Bayesian Statistics and Hierarchical Bayesian Modeling for Psychological Science

Lecture 12

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https://github.com/lei-zhang/BayesCog_Wien

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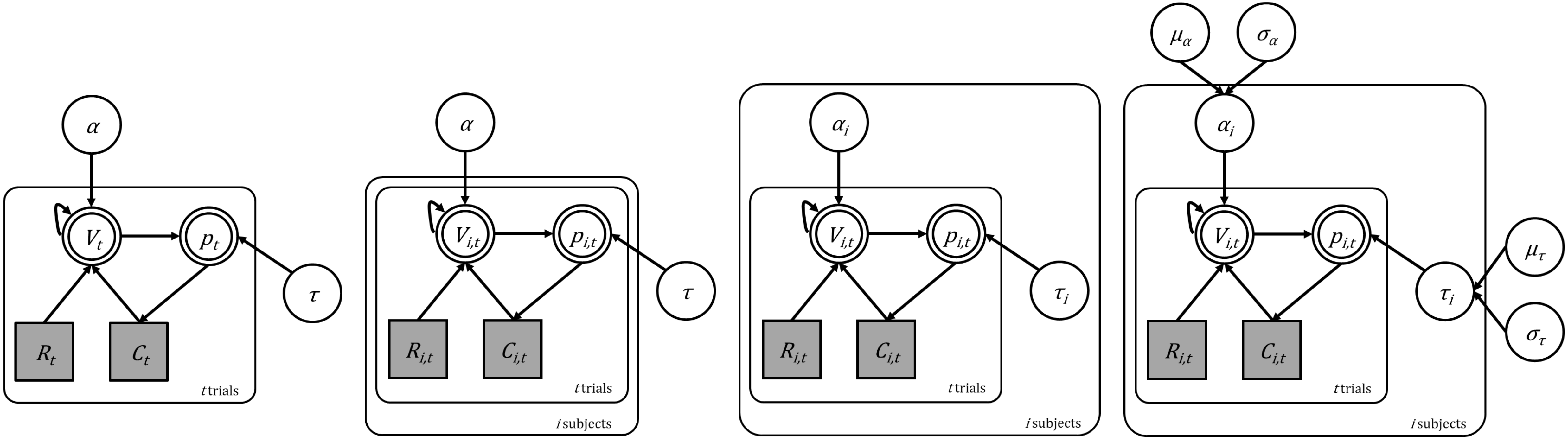
Bayesian warm-up?



constraint	reparameterization
$\theta \in (-\infty, +\infty)$	$\theta = \mu_\theta + \sigma_\theta \tilde{\theta}$
$\theta \in [0, N]$	$\theta = Probit^{-1}(\mu_\theta + \sigma_\theta \tilde{\theta}) \times N$
$\theta \in [M, N]$	$\theta = Probit^{-1}(\mu_\theta + \sigma_\theta \tilde{\theta}) \times (N - M) + M$
$\theta \in (0, +\infty)$	$\theta = exp(\mu_\theta + \sigma_\theta \tilde{\theta})$

Apply to Our Hierarchical RL Model

cognitive model
statistics
computing

```
parameters {
  real<lower=0,upper=1> lr_mu;
  real<lower=0,upper=3> tau_mu;

  real<lower=0> lr_sd;
  real<lower=0> tau_sd;

  real<lower=0,upper=1> lr[nSubjects];
  real<lower=0,upper=3> tau[nSubjects];
}
```

```
parameters {
  # group-level parameters
  real lr_mu_raw;
  real tau_mu_raw;
  real<lower=0> lr_sd_raw;
  real<lower=0> tau_sd_raw;

  # subject-level raw parameters
  vector[nSubjects] lr_raw;
  vector[nSubjects] tau_raw;
}

transformed parameters {
  vector<lower=0,upper=1>[nSubjects] lr;
  vector<lower=0,upper=3>[nSubjects] tau;

  for (s in 1:nSubjects) {
    lr[s]  = Phi_approx( lr_mu_raw  + lr_sd_raw * lr_raw[s] );
    tau[s] = Phi_approx( tau_mu_raw + tau_sd_raw * tau_raw[s] ) * 3;
  }
}
```

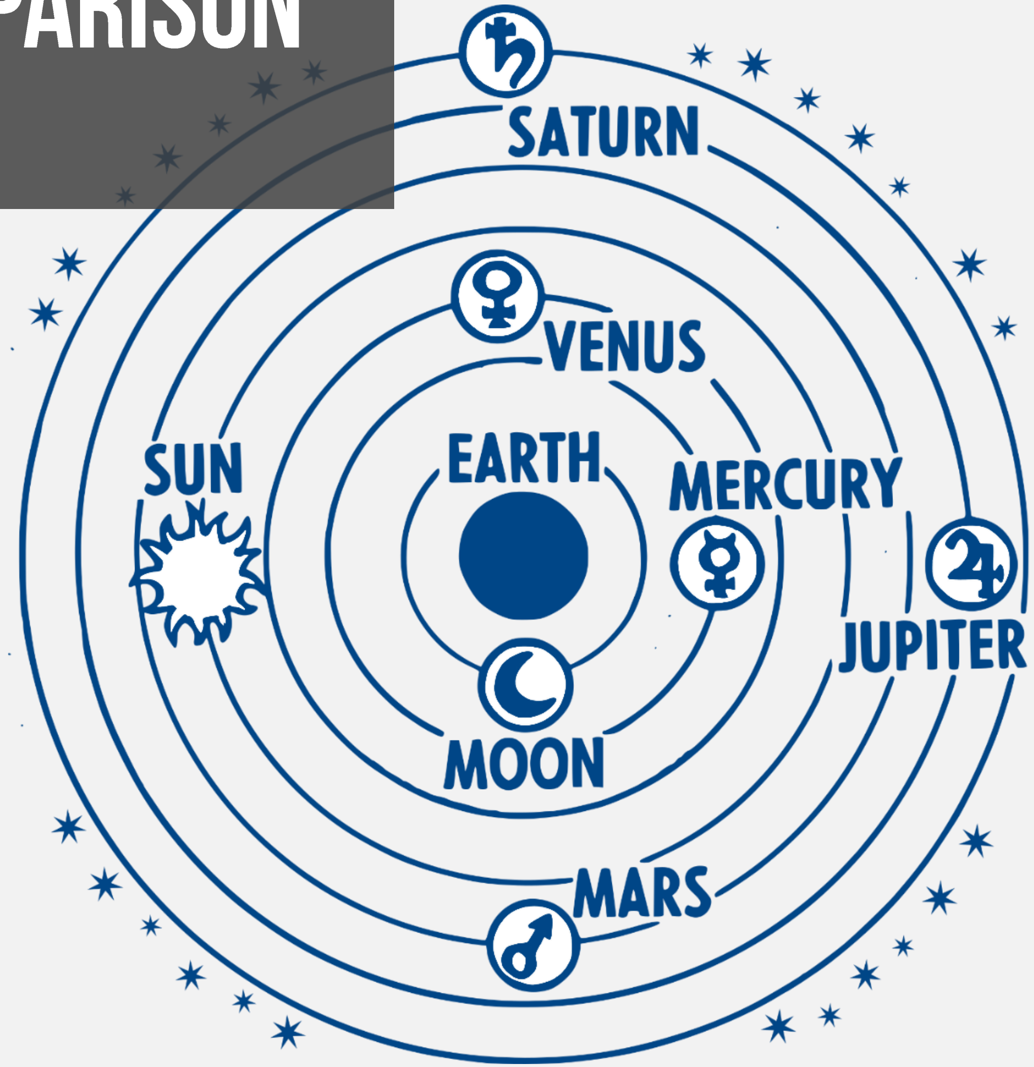
Apply to Our Hierarchical RL Model

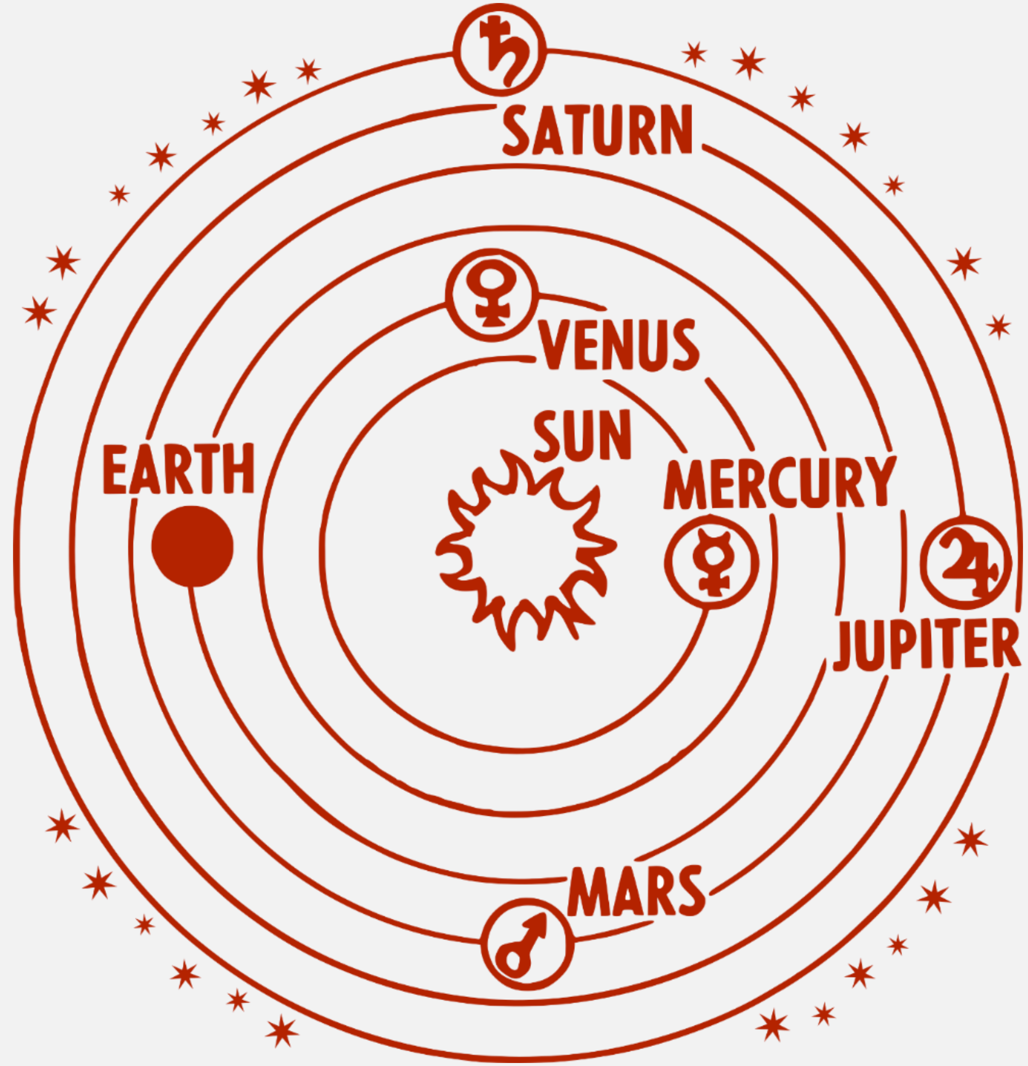

```
model {
  lr_sd  ~ cauchy(0,1);
  tau_sd ~ cauchy(0,3);
  lr     ~ normal(lr_mu, lr_sd) ;
  tau    ~ normal(tau_mu, tau_sd) ;

  for (s in 1:nSubjects) {
    vector[2] v;
    real pe;
    v = initV;

    for (t in 1:nTrials) {
      choice[s,t] ~ categorical_logit( tau[s] * v );
      pe = reward[s,t] - v[choice[s,t]];
      v[choice[s,t]] = v[choice[s,t]] + lr[s] * pe;
    }
  }
}
```

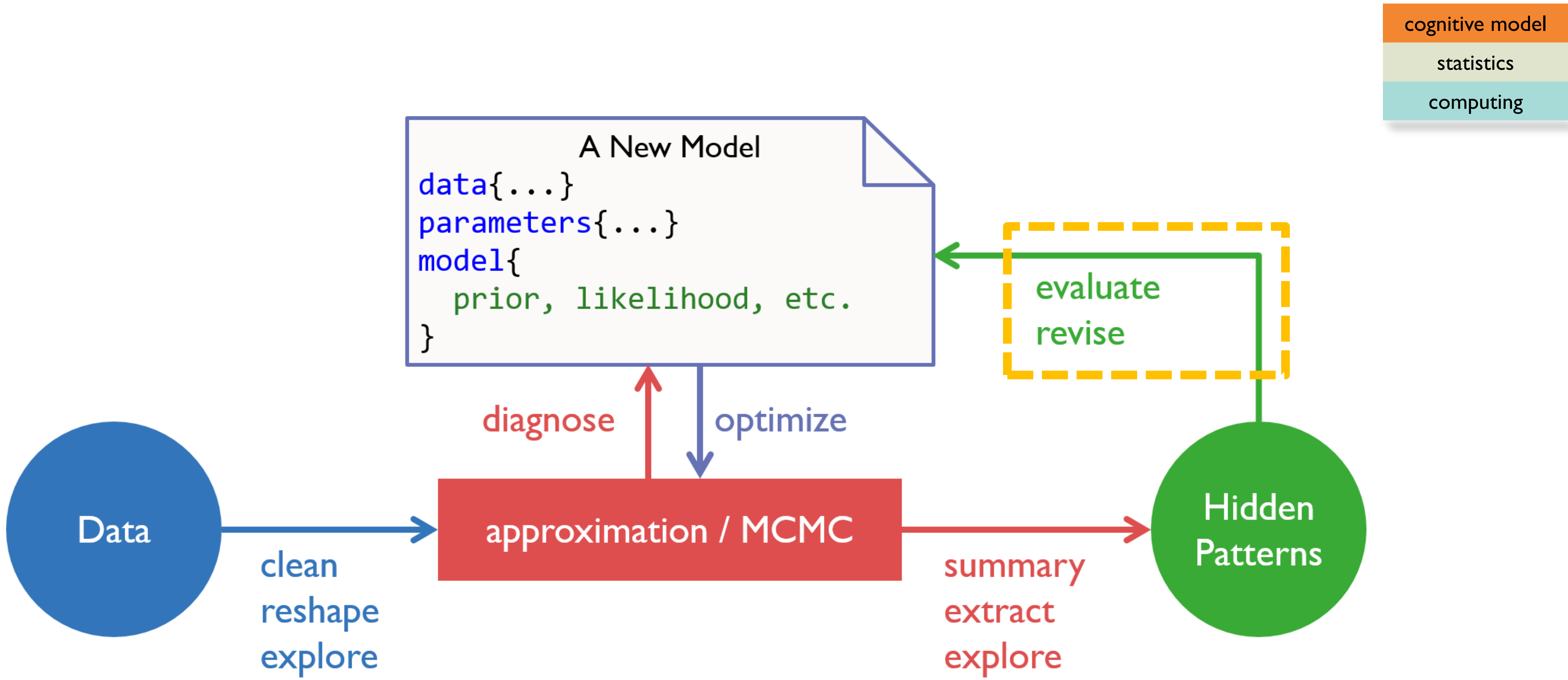
```
model {
  lr_mu_raw  ~ normal(0,1);
  tau_mu_raw ~ normal(0,1);
  lr_sd_raw  ~ cauchy(0,3);
  tau_sd_raw ~ cauchy(0,3);

  lr_raw  ~ normal(0,1);
  tau_raw ~ normal(0,1);

  for (s in 1:nSubjects) {
    ...
```

```
generated quantities {
  real<lower=0,upper=1> lr_mu;
  real<lower=0,upper=3> tau_mu;

  lr_mu  = Phi_approx(lr_mu_raw);
  tau_mu = Phi_approx(tau_mu_raw) * 3;
}
```

MODEL COMPARISON






cognitive model
statistics
computing
A New Model
data{...}
parameters{...}
model{
prior, likelihood, etc.
}
evaluate
revise
diagnose
optimize
Data
clean
reshape
explore
approximation / MCMC
summary
extract
explore
Hidden Patterns

Model Comparison

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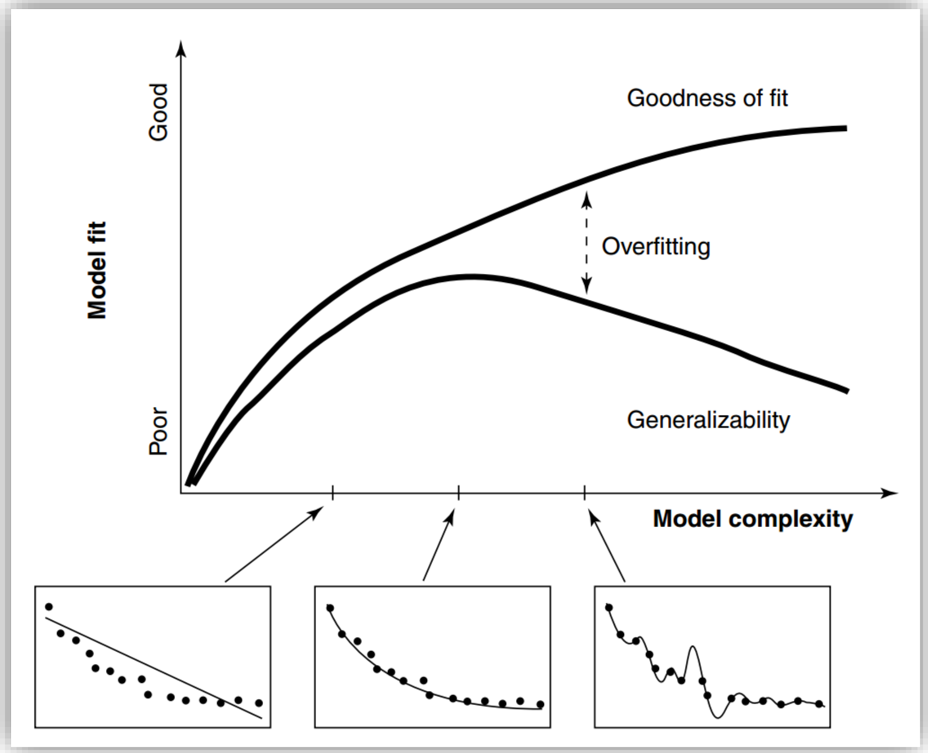
Which model provides the best fit?

Which model represents the best balance between model fit and model complexity?

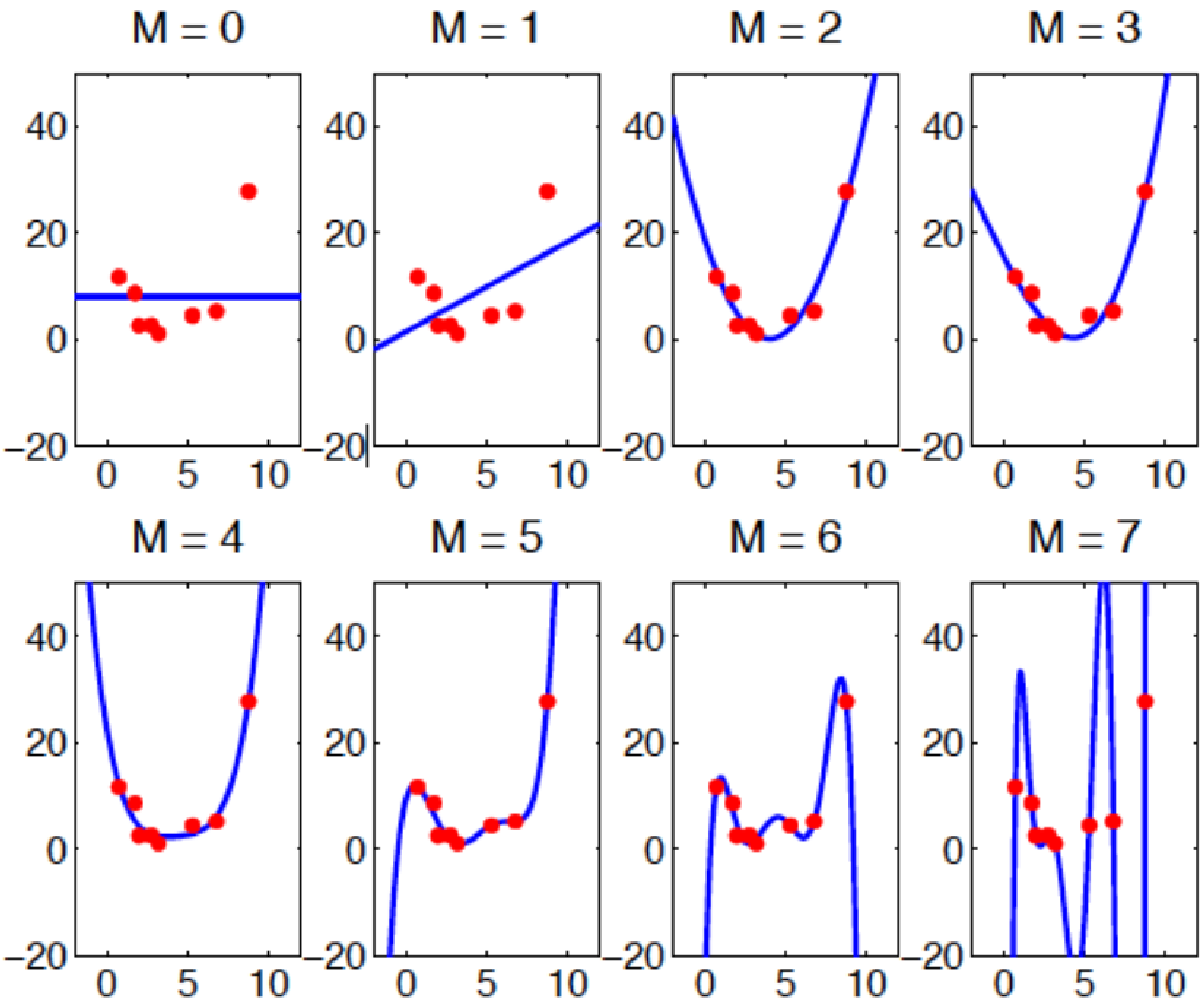
Ockham's razor:
Models with fewer assumptions are to be preferred



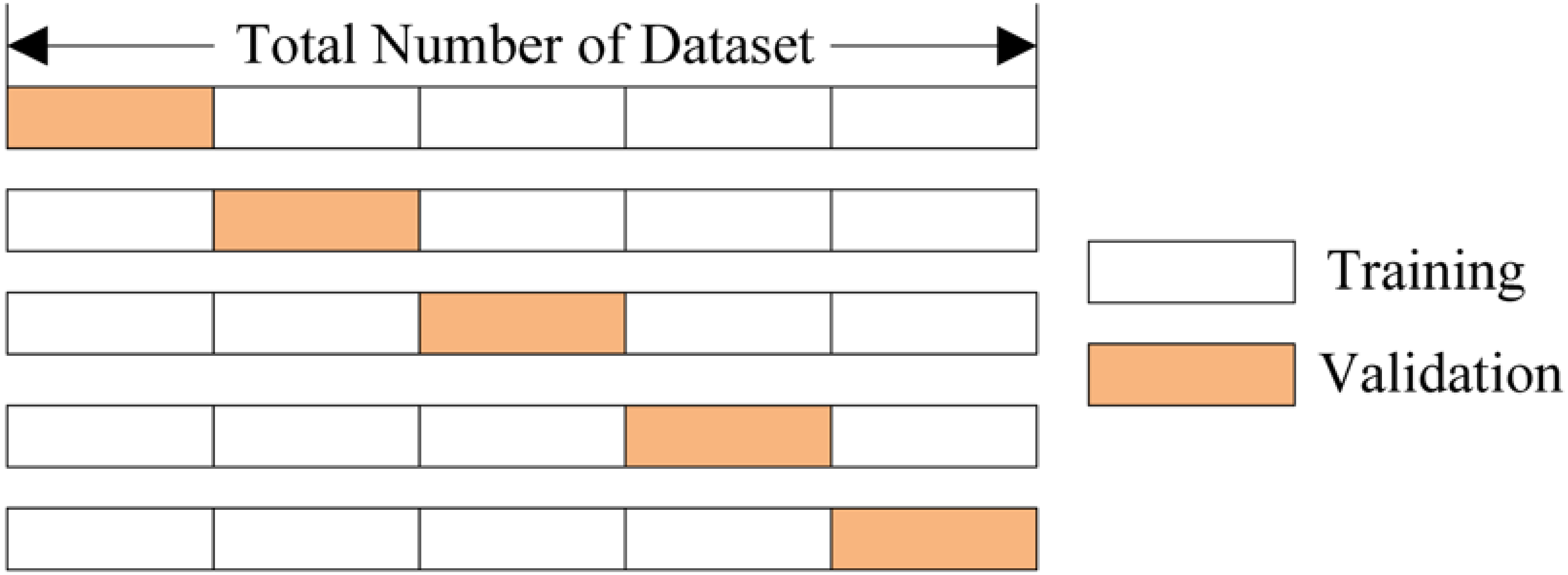
- overfitting: learn too much from the data
- underfitting: learn too little from the data

Pitt & Miyung (2002)

Which model has the highest predictive power?



Focusing on Predictive Accuracy



- Nothing prevents you from doing that in a Bayesian context but holding out data makes your posterior distribution more diffuse
- Bayesians usually condition on *all* the data and evaluate how well a model is *expected* to predict out of sample using "information criteria": model with the highest expected log predictive density (ELPD) for new data

Information Criteria

AIC – Akaike information criterion

DIC – Deviance Information Criterion

WAIC – Widely Applicable Information Criterion

finding the model that has the highest out-of-sample predictive accuracy

approximation to LOO

BIC – Bayesian Information Criterion

finding the "true" model

Compute WAIC from Likelihood

$$\text{WAIC} = -2\,\widehat{\text{elpd}}_{\text{waic}}$$

expected log pointwise predictive density

$$\widehat{\text{elpd}}_{\text{waic}} = \widehat{\text{lpd}} - \widehat{p}_{\text{waic}}$$

$$\begin{aligned}\widehat{\text{lpd}} &= \text{computed log pointwise predictive density} \\ &= \sum_{i=1}^{n} \log\left(\frac{1}{S}\sum_{s=1}^{S} p(y_i|\theta^s)\right).\end{aligned}$$

```
lpd <- log(colMeans(exp(log_lik)))
```

estimated effective number of parameters

$$\widehat{p}_{\text{waic}} = \sum_{i=1}^{n} V_{s=1}^{S}\left(\log p(y_i|\theta^s)\right)$$

```
p_waic <- colVars(log_lik)
```

Vehtari & Gelman (2014)

*IC comparisons

		No pooling ($\tau = \infty$)	Complete pooling ($\tau = 0$)	Hierarchical model (τ estimated)
AIC	$-2\,\text{lpd} = -2\log p(y\|\hat{\theta}_{\text{mle}})$	54.6	59.4	
	k	8.0	1.0	
	$\text{AIC} = -2\,\widehat{\text{elpd}}_{\text{AIC}}$	70.6	61.4	
DIC	$-2\,\text{lpd} = -2\log p(y\|\hat{\theta}_{\text{Bayes}})$	54.6	59.4	57.4
	p_{DIC}	8.0	1.0	2.8
	$\text{DIC} = -2\,\widehat{\text{elpd}}_{\text{DIC}}$	70.6	61.4	63.0
WAIC	$-2\,\text{lppd} = -2\sum_i \log p_{\text{post}}(y_i)$	60.2	59.8	59.2
	$p_{\text{WAIC}\,1}$	2.5	0.6	1.0
	$p_{\text{WAIC}\,2}$	4.0	0.7	1.3
	$\text{WAIC} = -2\,\widehat{\text{elppd}}_{\text{WAIC}\,2}$	68.2	61.2	61.8
LOO-CV	$-2\,\text{lppd}$		59.8	59.2
	$p_{\text{loo-cv}}$		0.5	1.8
	$-2\,\text{lppd}_{\text{loo-cv}}$		60.8	62.8

Gelman et al. (2013)

Recording the Log-Likelihood in Stan


```
generated quantities {
  ...
  real log_lik[nSubjects];
  ...

  { # local section, this saves time and space
    for (s in 1:nSubjects) {
      vector[2] v;
      real pe;

      log_lik[s] = 0;
      v = initV;

      for (t in 1:nTrials) {
        log_lik[s] = log_lik[s] + categorical_logit_lpmf(choice[s,t] | tau[s] * v);

        pe = reward[s,t] - v[choice[s,t]];
        v[choice[s,t]] = v[choice[s,t]] + lr[s] * pe;
```

The {loo} Package

```
> library(loo)
> LL1   <- extract_log_lik(stanfit)
> loo1  <- loo(LL1)    # PSIS leave-one-out
> waic1 <- waic(LL1)   # WAIC
```

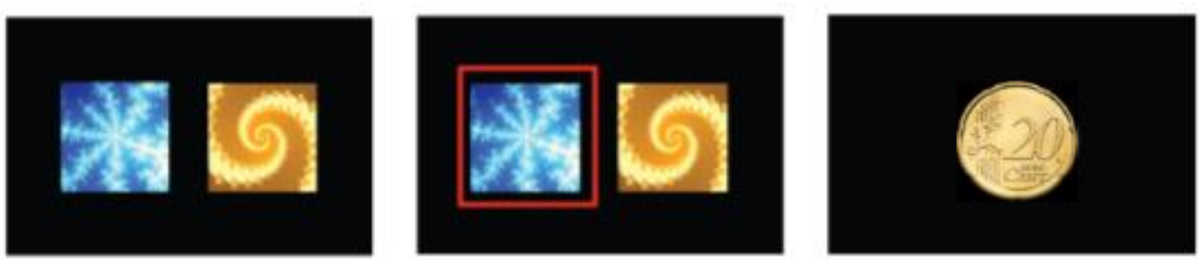
Pareto Smoothed Importance Sampling

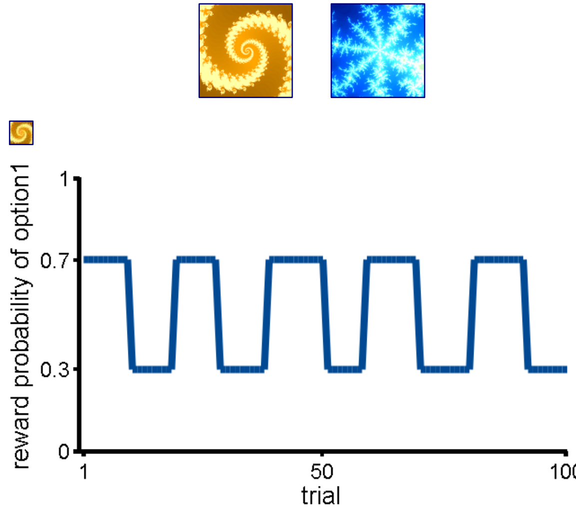
```
Computed from 4000 by 20 log-likelihood matrix

         Estimate  SE
elpd_loo    -29.5 3.3
p_loo         2.7 1.0
looic        58.9 6.7
```

Vehtari et al. (2015)

Reversal Learning Task





Fictitious RL
(Counterfactual RL)

Value update:

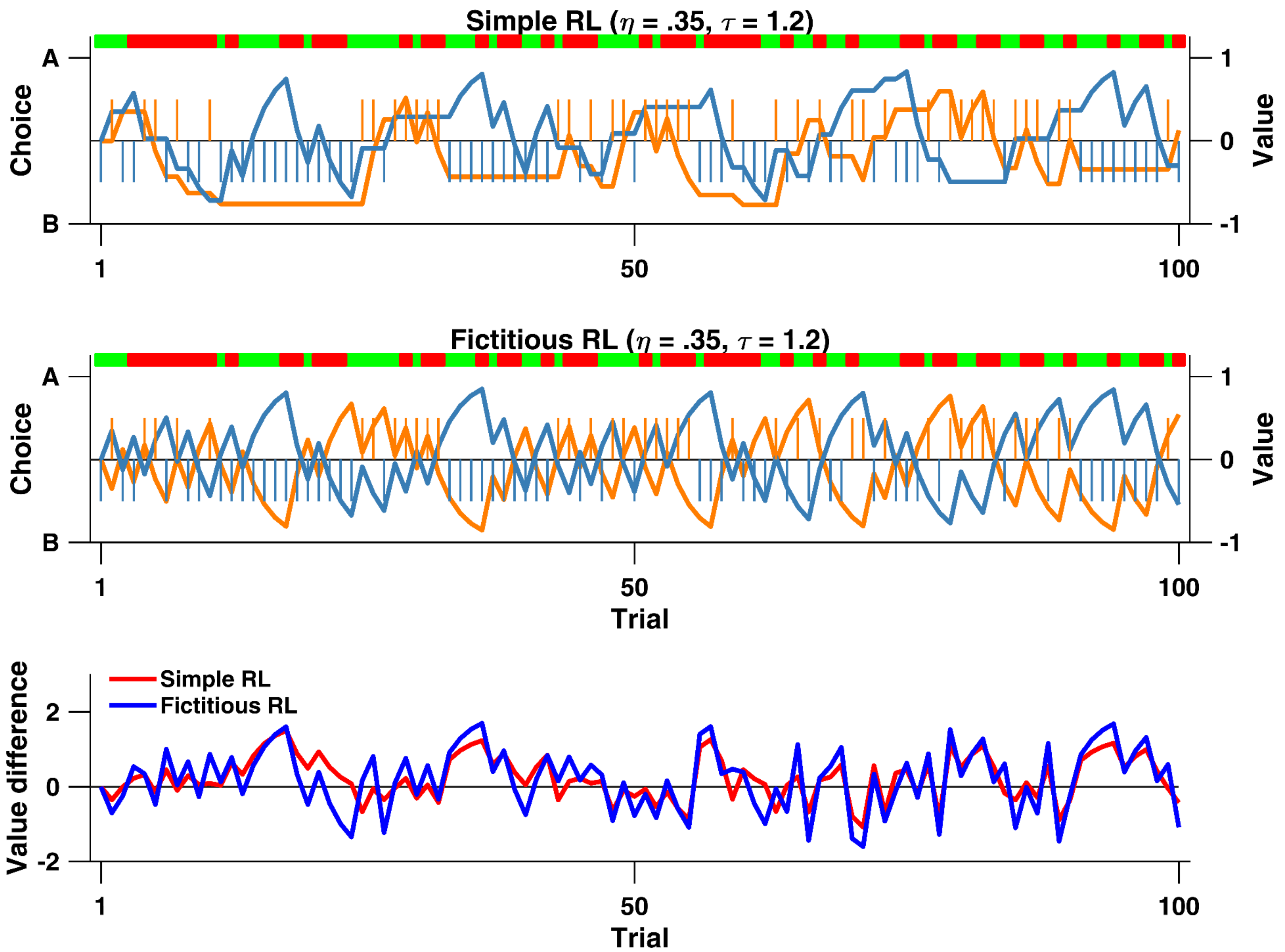
$$V_{t+1}^{c} = V_t^{c} + \alpha * PE$$
$$V_{t+1}^{nc} = V_t^{nc} + \alpha * PEnc$$

Prediction error:

$$PE = R_t - V_t^{c}$$
$$PEnc = -R_t - V_t^{nc}$$

Hampton et al. (2007)
Gläscher et al. (2009)

More on Fictitious RL



cognitive model
statistics
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Exercise XIII

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```
…/08.compare_models/_scripts/compare_models_main.R
```

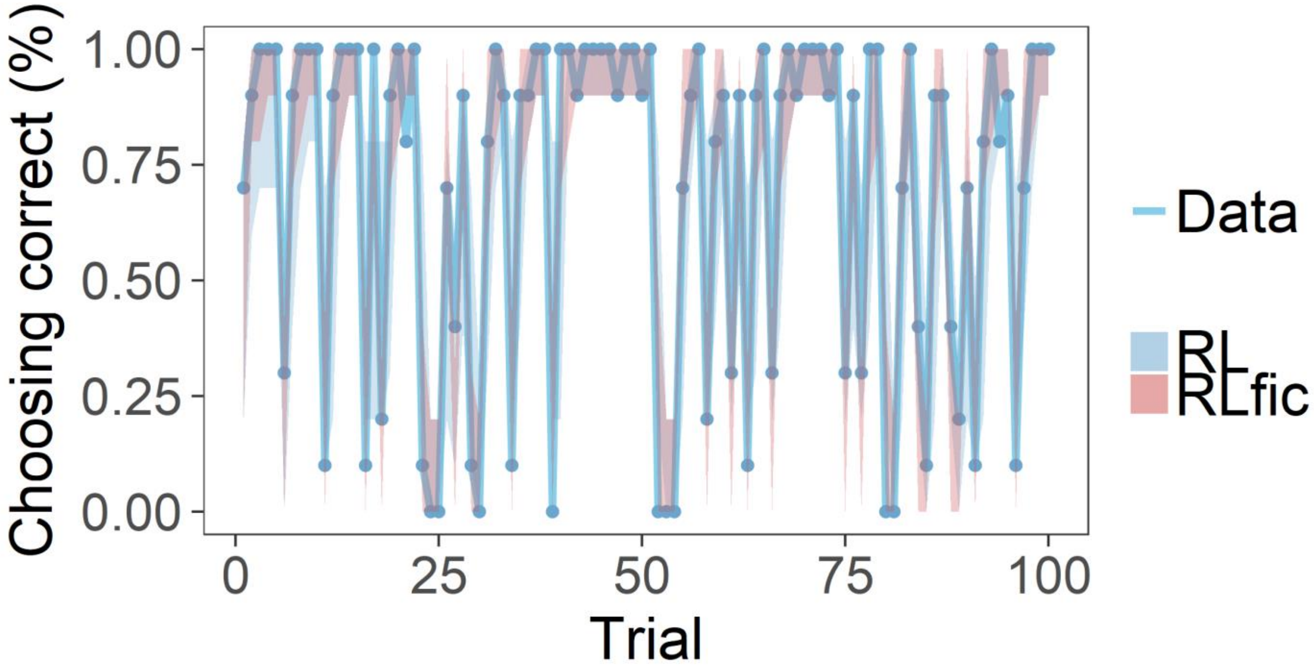
TASK: (1) complete the fictitious RL model (model2, `loglik`)

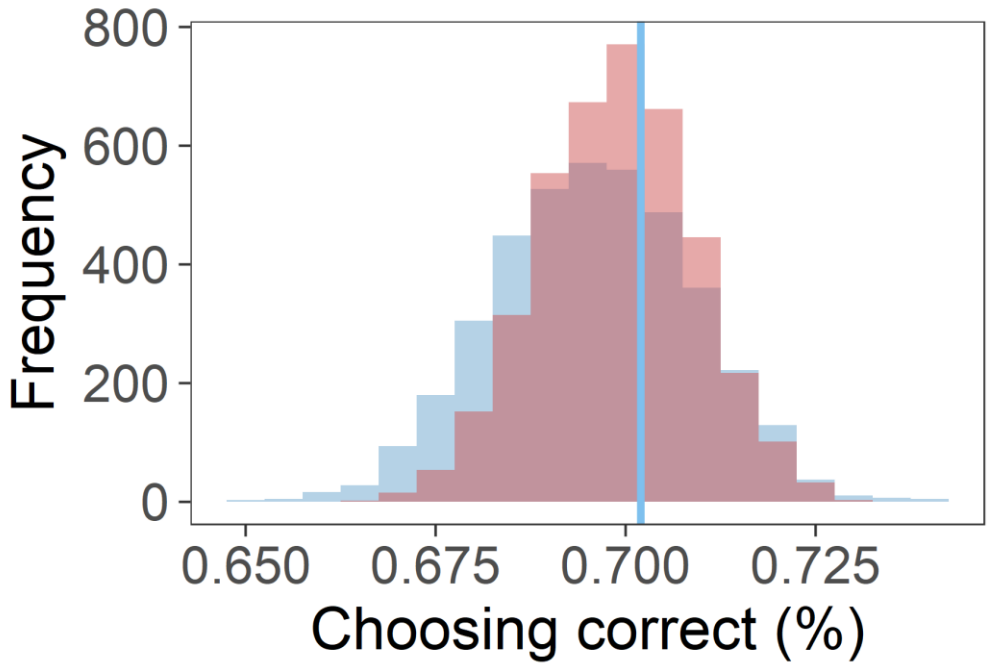
(2) fit and compare the 2 models

Exercise XIII – output


```
> LL1 <- extract_log_lik(fit_rl1)
> ( loo1 <- loo(LL1) )
Computed from 4000 by 10 log-likelihood matrix

         Estimate   SE
elpd_loo   -389.8 15.4
p_loo         3.8  0.8
looic       779.5 30.9

> ( loo2 <- loo(LL2) )
Computed from 4000 by 10 log-likelihood matrix

         Estimate   SE
elpd_loo   -281.3 17.5
p_loo         3.4  0.5
looic       562.6 35.0
```

Posterior Predictive Check






ANY
QUESTIONS
?

Happy Computing!