Bayesian Statistics and Bayesian Cognitive Modeling

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Schedule

		cognitive model
		statistics
		computing

DAY1	9:00 – 17:00	Introduction R Basics Probability Basics Bayes' theorem MCMC and Stan Single-Parameter Model – Binomial Model
DAY2	9:00 – 17:00	Multiple-Parameter Model – Linear Regression Inference, Posterior Predictive Check Reinforcement Learning Model Hierarchical Models Optimizing Stan Codes Model Comparison
DAY3	9:00 – 13:00	Stan Style Tip and Debugging Model-Based fMRI Capstone Project: Delay Discounting Task

DAY1

09:00 – 09:30	Introduction and overview
09:30 – 10:00	R Basics
10:00 – 10:30	Probability basics
10:30 – 10:45	Coffee break
10:45 – 11:30	Bayes' theorem
11:30 – 12:30	Calculation examples
12:30 – 13:30	Lunch break
13:30 – 14:00	Linking data and parameter
14:00 – 15:00	Binomial model
15:00 – 15:15	Coffee Break
15:15 – 16:15	MCMC and Stan
16:15 – 17:00	Binomial model in Stan

Overview

<u>What is your experience with…</u>

- Statistics?
- R? (and / or Matlab?)
- Cognitive Modeling?

<u>You would like to…</u>

- gain knowledge of Bayesian stats?
- be able to read "computational modeling" section in papers?
- write your own model?

About me

- Current position: Postdoc @ Gläscher Lab, ISN UKE
- Ph.D. Cognitive/computational neuroscience
- M.Sc. Cognitive neuroscience
- B.Sc. Psychology

- My journey through computational modeling
  - Started with MLE (@fminsearch in Matlab)
  - Switched to Bayesian: first JAGS, then Stan.

Overview

This workshop is NOT about…

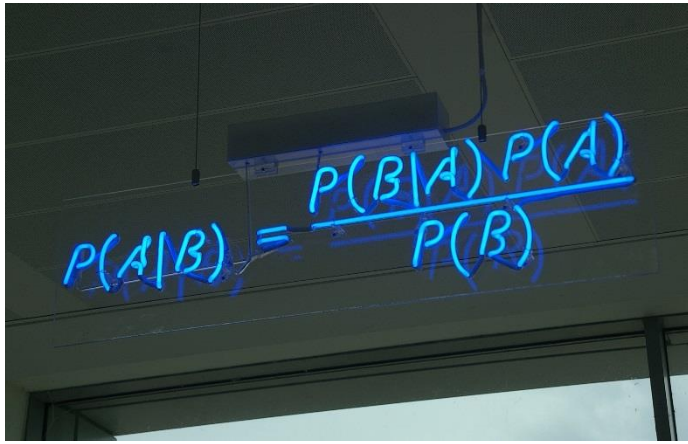
… Bayes in the brain (e.g. predictive coding)

… Cognitive process that are Bayesian themselves

… Bayesian statistics to supersede classic statistics

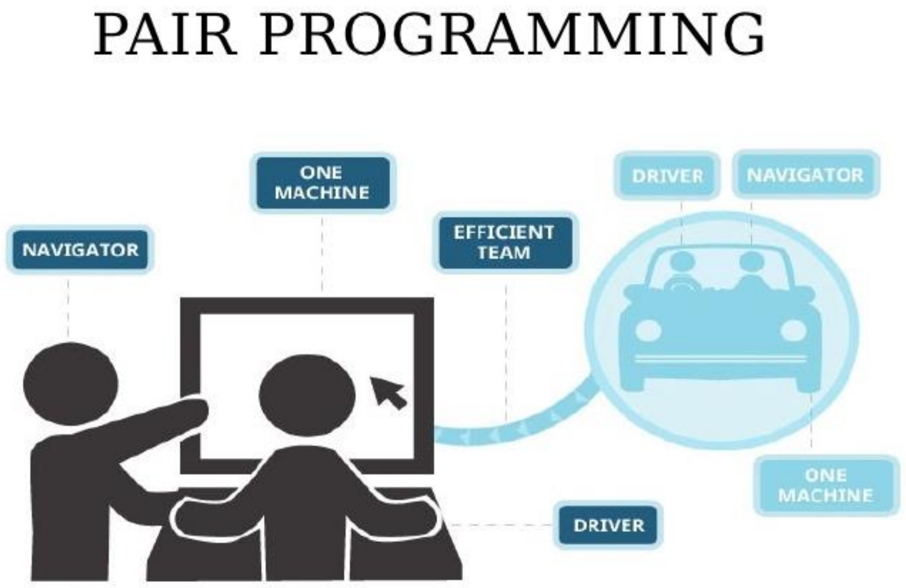
However, Bayesian statistics offer great tools to analyze cognitive processes!

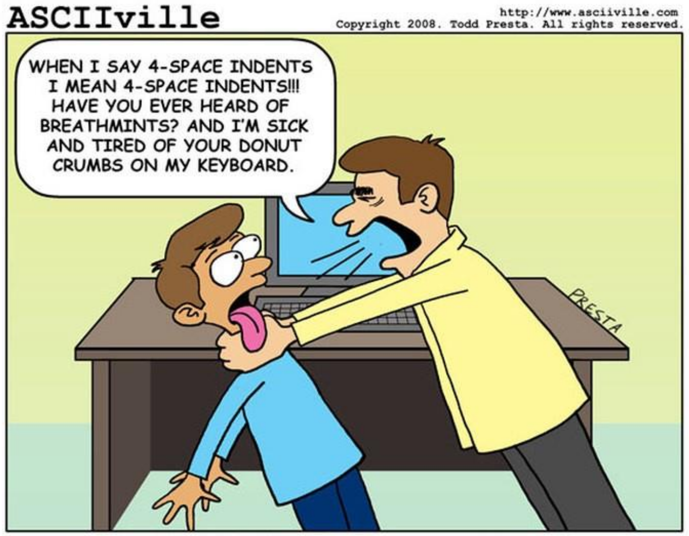
- Construct cognitive models
- Estimate posterior distributions of parameters
- Compare models: which is the best one, given the data
- Perform model-based analysis, e.g. model-based fMRI/EEG



How to Get the Most out of the Workshop

- Work in pairs: Talk to each other & help each other
- Ask questions
- Try the exercises





The dark side of pair programming.

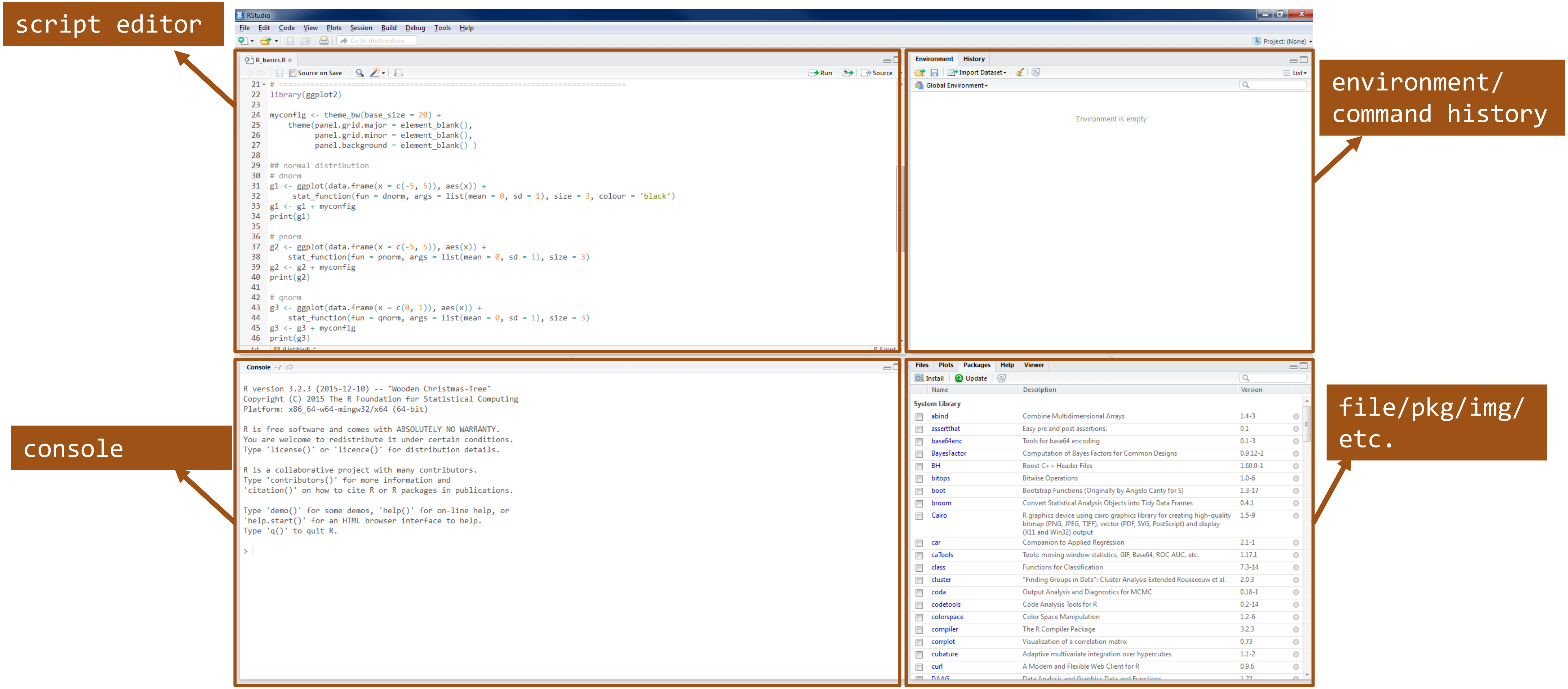
BASICS OF R PROGRAMMING

RStudio®

R Basics

- R
  - a programming language for statistical computing
  - R has its own user interface
  - freely available on Windows, Mac, and Linux

- R Studio
  - integrated development environment (IDE) for R
  - a more sophisticated R-friendly editor, with helpful syntax highlight


script editor
environment/ command history
console
file/pkg/img/ etc.
RStudio
File Edit Code View Plots Session Build Debug Tools Help
R_basics.R
Source on Save
Run
Source
# =====================
library(ggplot2)
myconfig <- theme_bw(base_size = 20) +
theme(panel.grid.major = element_blank(),
panel.grid.minor = element_blank(),
panel.background = element_blank())
normal distribution
# dnorm
g1 <- ggplot(data.frame(x = c(-5, 5)), aes(x)) +
stat_function(fun = dnorm, args = list(mean = 0, sd = 1), size = 3, colour = 'black')
g1 <- g1 + myconfig
print(g1)
# pnorm
g2 <- ggplot(data.frame(x = c(-5, 5)), aes(x)) +
stat_function(fun = pnorm, args = list(mean = 0, sd = 1), size = 3)
g2 <- g2 + myconfig
print(g2)
# qnorm
g3 <- ggplot(data.frame(x = c(0, 1)), aes(x)) +
stat_function(fun = qnorm, args = list(mean = 0, sd = 1), size = 3)
g3 <- g3 + myconfig
print(g3)
Environment History
Import Dataset
List
Global Environment
Environment is empty
Console ~/
R version 3.2.3 (2015-12-10) -- "Wooden Christmas-Tree"
Copyright (C) 2015 The R Foundation for Statistical Computing
Platform: x86_64-w64-mingw32/x64 (64-bit)
R is free software and comes with ABSOLUTELY NO WARRANTY.
You are welcome to redistribute it under certain conditions.
Type 'license()' or 'licence()' for distribution details.
R is a collaborative project with many contributors.
Type 'contributors()' for more information and
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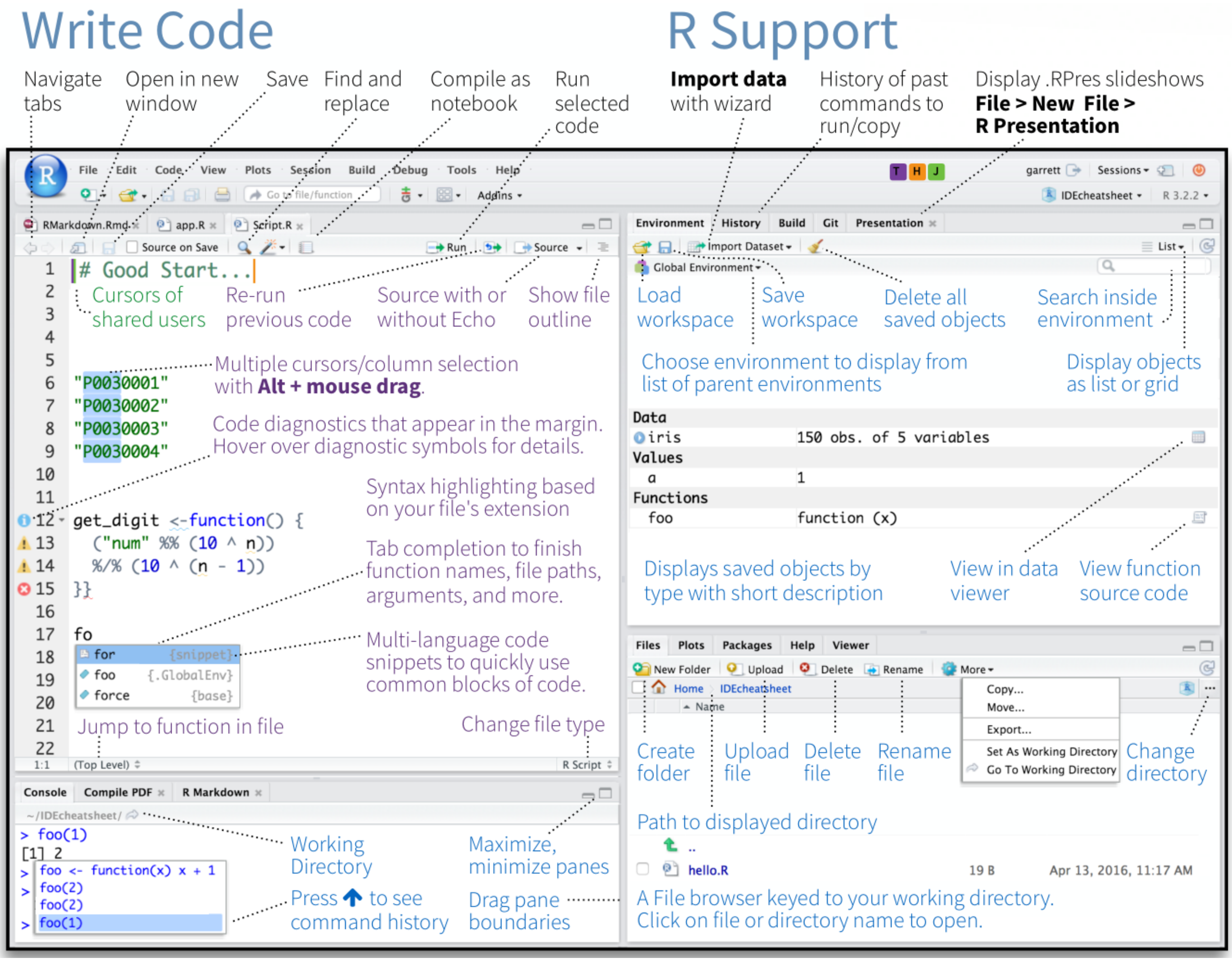
Type 'demo()' for some demos, 'help()' for on-line help, or
'help.start()' for an HTML browser interface to help.
Type 'q()' to quit R.
Files Plots Packages Help Viewer
Install Update
Name Description Version
System Library
abind Combine Multidimensional Arrays 1.4-3
assertthat Easy pre and post assertions. 0.1
base64enc Tools for base64 encoding 0.1-3
BayesFactor Computation of Bayes Factors for Common Designs 0.9.12-2
BH Boost C++ Header Files 1.60.0-1
bitops Bitwise Operations 1.0-6
boot Bootstrap Functions (Originally by Angelo Canty for S) 1.3-17
broom Convert Statistical Analysis Objects into Tidy Data Frames 0.4.1
Cairo R graphics device using cairo graphics library for creating high-quality bitmap (PNG, JPEG, TIFF), vector (PDF, SVG, PostScript) and display (X11 and Win32) output 1.5-9
car Companion to Applied Regression 2.1-1
caTools Tools: moving window statistics, GIF, Base64, ROC AUC, etc. 1.17.1
class Functions for Classification 7.3-14
cluster "Finding Groups in Data": Cluster Analysis Extended Rousseeuw et al. 2.0.3
coda Output Analysis and Diagnostics for MCMC 0.18-1
codetools Code Analysis Tools for R 0.2-14
colorspace Color Space Manipulation 1.2-6
compiler The R Compiler Package 3.2.3
corrplot Visualization of a correlation matrix 0.73
cubature Adaptive multivariate integration over hypercubes 1.1-2
curl A Modern and Flexible Web Client for R 0.9.6



cognitive model

statistics

computing

https://github.com/rstudio/cheatsheets/raw/master/rstudio-ide.pdf

R Console as a Calculator

Addition and Subtraction

```
> 3+2
[1] 5

> 3-2
[1] 1
```

Multiplication and Division

```
> 3*2
[1] 6

> 3/2
[1] 1.5
```

Exponents in R

```
> 3^2
[1] 9

> 2^3
[1] 8
```

Constants in R

```
> pi
[1] 3.141593

> exp(1)
```

base of the natural logarithm

```
[1] 2.718282
```

Special values

Infinite Values

```
> Inf
[1] Inf

> 1+Inf
[1] Inf
```

Machine Epsilon

```
> .Machine$double.eps
[1] 2.220446e-16

> 0>.Machine$double.eps
[1] FALSE
```

Empty Values

```
> NULL
NULL

> 1+NULL
numeric(0)
```

Missing Values

```
> NA
[1] NA

> 1+NA
[1] NA
```

Storing and manipulating variables

Define objects `x` and `y` with values of 3 and 2, respectively:

```
> x=3
> y=2
```

Some calculations with the defined objects `x` and `y`:

```
> x+y
[1] 5
```

```
> x*y
[1] 6
```

Warning: R is case sensitve, so `x` and `X` are not the same object.

Basic R functions

Combine

```
> c(1,3,-2)
[1]  1  3 -2
```

```
> c("a","a","b","b","a")
[1] "a" "a" "b" "b" "a"
```

Sum and Mean

```
> sum(c(1,3,-2))
[1] 2
```

```
> mean(c(1,3,-2))
[1] 0.6666667
```

Variance and Std. Dev.

```
> var(c(1,3,-2))
[1] 6.333333
```

```
> sd(c(1,3,-2))
[1] 2.516611
```

Minimum and Maximum

```
> min(c(1,3,-2))
[1] -2
```

```
> max(c(1,3,-2))
[1] 3
```

Basic R functions (cont.)

Define objects `x` and `y`:

```
> x=c(1,3,4,6,8)
> y=c(2,3,5,7,9)
```

Calculate the correlation:

```
> cor(x,y)
[1] 0.988765
```

Calculate the covariance:

```
> cov(x,y)
[1] 7.65
```

Combine as columns

```
> cbind(x,y)
     x y
[1,] 1 2
[2,] 3 3
[3,] 4 5
[4,] 6 7
[5,] 8 9
```

Combine as rows

```
> rbind(x,y)
  [,1] [,2] [,3] [,4] [,5]
x    1    3    4    6    8
y    2    3    5    7    9
```

Basic Commands

```
getwd()
setwd('E:/teaching/BayesCog/')
dir() # folders/files in the wd
ls()  # anything in the environment/workspace
print('Hello World!')
cat('Hello', 'World!')
paste0('C:/', 'Group1')
help(func)
? func
a <- 5
a = 5
head(d) # first 6 entries
tail(d) # last 6 entries
save(varname, file = "pathname/varname.RData")
load("pathname/varname.RData")
rm(list = ls())
q()
```

RStudio - Shortcuts

Ctrl + L: clean console

Ctrl + Shift + N: create a new script

↑: command history

Ctrl(hold) + ↑: command history with certain starts

Ctrl + Enter: execute selected codes (in a script)

Editor (WIN general) - Shortcuts

`Ctrl + home/Pos`: go to the very top of a script

`Ctrl + end/Ende`: go to the very end of a script

`Shift(hold) + ↑/↓`: select line(s)

`Ctrl(hold) + ←/→`: select word(s)

Data Classes

<u>numeric</u>: 1.1 2.0

<u>integer</u>: 1 2 3

<u>character</u> / <u>string</u>: "hello world!"

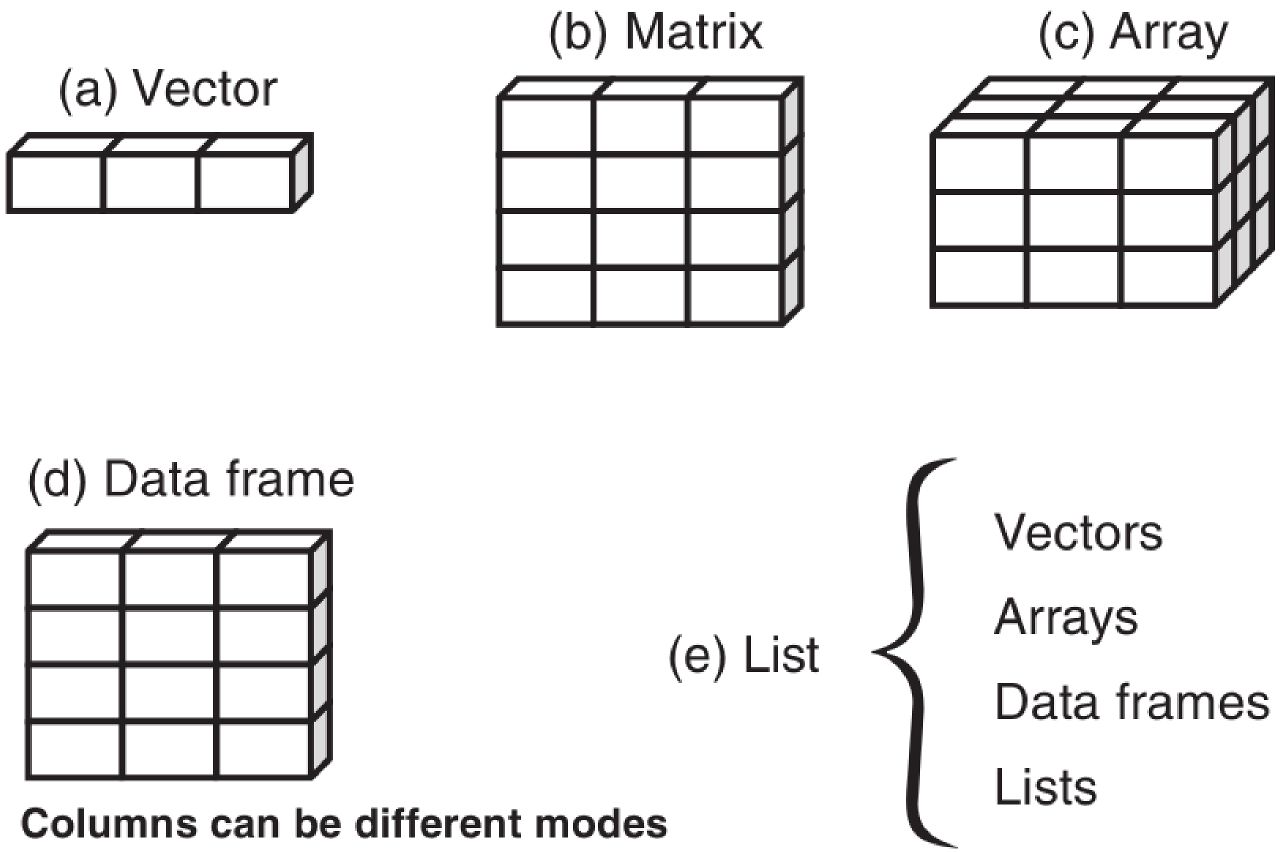
<u>logical</u>: TRUE FALSE

<u>factors</u>: "male" / "female"

(<u>complex</u>: 1+2i)

Data Types

- Vector
- Matrix
- Array
- Data Frame
- List



Kabacoff (2015)

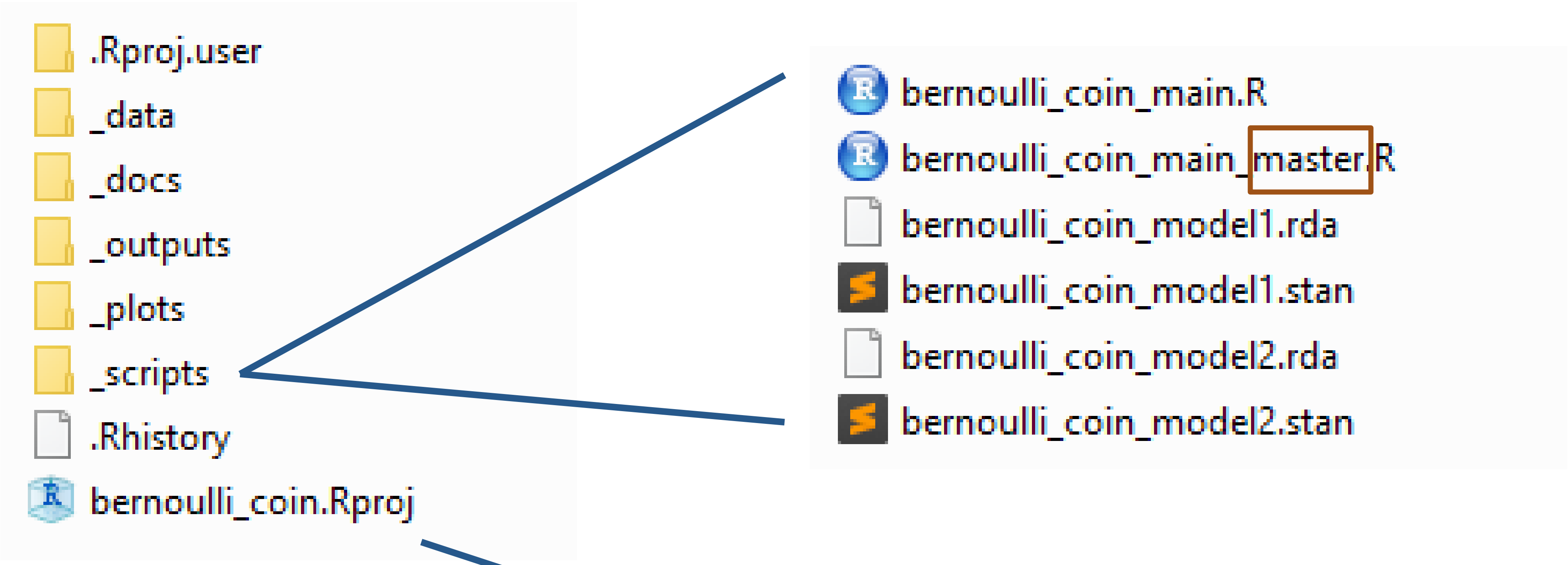
Exercise I

```
…/BayesCog/01.R_basics/_scripts/R_basics.R

up to "Control Flow"
```

TASK: practise basic R commands and data type

TIP: `class()`, `str()`

Side note: folder structure



click this to start each exercise,
then no need to set directory

Logical Operators

Operator	Summary
$<$	Less than
$>$	Greater than
$<=$	Less than or equal to
$>=$	Greater than or equal to
$==$	Equal to
$!=$	Not equal to
$!x$	NOT x
$x \| y$	x OR y
$x\&y$	x AND y

Control Flow

- if-else

```
if (cond) {
   ..statement..
} else {
   ..statement..
}
```

```
if (cond) {
   ..statement..
} else if (cond) {
   ..statement..
} else {
   ..statement..
}
```

- for-loop

```
for ( j in 1:n) {
   ..statement..
}
```

```
for ( j in 1:J ) {
   for ( k in 1:K ) {
      ..statement..
   }
}
```

User-defined Function

```
funname <- function (input_arges) {
    .. function body ..
    .. function body ..
    return(output_arges)
}
```

$$sem = \sqrt{\frac{s^2}{n-1}}$$

```
sem <- function(x) {
    sqrt( var(x,na.rm=TRUE) / (length(na.omit(x))-1) )
}
```

Exercise II

```
…/BayesCog/01.R_basics/_scripts/R_basics.R
```

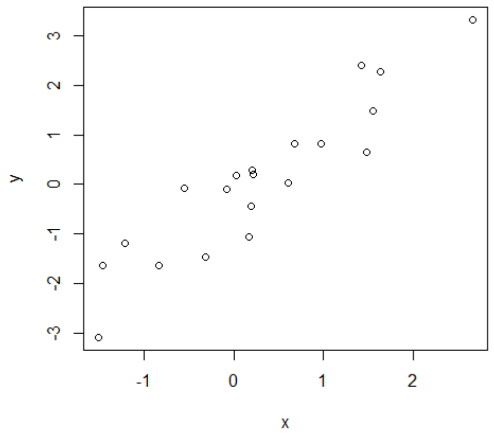
TASK: practise control flow and user-defined function

Visualization

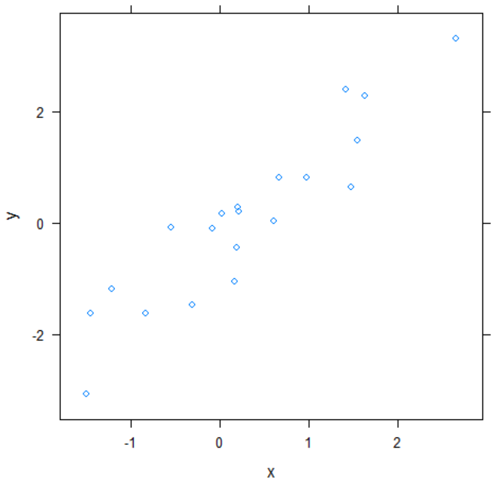
cognitive model
statistics
computing

- built-in plotting functions – first attempt / quick look / exploratory
- {lattice} – making nicer, similar to basic plotting functions
- {ggplot2} – making nicer, a layering philosophy

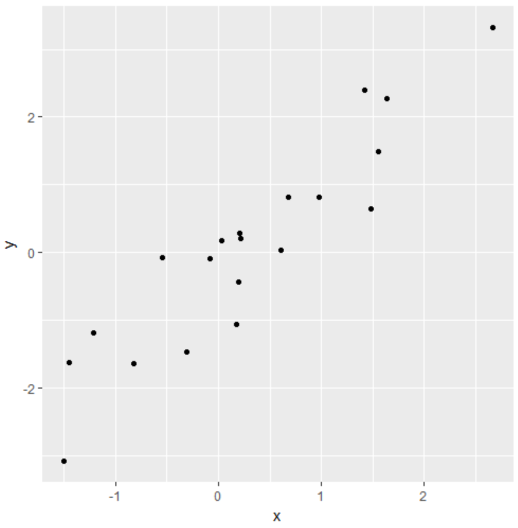
`plot(x,y)`



`lattice::xyplot(y~x)`



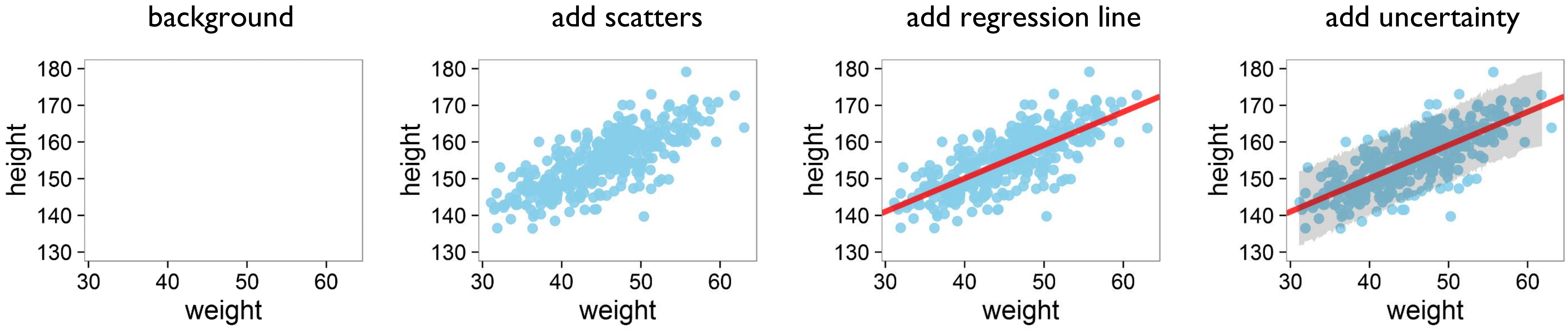
`ggplot2::ggplot(x,y)`



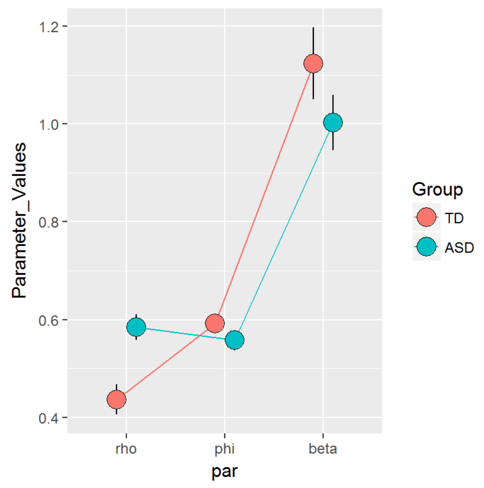
Brief Intro to ggplot2

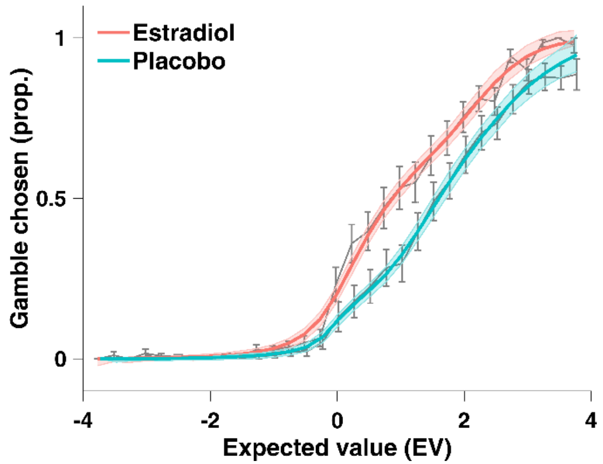
`plot = geometric (points, lines, bars) + aesthetic (color, shape, size)`

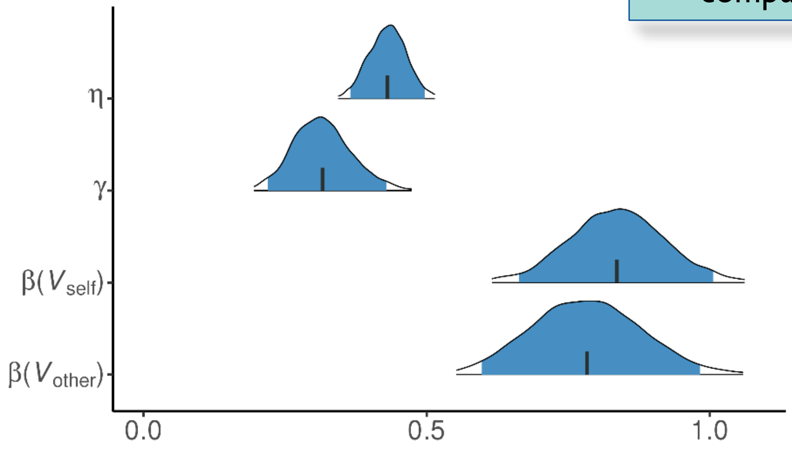
game of adding layers!

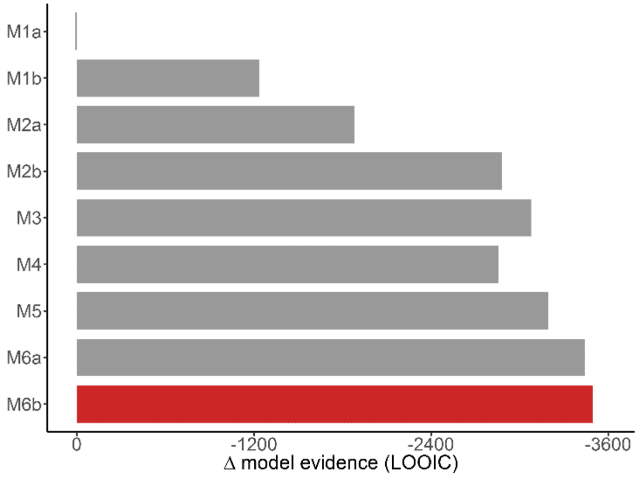


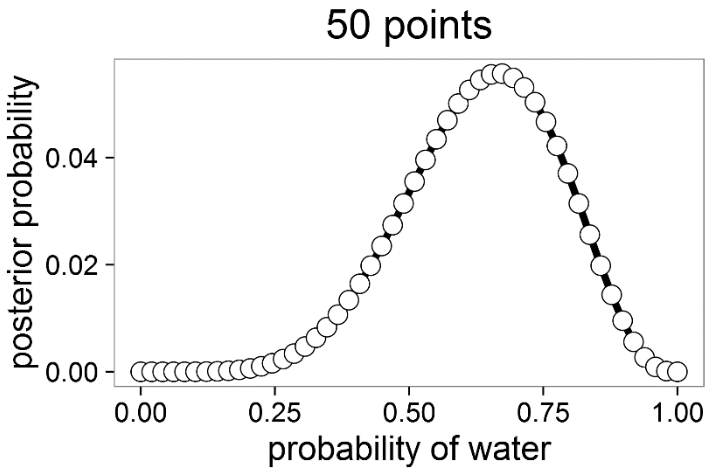
A taste of ggplot2

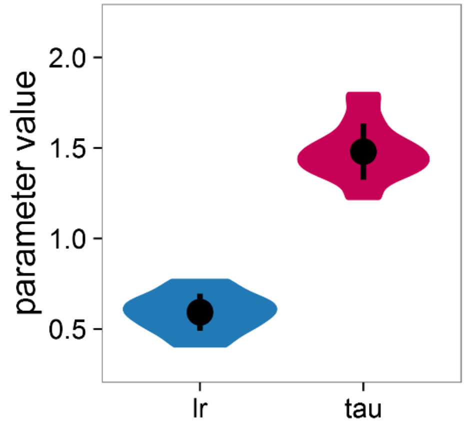


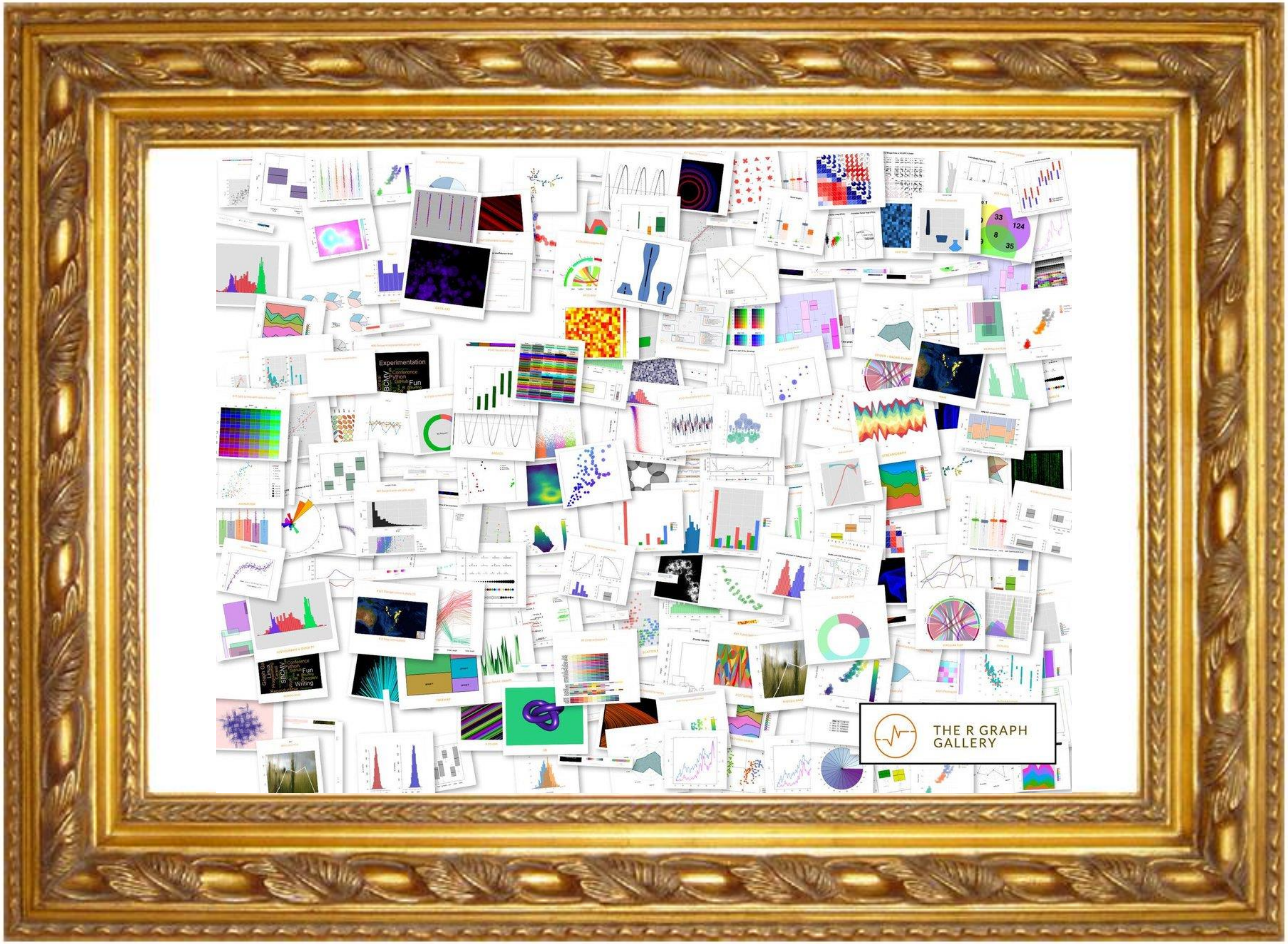












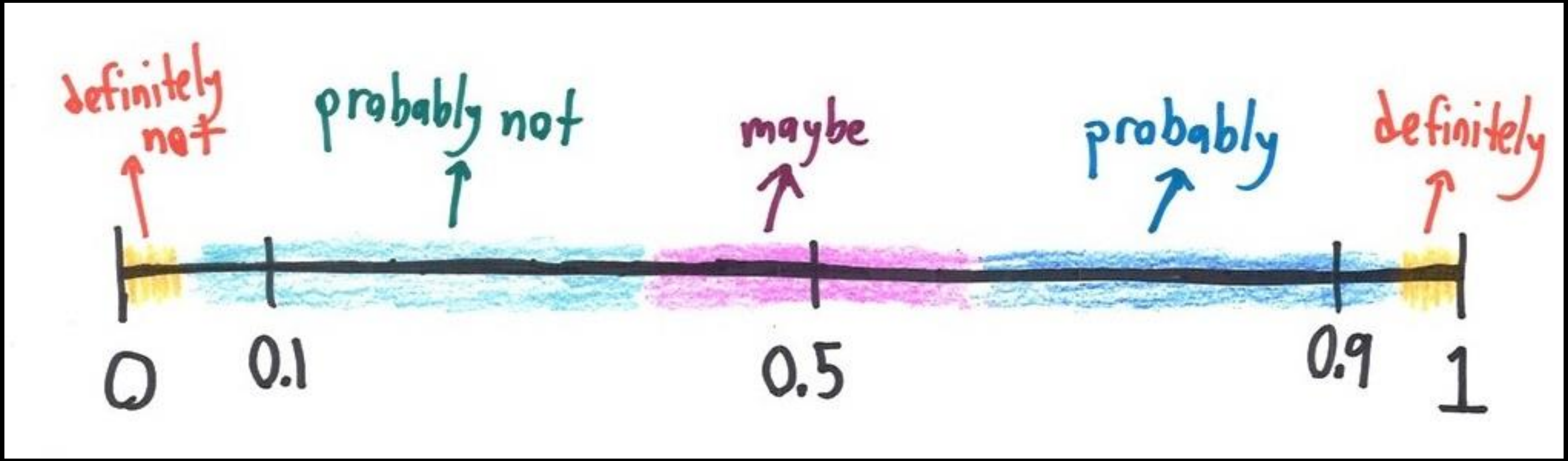
cognitive model

statistics

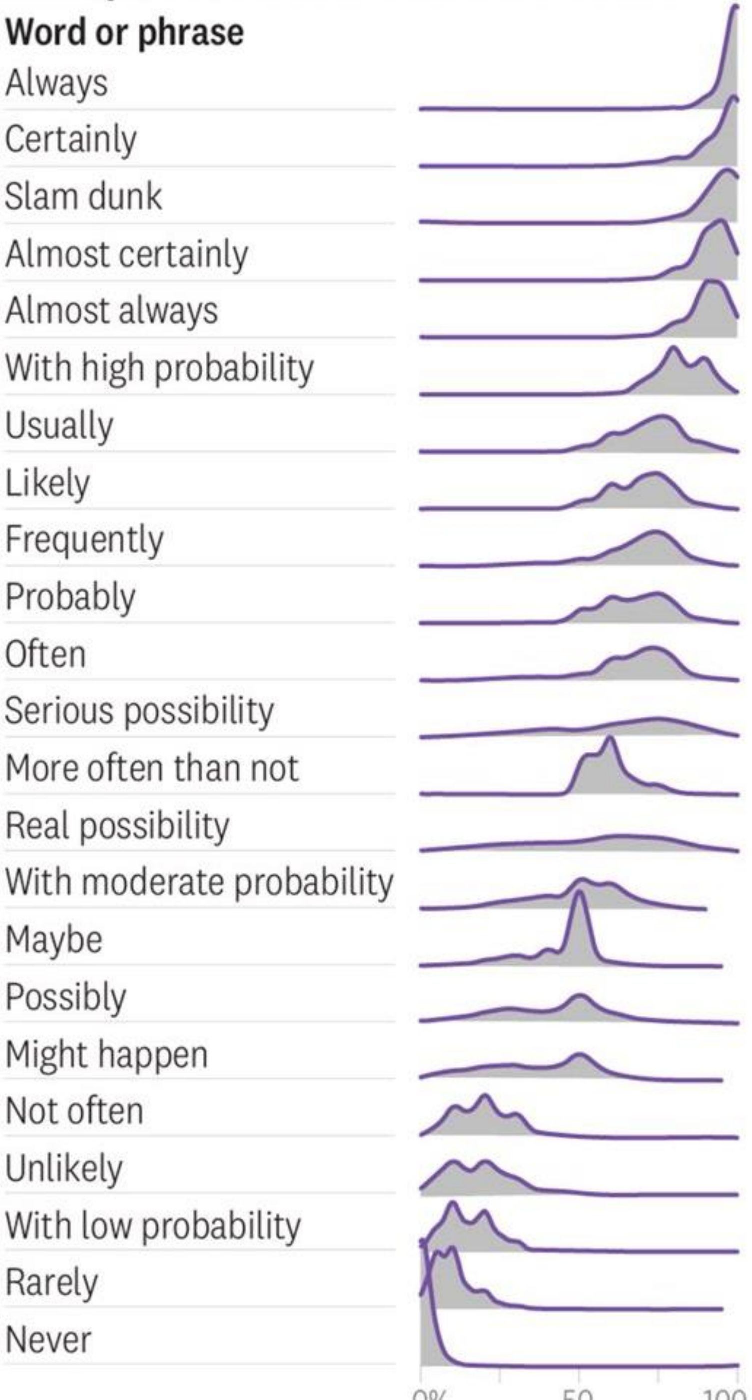
computing

https://www.r-graph-gallery.com/

BASICS OF PROBABILITY



to respondents' estimate of likelihood



Probability

…assigning numbers to a set of possibilities

Properties (Kolmogorov, 1956)

- $p \in [0,1]$
- $\Sigma p = 1$
- $p(\mathrm{A} \cup \mathrm{B}) = p(\mathrm{A}) + p(\mathrm{B})$, when A and B are *mutually exclusive*

Joint Probability and Conditional Probability

Joint Probability

$p(A, B) = p(B, A)$

- e.g., $p(\text{raining})$ and $p(\text{cold})$

Conditional Probability

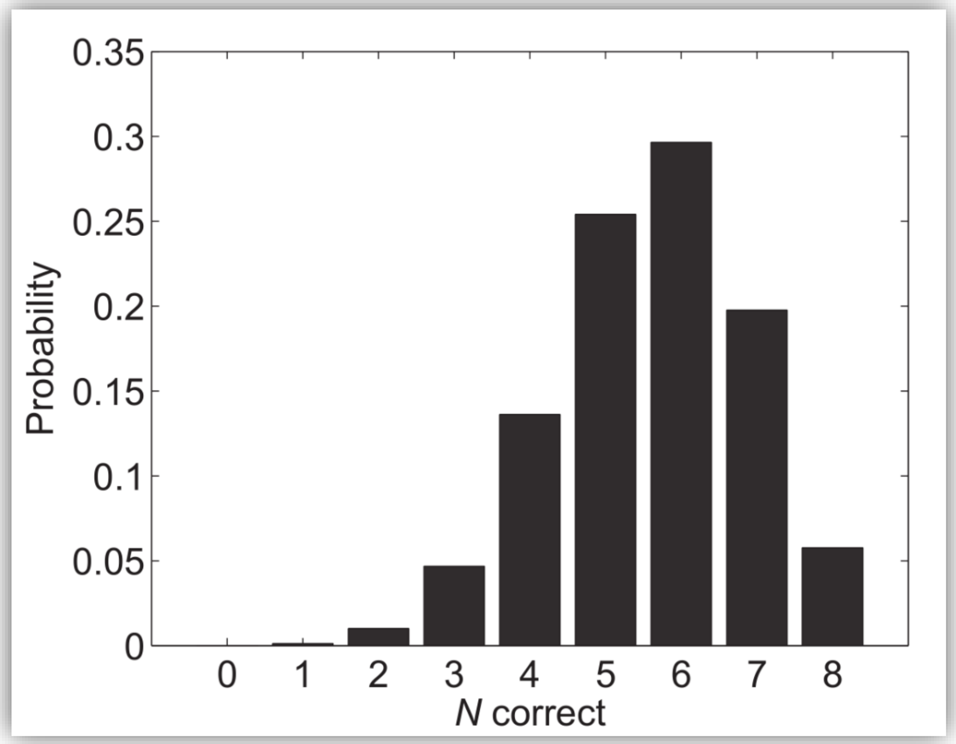
$p(A|B)$ – 'p of A given B'

$p(A,B) = p(A|B)\,p(B)$, when A and B are *independent*

- e.g., $p(\text{raining, cold}) = p(\text{raining}|\text{cold})\,p(\text{cold})$

Probability Functions

discrete events – we talk about mass

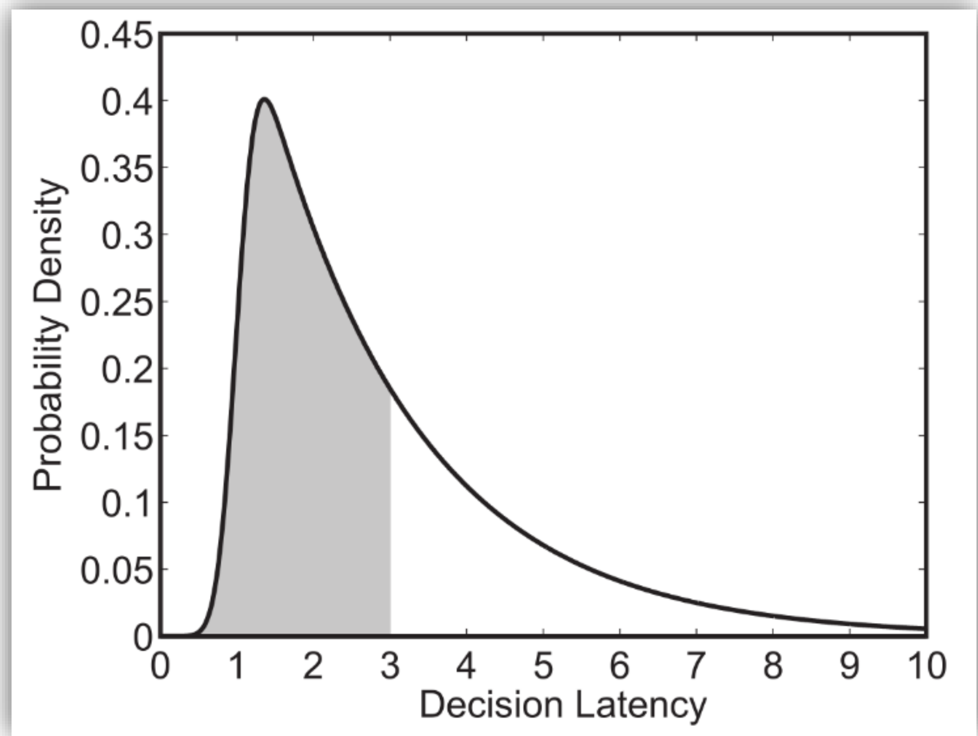


Lewandowsky & Farrell (2011)

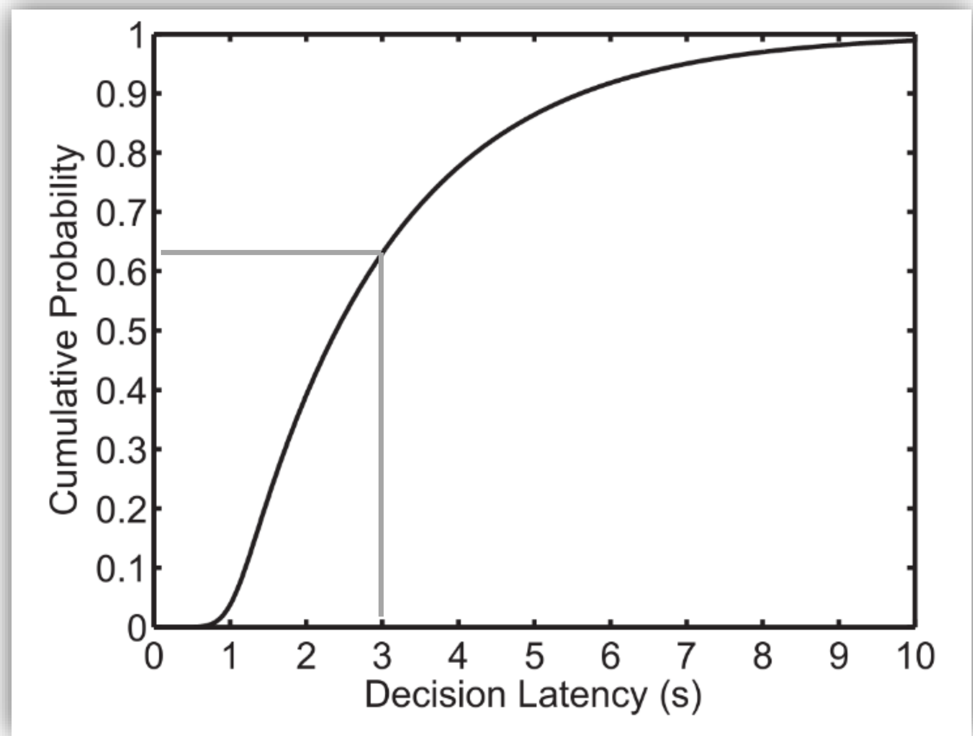
Probability Functions

continuous events – we talk about density

probability density function (PDF)



cumulative distribution function (CDF)



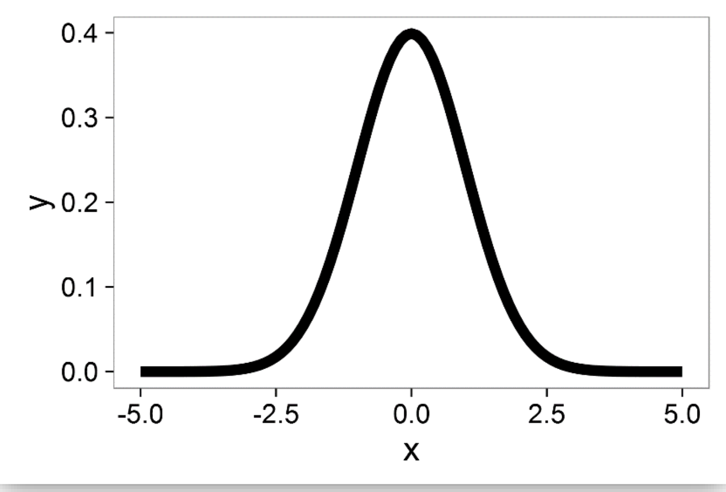
Lewandowsky & Farrell (2011)

Playing with Probability Functions in R

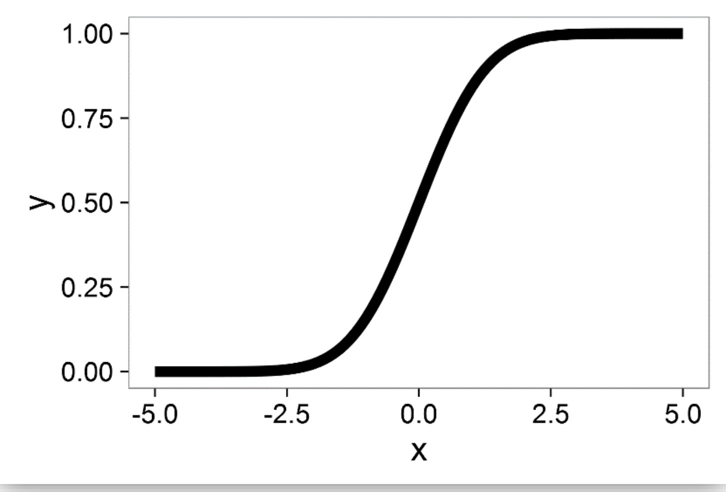

```
dnorm() – PDF
pnorm() – CDF
qnorm() – quantile, inverse cdf
rnorm() – random number generator
```

Example: Normal(0,1)

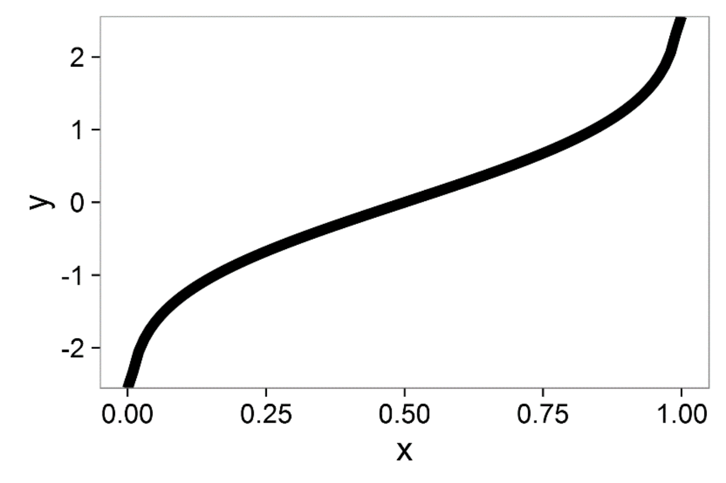
`dnorm()`



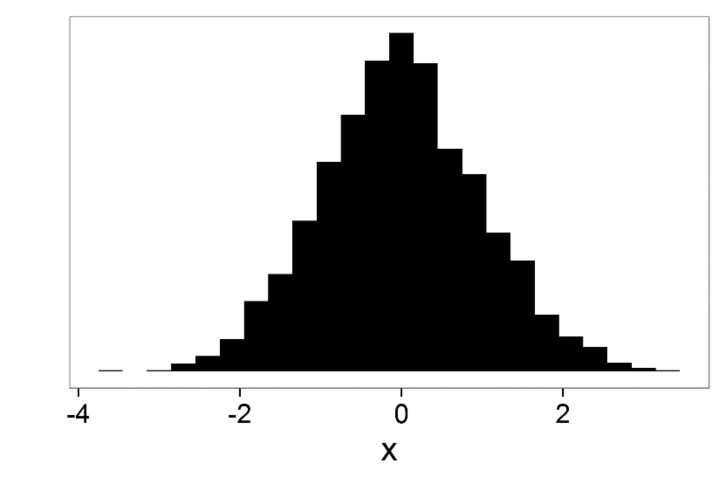
`pnorm()`



`qnorm()`



`rnorm()`



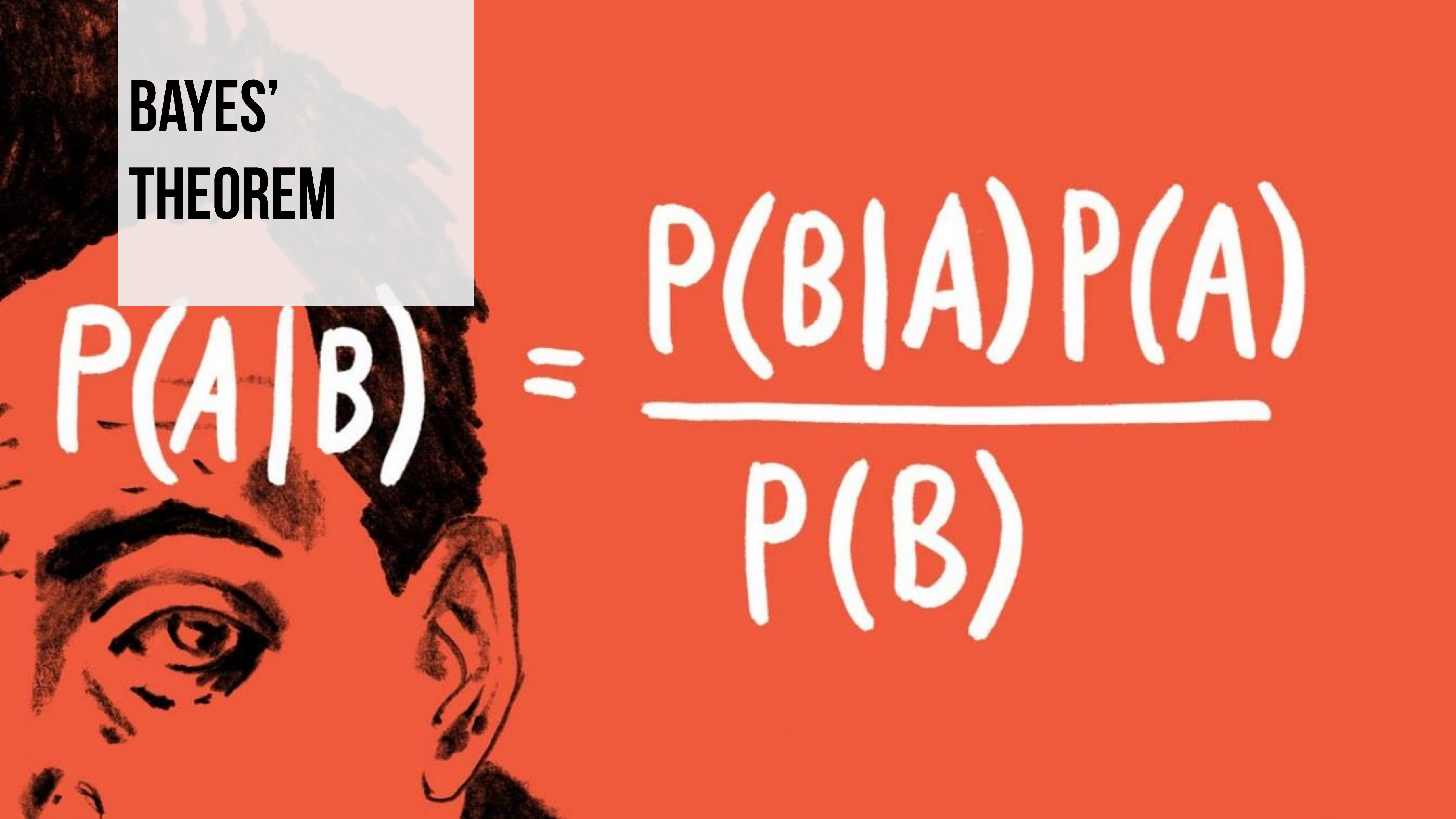
Exercise III

```
…/BayesCog/01.R_basics/_scripts/R_basics.R
```

TASK: produce `dnorm`, `pnorm`, `qnorm`, `rnorm` figures for

$$\text{Normal}(0.5, 2)$$

BAYES' THEOREM



Bayes' theorem

$$p(A,B) = p(B,A)$$

$$p(A,B) = p(A|B)\,p(B)$$

$$p(B,A) = p(B|A)\,p(A)$$

$$p(A|B)\,p(B) = p(B|A)\,p(A)$$

$$\boxed{p(A \mid B) = \frac{p(B \mid A)\,p(A)}{p(B)}}$$

$$p(A \mid B) = \frac{p(B \mid A)\,p(A)}{\sum_{A^*} p(B \mid A^*)\,p(A^*)}$$

A* is a variable that takes on all possible values

One Example

Row	Column			Marginal
	…	c	…	
⋮		⋮		
r	…	$p(r, c) = p(r\|c)\, p(c)$	…	$p(r) = \sum_{c^*} p(r\|c^*)\, p(c^*)$
⋮		⋮		
Marginal		$p(c)$		

Kruschke (2015)

One Example

Eye color	Hair color: Black	Brunette	Red	Blond	Marginal (Eye color)
Brown	0.11	0.20	0.04	0.01	0.37
Blue	0.03	0.14	0.03	0.16	0.36
Hazel	0.03	0.09	0.02	0.02	0.16
Green	0.01	0.05	0.02	0.03	0.11
Marginal (hair color)	0.18	0.48	0.12	0.21	1.0

$$p(A \mid B) = \frac{p(B \mid A)\, p(A)}{\sum_{A^*} p(B \mid A^*)\, p(A^*)}$$

Eye color	Hair color: Black	Brunette	Red	Blond	Marginal (Eye color)
Blue	$0.03/0.36 = 0.08$	$0.14/0.36 = 0.39$	$0.03/0.36 = 0.08$	$0.16/0.36 = 0.45$	$0.36/0.36 = 1.0$

Kruschke (2015)

Exercise IV

Suppose that in the general population, the probability of having a rare disease is 1/1000. We denote the true presence or absence of the disease as the value of a parameter, ϑ, that can have the value ϑ = ☹ if disease is present in a person, or the value ϑ = ☺ if the disease is absent. The base rate of the disease is therefore denoted $p(\vartheta =$ ☹ $) = 0.001$.

Suppose(1): a test for the disease that has a 99% hit rate: $p(T = + | \vartheta =$ ☹ $) = 0.99$

Suppose(2): the test has a false alarm rate of 5%: $p(T = + | \vartheta =$ ☺ $) = 0.05$

Q: Suppose we sample a person at random from the population, administer the test, and it comes up positive. What is the posterior probability that the person has the disease?

Kruschke (2015)

Exercise IV

Q: What is the posterior probability that the person has the disease?

→ $p(\vartheta = ☹ \mid T = +)$

Kruschke (2015)

Exercise IV

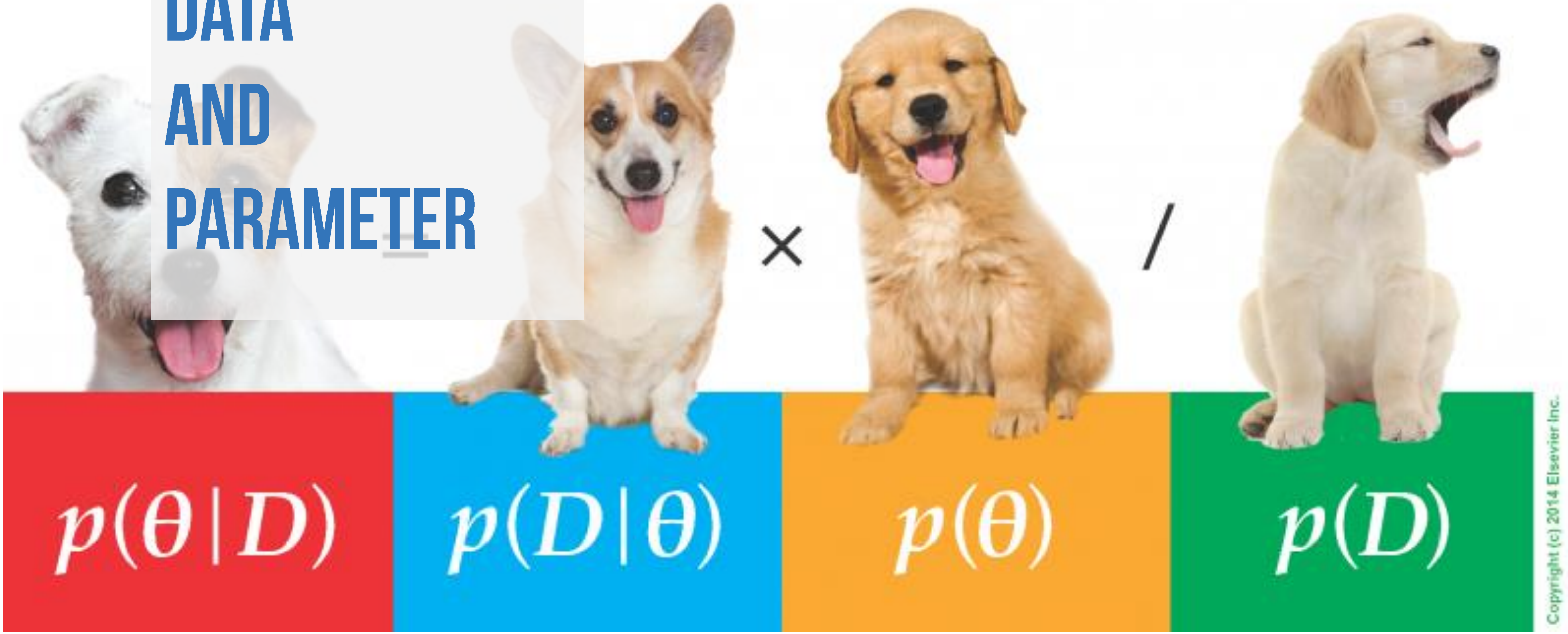
Test result	Disease: $\theta = $ ☹ (present)	Disease: $\theta = $ ☺ (absent)	Marginal (test result)
$T = +$	$p(+\mid$☹$)\, p($☹$)$ $= 0.99 \cdot 0.001$	$p(+\mid$☺$)\, p($☺$)$ $= 0.05 \cdot (1 - 0.001)$	$\sum_\theta p(+\mid\theta)\, p(\theta)$
$T = -$	$p(-\mid$☹$)\, p($☹$)$ $= (1 - 0.99) \cdot 0.001$	$p(-\mid$☺$)\, p($☺$)$ $= (1 - 0.05) \cdot (1 - 0.001)$	$\sum_\theta p(-\mid\theta)\, p(\theta)$
Marginal (disease)	$p($☹$) = 0.001$	$p($☺$) = 1 - 0.001$	1.0

$$p(\theta = \text{☹} \mid T = +) = \frac{p(T = + \mid \theta = \text{☹})\, p(\theta = \text{☹})}{\sum_\theta p(T = + \mid \theta)\, p(\theta)}$$

$$= \frac{0.99 \cdot 0.001}{0.99 \cdot 0.001 + 0.05 \cdot (1 - 0.001)}$$

$$= 0.019$$

LINKING DATA AND PARAMETER



Linking Data and Parameter

θ D

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}$$

Linking Data and Parameter

$$p(\theta|D) = \frac{p(D|\theta)p(\theta)}{p(D)}$$

Linking Data and Parameter

Likelihood
How plausible is the data given our parameter is true?

Prior
How plausible is our parameter before observing the data?

$$p(\theta|D) = \frac{p(D|\theta)p(\theta)}{p(D)}$$

Posterior
How plausible is our parameter given the observed data?

Evidence
How plausible is the data under all possible parameters?

What is *p*(*Data*)?

discrete parameters

$$p(\theta \mid D) = \frac{p(D \mid \theta)\, p(\theta)}{\boxed{\sum_{\theta^*} p(D \mid \theta^*)\, p(\theta^*)}}$$

continuous parameters

$$p(\theta \mid D) = \frac{p(D \mid \theta)\, p(\theta)}{\boxed{\int p(D \mid \theta^*)\, p(\theta^*)\, d\theta^*}}$$

BINOMIAL MODEL

Binomial Model

cognitive model
statistics
computing

- You are curious how much of the surface is covered in water.
- You will toss the globe up in the air.
- You will record whether or not the surface under your right index finger is water (W) or land (L).
- You might observe: W L W W W L W L W
- → 6/9 = 0.666667?
- Is it right? If not, what to do next?

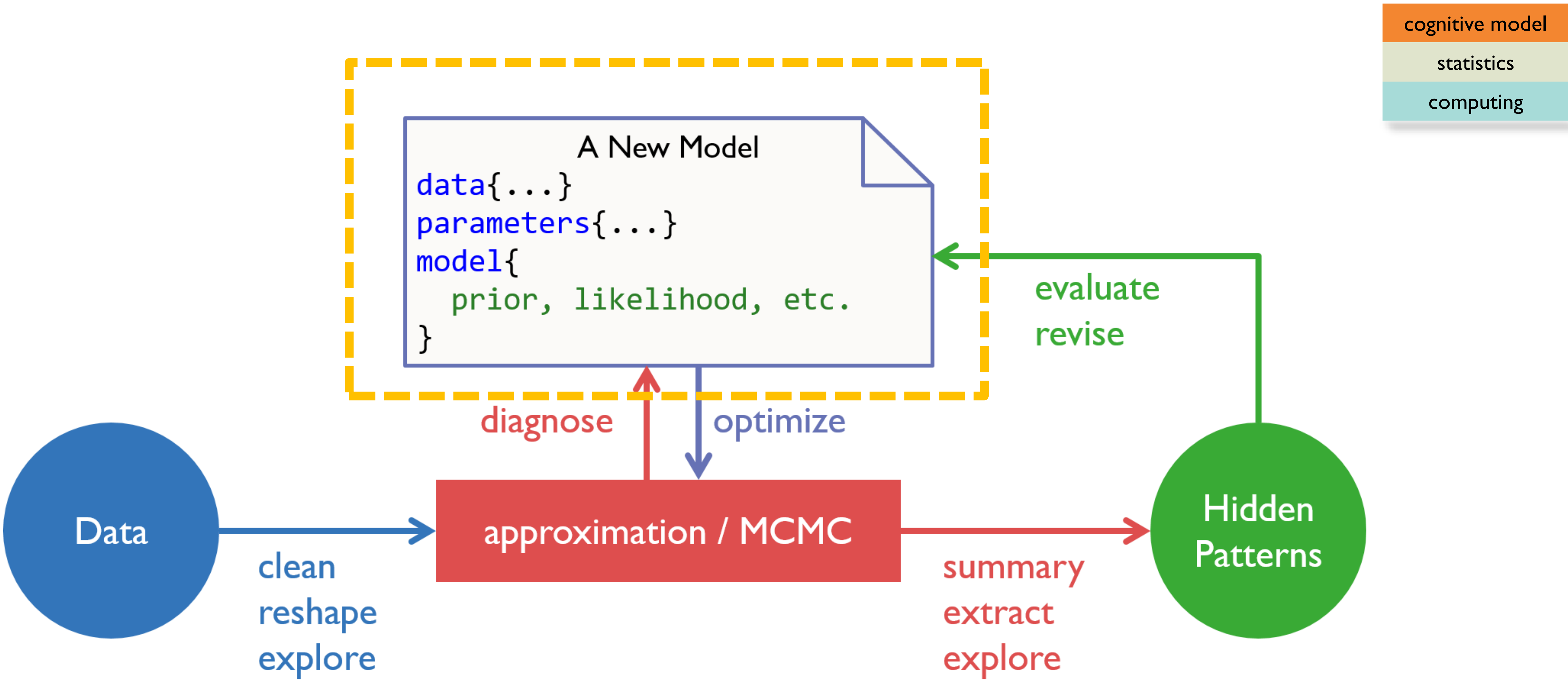
McElreath (2016)

Steps of Bayesian Modeling?

cognitive model
statistics
computing

A data story	Think about how the data might arise. It can be *descriptive* or even *causal*.
Update	Educate your model by feeding it with data. Bayesian Update: update the prior, in light of data, to produce posterior the updated posterior then becomes the prior of next update
Evaluate	Compare model with reality. Revise your model.

McElreath (2016)

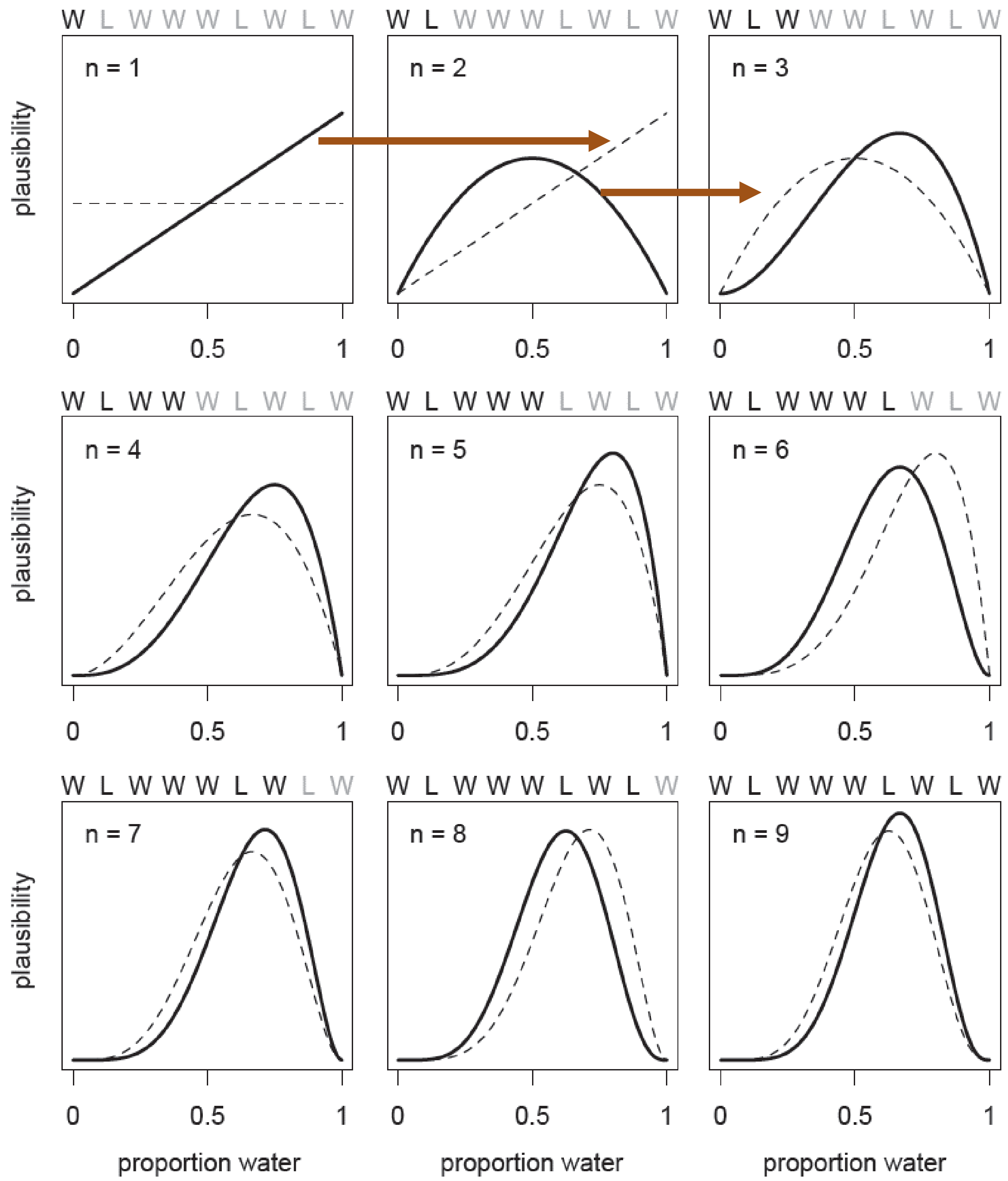

cognitive model
statistics
computing
A New Model
data{...}
parameters{...}
model{
prior, likelihood, etc.
}
evaluate
revise
diagnose
optimize
Data
approximation / MCMC
Hidden Patterns
clean
reshape
explore
summary
extract
explore

A Data Story of the Globe

- The true proportion of water covering the globe is p.
- A single toss of the globe has a probability p of producing a water (W) observation.
- It has a probability $1 - p$ of producing a land (L) observation.
- Each toss of the globe is independent of the others.

McElreath (2016)

Update



- order doesn't matter
- 2/3 is most likely
- others are not ruled out

cognitive model
statistics
computing

McElreath (2016)

Components of a Model

think about the likelihood function (of Binomial):

$$p(\theta \mid D) = \frac{\boxed{p(D \mid \theta)}\, p(\theta)}{\int p(D \mid \theta^*)\, p(\theta^*)\, d\theta^*}$$

$$p(w \mid N, p) = \binom{N}{w} p^w (1-p)^{N-w}$$

N: total number of observations
w: number of water $\Big\}$ known (data)

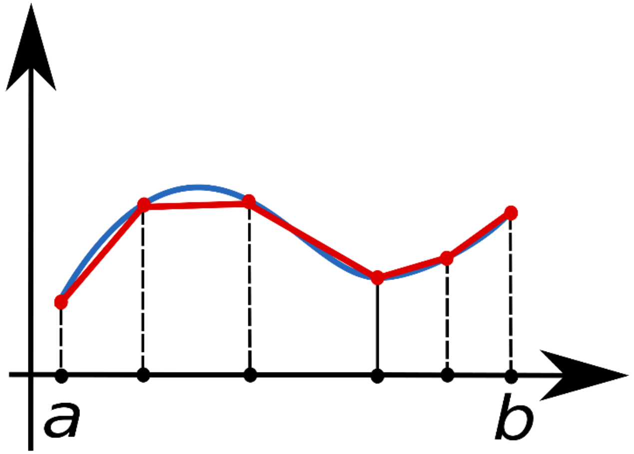
p: proportion of water — unknown (parameter)

Binomial Model – Grid Approximation


```
p_start <- 0; p_end <- 1; n_grid <- 20
w <- 6; N <- 9

# define grid
p_grid <- seq( from = p_start ,
          to = p_end , length.out = n_grid )

# define prior
prior <- rep(1 , n_grid)

# compute likelihood at each value in grid
likelihood <- dbinom(w , size = N , prob = p_grid )

# compute product of likelihood and prior
unstd.posterior <- likelihood * prior

# standardize the posterior, so it sums to 1
posterior <- unstd.posterior / sum(unstd.posterior)
```

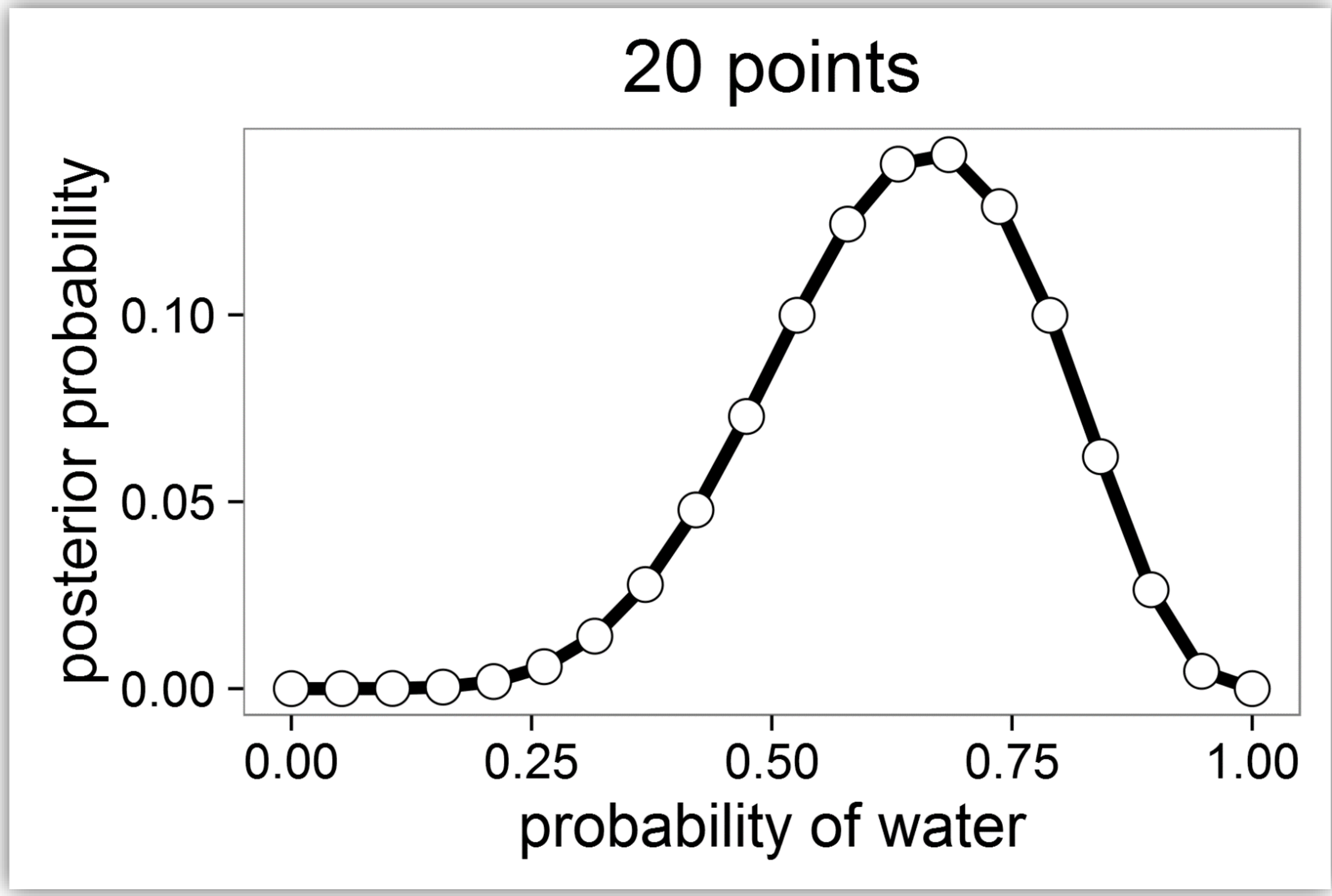
$$p(\theta \mid D) = \frac{p(D \mid \theta)\, p(\theta)}{\int p(D \mid \theta^*)\, p(\theta^*)\, d\theta^*}$$

$$p(w \mid N, p) = \binom{N}{w} p^w (1-p)^{N-w}$$

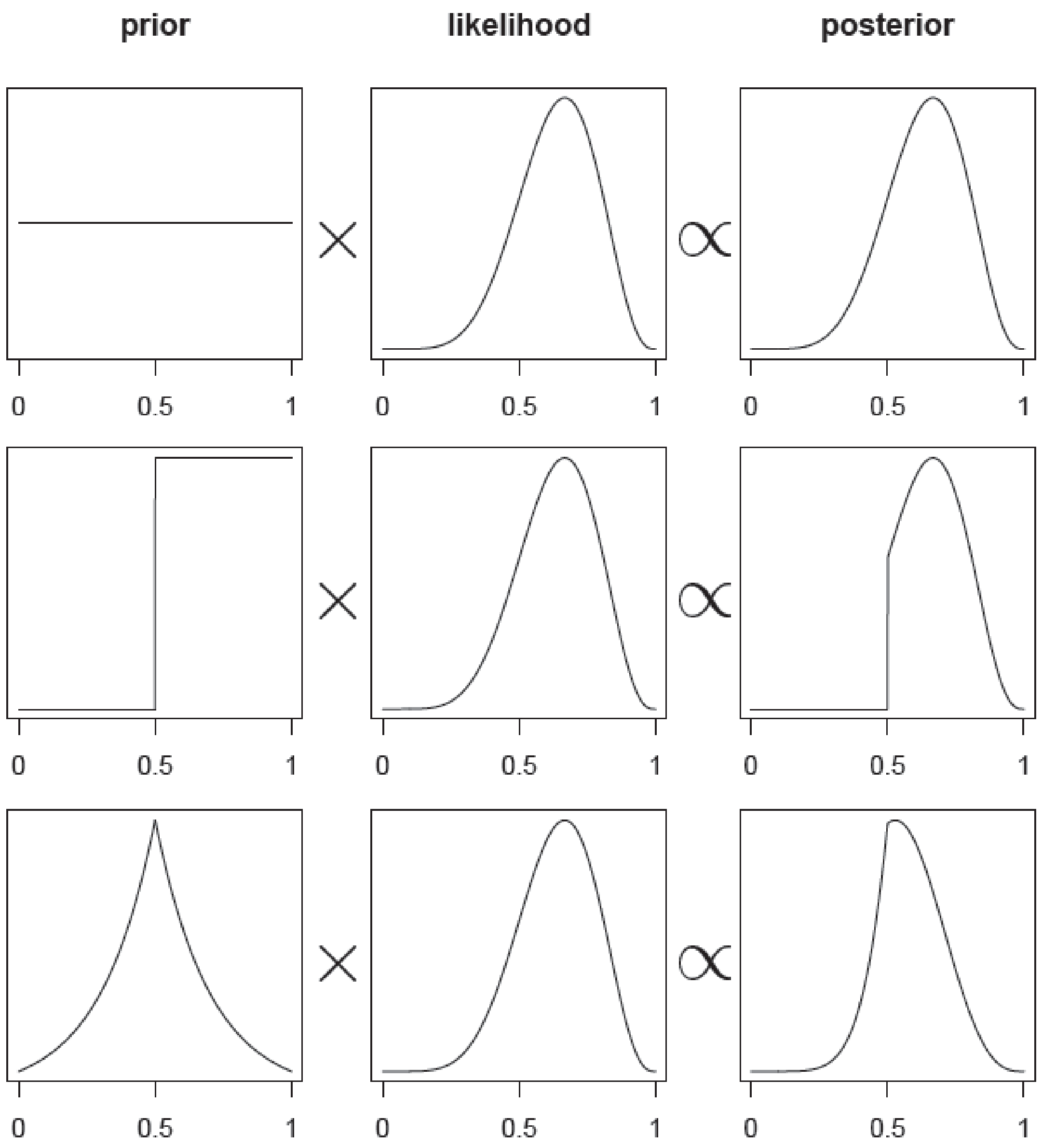


Binomial Model – Grid Approximation

cognitive model
statistics
computing



Impact of Prior



McElreath (2016)

Exercise V


```
…/BayesCog/02.binomial_globe/_scripts/binomial_globe_grid.R
```

TASK: run a grid approximation with `grid_size = 50`

Components of a Model

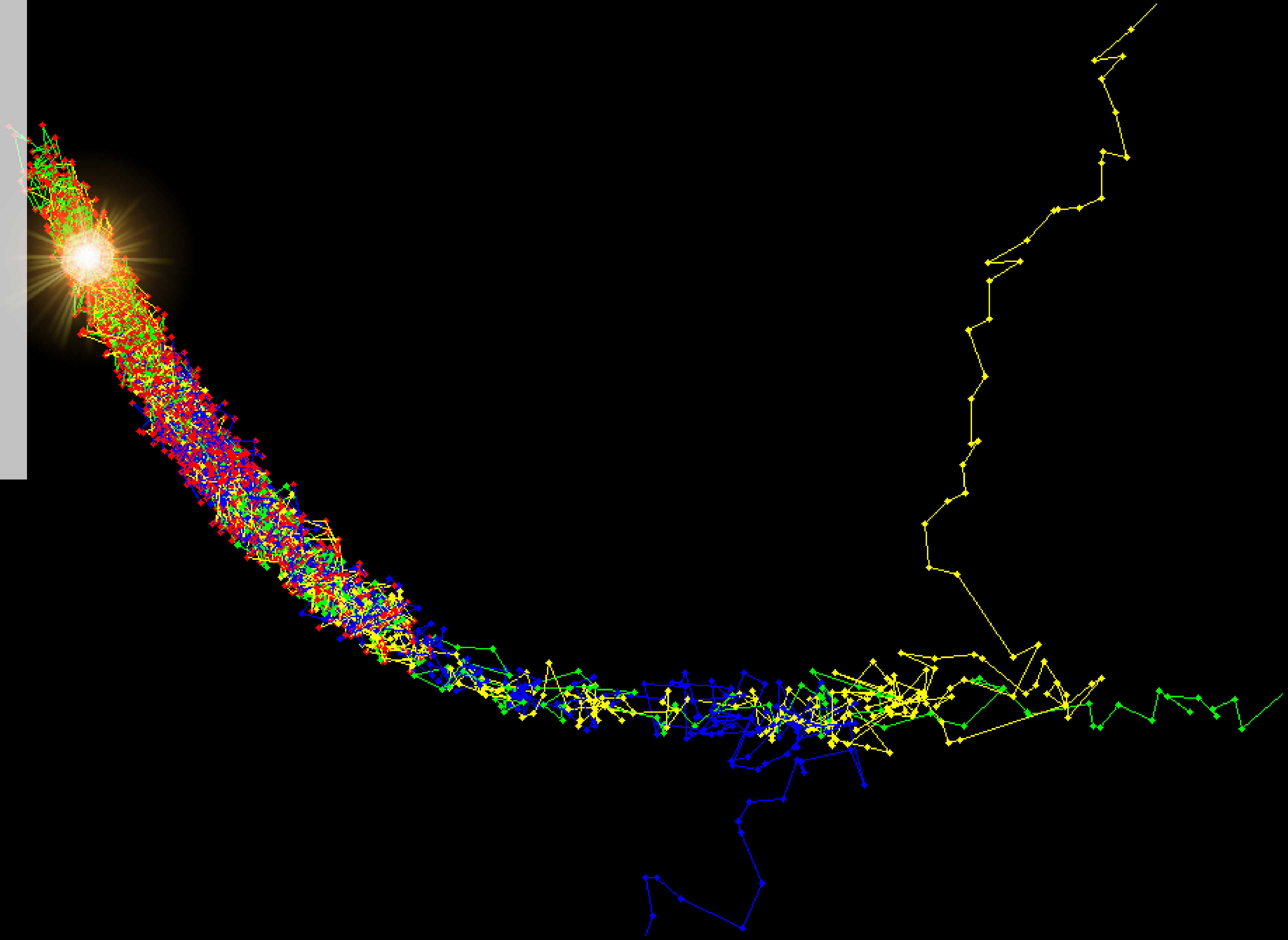
grid approximation for
2 parameters?
5 parameters?
10 parameters?

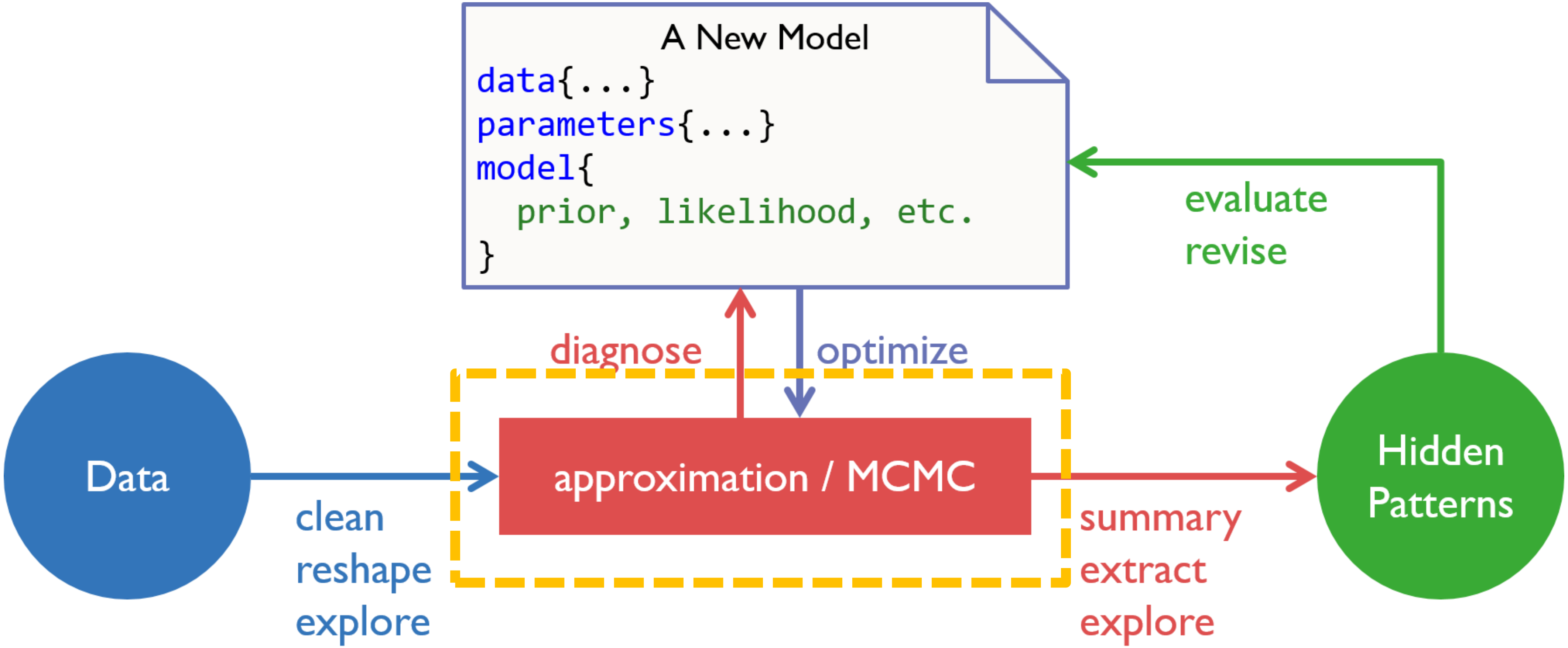
$$p(\theta \mid D) = \frac{p(D \mid \theta)\, p(\theta)}{\boxed{\int p(D \mid \theta^*)\, p(\theta^*)\, d\theta^*}}$$

$$p(w \mid N, p) = \binom{N}{w} p^w (1-p)^{N-w}$$

$$\boxed{p(\theta \mid D) \propto p(D \mid \theta)\, p(\theta)}$$

MARKOV CHAIN MONTE CARLO




A New Model
data{...}
parameters{...}
model{
prior, likelihood, etc.
}
evaluate
revise
diagnose
optimize
Data
clean
reshape
explore
approximation / MCMC
summary
extract
explore
Hidden Patterns

Solving the Problem by Approximation

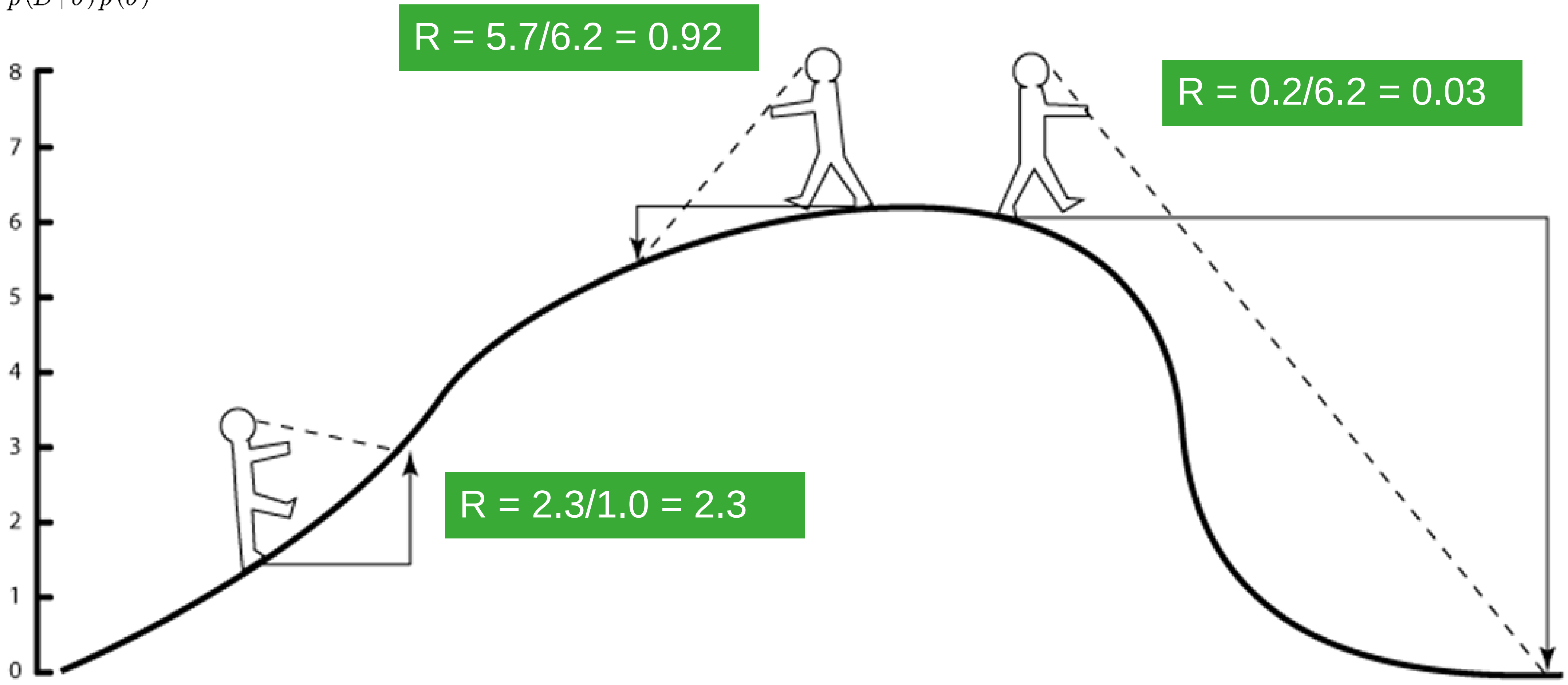
$$p(\theta \mid D) \propto p(D \mid \theta)\, p(\theta)$$

Deterministic Approximation	Stochastic Approximation
→ Variational Bayes	→ Sampling Methods

An MCMC Robot

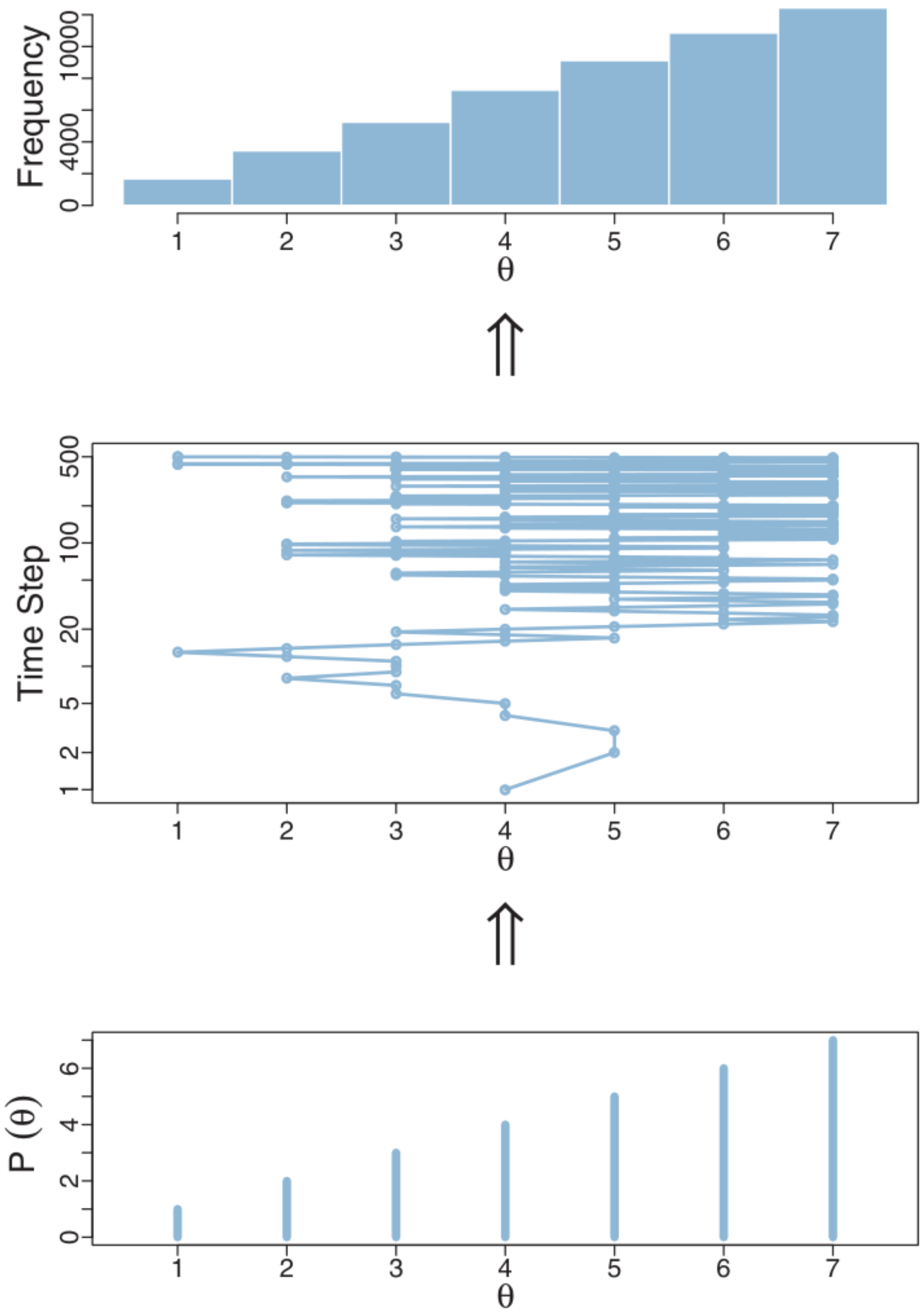
cognitive model
statistics
computing

$p(\theta \mid D) \propto p(D \mid \theta)p(\theta)$



http://marple.eeb.uconn.edu/mcmcrobot/

Sampling Example



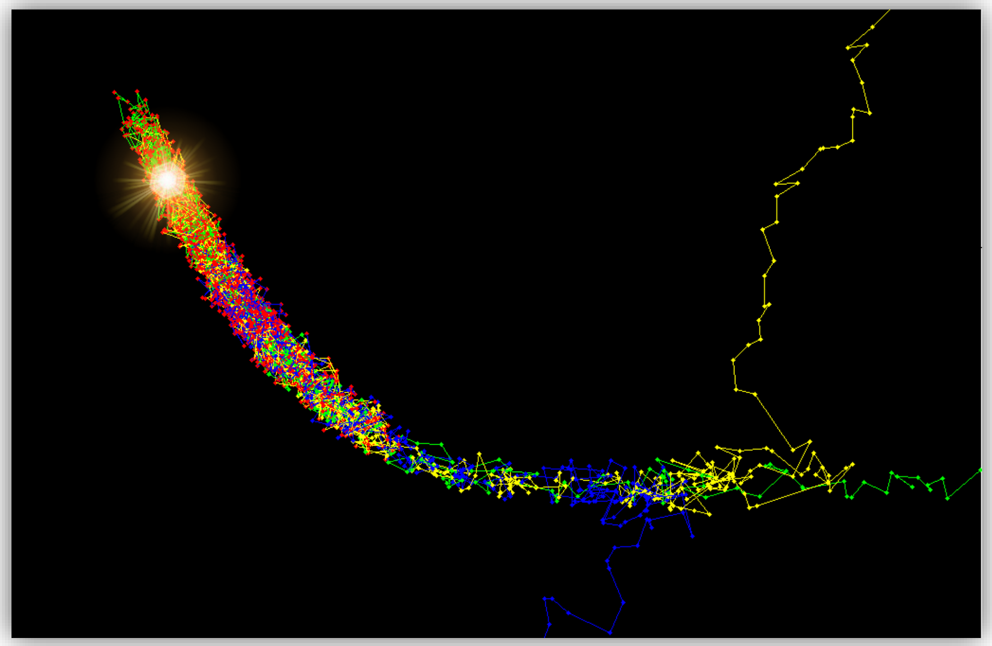
MCMC summary

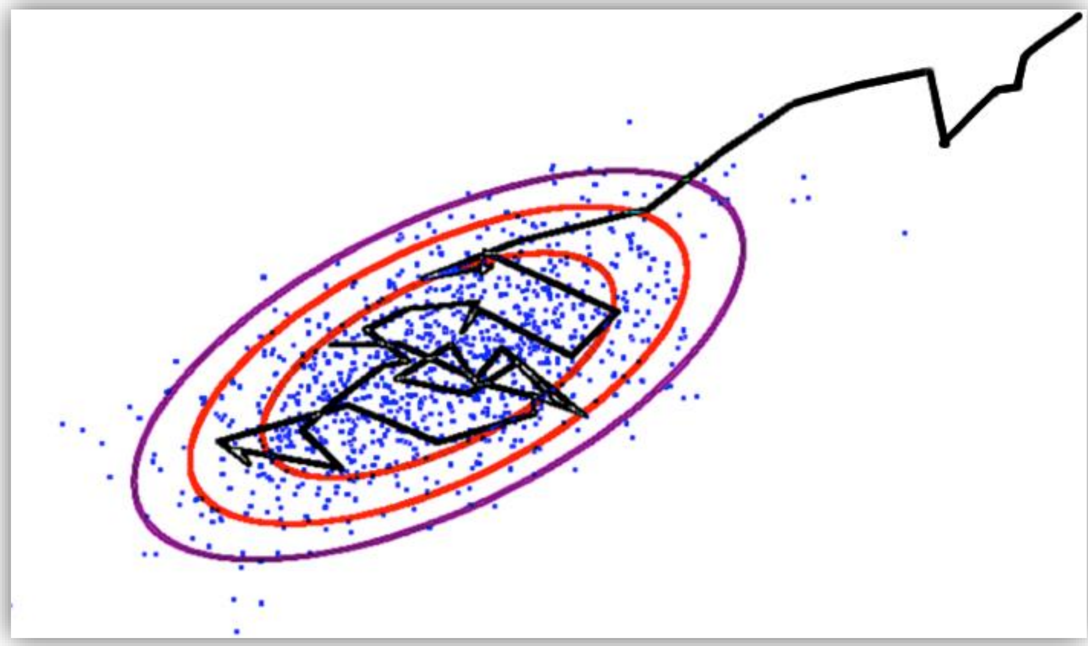
MCMC trace

True distribution

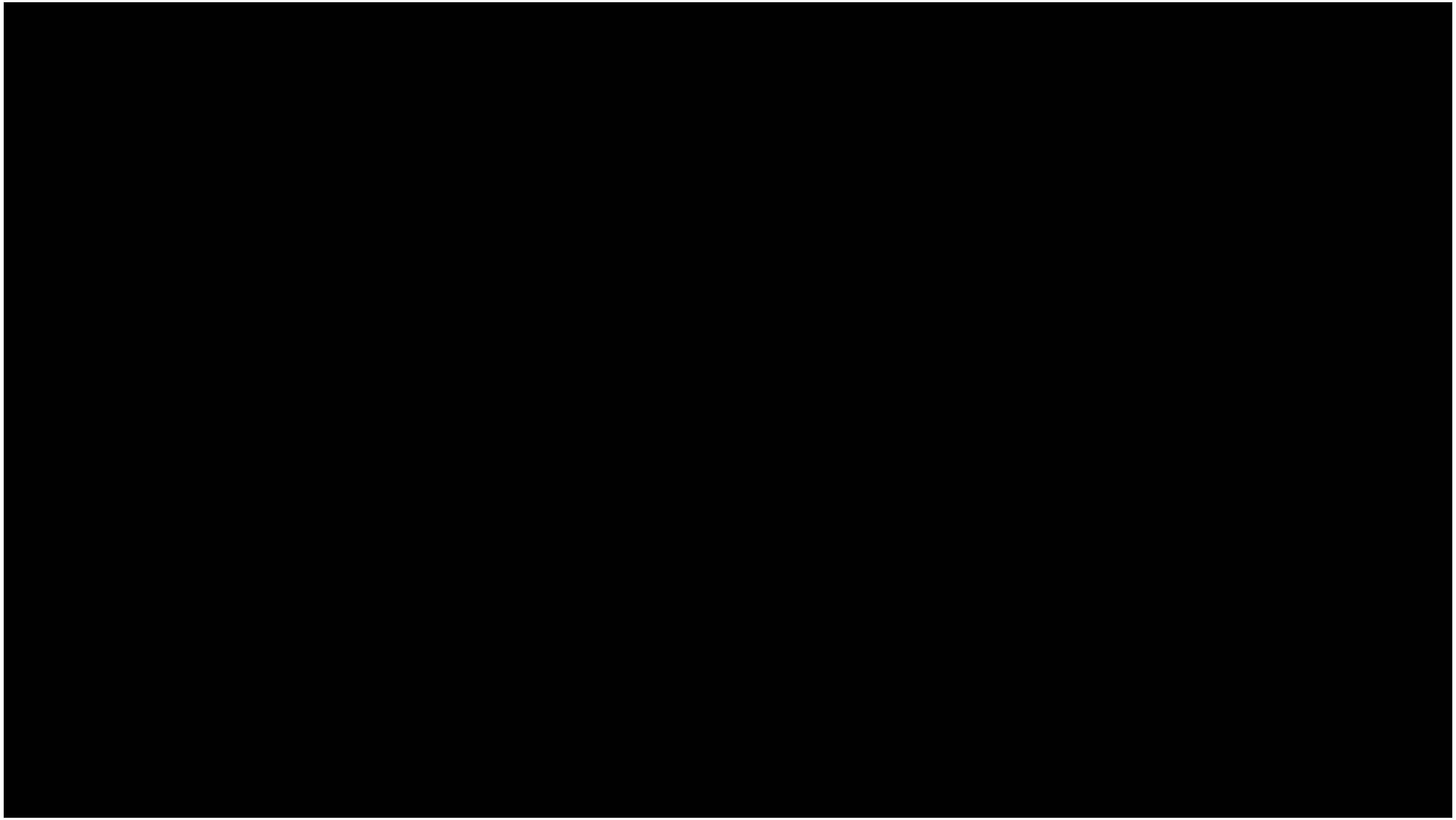
Kruschke (2015)

Visual Example





Let’s watch a video!



Credit: Thomas Wiecki

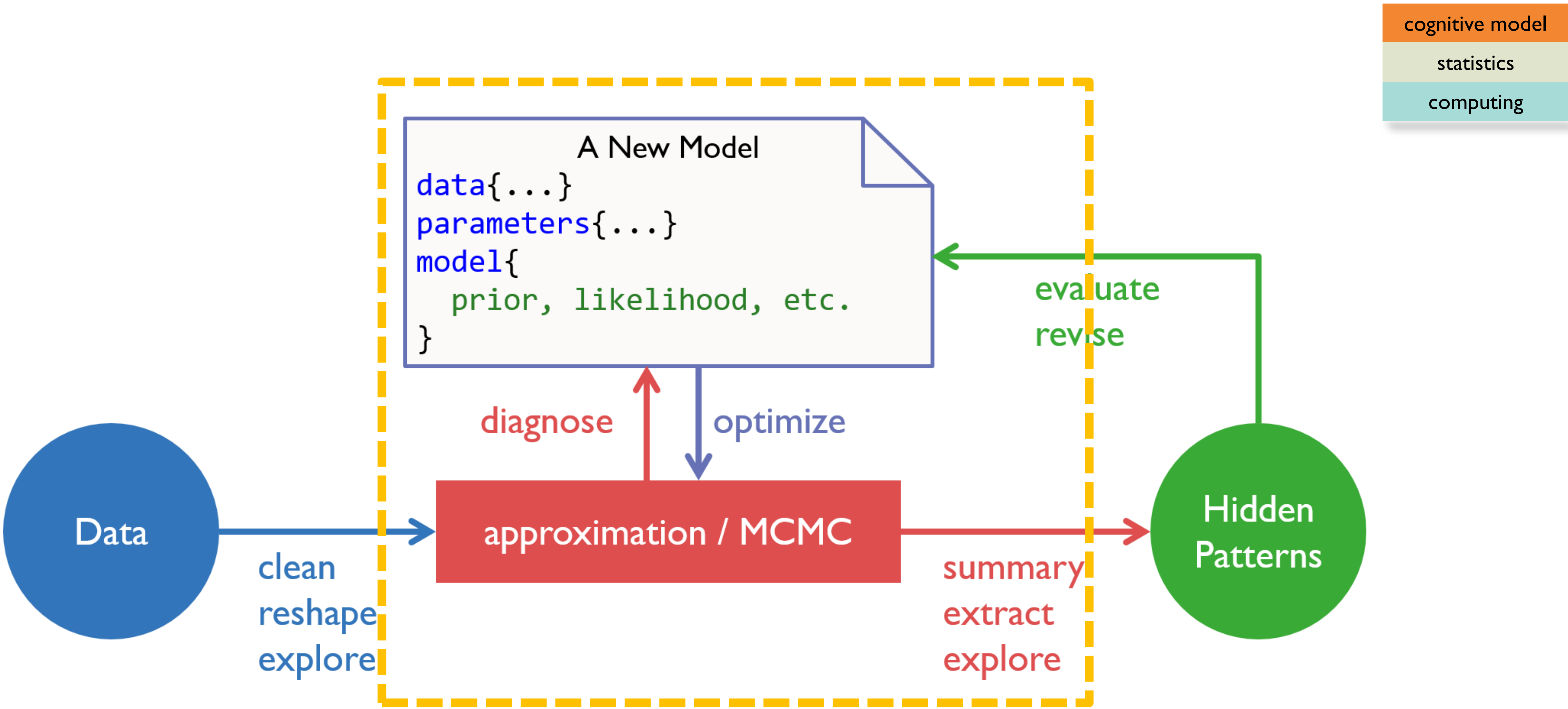
MCMC Sampling Algorithms

- Rejection sampling
- Importance sampling
- Metropolis algorithm
- Gibbs sampling → JAGS
- HMC sampling*

Stan!

*: also PyMC3 PyMC3

STAN
PROGRAMMING
LANGUAGE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Why Use Stan?

vs. BUGS and JAGS

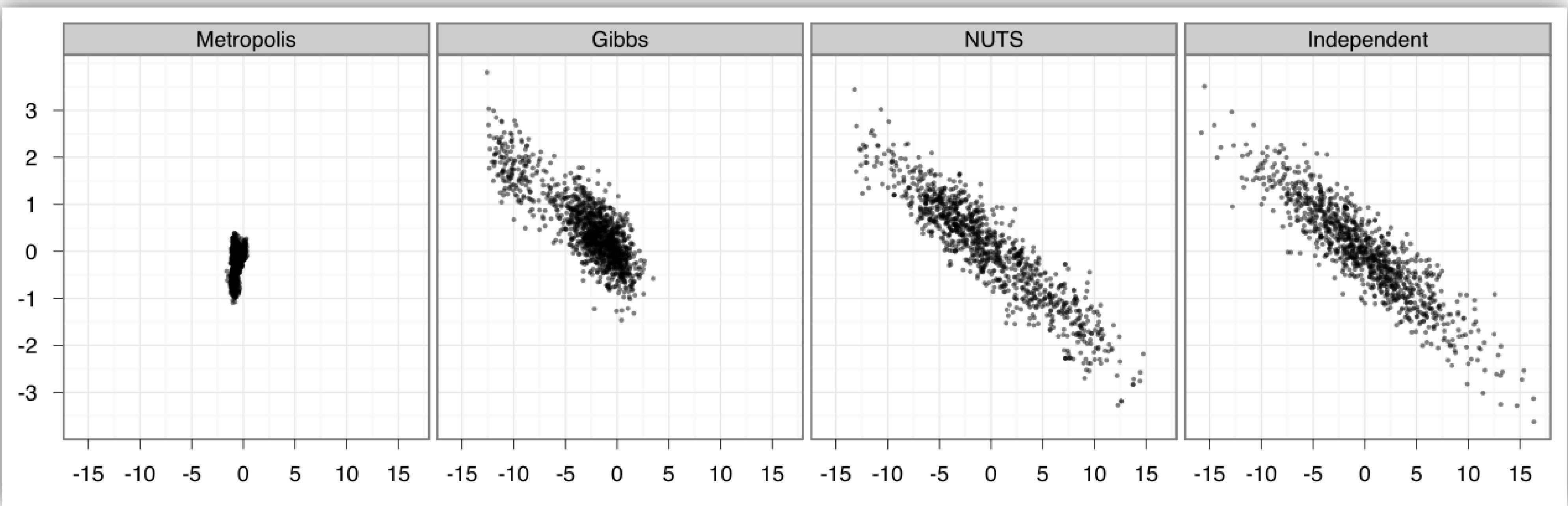
- Time to converge and per effective sample size:

  0.5 - ∞ times faster
- Memory usage: 1 - 10%
- Language features
  - variable overwrite: `a = 4, then a = 5`
  - formal control flow
  - full support of vectorizing

http://mc-stan.org/
https://discourse.mc-stan.org/t/selling-stan/3693/6

NUTS vs. Gibbs and Metropolis

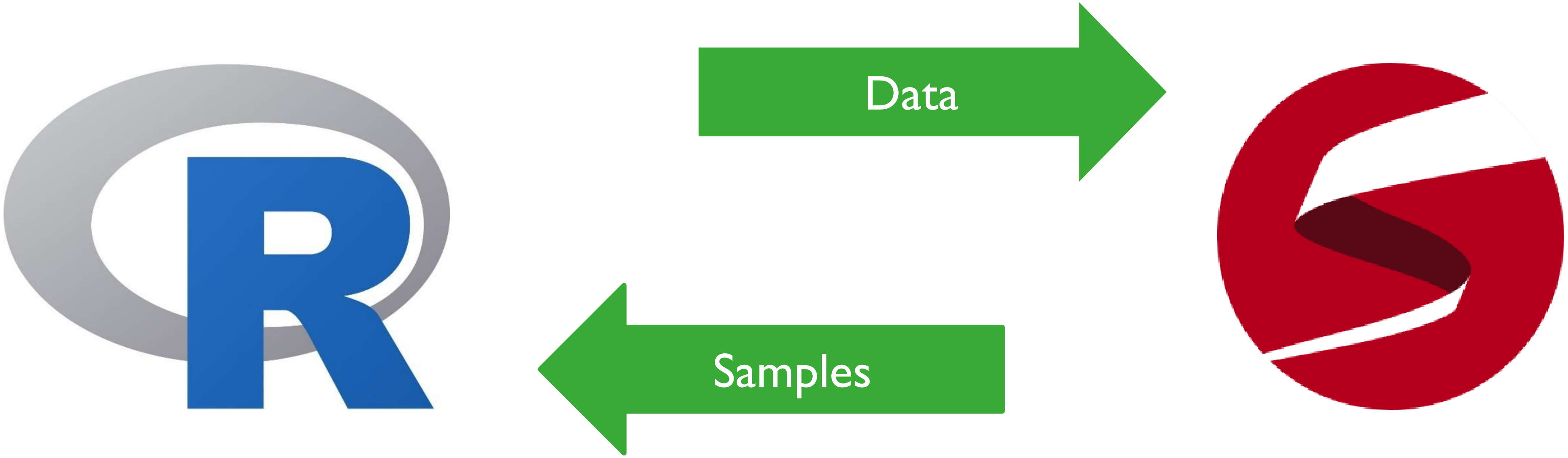
Hamilton MC (HMC) implements No-U-Turn Sampler (NUTS)



- Two dimensions of highly correlated 250-dim normal
- 1,000,000 draws from Metropolis and Gibbs (thin to 1000)
- 1,000 draws from NUTS; 1000 independent draws

http://mc-stan.org/

Stan and RStan



Steps of Bayesian Modeling, with Stan

A data story	Think about how the data might arise. It can be *descriptive* or even *causal*. Write a Stan program.
Update	Educate your model by feeding it the data. Bayesian Update: update the prior, in light of data, to produce posterior. Run Stan using RStan (PyStan, MatlabStan etc.)
Evaluate	Compare model with reality. Revise your model. Evaluate in RStan and ShinyStan.

McElreath (2016)

Steps of Using Stan

1. Stan program read into memory
2. Source-to-source transformation into C++
3. C++ compiled and linked (takes a while)
4. Run Stan program
5. Posterior analysis / interface

```
data {
  int<lower=0> N;
  int<lower=0,upper=1> y[N];
}
parameters {
  real<lower=0,upper=1> theta;
}
model {
  y ~ bernoulli(theta);
}
```

Stan Language

model blocks

```
data {
 //... read in external data...
}

transformed data {
 //... pre-processing of data ...
}

parameters {
 //... parameters to be sampled by HMC ...
}

transformed parameters {
 //... pre-processing of parameters ...
}

model {
 //... statistical/cognitive model ...
}

generated quantities {
 //... post-processing of the model ...
}
```

General Properties of Stan Language

- Whitespace does not matter
- Comments
  - //
  - /* … */
- Must use semicolon (;)
- Variables are typed and scoped

Variable's Scope

	data	transformed data	parameters	transformed parameters	model	generated quantities
Variable Declarations	Yes	Yes	Yes	Yes	Yes	Yes
Variable Scope	Global	Global	Global	Global	Local	Local
Variables Saved?	No	No	Yes	Yes	No	Yes
Modify Posterior?	No	No	No	No	Yes	No
Random Variables	No	No	No	No	No	Yes

Variable Declaration

- Each variable has a type (static type; scalar, vector, matrix etc.)
- Only values of that type can be assigned to the variable
  - e.g. cannot assign [1 2 3] to a (declared as a scalar)
- Declaration of variables happen at the top of a block (including local blocks)

Scalar Variables

`real`

- scalar
- continuous

```
data {
  real y;
}
```

`int`

- scalar
- integer
- can't be used in `parameters` or `transformed parameters` blocks

```
data {
  int n;
}
```

Constraining Scalar Variables

```
data {
  int<lower=1> m;
  int<lower=0,upper=1> n;
  real<lower=0> x;
  real<upper=0> y;
  real<lower=-1,upper=1> rho;
}
```

Vector & Matrix

```
vector[3] a;
// column vector

row_vector[4] b;
// row vector

matrix[3,4] A;
// A is a 3x4 matrix
// A[1] returns a 4-element row vector

vector<lower=0,upper=1>[5] rhos;
row_vector<lower=0>[4] sigmas;
matrix<lower=-1, upper=1>[3,4] Sigma;
```

Control Flow

- if-else

```
if (cond) {
   ..statement..
} else {
   ..statement..
}
```

```
if (cond) {
   ..statement..
} else if (cond) {
   ..statement..
} else {
   ..statement..
}
```

- for-loop

```
for ( j in 1:n) {
   ..statement..
}
```

```
for ( j in 1:J ) {
   for ( k in 1:K ) {
      ..statement..
   }
}
```

same as the R syntax, but terminate each line with ;

REVISIT BINOMIAL MODEL

Binomial Model

cognitive model
statistics
computing

W L W W W L W L W

$$p(w \mid N, p) = \binom{N}{w} p^{w} (1-p)^{N-w}$$

$$w \sim \text{Binomial}(N, p)$$

reads as:
w is distributed as a binomial distribution, with number of trials *N*, and success rate *p*.

McElreath (2016)

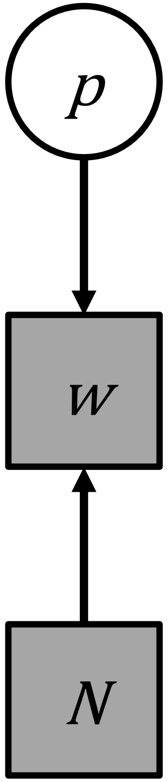
Graphical Model Notations

	continuous	discrete
unobserved	p	δ
observed	y	N

Binomial Model

W L W W W L W L W

$$p(w \mid N, p) = \binom{N}{w} p^{w} (1-p)^{N-w}$$



$$p \sim \text{Uniform}(0, 1)$$

$$w \sim \text{Binomial}(N, p)$$

	continuous	discrete
unobserved	p	δ
observed	y	N

McElreath (2016)

Binomial Model

cognitive model
statistics
computing

W L W W W L W L W

$$p(w \mid N, p) = \binom{N}{w} p^{w} (1-p)^{N-w}$$

```
data {
  int<lower=0> w;
  int<lower=0> N;
}

parameters {
  real<lower=0,upper=1> p;
}

model {
  w ~ binomial(N, p);
}
```

McElreath (2016)

Running Binomial Model with Stan


```
…/BayesCog/02.binomial_globe/_scripts/binomial_globe_main.R
```

```
> R.version
R version 3.5.0 (2018-04-23)

> stan_version()
[1] "2.17.0"
```

Model Summary

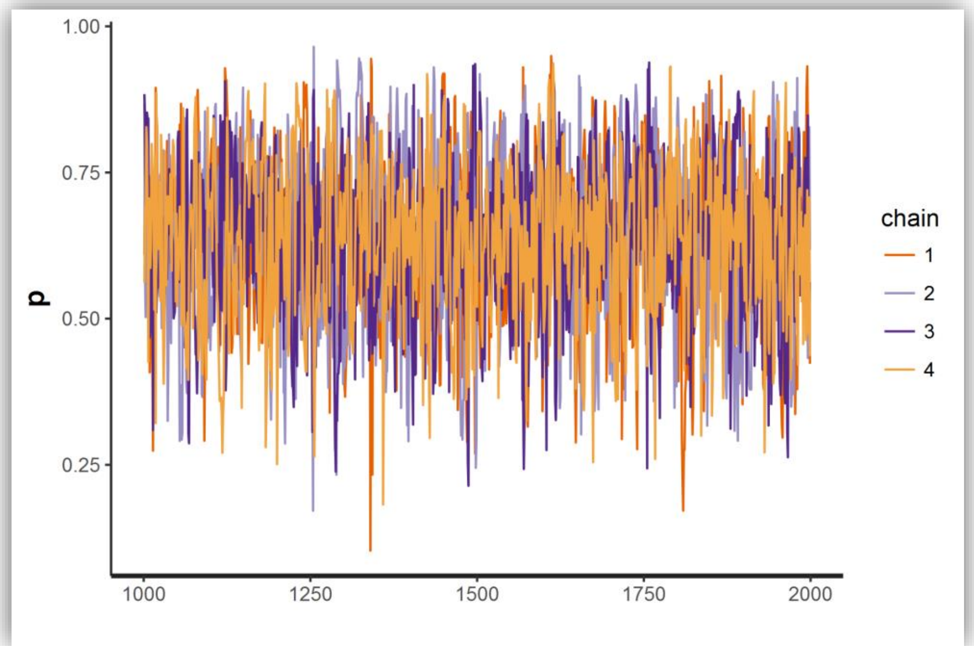
cognitive model
statistics
computing

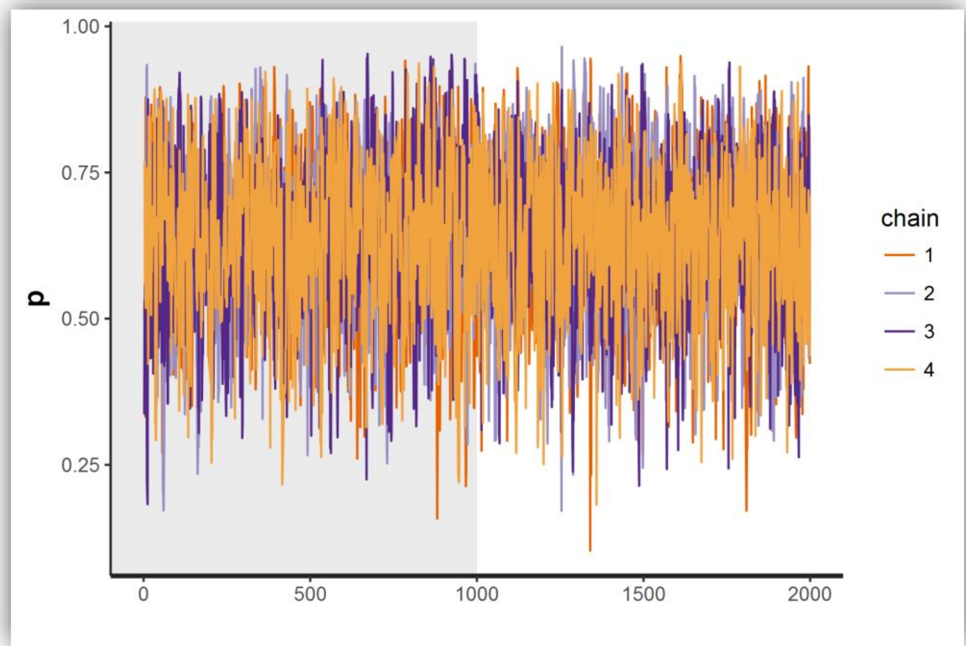
```
> print(fit_globe)
Inference for Stan model: binomial_globe_model.
4 chains, each with iter=2000; warmup=1000; thin=1;
post-warmup draws per chain=1000, total post-warmup draws=4000.

      mean se_mean   sd  2.5%   25%   50%   75% 97.5% n_eff Rhat
p     0.64    0.00 0.14  0.36  0.54  0.64  0.74  0.88  1699    1
lp__ -7.73    0.02 0.73 -9.92 -7.91 -7.44 -7.27 -7.21  1368    1

Samples were drawn using NUTS(diag_e) at Thu Jun 28 02:04:11 2018.
For each parameter, n_eff is a crude measure of effective sample size,
and Rhat is the potential scale reduction factor on split chains (at
convergence, Rhat=1).
```

Gelman-Rubin convergence diagnostic (Gelman & Rubin, 1992)

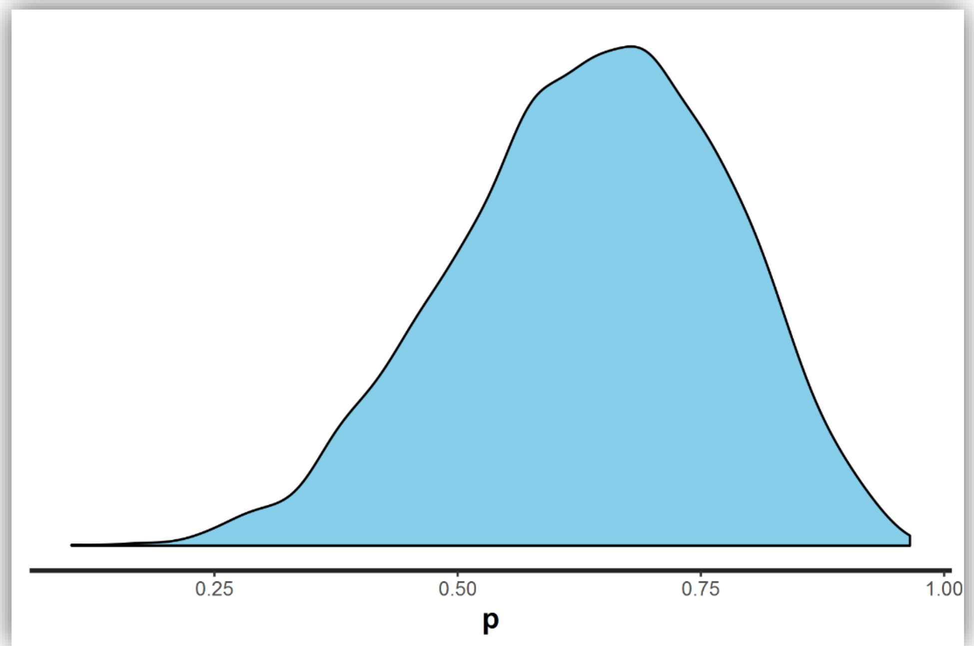
Diagnostics - traceplot

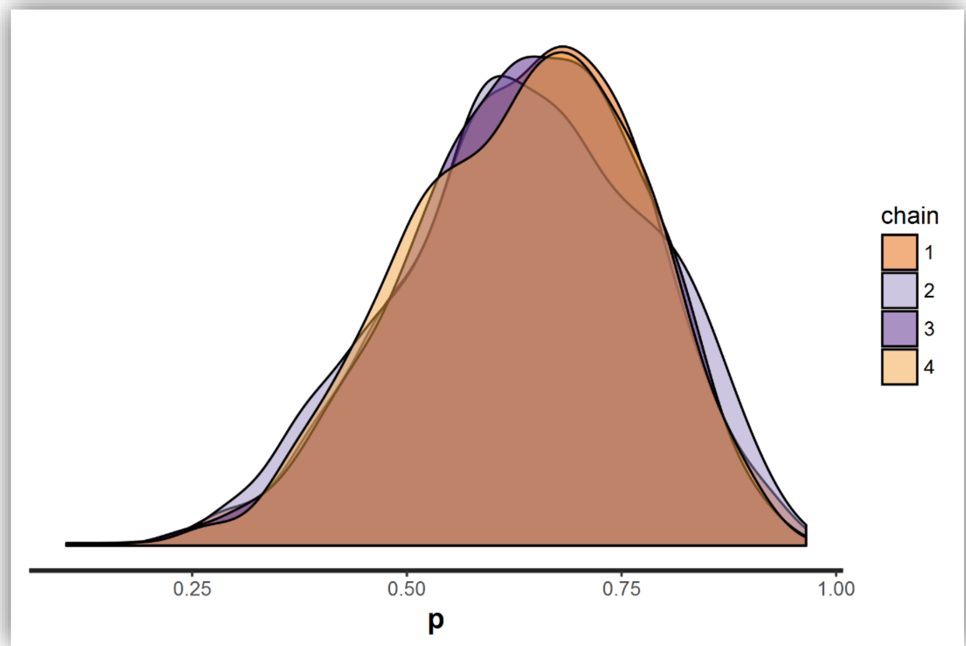
cognitive model
statistics
computing





Diagnostics - density

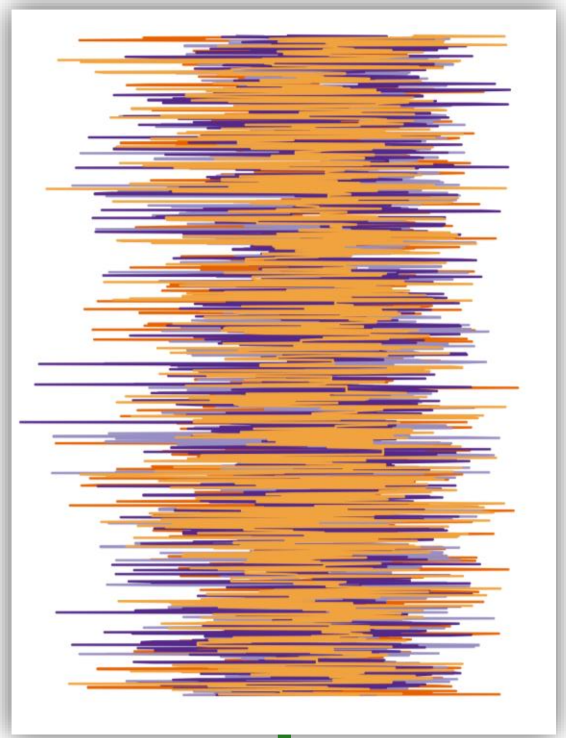


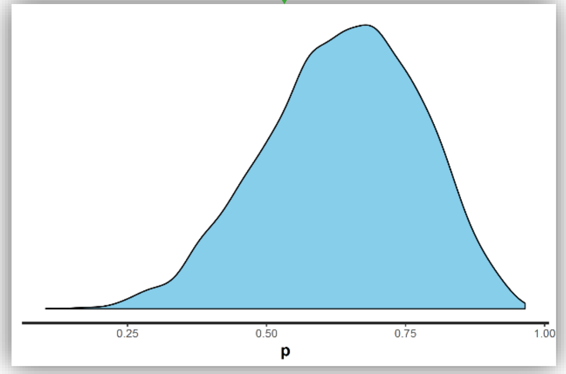


Diagnostics

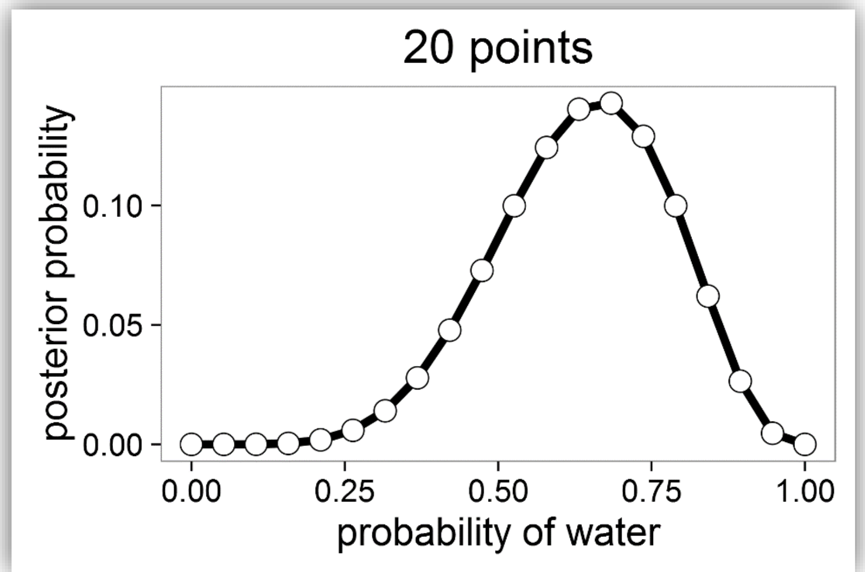
cognitive model
statistics
computing

MCMC





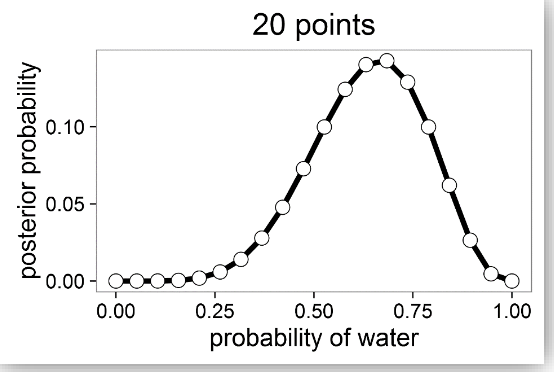
Grid Approximation

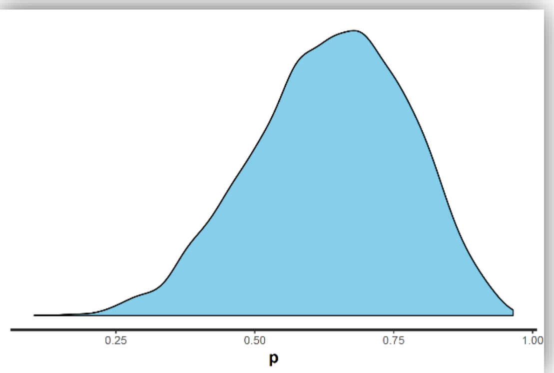


Draw a Conclusion?

cognitive model
statistics
computing

- W = 6 out of N = 9
- uncertainty (relative plausibility) of all p values
- the relative plausibility of p = 0.63 is the highest, but it never rules out the possibility of p being other values, e.g., 0.5, 0.75
- → when p = 0.5, you may still observe 6W / 9 trials





Is Anything Missing? – NO

cognitive model

statistics

computing

```
data {
  int<lower=0> w;
  int<lower=0> N;
}

parameters {
  real<lower=0,upper=1> p;
}

model {
  p ~ uniform(0,1);
  w ~ binomial(N, p);
}
```

```
data {
  int<lower=0> w;
  int<lower=0> N;
}

parameters {
  real<lower=0,upper=1> p;
}

model {
  w ~ binomial(N, p);
}
```


ANY QUESTIONS
?

Stay tuned and
bis morgen!