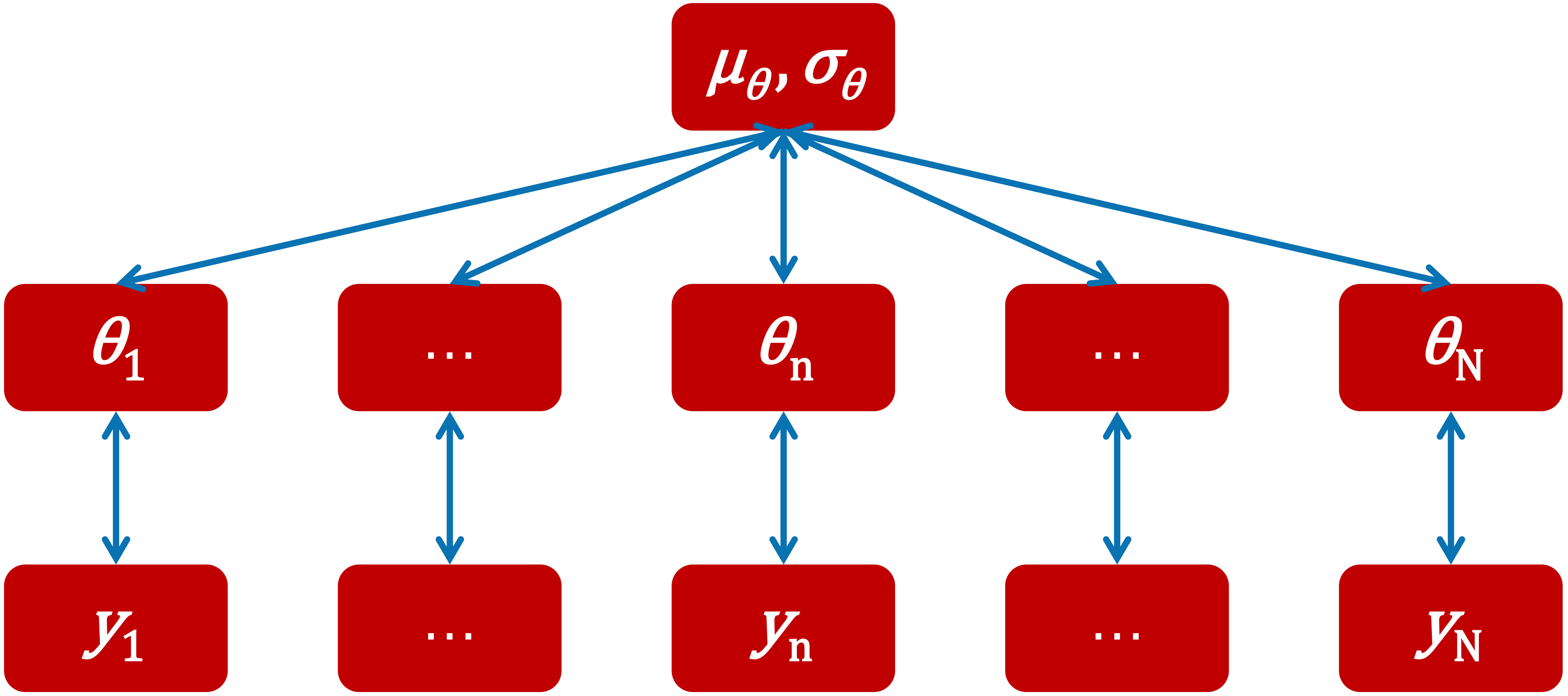
Hierarchical Structure

cognitive model
statistics
computing



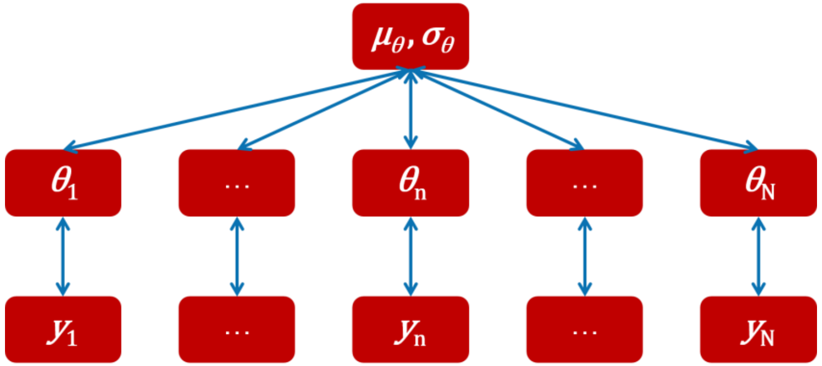
Hierarchical Structure

cognitive model
statistics
computing

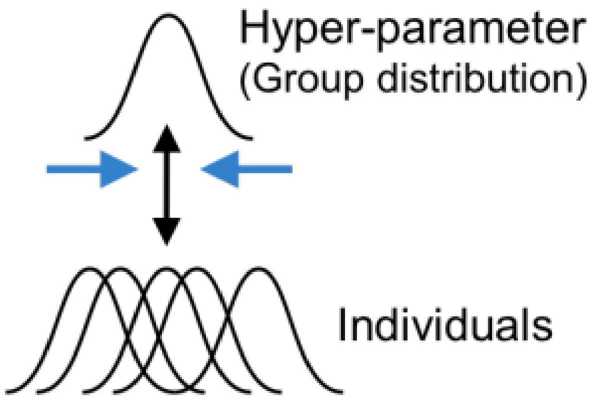


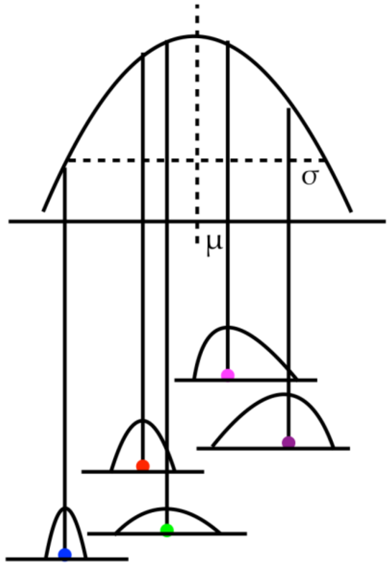
Hierarchical Structure

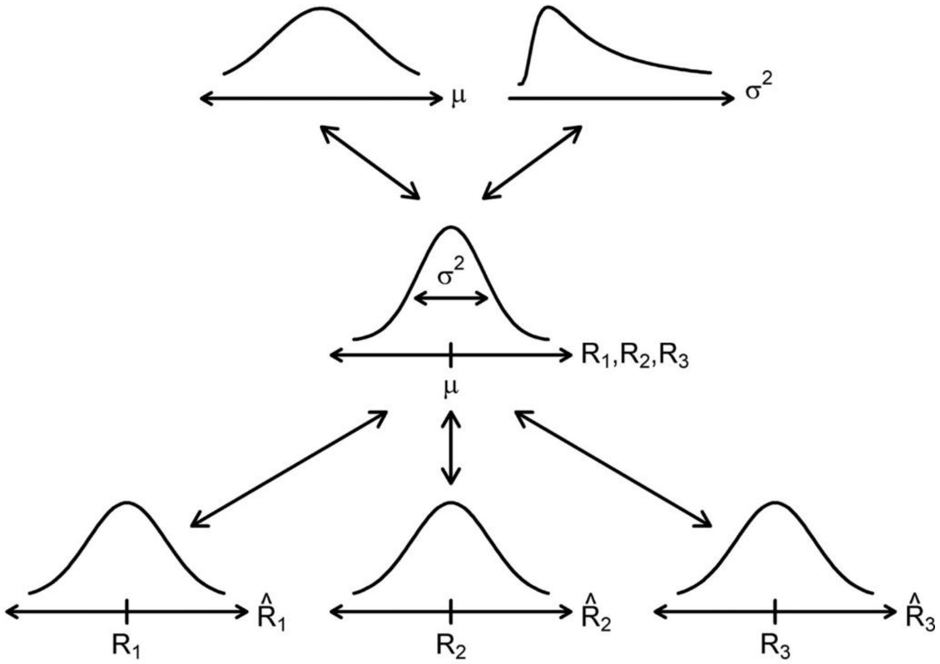
cognitive model
statistics
computing



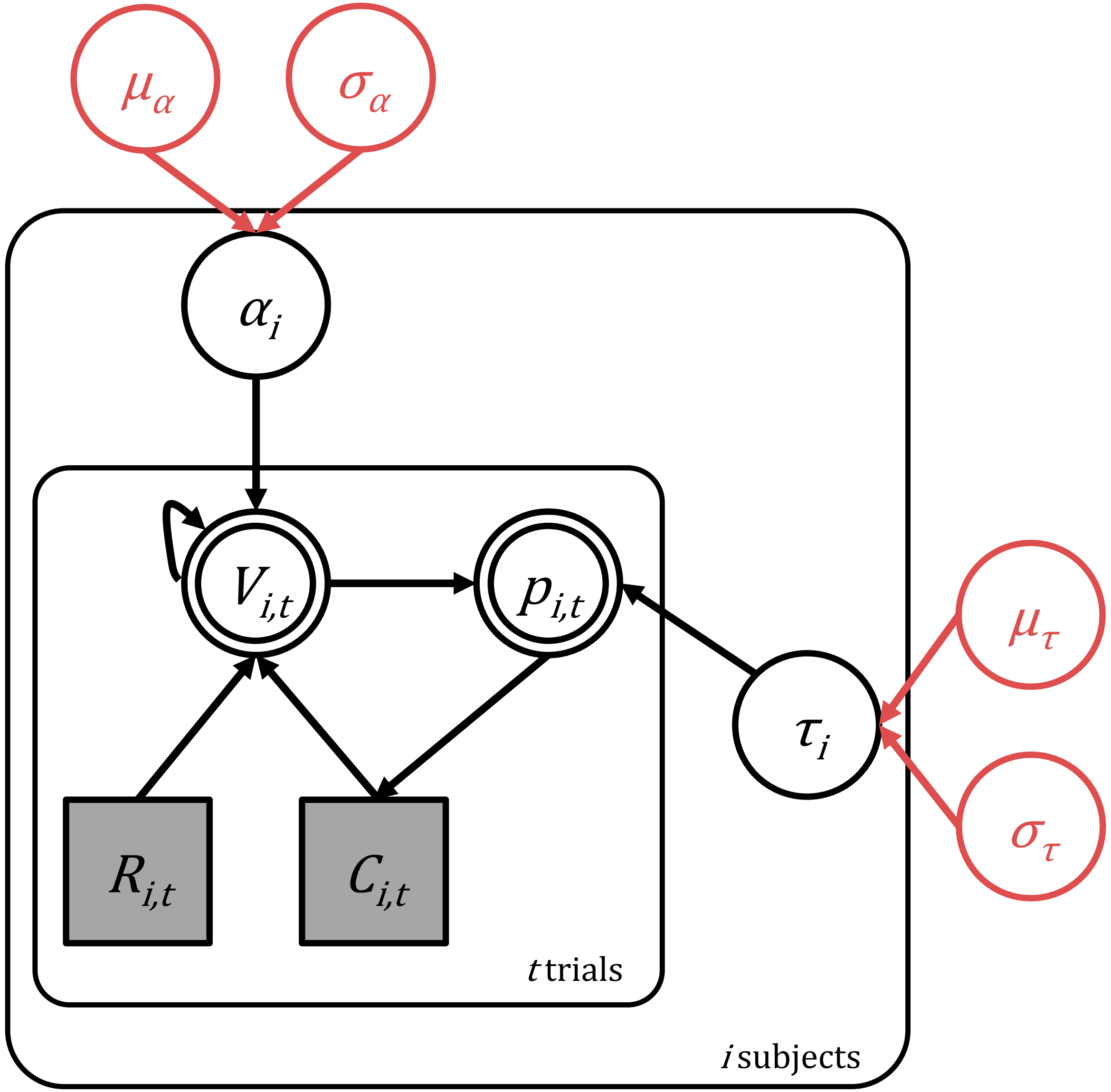
$$P(\Theta,\Phi \mid D) = \frac{P(D \mid \Theta,\Phi)P(\Theta,\Phi)}{P(D)} \propto P(D \mid \Theta)P(\Theta \mid \Phi)P(\Phi)$$





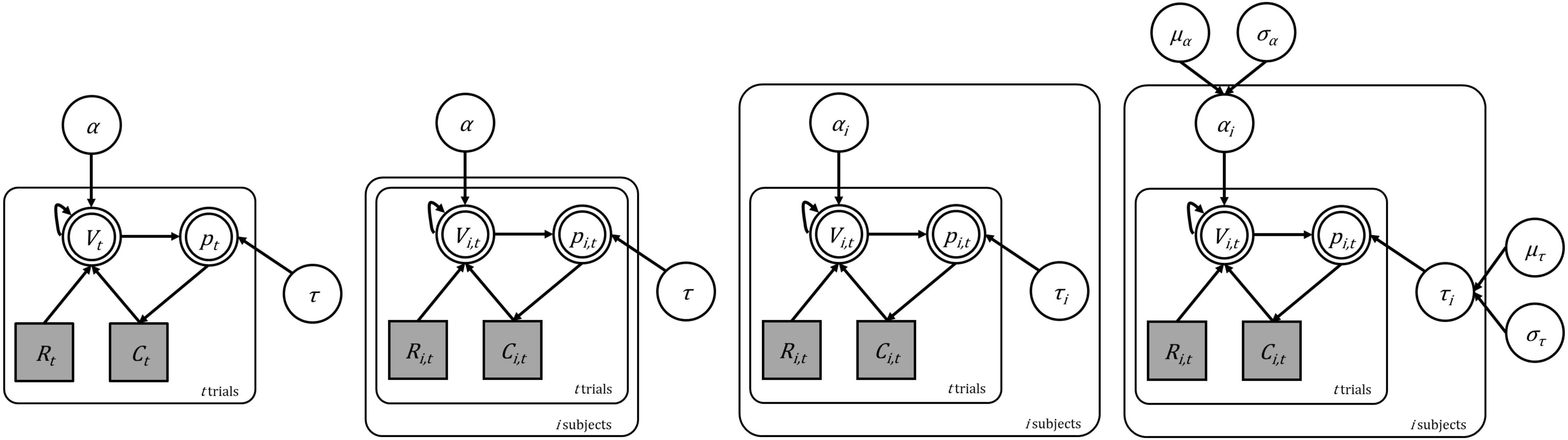


Hierarchical RL Model



HOW DID WE GET HERE?

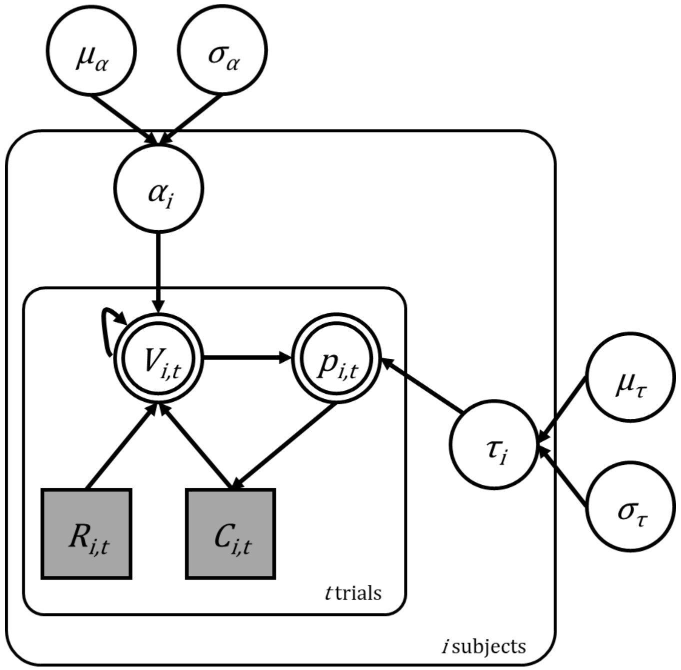
cognitive model
statistics
computing



The cognitive model *per se* is the same!

cognitive model
statistics
computing

Implementing Hierarchical RL Model



$$\mu_\alpha \sim Uniform(0,1)$$
$$\sigma_\alpha \sim halfCauchy(0,1)$$
$$\mu_\tau \sim Uniform(0,3)$$
$$\sigma_\tau \sim halfCauchy(0,3)$$
$$\alpha_i \sim Normal(\mu_\alpha, \sigma_\alpha)_{\mathcal{T}(0,1)}$$
$$\tau_i \sim Normal(\mu_\tau, \sigma_\tau)_{\mathcal{T}(0,3)}$$

$$p_{i,t}(C=A) = \frac{1}{1+e^{\tau_i(V_{i,t}(B)-V_{i,t}(A))}}$$

$$V^c_{i,t+1} = V^C_{i,t} + \alpha_i(R_{i,t} - V^C_{i,t})$$

```
parameters {
  real<lower=0,upper=1> lr_mu;
  real<lower=0,upper=3> tau_mu;

  real<lower=0> lr_sd;
  real<lower=0> tau_sd;

  real<lower=0,upper=1> lr[nSubjects];
  real<lower=0,upper=3> tau[nSubjects];
}

model {
  lr_sd  ~ cauchy(0,1);
  tau_sd ~ cauchy(0,3);
  lr     ~ normal(lr_mu, lr_sd) ;
  tau    ~ normal(tau_mu, tau_sd) ;

  for (s in 1:nSubjects) {
    vector[2] v;
    real pe;
    v = initV;

    for (t in 1:nTrials) {
      choice[s,t] ~ categorical_logit( tau[s] * v );
      pe = reward[s,t] - v[choice[s,t]];
      v[choice[s,t]] = v[choice[s,t]] + lr[s] * pe;
    }
  }
}
```

Exercise XI


```
…/06.reinforcement_learning/_scripts/reinforcement_learning_multi_parm_main.R
```

TASK: (1) complete the model (TIP: individual ~ group)

(2) fit the hierarchical RL model

```
> source('_scripts/reinforcement_learning_multi_parm_main.R')

> fit_rl3 <- run_rl_mp( modelType ='hrch' )
```

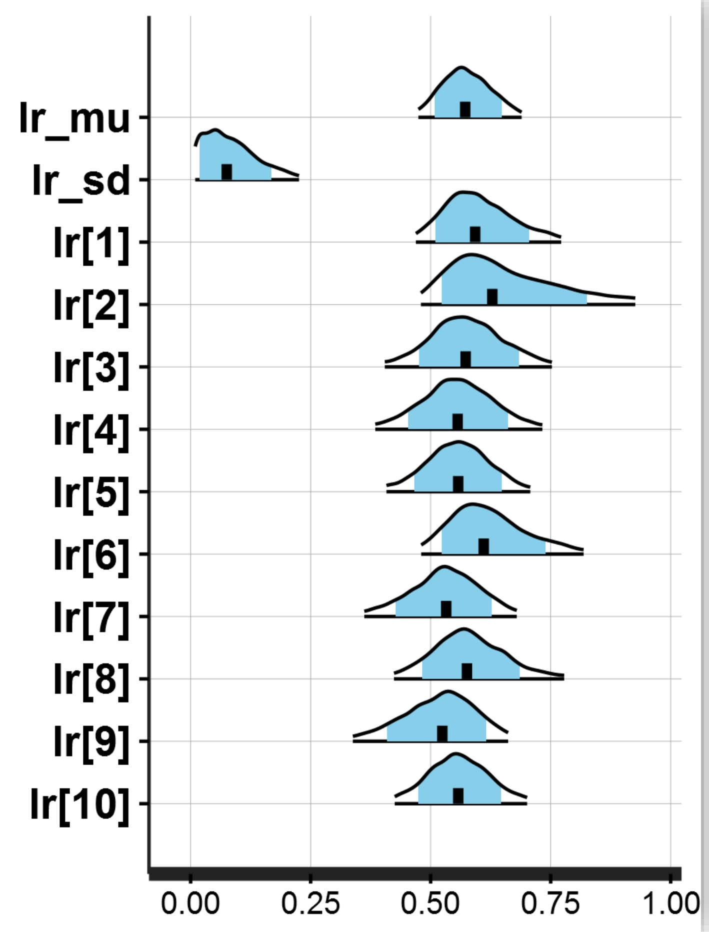
```
In addition: Warning messages:
1: There were 97 divergent transitions after warmup. Increasing adapt_delta above 0.8 may help. See
http://mc-stan.org/misc/warnings.html#divergent-transitions-after-warmup
2: Examine the pairs() plot to diagnose sampling problems
```

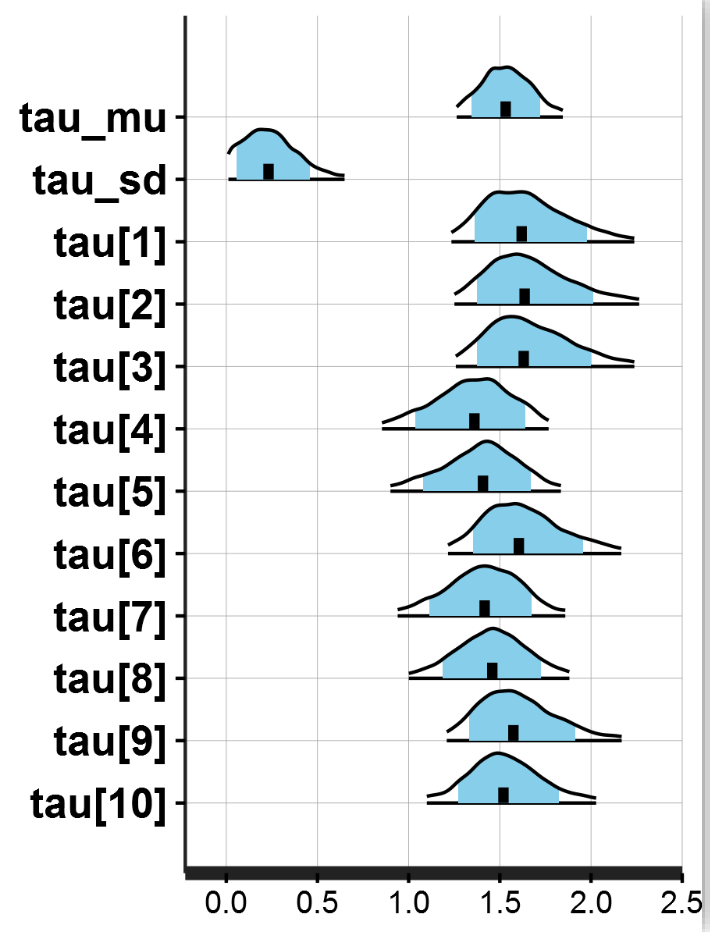
Hierarchical Fitting*

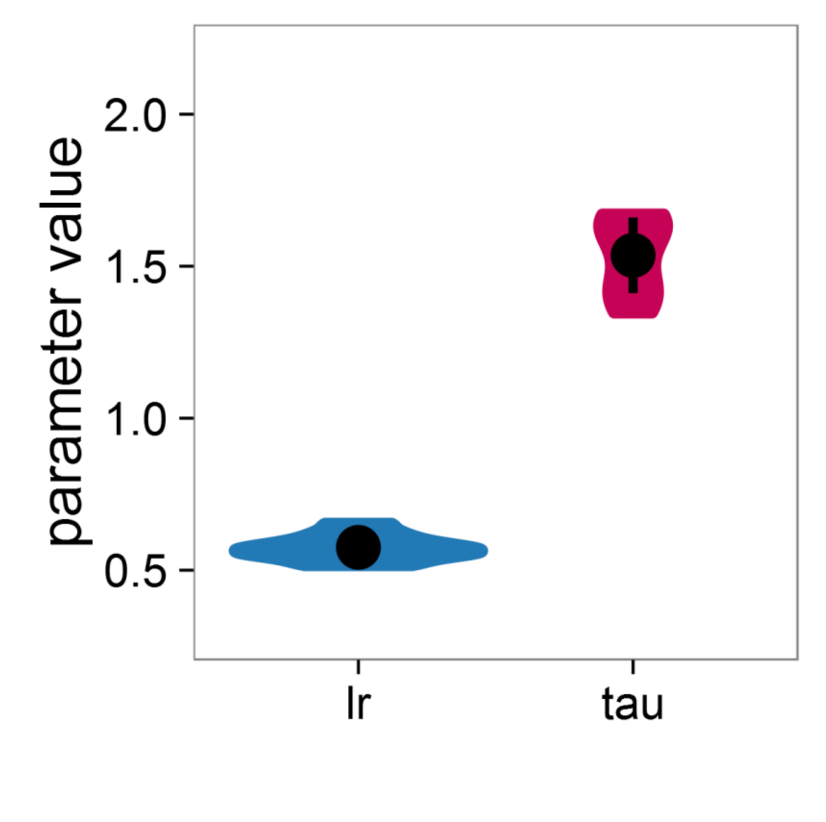
cognitive model

statistics

computing





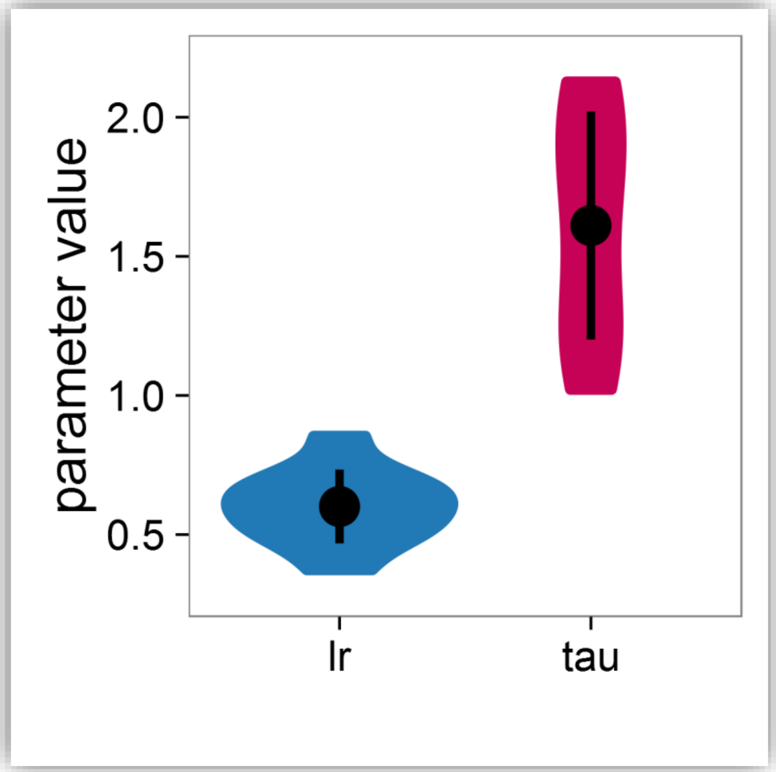


*: adapt_delta=0.999, max_treedepth=100

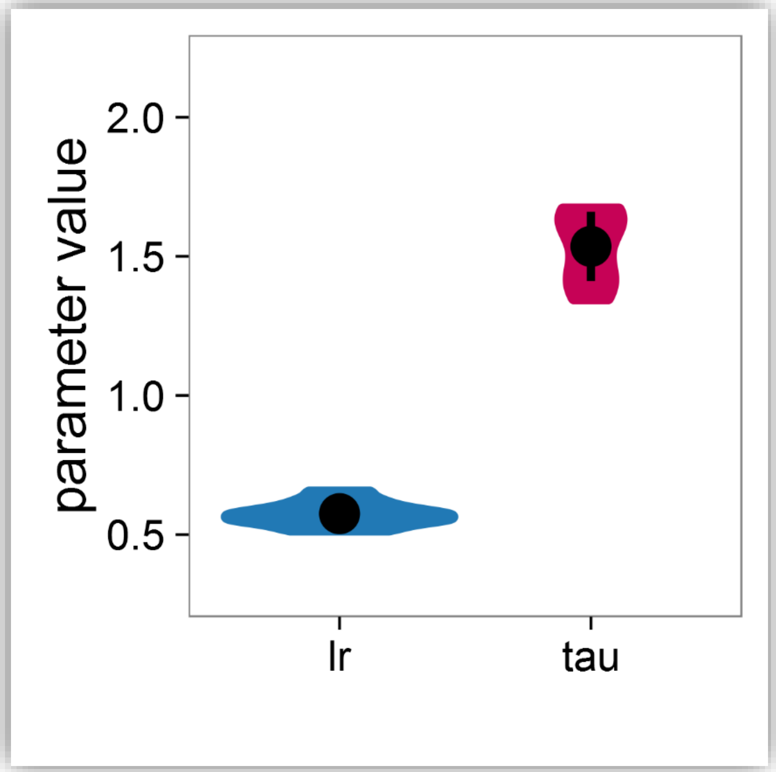
Comparing with True Parameters

cognitive model
statistics
computing

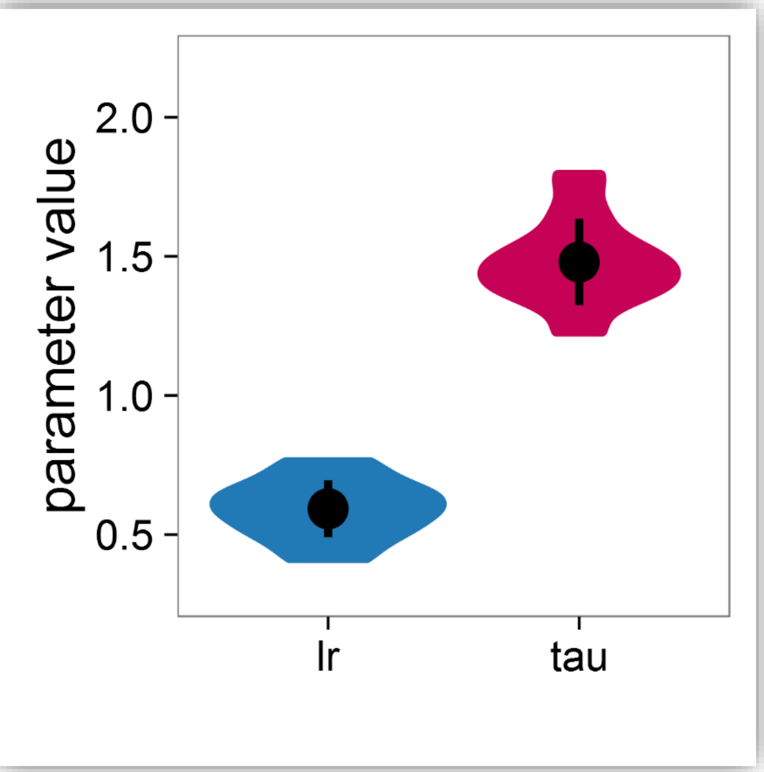
Posterior Means (indv)



Posterior Means (hrch)*



True Parameters



*: adapt_delta=0.999, max_treedepth=100

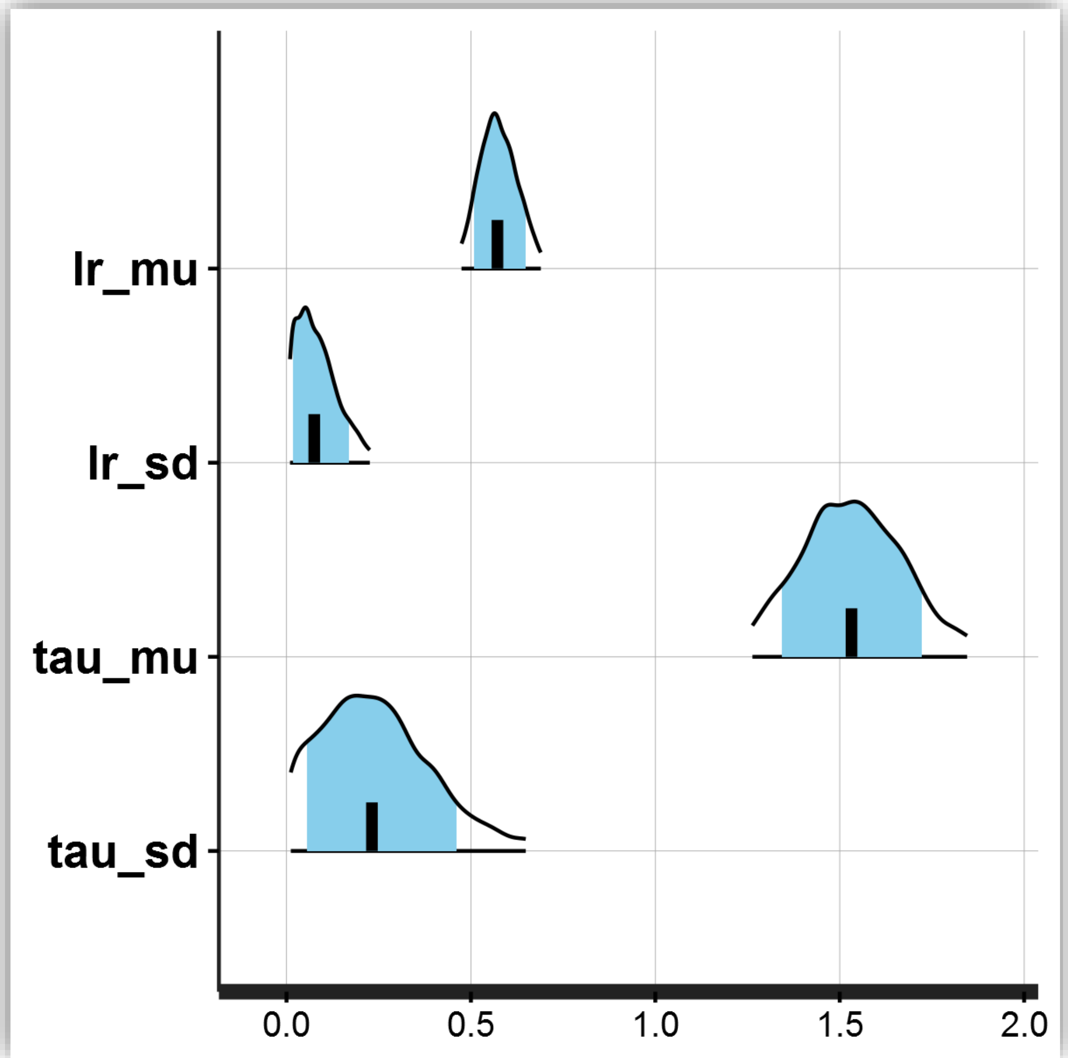
Group-level Parameters

True group parameters

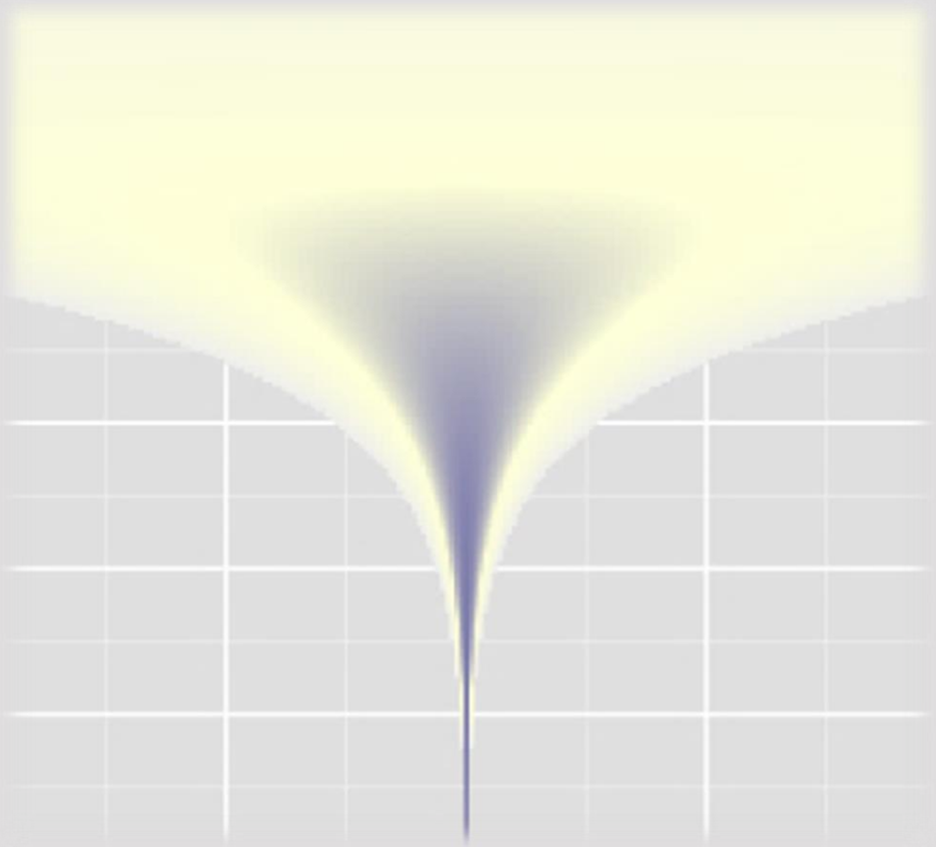
```
lr  = rnorm(10, mean=0.6, sd=0.12)
tau = rnorm(10, mean=1.5, sd=0.2)
```

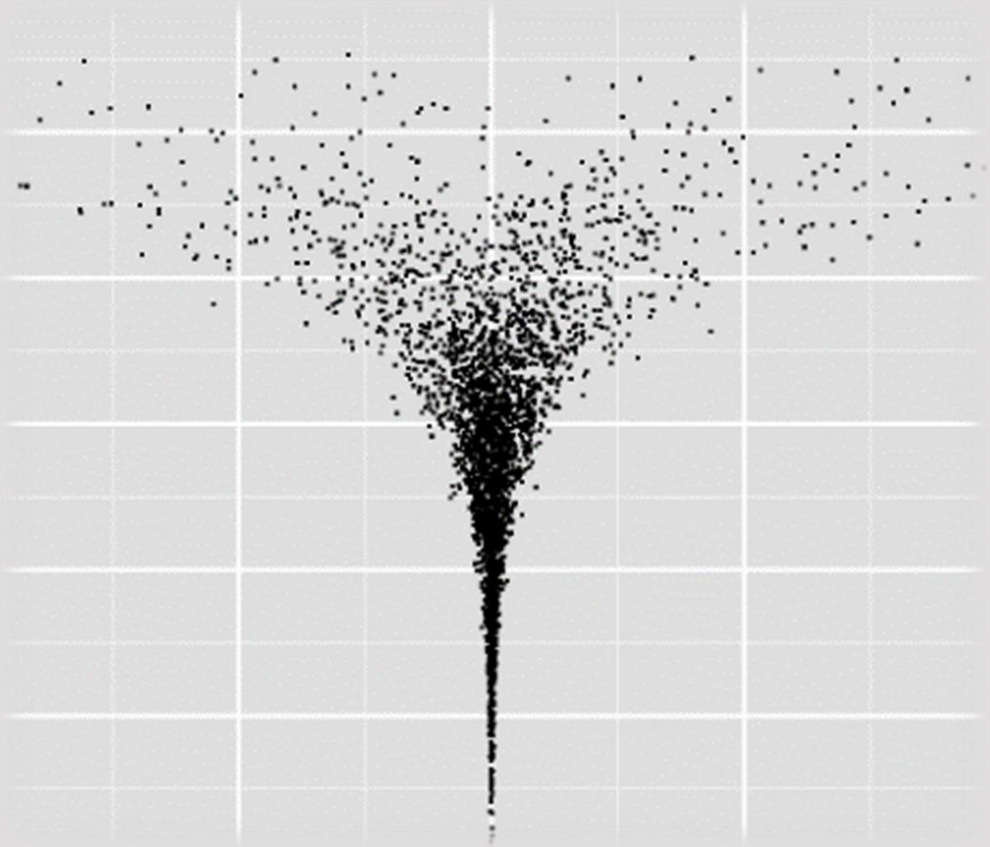
Estimated group parameters

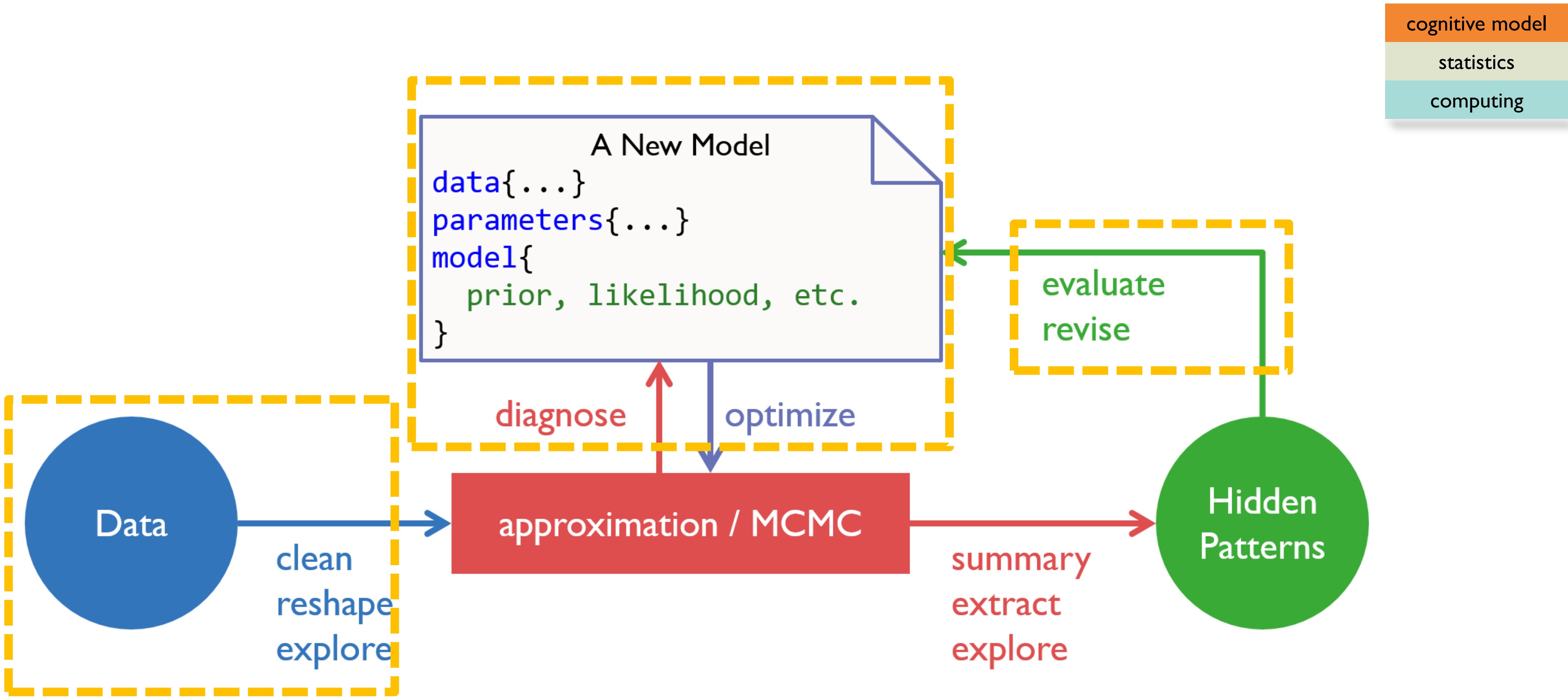
	mean	2.5%	25%	50%	75%	97.5%
lr_mu	0.58	0.47	0.54	0.57	0.61	0.69
lr_sd	0.09	0.01	0.04	0.08	0.12	0.23
tau_mu	1.54	1.26	1.43	1.53	1.63	1.85
tau_sd	0.25	0.01	0.13	0.23	0.34	0.65



OPTIMIZING STAN CODES






cognitive model
statistics
computing
A New Model
data{...}
parameters{...}
model{
prior, likelihood, etc.
}
diagnose
optimize
evaluate
revise
Data
clean
reshape
explore
approximation / MCMC
summary
extract
explore
Hidden Patterns

Optimizing Stan Code

cognitive model

statistics

computing

Preprocess data

run as many calculations as you can outside Stan

Specify a proper model

follow literature, supervision, experience, etc.

Vectorizing

vectorize Stan code whenever you can

Reparameterizing

reparameterize target parameter to simple distributions

Preprocess Data

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statistics
computing

$$\overline{\text{height}} = \alpha + \beta 1 * \text{weight} + \beta 2 * \text{weight}^2$$

```
d$weight_sq <- d$weight^2
```

```
data {
  int<lower=0> N;
  vector<lower=0>[N] height;
  vector<lower=0>[N] weight;
  vector<lower=0>[N] weight_sq;
}
```

Specify a Proper Model

- Visualize your data
- Follow Literatures
- Start from simple and then build complexities
- Simulate data and run model recovery

A New Model

```
data{...}
parameters{...}
model{
  prior, likelihood, etc.
}
```

Vectorization

cognitive model
statistics
computing

```
model {
  for (n in 1:N) {
    flip[n] ~ bernoulli(theta);
  }
}
```

```
model {
  flip ~ bernoulli(theta);
}
```

```
parameters {
  ...
  real<lower=0,upper=1> lr[nSubjects];
  real<lower=0,upper=3> tau[nSubjects];
}

model {
  ...
  lr     ~ normal(lr_mu, lr_sd) ;
  tau    ~ normal(tau_mu, tau_sd) ;
  ...
}
```

```
model {
  vector[N] mu;
  for (i in 1:N) {
    mu[i] = alpha + beta * weight[i];
    height[i] ~ normal(mu[i], sigma)
  }
}
```

```
model {
  vector[N] mu;
  mu = alpha + beta * weight;
  height ~ normal(mu, sigma);
}
```

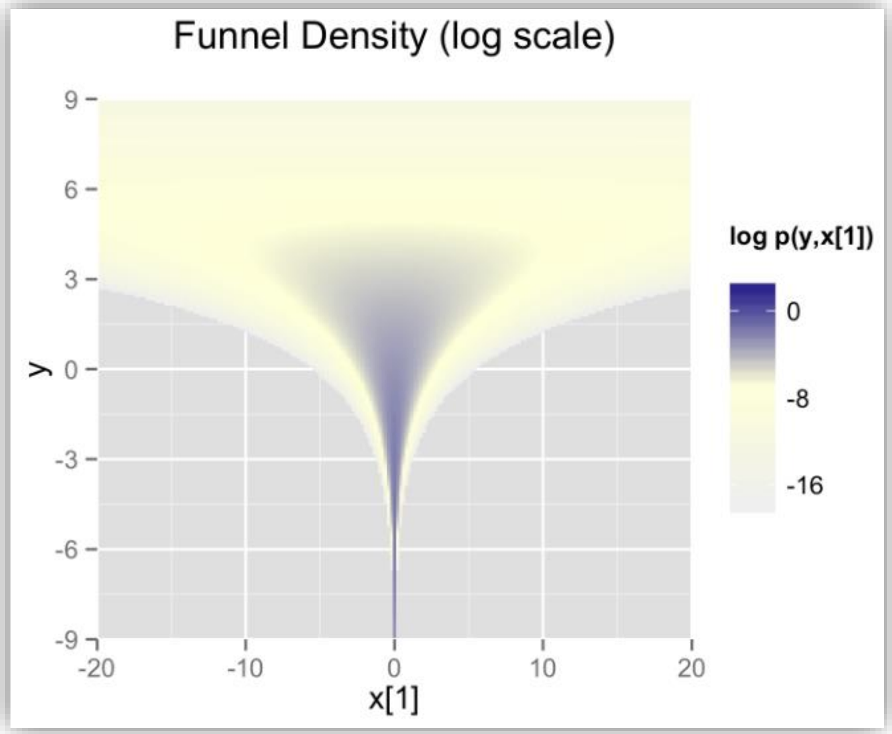
```
model {
  height ~ normal(alpha + beta * weight, sigma);
}
```

Reparameterization

Neal's Funnel

$$p(y,x) = \text{Normal}(y|0,3) \times \prod_{n=1}^{9} \text{Normal}(x_n|0,\exp(y/2))$$

```
parameters {
  real y;
  vector[9] x;
}
model {
  y ~ normal(0,3);
  x ~ normal(0,exp(y/2));
}
```



Non-centered Reparameterization*

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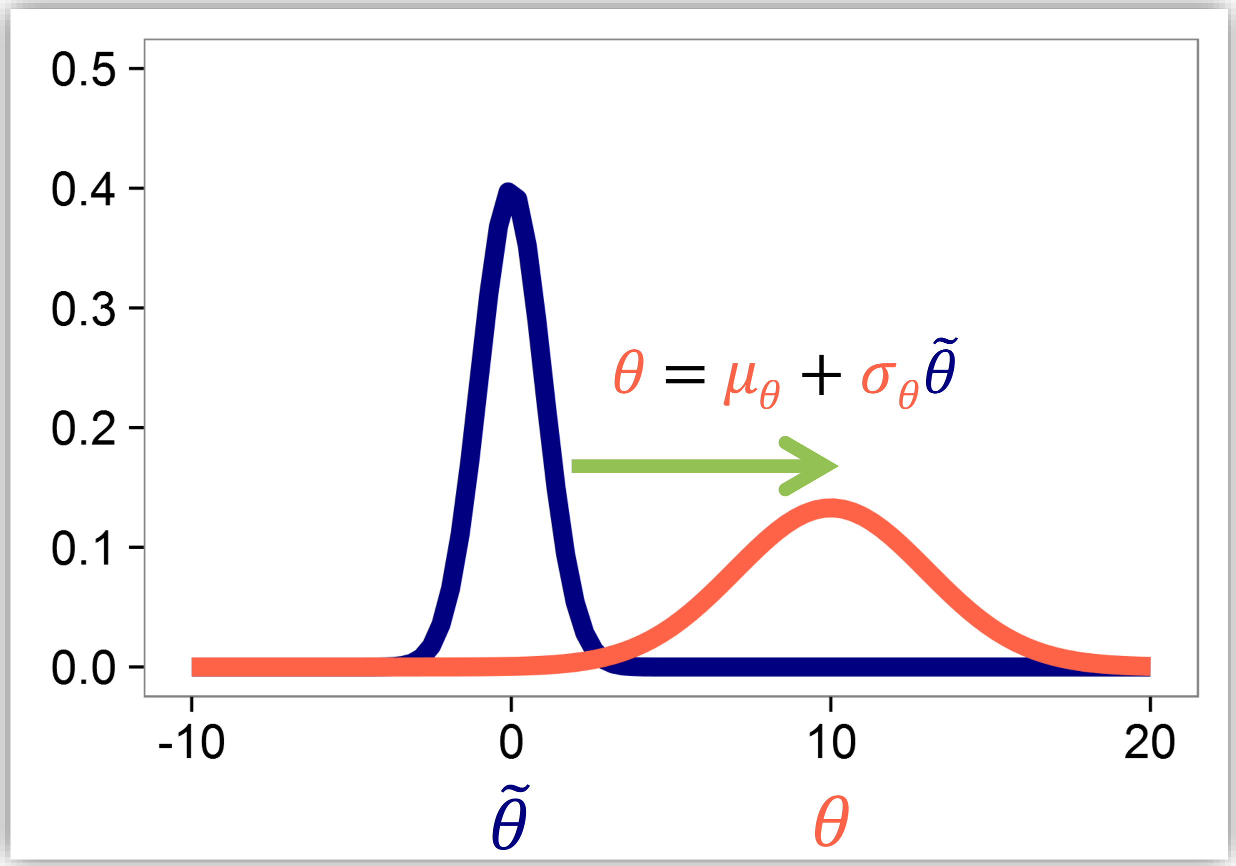
statistics
computing

$$\theta \sim Normal(\mu_\theta, \sigma_\theta)$$

$$\tilde{\theta} \sim Normal(0, 1)$$

$$\theta = \mu_\theta + \sigma_\theta \tilde{\theta}$$

Stan likes **simple distributions!**



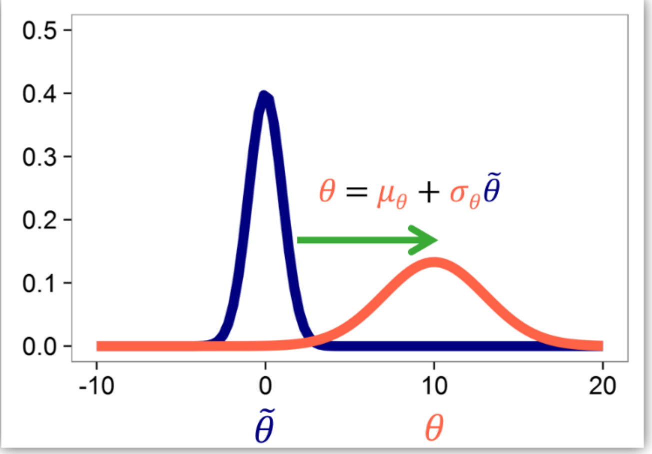
*:Stan Development Team (2016), aka, Matt Trick

Reparameterization

Neal's Funnel

$$p(y, x) = \text{Normal}(y|0, 3) \times \prod_{n=1}^{9} \text{Normal}(x_n|0, \exp(y/2))$$

```
parameters {
  real y;
  vector[9] x;
}
model {
  y ~ normal(0,3);
  x ~ normal(0,exp(y/2));
}
```




```
parameters {
  real y_raw;
  vector[9] x_raw;
}
transformed parameters {
  real y;
  vector[9] x;

  y = 3.0 * y_raw;
  x = exp(y/2) * x_raw;
}
model {
  y_raw ~ normal(0,1);
  x_raw ~ normal(0,1);
}
```

Stan Sampling Parameters

parameter	description	constraint	default
iterations	number of MCMC samples (per chain)	int, > 0	2000
delta: δ	target Metropolis acceptance rate	$\delta \in [0,1]$	0.80
stepsize: ε	initial HMC step size	real, $\varepsilon > 0$	2.0
max_treedepth: L	maximum HMC steps per iteration	int, $L > 0$	10

Typical adjustments

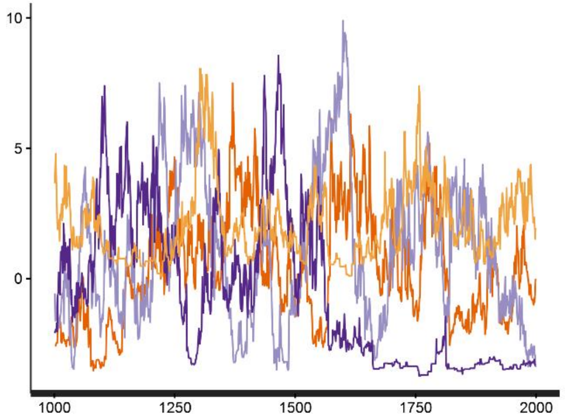
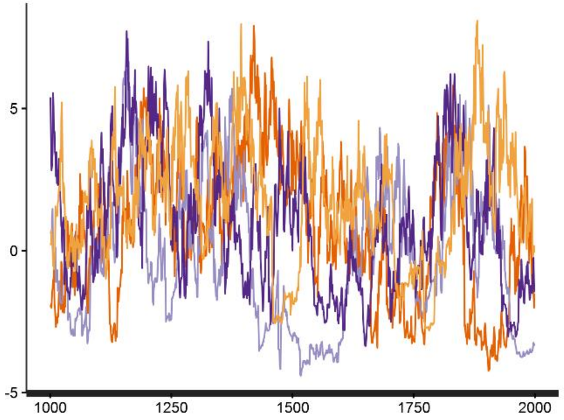
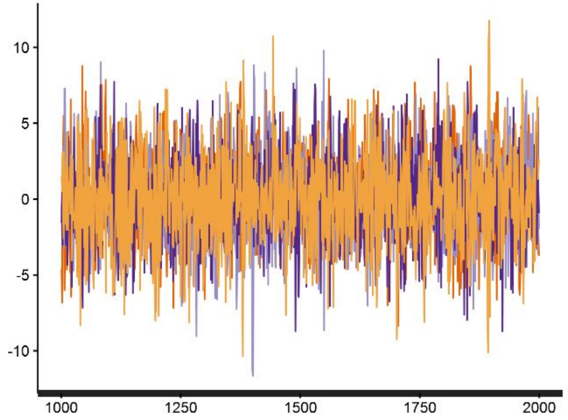
- Increase iterations
- Increase delta
- Decrease stepsize
- Might have to increase max_treedepth

```
funnel_fit2 <- stan("_scripts/funnel.stan",
          iter = 4000,
          control = list(adapt_delta = 0.999,
                         stepsize = 1.0,
                         max_treedepth = 20))
```

Stan Development Team (2016)

cognitive model
statistics
computing

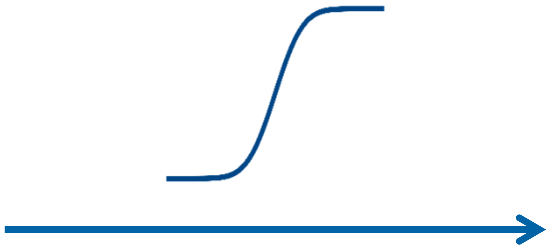
Neal's Funnel: Comparing Performance

	direct model	adjusted direct model	reparameterized model
Rhat (*y*)	1.22	1.1	1.0
n_eff (*y*)	18	42	3886
runtime*	48.50 sec	50.76 sec	50.12 sec
n_eff (*y*) / runtime	**0.37 / sec**	**0.82 / sec**	**77.53 / sec**
n_divergent	53	0	0
traceplot (*y*)			

*: 2 cores in parallel, including compiling time

How about **Bounded** Parameters?

$$\tilde{\theta} \sim Normal(0,1)$$
$$\theta = \mu_\theta + \sigma_\theta \tilde{\theta}$$
$$\theta \in (-\infty, +\infty)$$



$$\tilde{\theta} \sim Normal(0,1)$$
$$\theta = Probit^{-1}(\mu_\theta + \sigma_\theta \tilde{\theta})$$
$$\theta \in [0,1]$$

constraint	reparameterization
$\theta \in (-\infty, +\infty)$	$\theta = \mu_\theta + \sigma_\theta \tilde{\theta}$
$\theta \in [0, N]$	$\theta = Probit^{-1}(\mu_\theta + \sigma_\theta \tilde{\theta}) \times N$
$\theta \in [M, N]$	$\theta = Probit^{-1}(\mu_\theta + \sigma_\theta \tilde{\theta}) \times (N - M) + M$
$\theta \in (0, +\infty)$	$\theta = exp(\mu_\theta + \sigma_\theta \tilde{\theta})$

* $Probit^{-1}$: Normal cumulative distribution function (normcdf)

Apply to Our Hierarchical RL Model

cognitive model

statistics

computing

```
parameters {
  real<lower=0,upper=1> lr_mu;
  real<lower=0,upper=3> tau_mu;

  real<lower=0> lr_sd;
  real<lower=0> tau_sd;

  real<lower=0,upper=1> lr[nSubjects];
  real<lower=0,upper=3> tau[nSubjects];
}
```

→

```
parameters {
  # group-level parameters
  real lr_mu_raw;
  real tau_mu_raw;
  real<lower=0> lr_sd_raw;
  real<lower=0> tau_sd_raw;

  # subject-level raw parameters
  vector[nSubjects] lr_raw;
  vector[nSubjects] tau_raw;
}

transformed parameters {
  vector<lower=0,upper=1>[nSubjects] lr;
  vector<lower=0,upper=3>[nSubjects] tau;

  for (s in 1:nSubjects) {
    lr[s]  = Phi_approx( lr_mu_raw  + lr_sd_raw * lr_raw[s] );
    tau[s] = Phi_approx( tau_mu_raw + tau_sd_raw * tau_raw[s] ) * 3;
  }
}
```

Apply to Our Hierarchical RL Model

cognitive model
statistics
computing

```
model {
  lr_sd  ~ cauchy(0,1);
  tau_sd ~ cauchy(0,3);
  lr     ~ normal(lr_mu, lr_sd) ;
  tau    ~ normal(tau_mu, tau_sd) ;

  for (s in 1:nSubjects) {
    vector[2] v;
    real pe;
    v = initV;

    for (t in 1:nTrials) {
      choice[s,t] ~ categorical_logit( tau[s] * v );
      pe = reward[s,t] - v[choice[s,t]];
      v[choice[s,t]] = v[choice[s,t]] + lr[s] * pe;
    }
  }
}
```

```
model {
  lr_mu_raw  ~ normal(0,1);
  tau_mu_raw ~ normal(0,1);
  lr_sd_raw  ~ cauchy(0,3);
  tau_sd_raw ~ cauchy(0,3);

  lr_raw  ~ normal(0,1);
  tau_raw ~ normal(0,1);

  for (s in 1:nSubjects) {
    ...
```

```
generated quantities {
  real<lower=0,upper=1> lr_mu;
  real<lower=0,upper=3> tau_mu;

  lr_mu  = Phi_approx(lr_mu_raw);
  tau_mu = Phi_approx(tau_mu_raw) * 3;
}
```

Exercise XII

cognitive model
statistics
computing

```
…/07.optm_rl/_scripts/reinforcement_learning_hrch_main.R
```

TASK: (1) Complete the Matt Trick

(2) fit the optimized hierarchical RL model

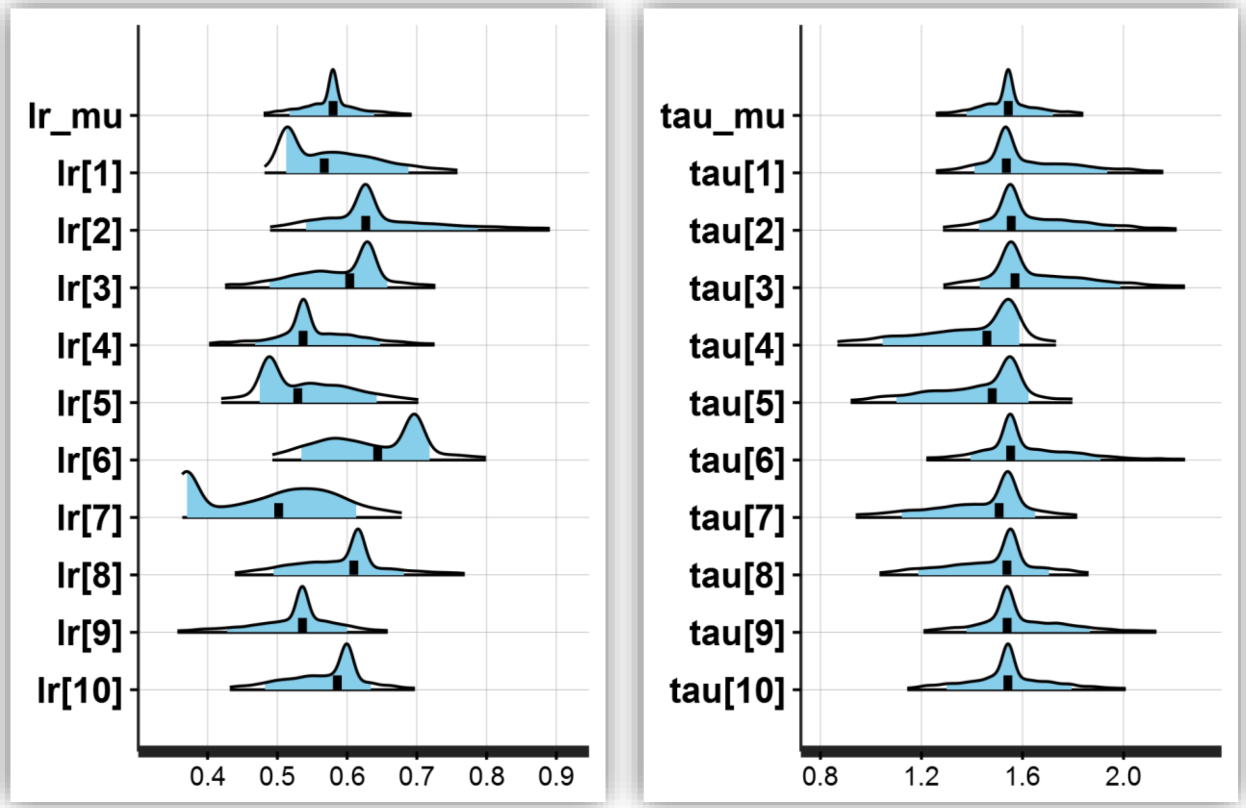
```
> source('_scripts/reinforcement_learning_hrch_main.R')

> fit_rl4 <- run_rl_mp2(optimized = TRUE)
```

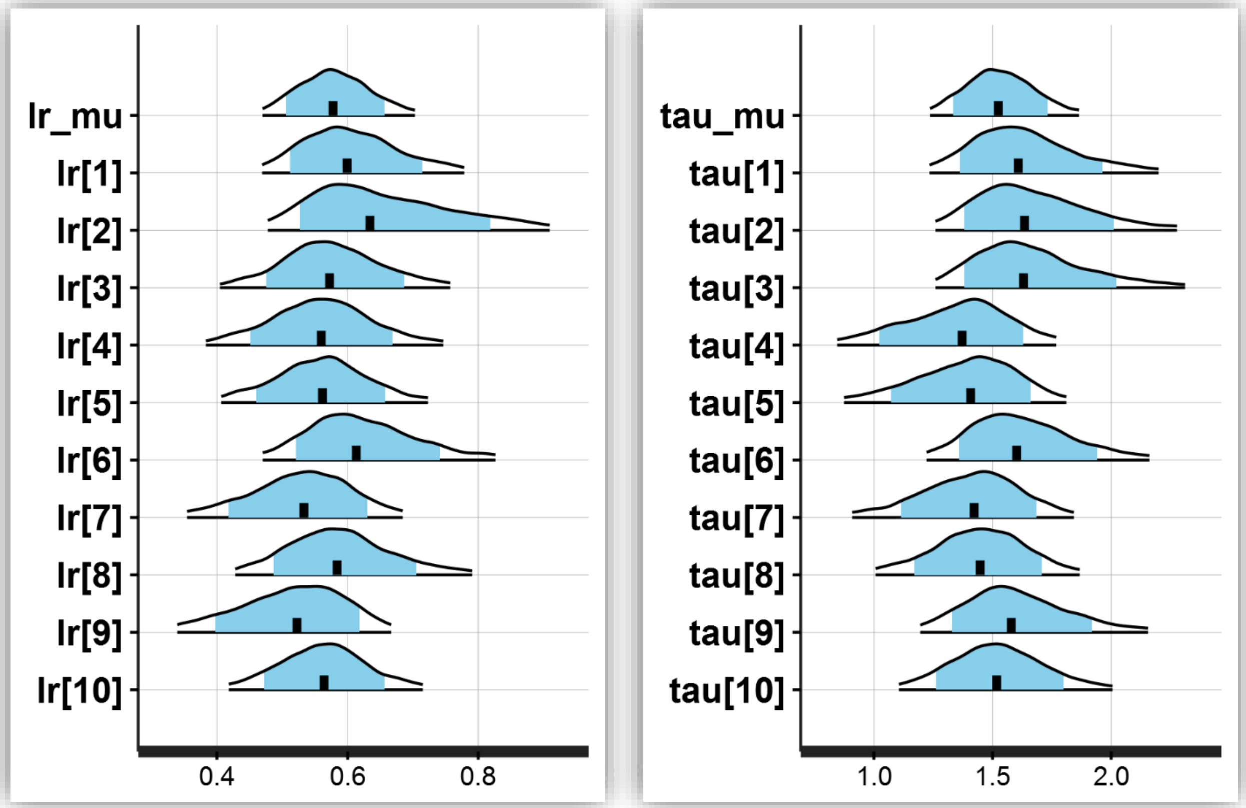
Hierarchical Fitting – Optimized

cognitive model
statistics
computing

Posterior Means (hrch)



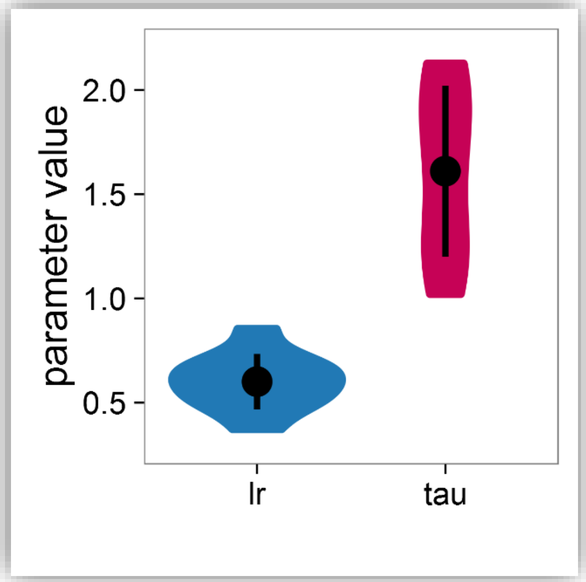
Posterior Means (hrch + optm)



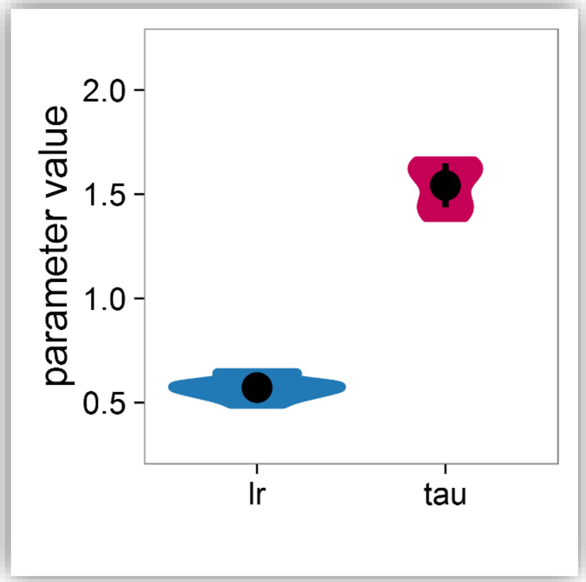
Comparing with True Parameters

cognitive model
statistics
computing

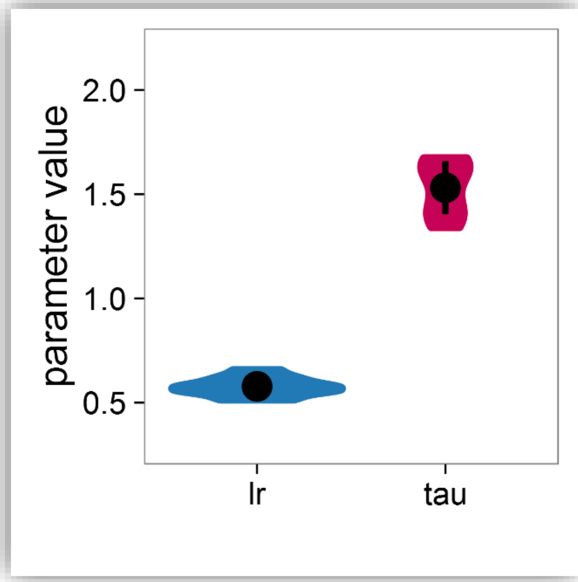
Posterior Means (indv)



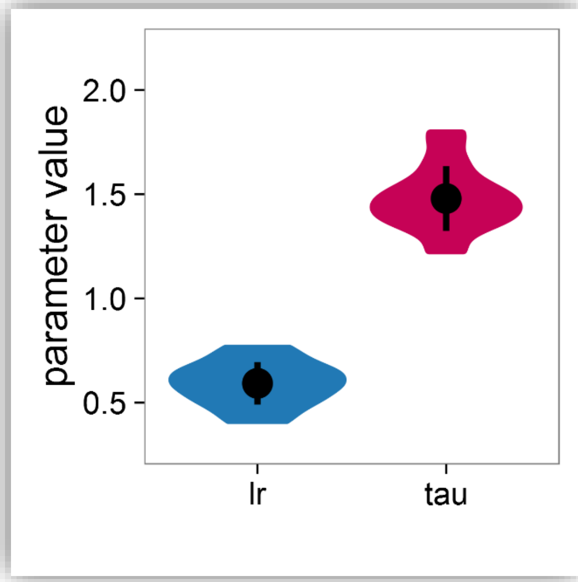
Posterior Means (hrch)

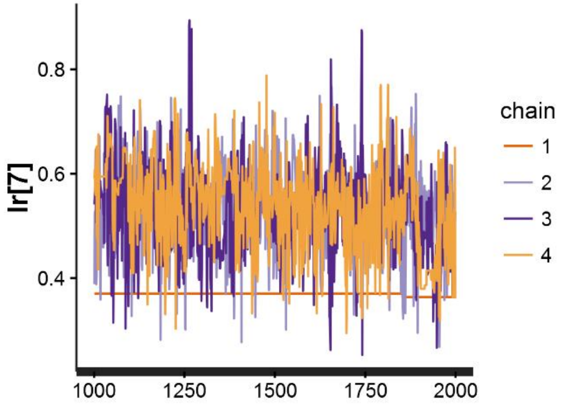


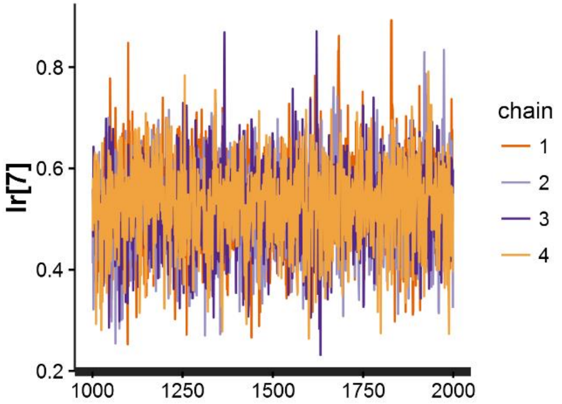
Posterior Means (hrch+optm)



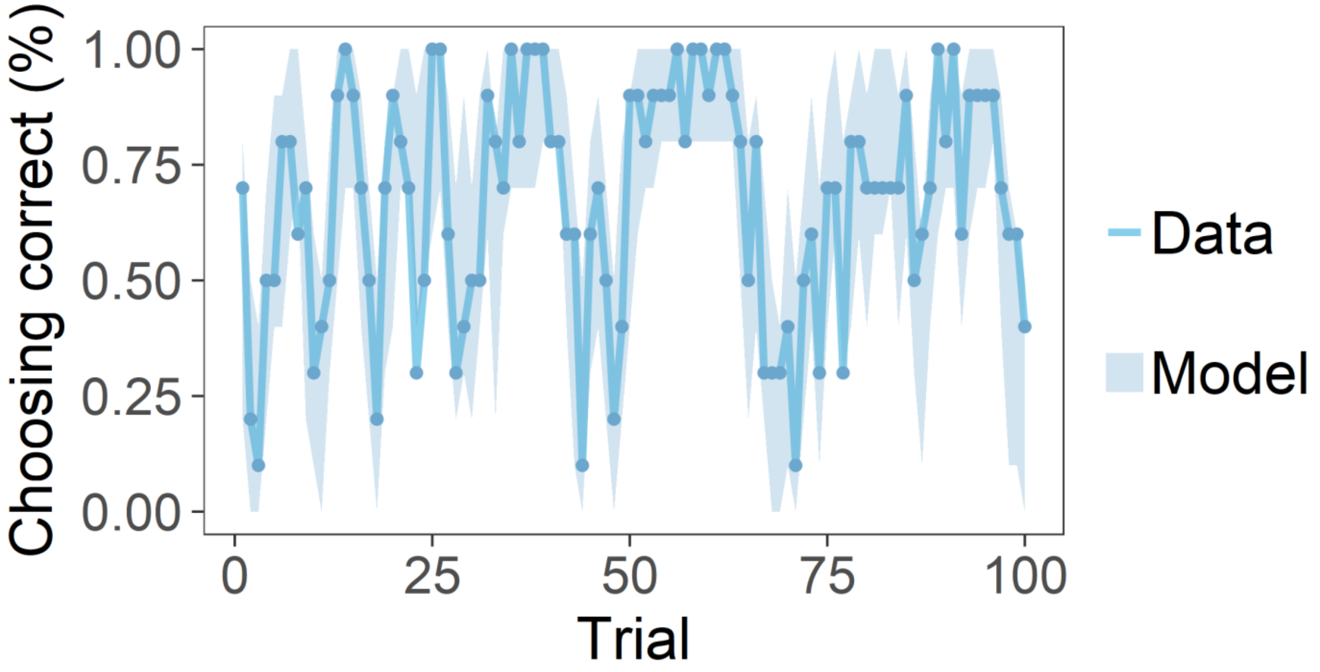
True Parameters

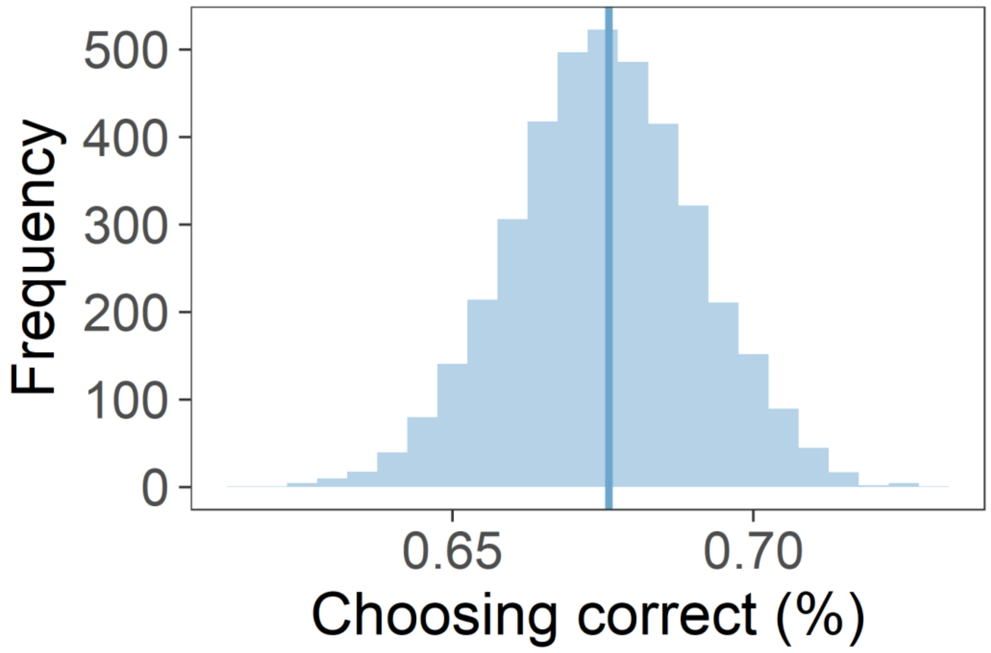






Posterior Predictive Check






ANY
QUESTIONS
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Happy Computing!