Bayesian Statistics and Bayesian Cognitive Modeling

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Recap

What we've learned…

- R Basics
- probability distributions
- Bayes' theorem, $p(\theta|D)$
- Binomial model
- MCMC and Stan

BERNOULLI MODEL

Bernoulli Model

- You are interested in if a coin is biased.
- You will flip the coin.
- You will record whether it comes up a head (h) or a tail (t).
- You might observe 15 heads out of 20 flips.
- What is your degree of belief about the biased parameter ϑ?

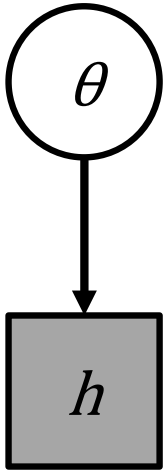
Kruschke (2015)

Bernoulli Model

$$p(w \mid N, p) = \binom{N}{w} p^{w} (1-p)^{N-w}$$

$N = 1$

$$p(h \mid \theta) = \theta^{h} (1-\theta)^{1-h}$$



$$\theta \sim \text{Uniform}(0, 1)$$

$$h \sim \text{Bernoulli}\,(\theta)$$

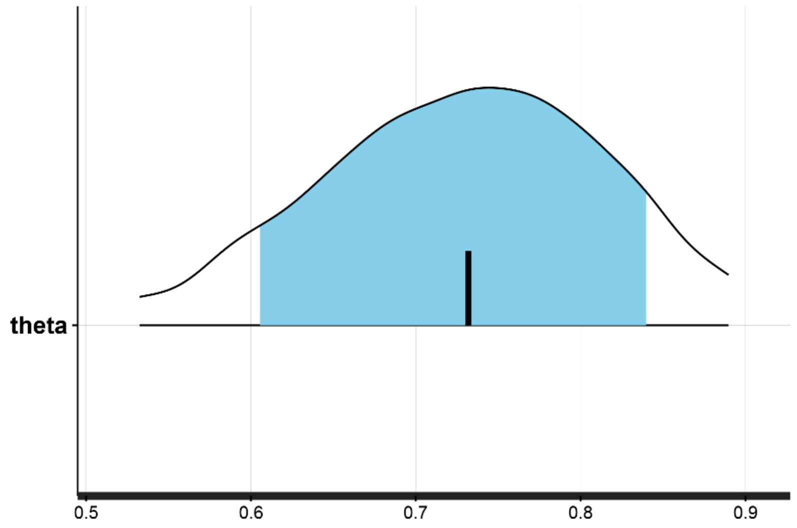
Exercise I

cognitive model
statistics
computing

```
…/BayesCog/03.bernoulli_coin/_scripts/bernoulli_coin_main.R
```

TASK: fit the Bernoulli model

```
> dataList
$`flip`
 [1] 1 1 1 0 1 1 1 1 1 0 0 1 1 0 1 1 1 1 0 1

$N
[1] 20
```



Possible Optimization?

```
model {
  for (n in 1:N) {
    flip[n] ~ bernoulli(theta);
  }
}
```

61.59 secs*

```
model {
  flip ~ bernoulli(theta);
}
```

53.25 secs*

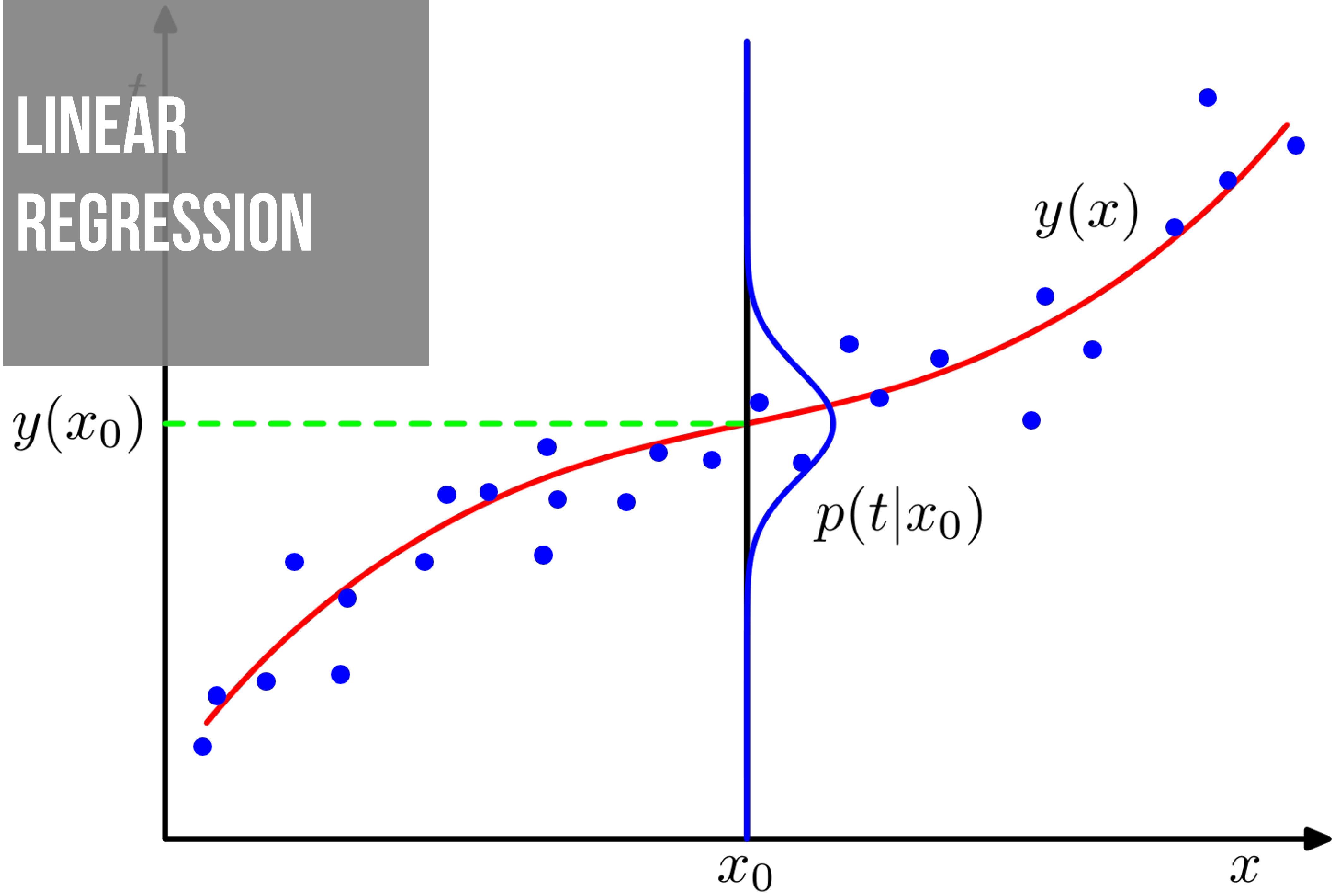
Thinking before looping!

* compiling time included

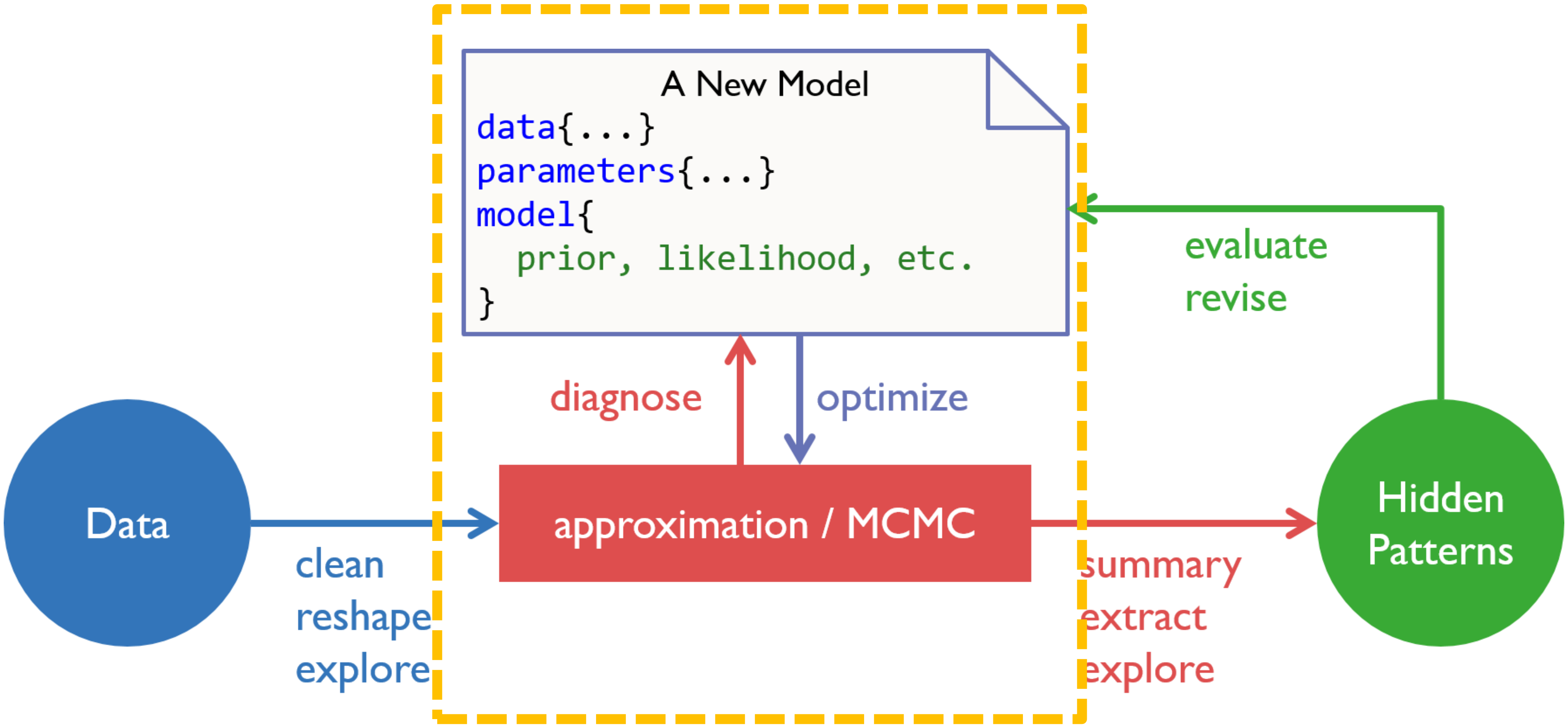
DAY2

09:00 – 09:30	Warmup – Bernoulli Model
09:30 – 10:00	Linear Regression Model
10:00 – 10:30	Predictive Check
10:30 – 10:45	Coffee break
10:45 – 11:30	Cognitive Modeling
11:30 – 12:30	Reinforcement Learning Model
12:30 – 13:30	Lunch break
13:30 – 14:00	Fitting Multiple Participants
14:00 – 15:00	Hierarchical Modeling
15:00 – 15:15	Coffee Break
15:15 – 16:15	Optimizing Stan Codes
16:15 – 17:00	Model Comparison

LINEAR REGRESSION



cognitive model
statistics
computing



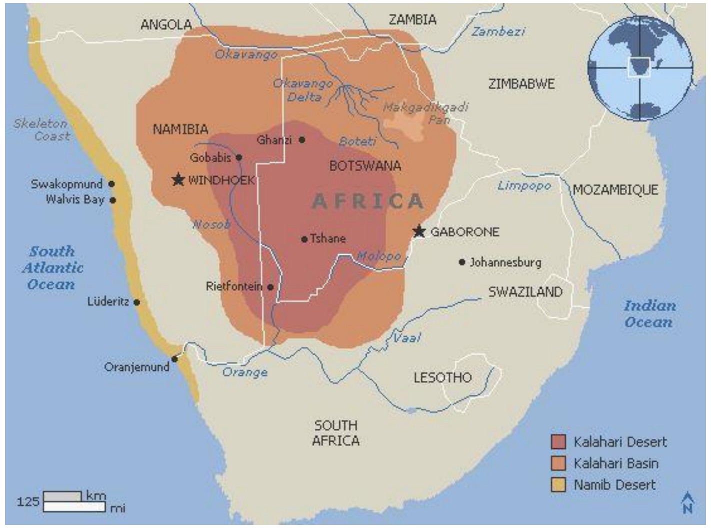
Linear Regression: height ~ weight

cognitive model
statistics
computing

```
…/BayesCog/04.regression_height/_scripts/regression_height_main.R
```

make scatter plot and fit the model with `lm()`

```
>load('_data/height.RData')
>d <- Howell1
>d <- d[ d$age >= 18 , ]
>head(d)
height   weight age male
1 151.765 47.82561  63    1
2 139.700 36.48581  63    0
3 136.525 31.86484  65    0
4 156.845 53.04191  41    1
5 145.415 41.27687  51    0
6 163.830 62.99259  35    1
```



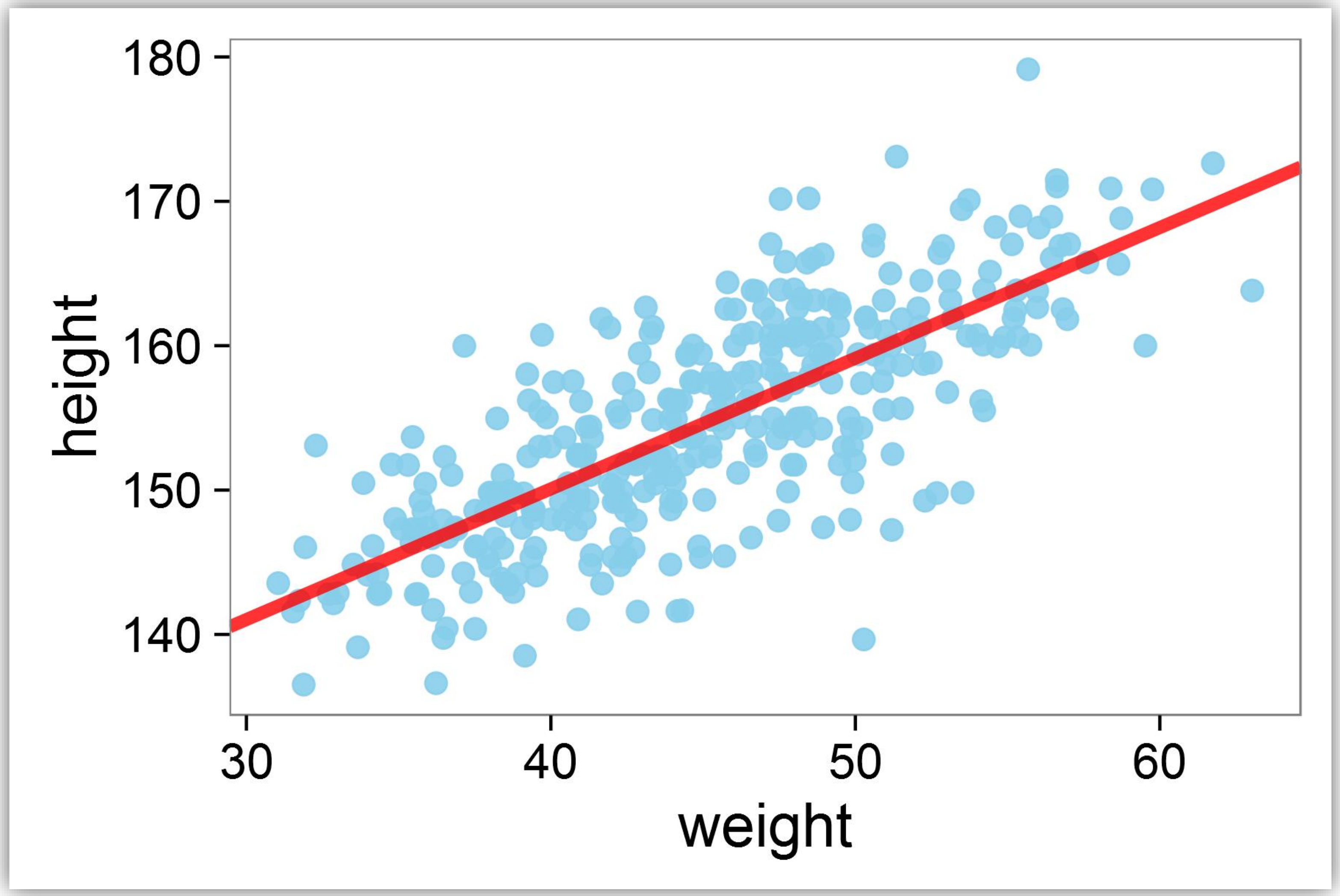
McElreath (2016)

Results with lm()


```
> L <- lm( height ~ weight, d) # estimate model by minimizing least squares errors
> summary(L)

Call:
lm(formula = height ~ weight, data = d)

Residuals:
     Min       1Q   Median       3Q      Max 
-19.7464  -2.8835   0.0222   3.1424  14.7744 

Coefficients:
             Estimate Std. Error t value Pr(>|t|)    
(Intercept) 113.87939    1.91107   59.59   <2e-16 ***
weight        0.90503    0.04205   21.52   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 5.086 on 350 degrees of freedom
Multiple R-squared:  0.5696, Adjusted R-squared:  0.5684 
F-statistic: 463.3 on 1 and 350 DF,  p-value: < 2.2e-16
```

height ~ weight

cognitive model
statistics
computing



McElreath (2016)

Rethinking Regression Model

cognitive model
statistics
computing

$$\mu_i = \alpha + \beta x_i$$

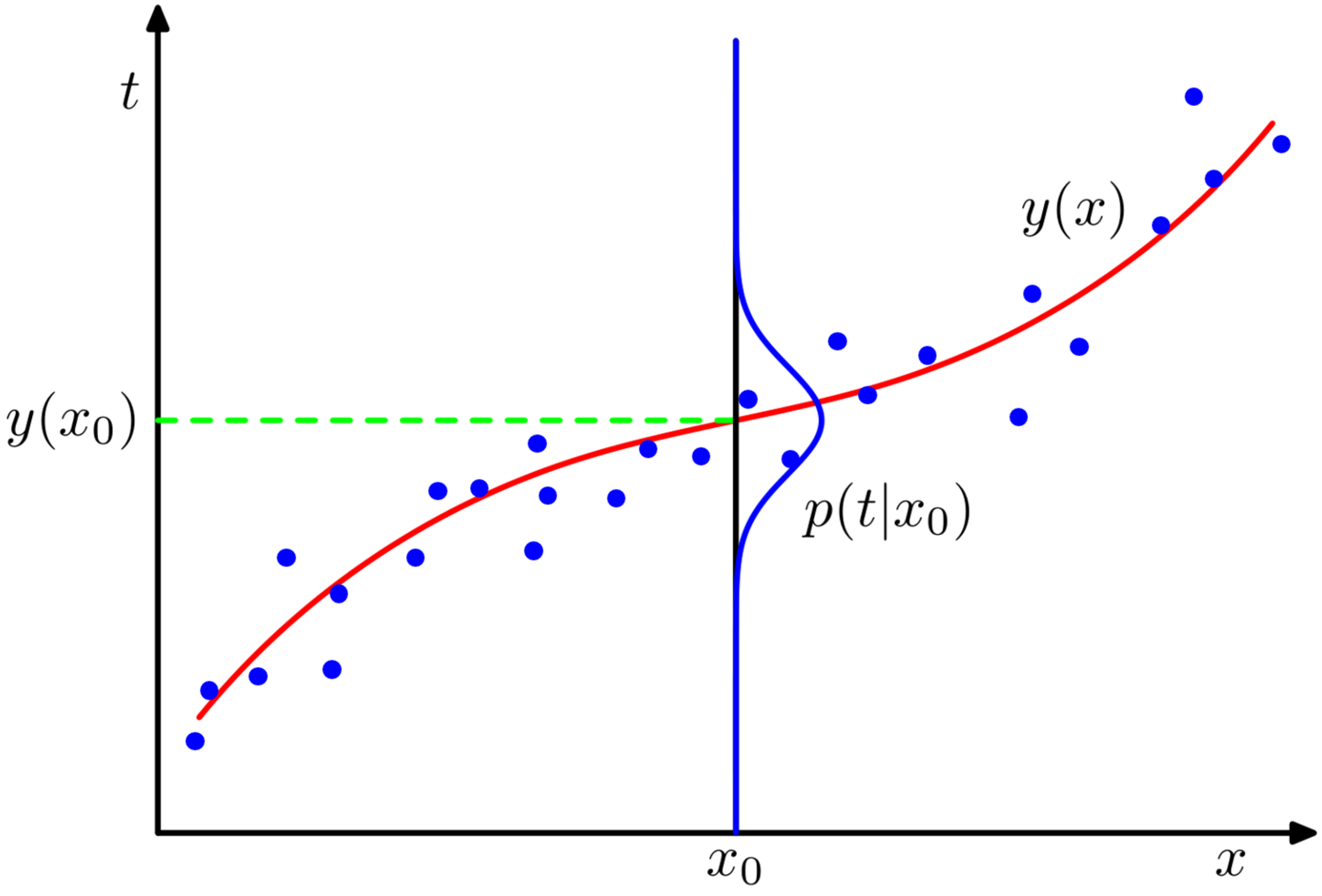
$$\cancel{y_i = \mu_i + \varepsilon}$$

$$\cancel{\varepsilon \sim Normal(0, \sigma)}$$

$$\boxed{y_i \sim Normal(\mu_i, \sigma)}$$

Rethinking Regression Model

$$\mu_i = \alpha + \beta x_i$$
$$y_i \sim Normal(\mu_i, \sigma)$$



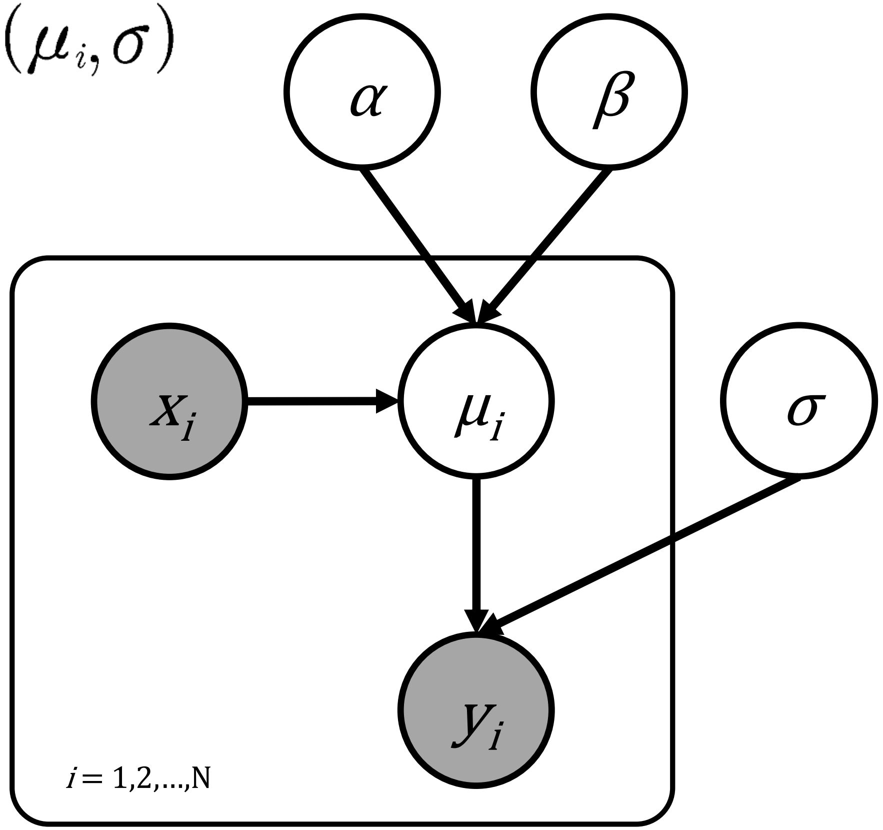
Bishop (2006)

Rethinking Regression Model

cognitive model
statistics
computing

$$\mu_i = \alpha + \beta x_i$$

$$y_i \sim Normal(\mu_i, \sigma)$$




```
model {
  vector[N] mu;
  for (i in 1:N) {
    mu[i] = alpha + beta * weight[i];
    height[i] ~ normal(mu[i], sigma);
  }
}
```

```
model {
  vector[N] mu;
  mu = alpha + beta * weight;
  height ~ normal(mu, sigma);
}
```

```
model {
  height ~ normal(alpha + beta * weight, sigma);
}
```

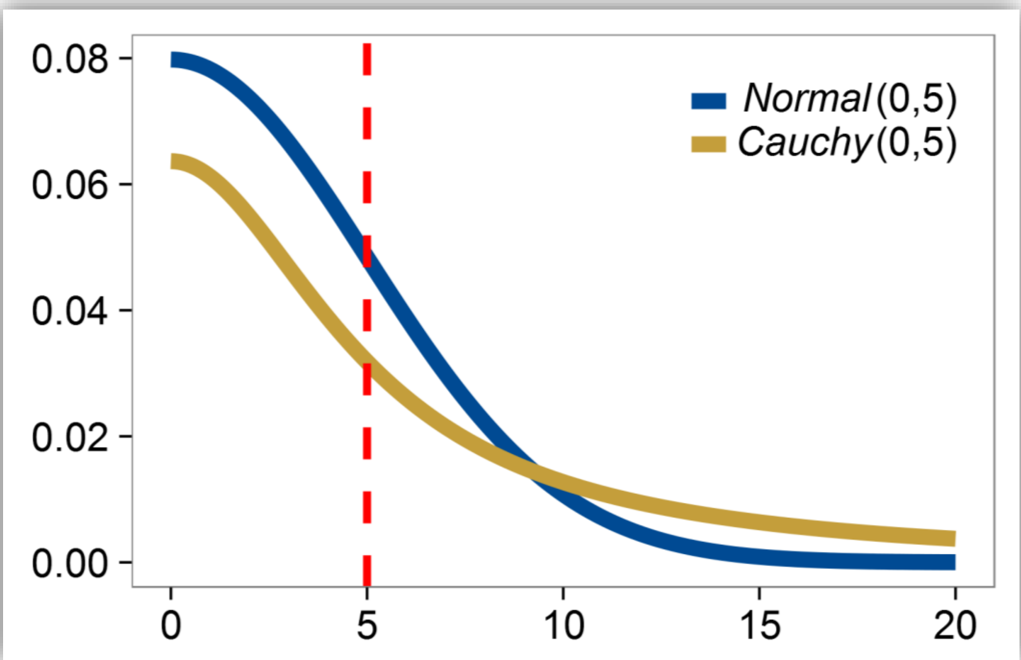
Thinking about Priors?

$\alpha \sim Normal(170, 100)$ $\beta \sim Normal(0,20)$

$$\overline{\text{height}} = \alpha + \beta * \text{weight}$$

$\sigma \sim halfCauchy(0,20)$

$$\text{height} \sim Normal(\overline{\text{height}}, \sigma)$$

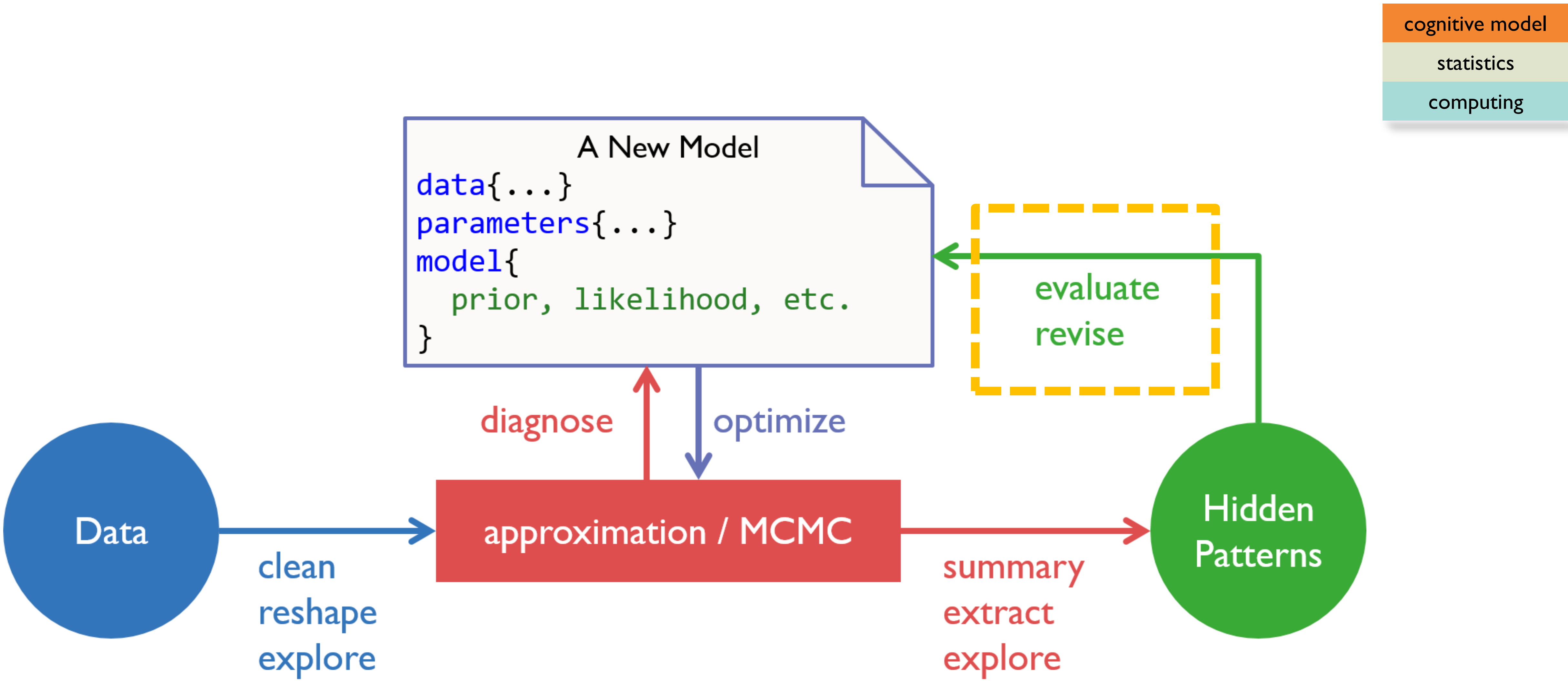


Exercise II

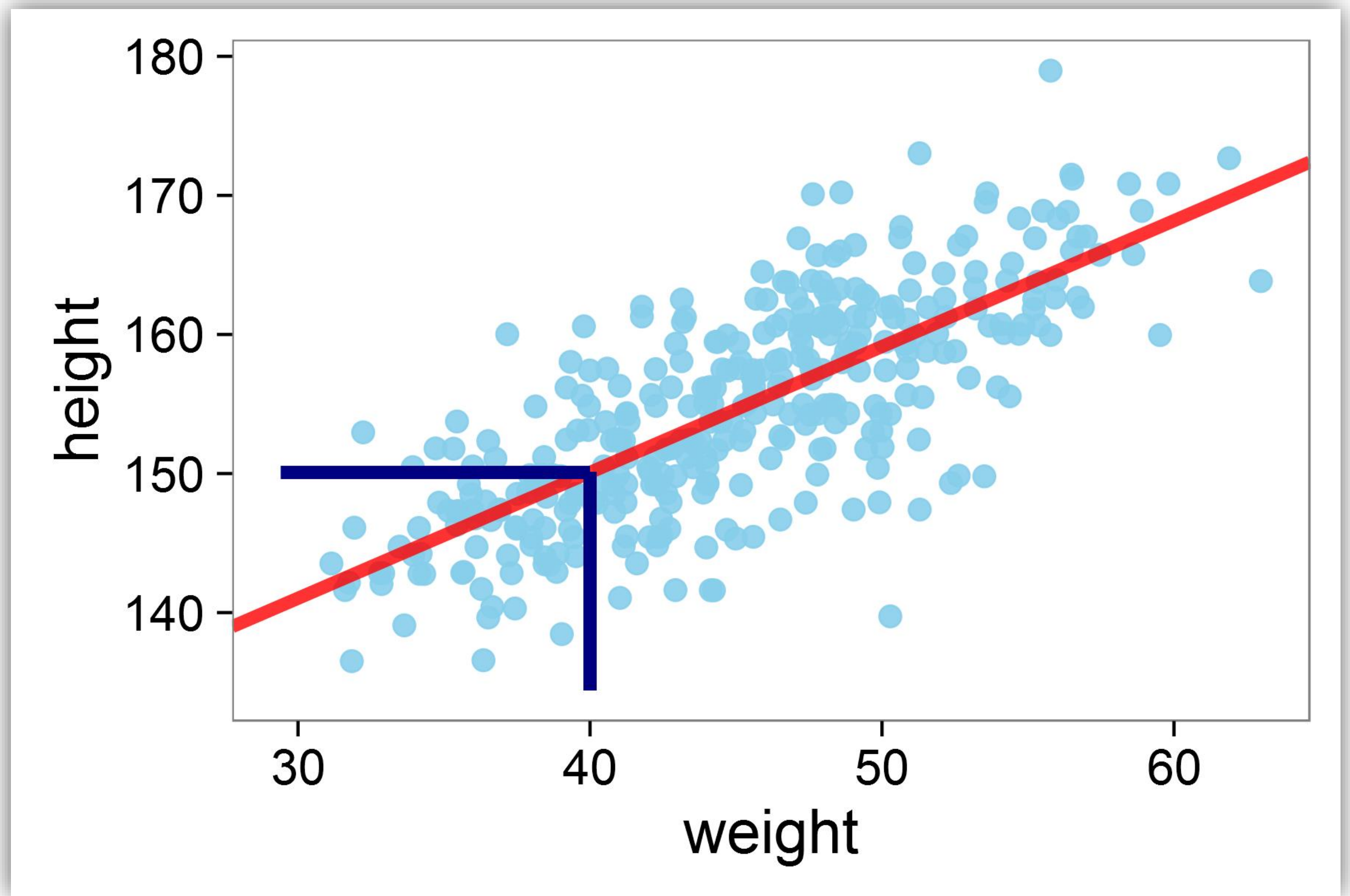

```
…/BayesCog/04.regression_height/_scripts/regression_height_main.R
```

TASK: estimate the model and produce the results

```
Inference for Stan model: regression_height_model.
4 chains, each with iter=2000; warmup=1000; thin=1;
post-warmup draws per chain=1000, total post-warmup draws=4000.

         mean se_mean   sd    2.5%     25%     50%     75%   97.5% n_eff Rhat
alpha  113.97    0.06 1.86  110.27  112.76  113.93  115.20  117.66   934    1
beta     0.90    0.00 0.04    0.82    0.88    0.90    0.93    0.99   922    1
sigma    5.11    0.01 0.19    4.74    4.97    5.10    5.24    5.50  1437    1
lp__  -747.61    0.04 1.23 -750.80 -748.15 -747.28 -746.72 -746.24   993    1
```


cognitive model
statistics
computing
A New Model
data{...}
parameters{...}
model{
prior, likelihood, etc.
}
evaluate
revise
diagnose
optimize
Data
clean
reshape
explore
approximation / MCMC
summary
extract
explore
Hidden Patterns

What does the Model Predict?



$$p(y_{rep} \mid y) = \int p(y_{rep} \mid \theta)\, p(\theta \mid y)\, d(\theta)$$

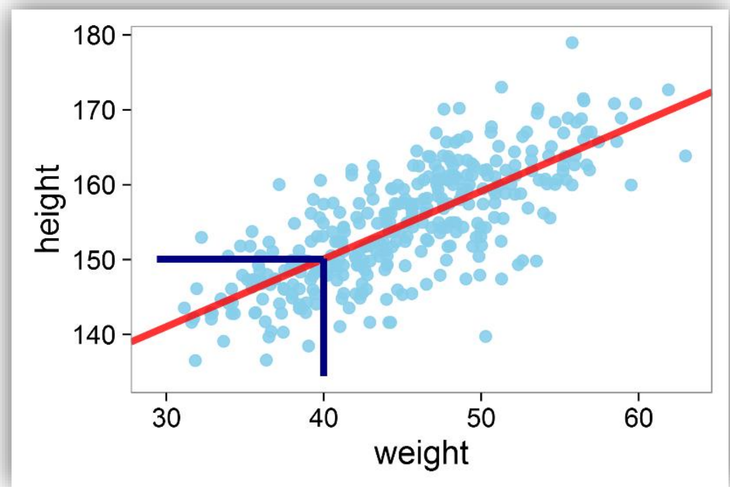
McElreath (2016)

Posterior Predictive Check (PPC)

cognitive model
statistics
computing

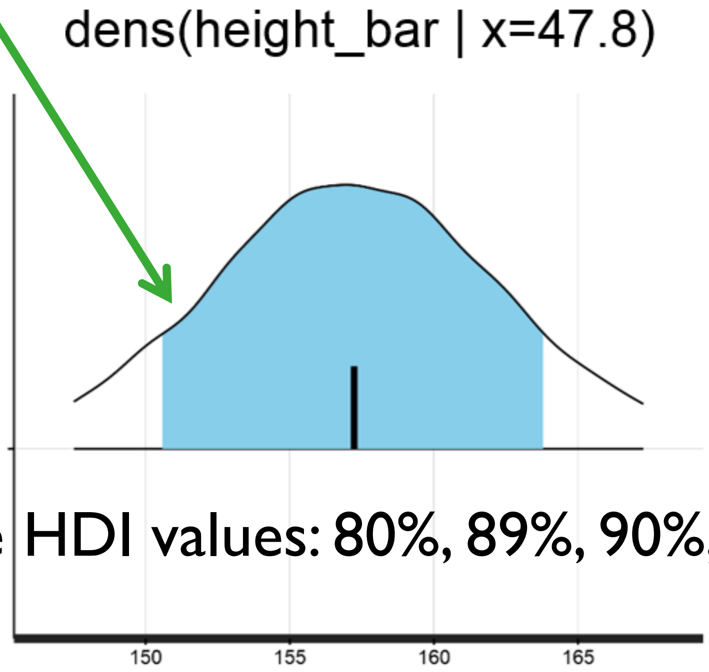
```
generated quantities {
  vector[N] height_bar;
  for (n in 1:N) {
      height_bar[n] = normal_rng(alpha + beta * weight[n], sigma);
  }
}
```

the generated quantities block runs only AFTER the sampling, and the time it costs can be essentially ignored!

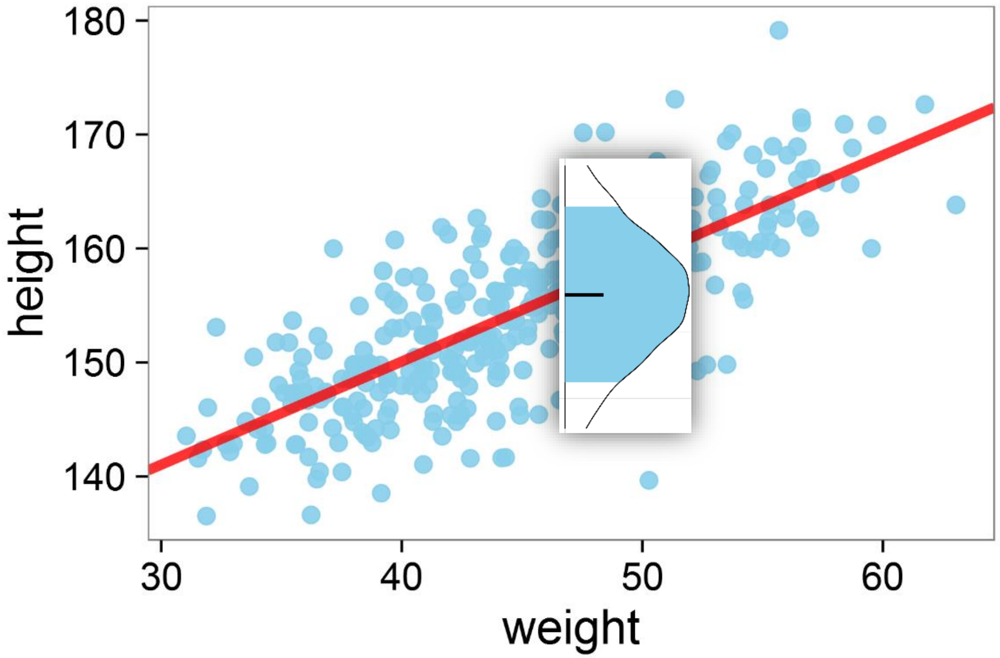


Posterior Predictive Check (PPC)

Highest density interval (HDI)

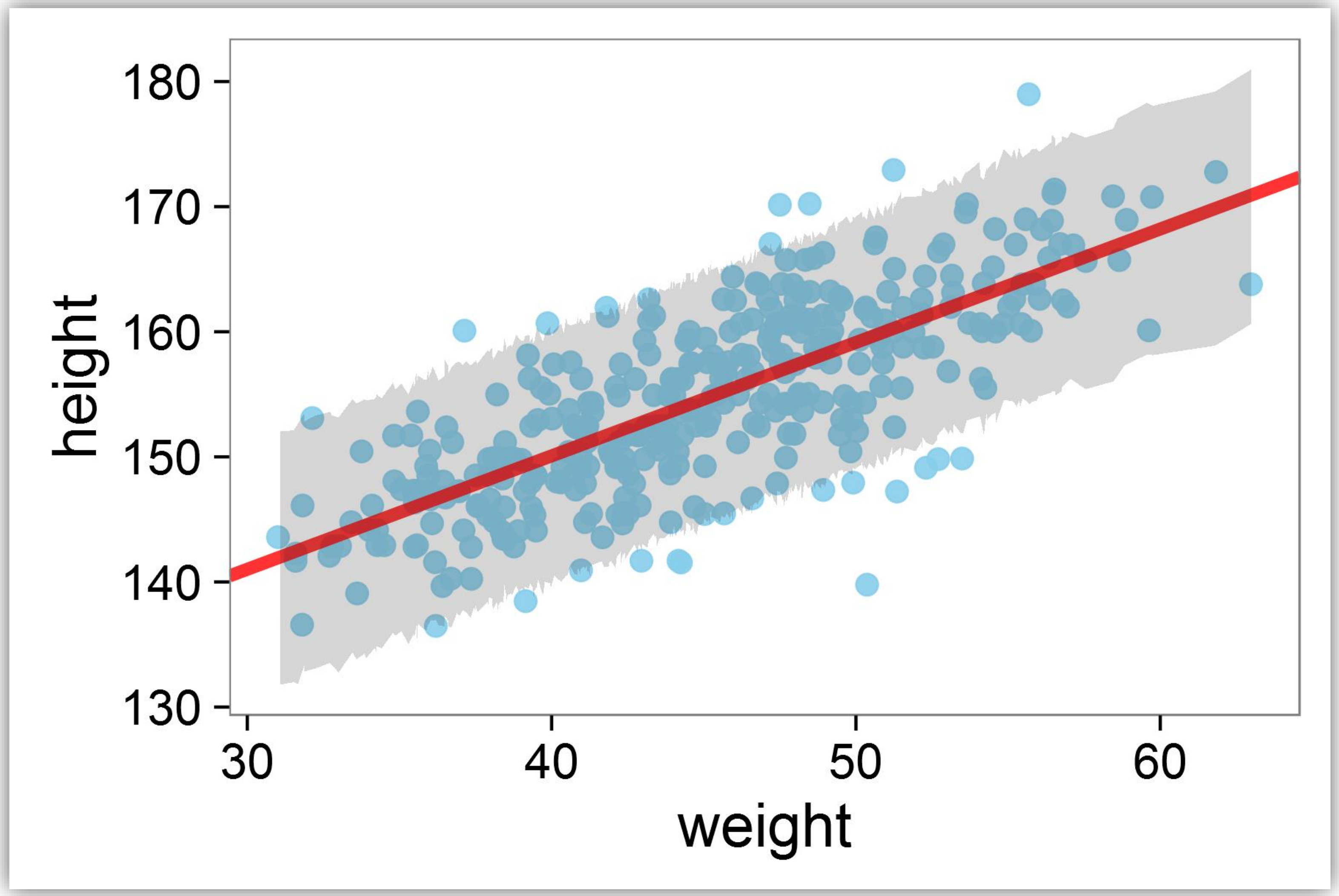


possible HDI values: 80%, 89%, 90%, 95%




```
height_bar <- extract(fit_reg_ppc, pars = 'height_bar',
                      permuted = FALSE)$height_bar
height_HDI <- apply(height_bar, 2, HDIofMCMC)
```

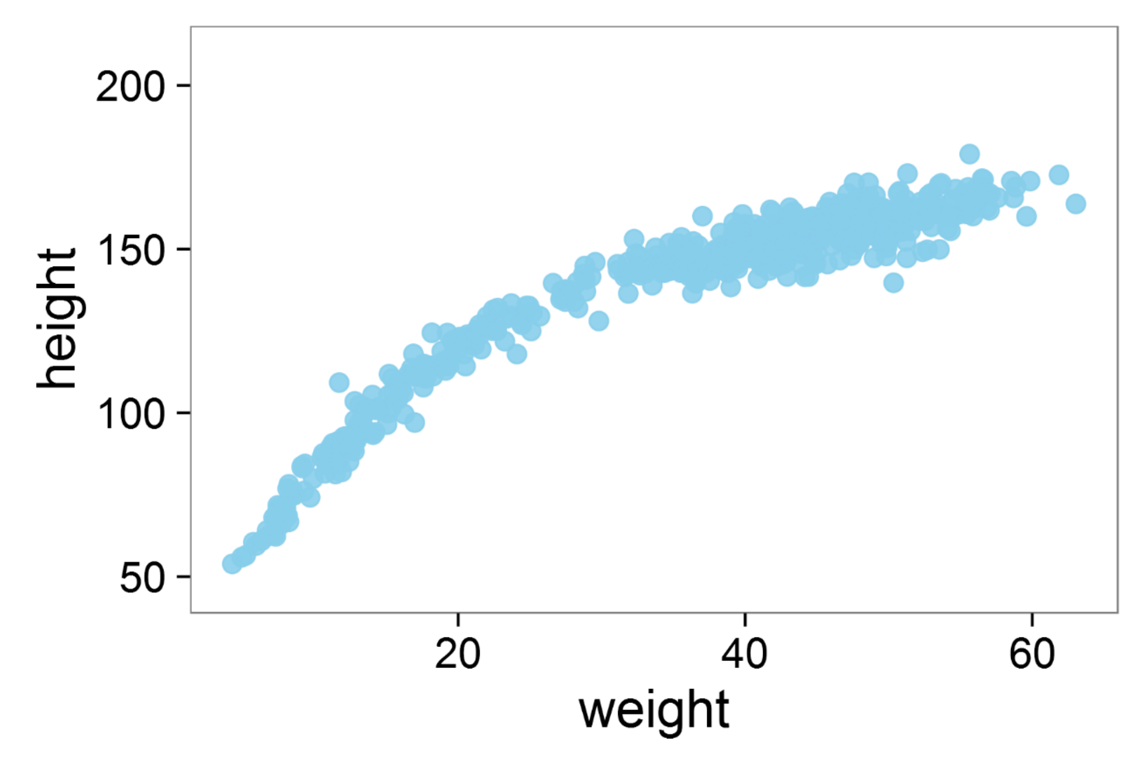
Posterior Predictive Check (PPC)



Exercise III


```
…/BayesCog/05.regression_height_poly/_scripts
/regression_height_poly_main.R
```

TASK: produce PPC plot for both 1st order and 2nd order polynomial fit



Exercise III – Tips

cognitive model

statistics

computing

```
> source('_scripts/regression_height_poly_main.R')

> out1 <- reg_poly(poly_order = 1)
```

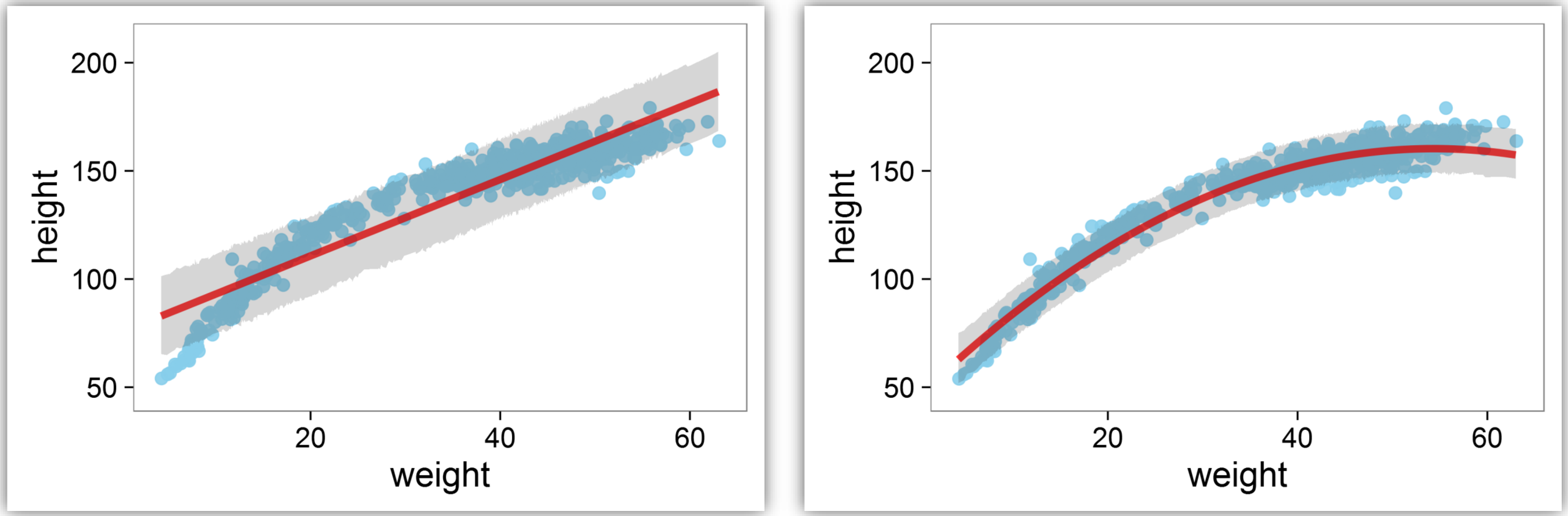
$$\overline{\text{height}} = \alpha + \beta 1 * \text{weight} + \beta 2 * \text{weight}^2$$

$$\text{height} \sim Normal(\overline{\text{height}}, \sigma)$$

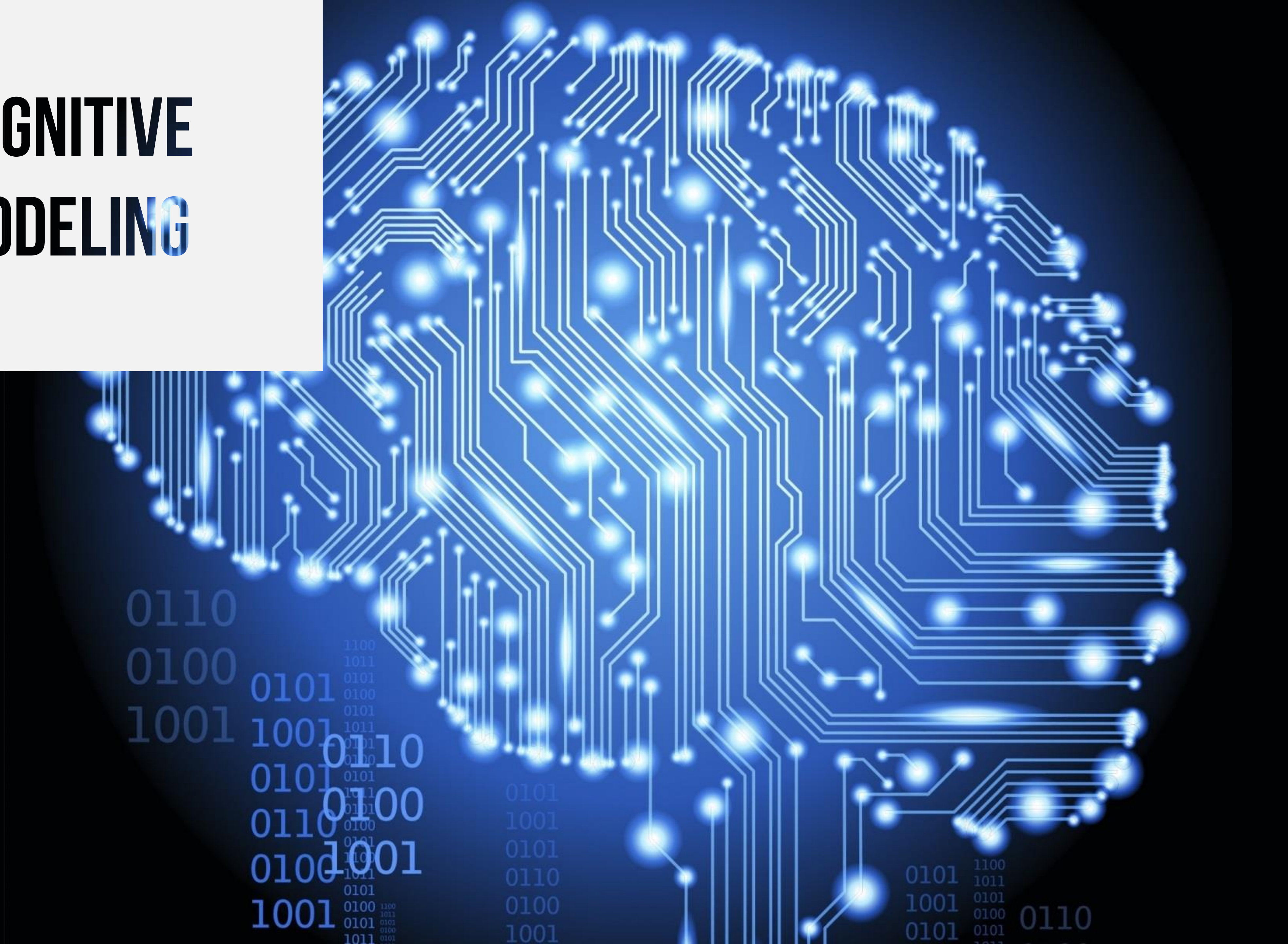
```
data {
  int<lower=0> N;
  vector<lower=0>[N] height;
  vector<lower=0>[N] weight;
  vector<lower=0>[N] weight_sq;
}
```

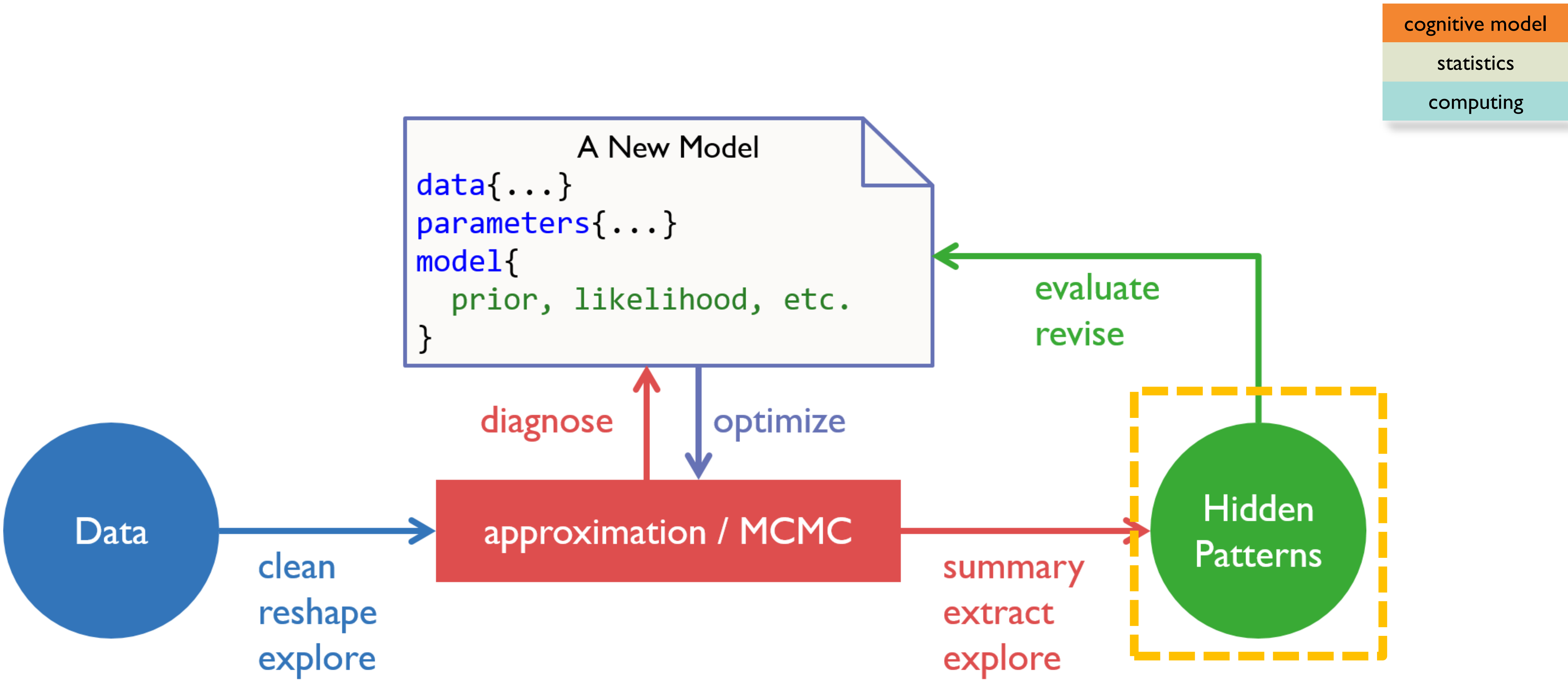
```
height ~ normal(alpha + beta1 * weight + beta2 * weight_sq, sigma);
```

Exercise III – output2



COGNITIVE MODELING




cognitive model
statistics
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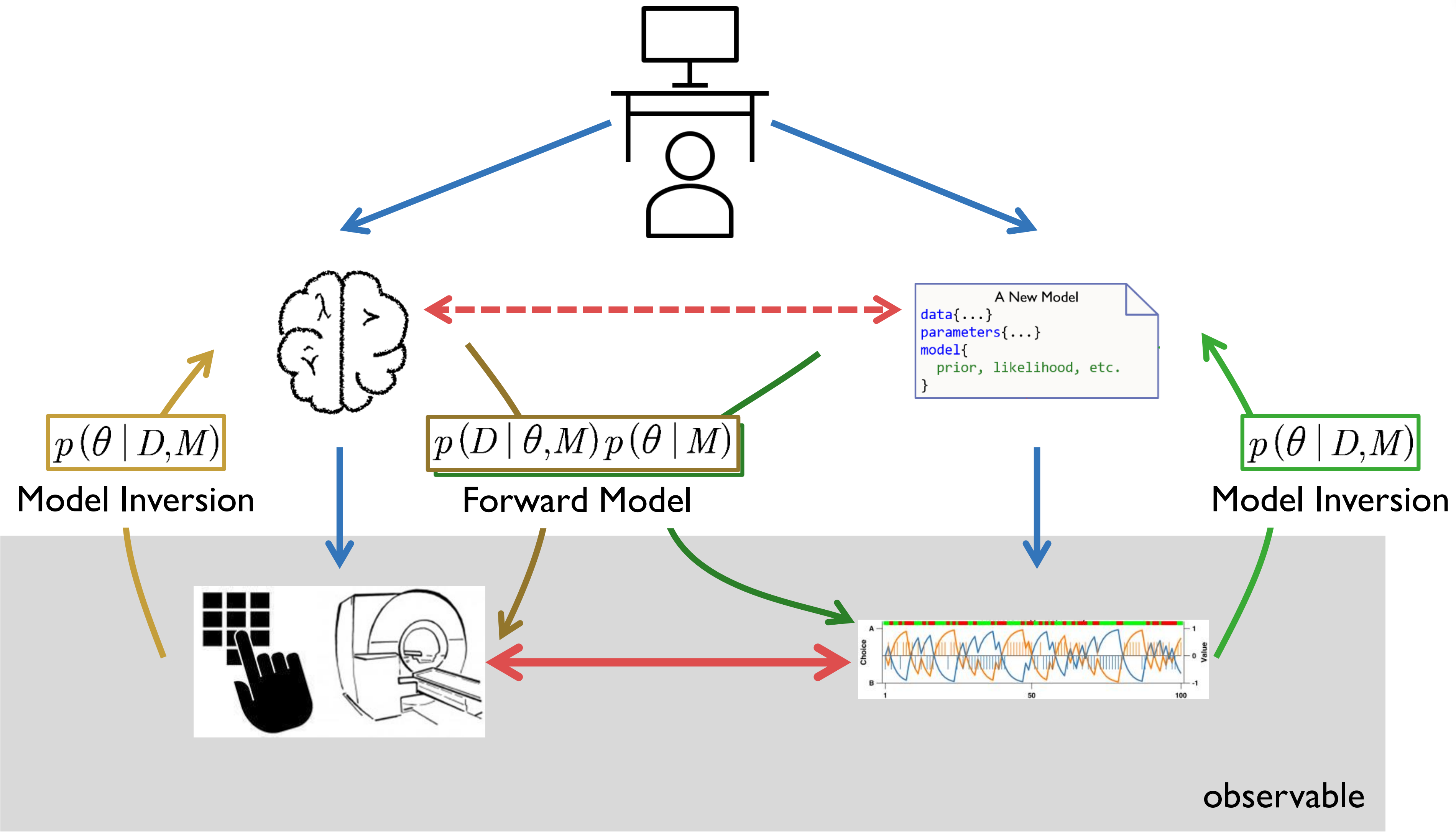
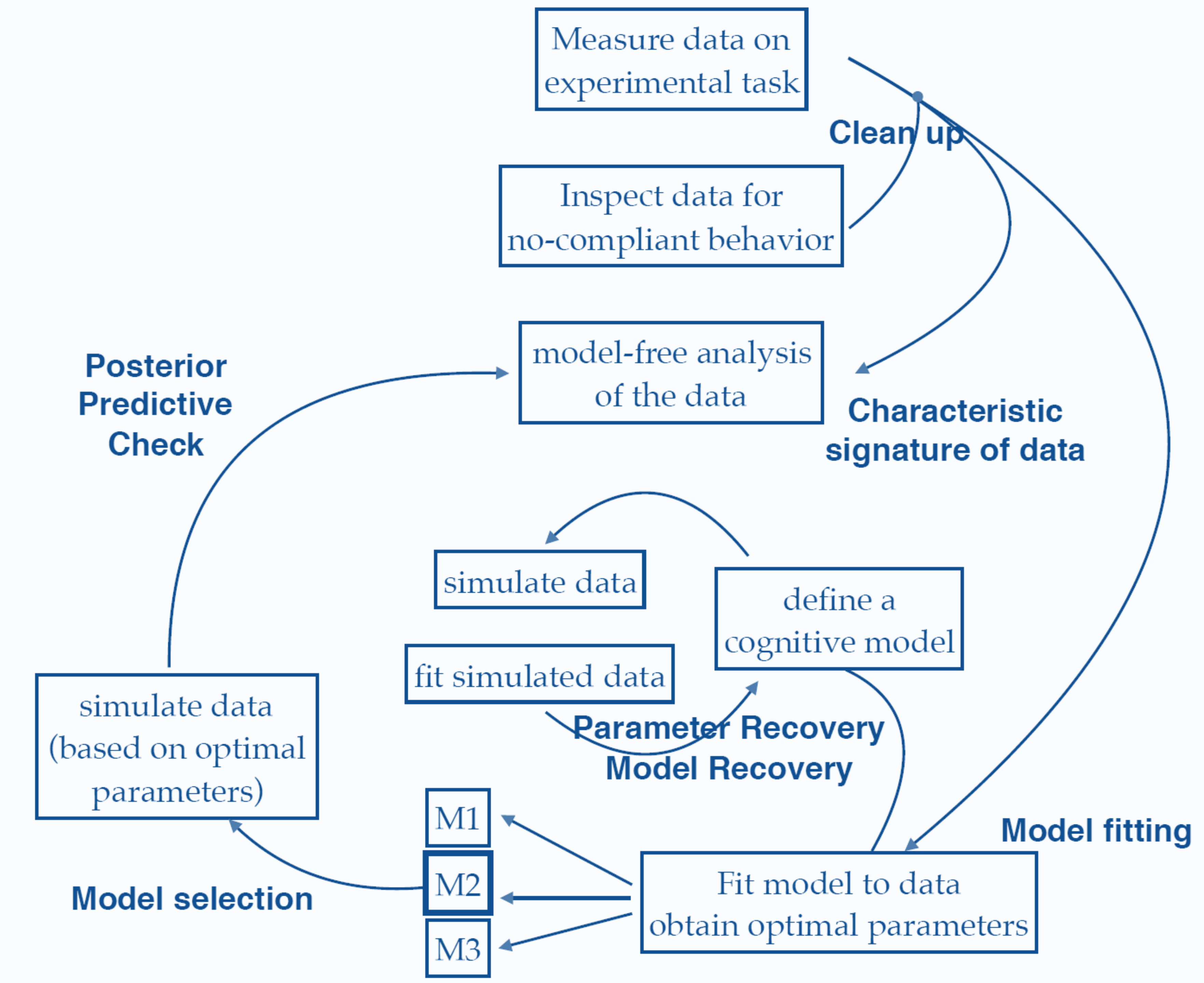
A New Model
data{...}
parameters{...}
model{
prior, likelihood, etc.
}
evaluate
revise
diagnose
optimize
Data
approximation / MCMC
Hidden Patterns
clean
reshape
explore
summary
extract
explore

What is Cognitive Modeling?



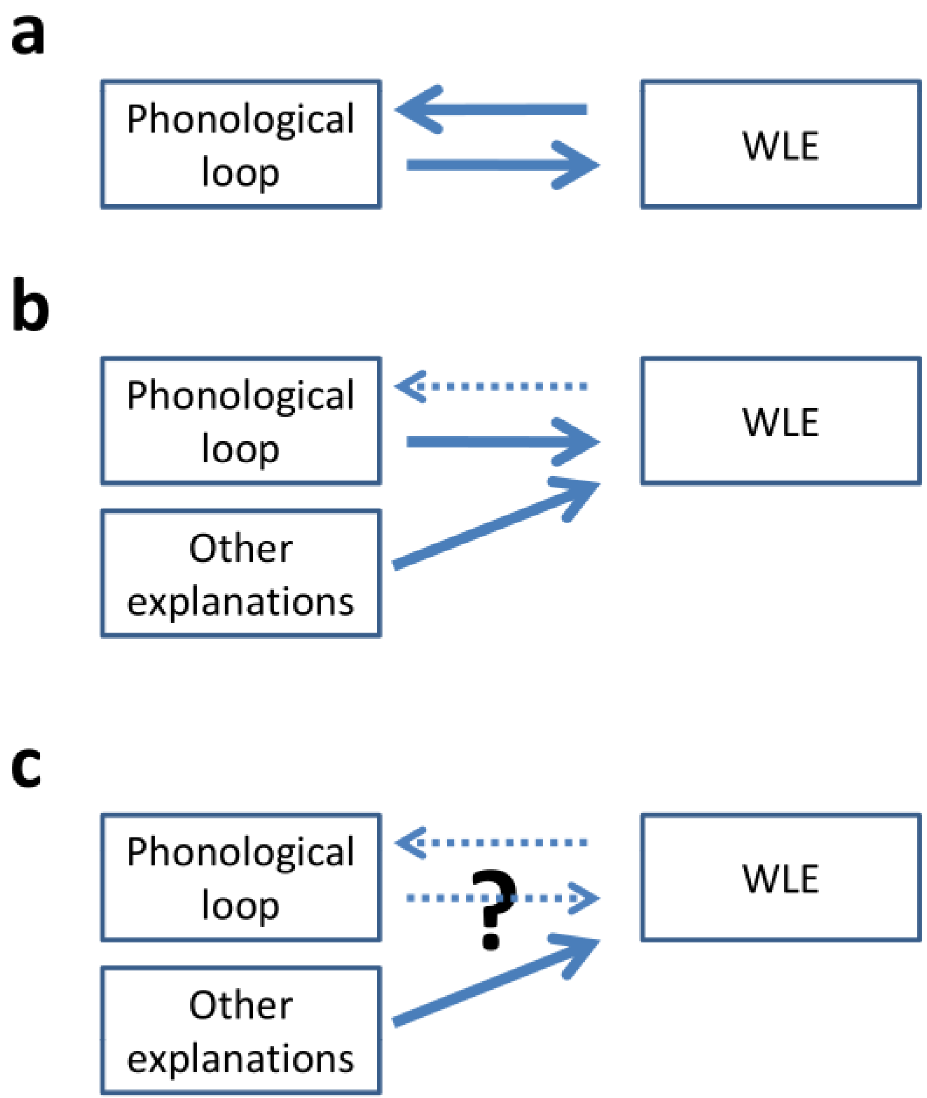
Lewandowsky & Farrell (2011); Stephan et al. (2016); Ahn et al., (2017)

cognitive model
statistics
computing



Adapted from Jan Gläscher's workshop

Mapping Model onto Cognitive Process?



WLE = word length effect

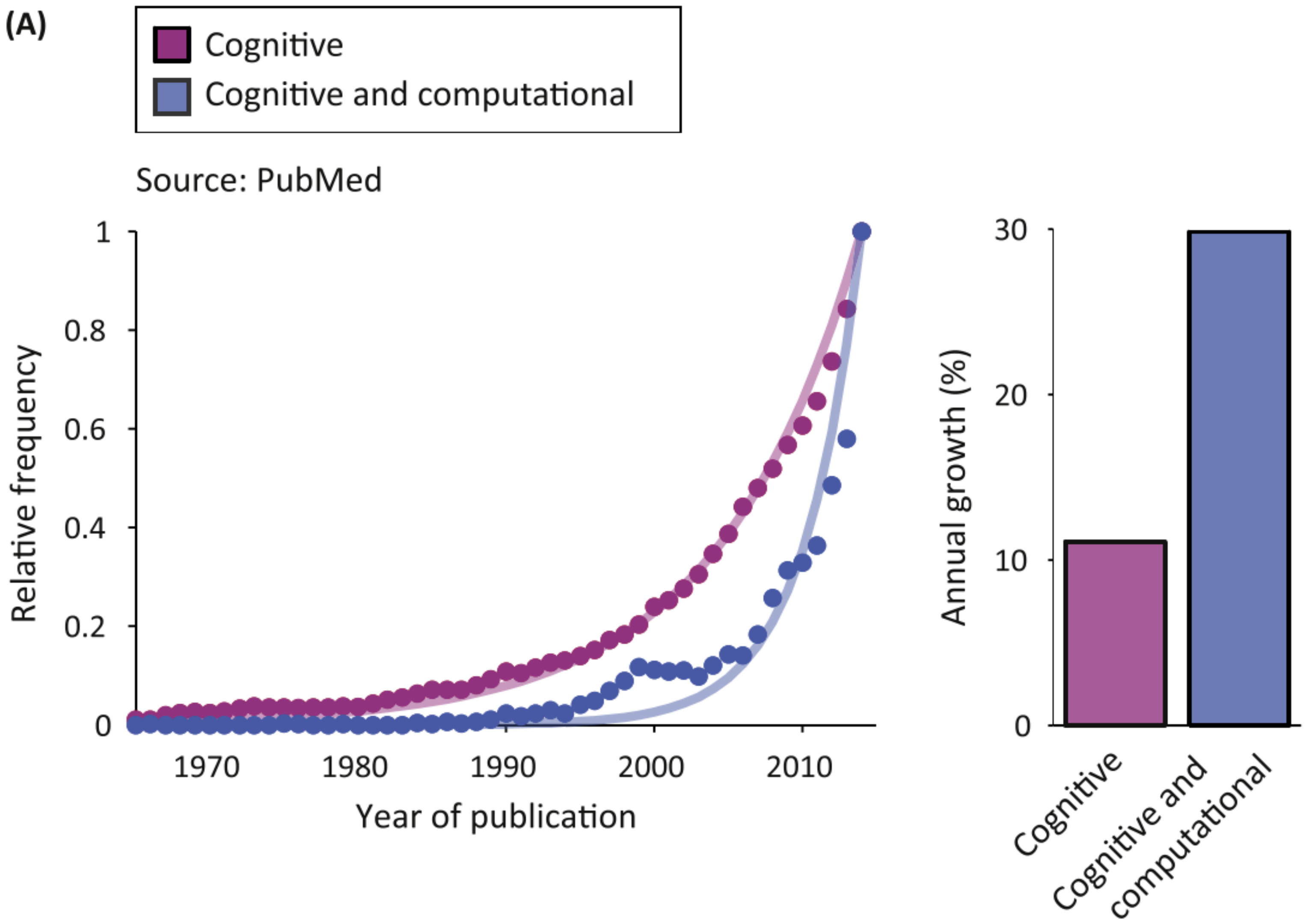
Lewandowsky & Farrell (2011)

Essentially, all the models are wrong, but some are useful.

– George E. P. Box

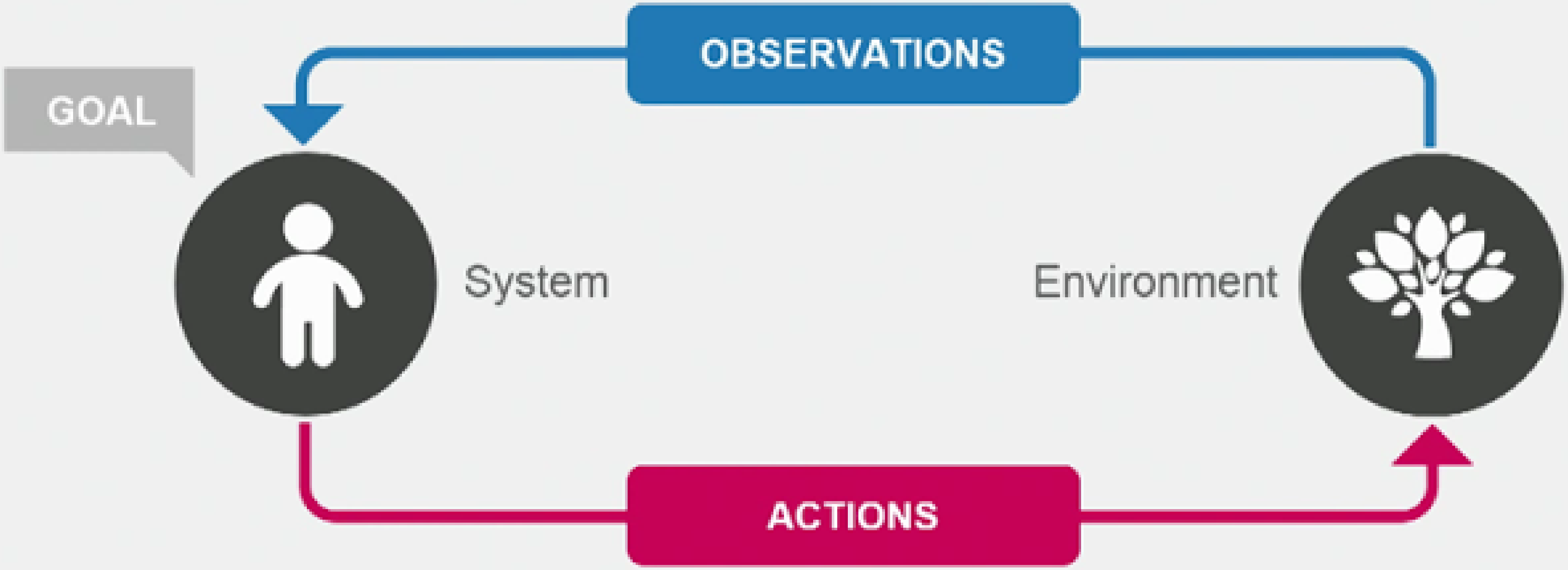
Essentially, all the models are ~~wrong~~ imperfect, but some are useful.

Boom in Cognitive Modeling

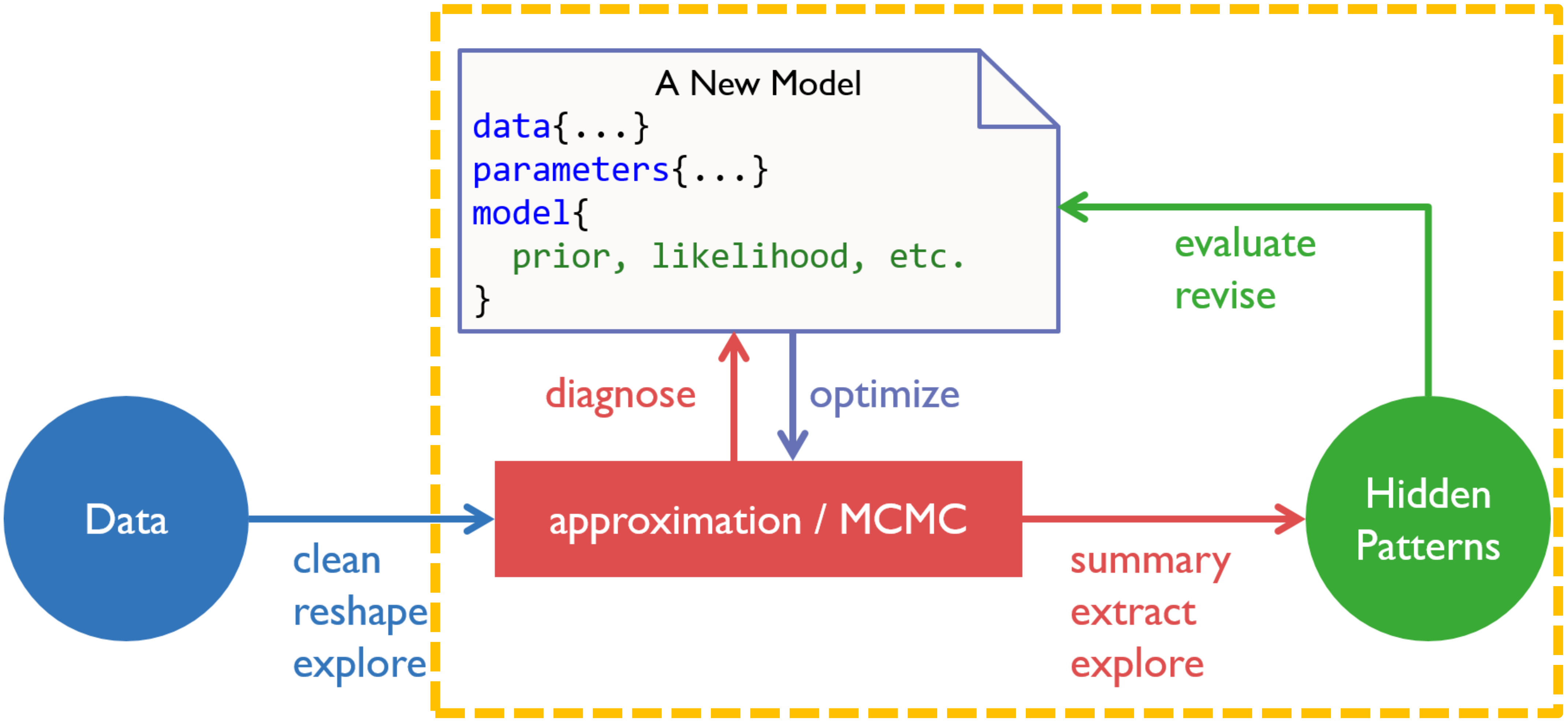


Palminteri et al. (2017)

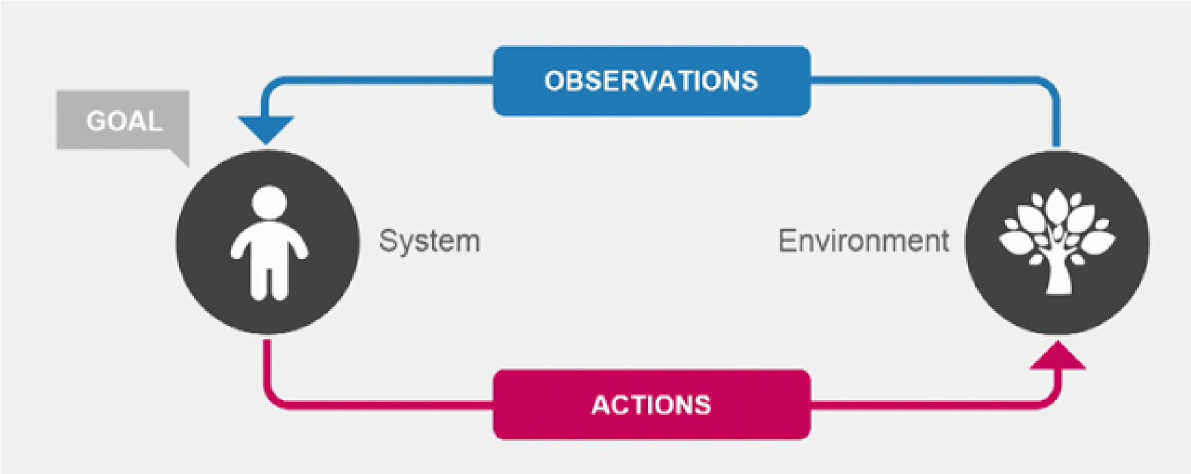
REINFORCEMENT LEARNING FRAMEWORK

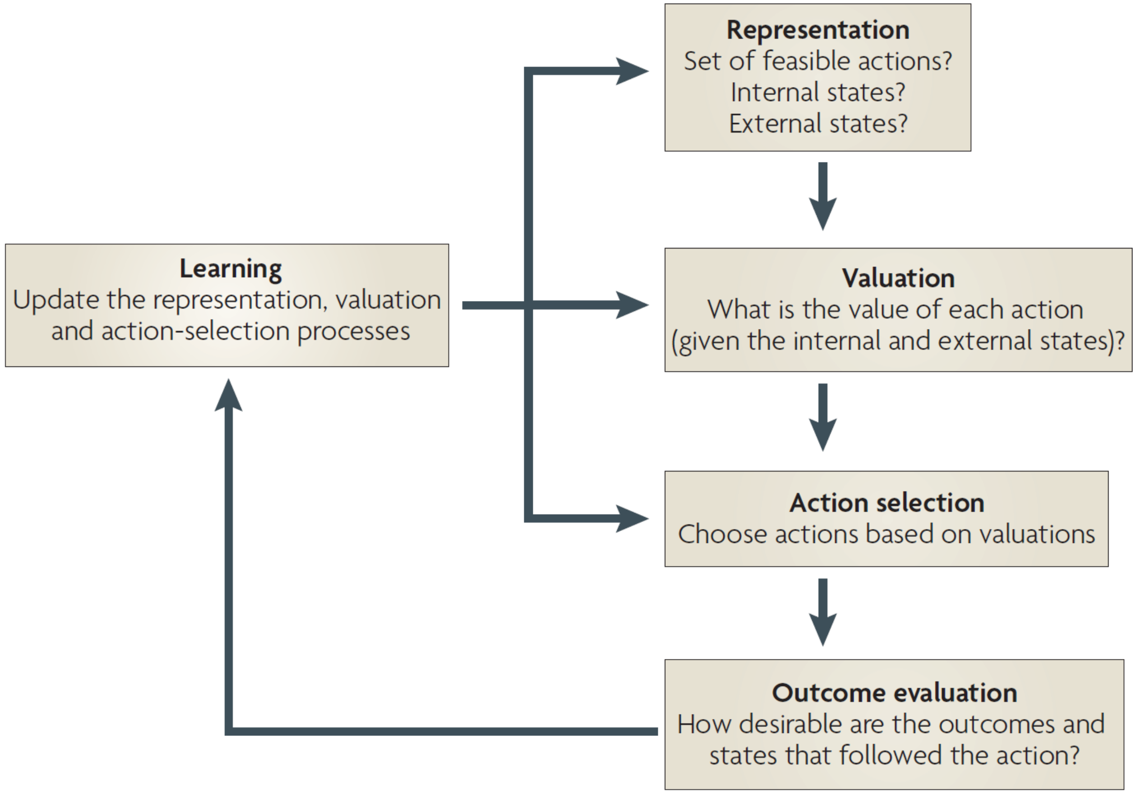


cognitive model
statistics
computing



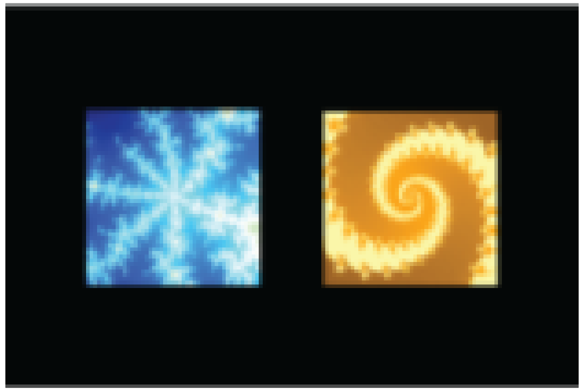
Reinforcement Learning



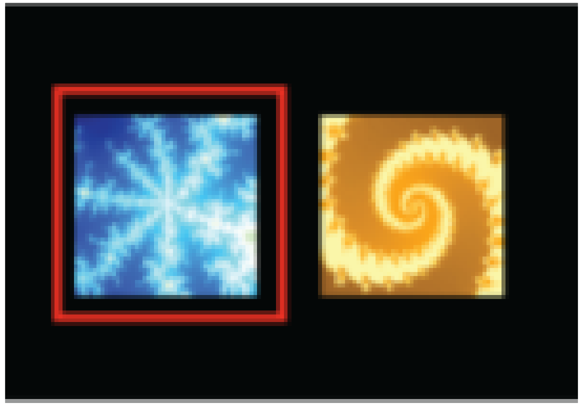


Sutton & Barto (1998); Rangel et al (2008)

One simple experiment: two choice task



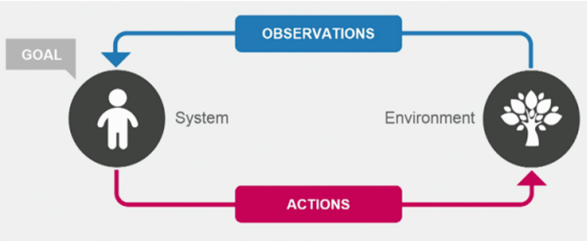
choice presentation



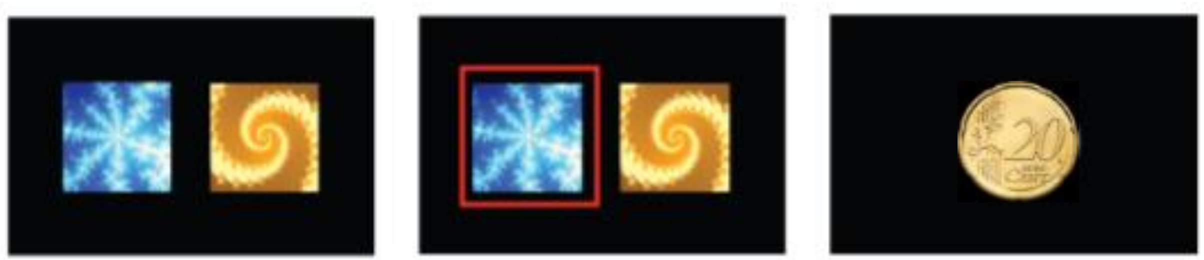
action selection

outcome

reward contingency – 80:20



Rescorla-Wagner Value Update



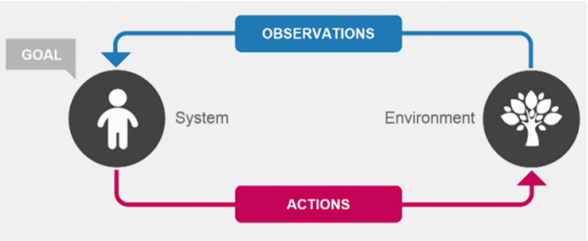
Cognitive Model

- cognitive process
- using internal variables and free parameters

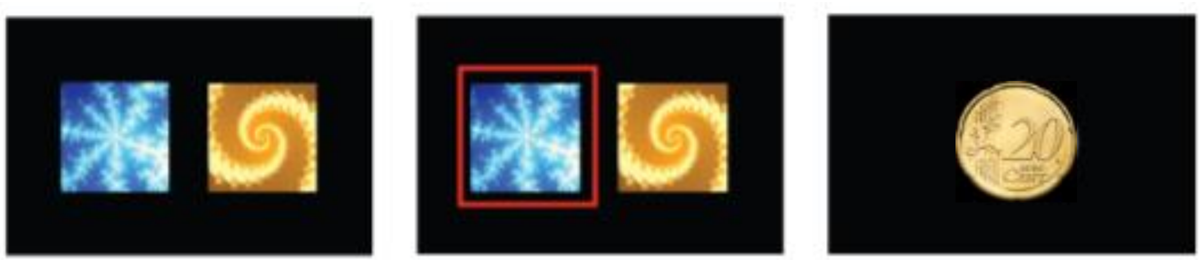
Observation Model (Data Model)

- relate model to observed data
- has to account for noise

Rescorla & Wagner (1972)



Rescorla-Wagner Value Update



Value update:

$$V_{t+1} = V_t + \alpha * PE$$

Prediction error:

$$PE = R_t - V_t$$

choice rule (sigmoid /softmax):

$$p(\mathrm{C=a}) = \frac{1}{1+e^{\tau*(v(b)-v(a))}}$$

Data:

choice & outcome

Parameters:

α & τ

```
α  - learning rate
PE - reward prediction error
V  - value
R  - reward
τ  - softmax temperature
```

Rescorla & Wagner (1972)

Understand the learning rate

Value update:

$$V_{t+1} = V_t + \alpha * PE$$

Prediction error:

$$PE = R_t - V_t$$

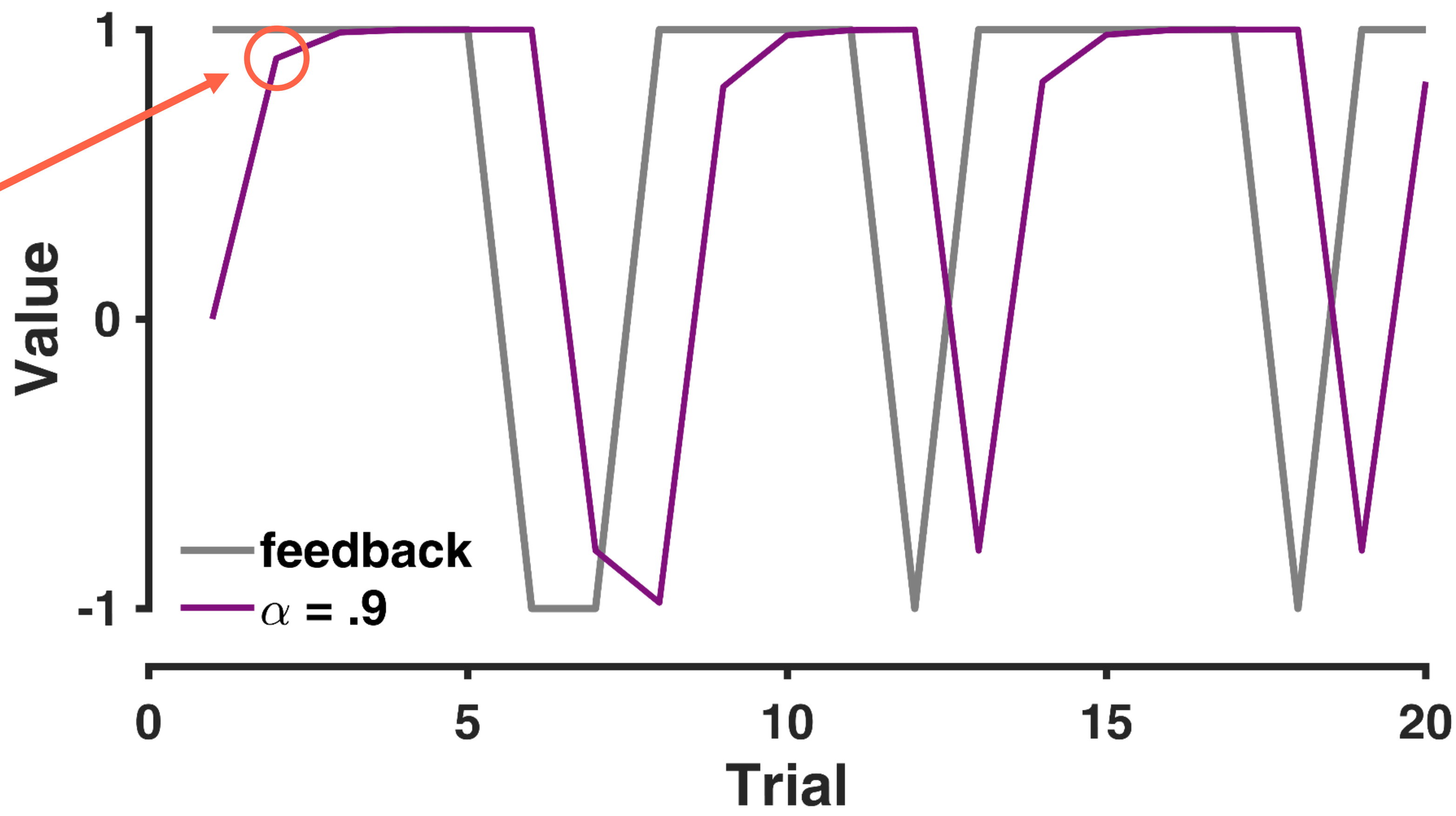
if $\eta = 0.9$

$v_1 = 0$

$v_2 = v_1 + 0.9 * (1 - 0)$

$= 0 + 0.9$

$= 0.9$



reward contingency – 80:20

Understand the learning rate

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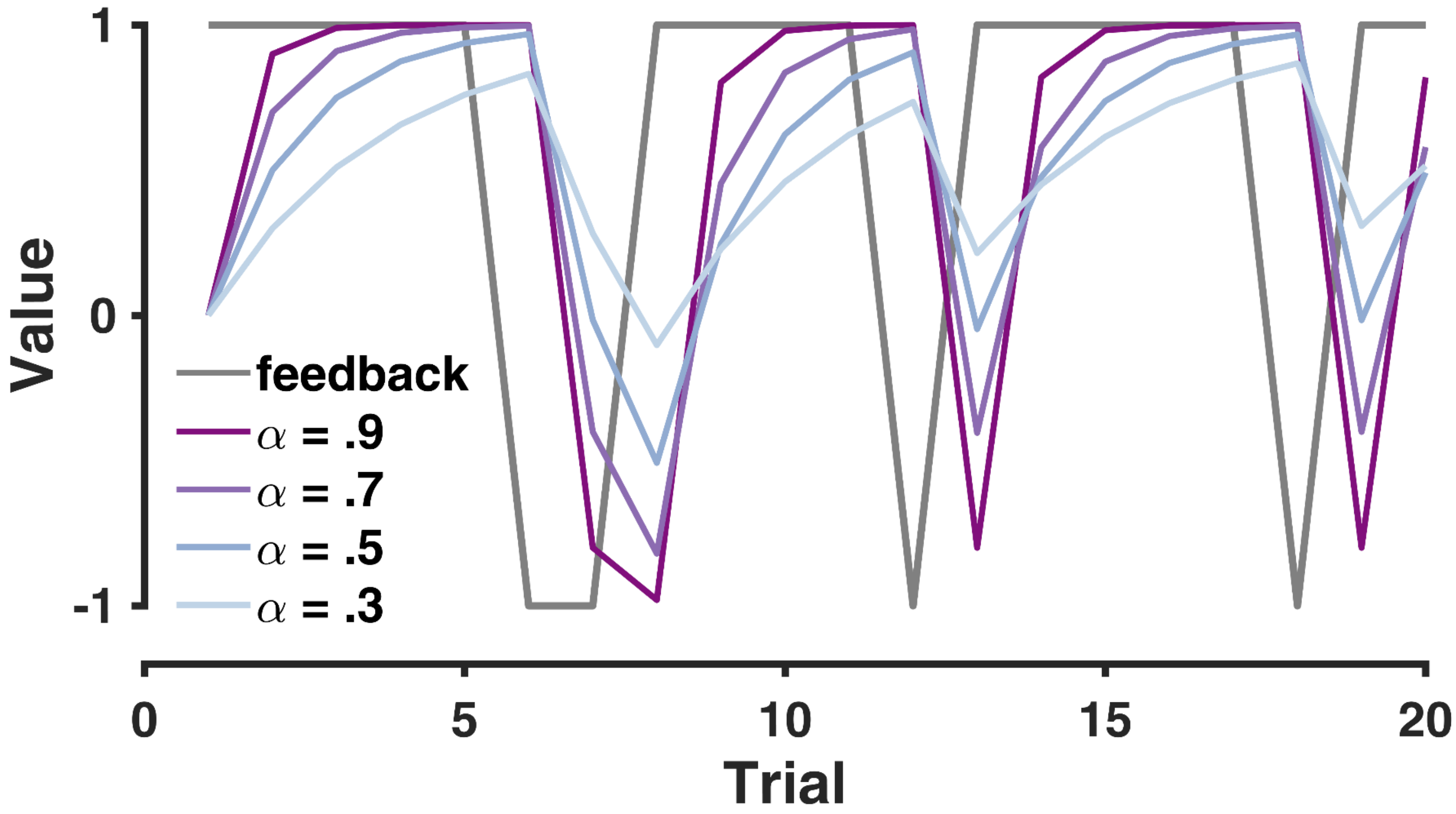

reward contingency – 80:20

Understand the learning rate

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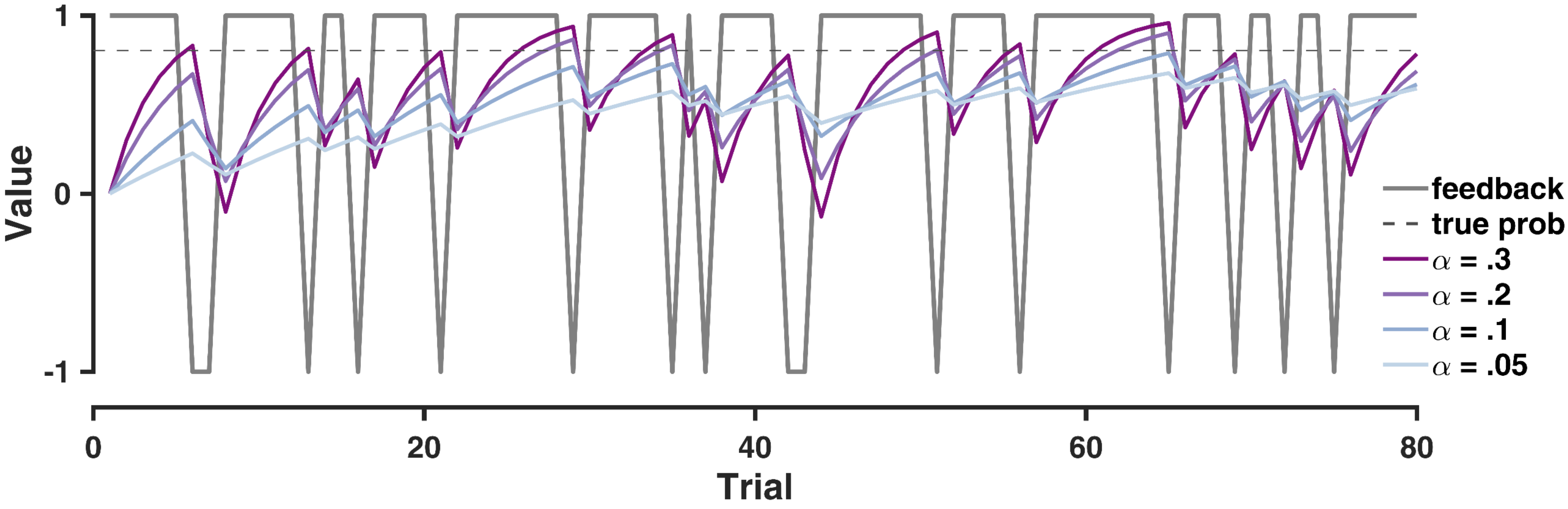


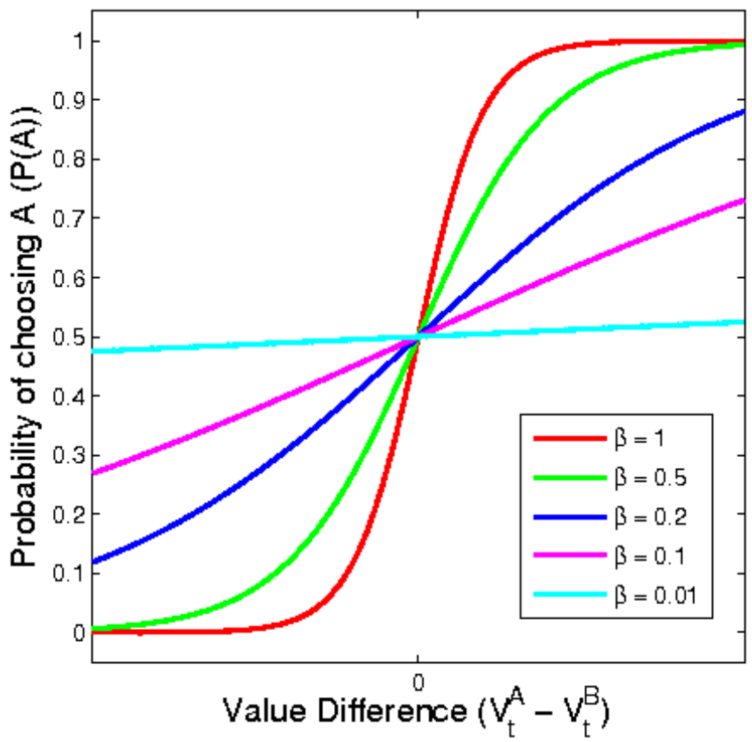
reward contingency – 80:20

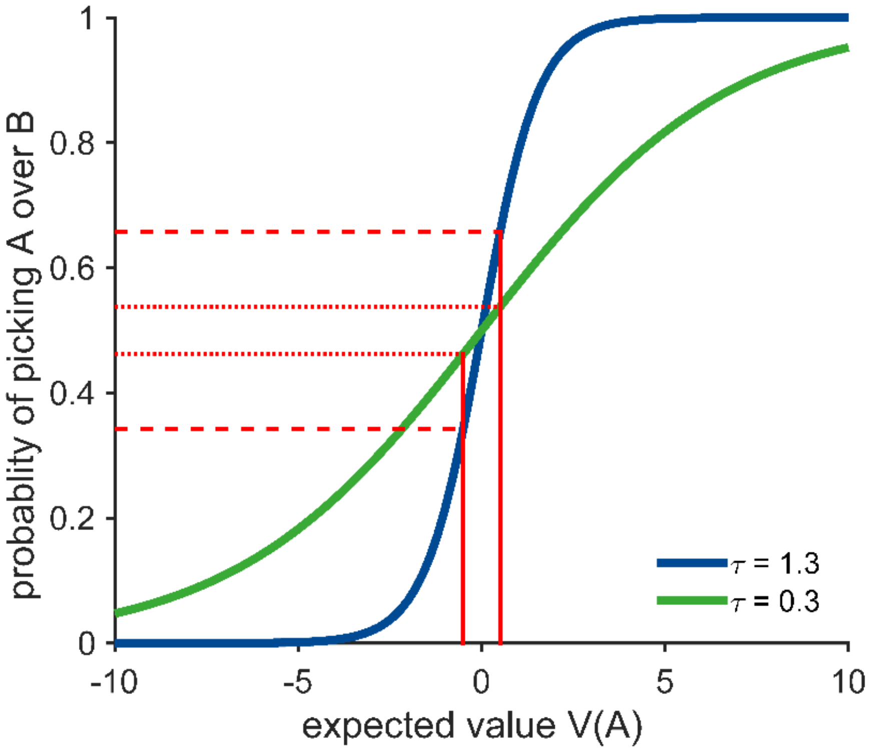
Understand the Softmax rule

choice rule (sigmoid /softmax):

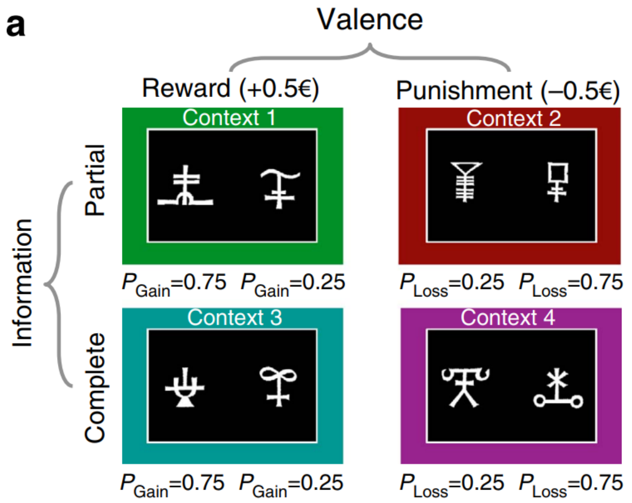
$$p(\mathrm{C{=}a}) = \frac{1}{1+e^{\tau*(v(b)-v(a))}}$$

$$p(\mathrm{C{=}a}) = \frac{e^{\tau*(v(a))}}{e^{\tau*(v(a))}+e^{\tau*(v(b))}}$$

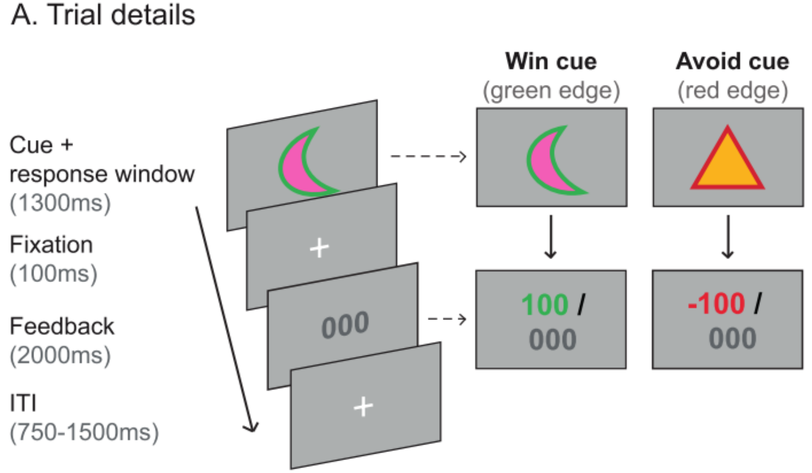




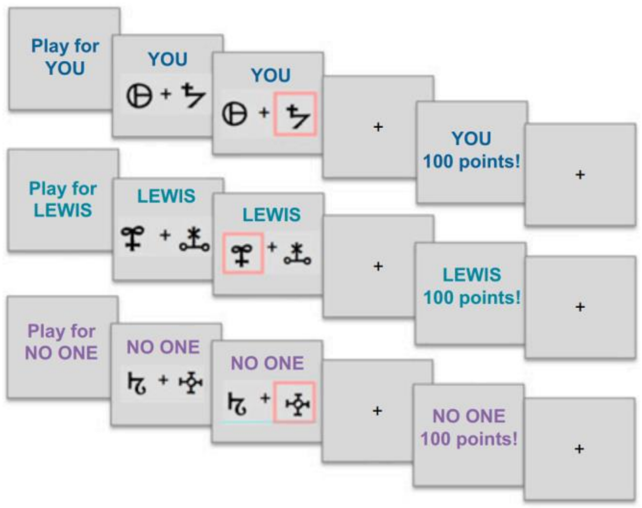
Generalizing RL framework



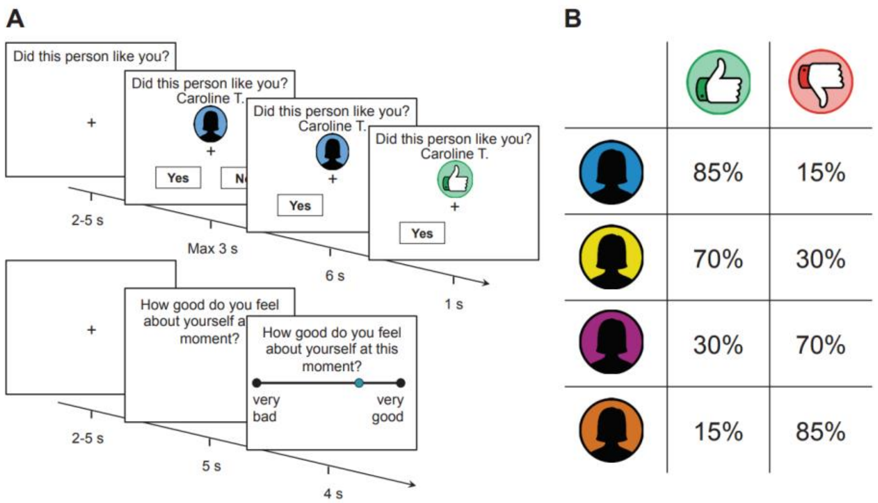
Stefano Palminteri et al. (2005)



Swart et al. (2017)

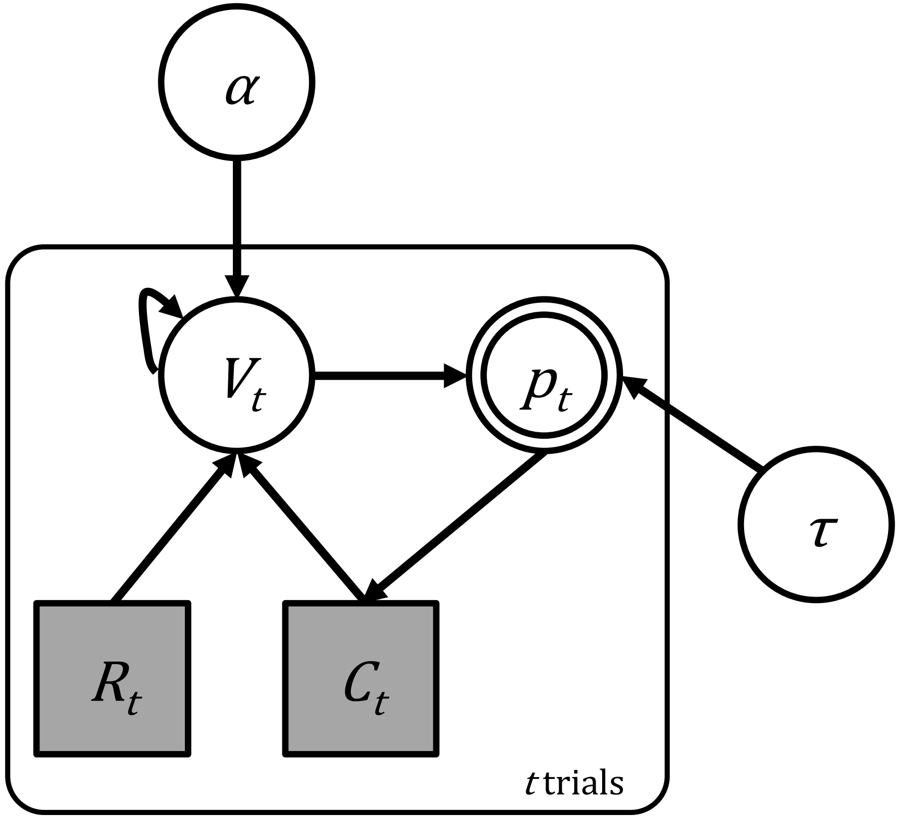


Lockwood et al. (2006)



Will et al. (2017)

RL – Implementation



$$\alpha \sim Uniform(0,1)$$

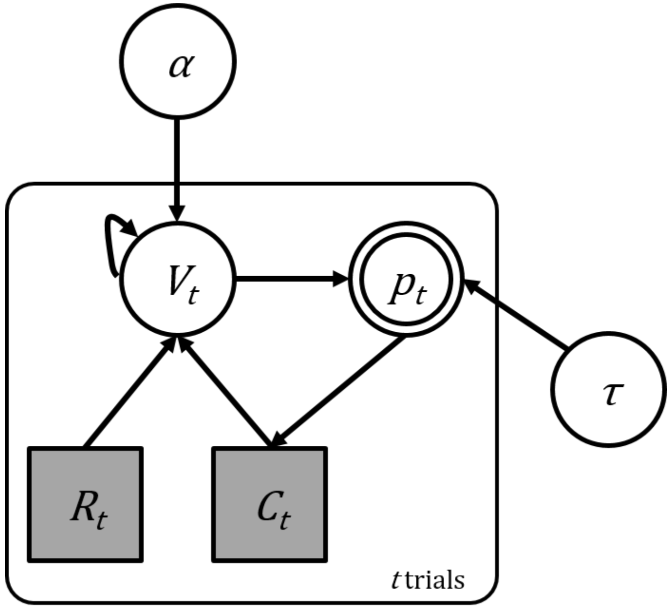
$$\tau \sim Uniform(0,3)$$

$$p_t(C = A) = \frac{1}{1 + e^{\tau(V_t(B) - V_t(A))}}$$

$$V_{t+1}^{c} = V_t^{C} + \alpha\left(R_t - V_t^{C}\right)$$

RL – Implementation

cognitive model
statistics
computing



$$\alpha \sim Uniform(0,1)$$

$$\tau \sim Uniform(0,3)$$

$$p_t(C=A) = \frac{1}{1+e^{\tau(V_t(B)-V_t(A))}}$$

$$V_{t+1}^{c} = V_t^{C} + \alpha(R_t - V_t^{C})$$

```
transformed data {
  vector[2] initV;
  initV = rep_vector(0.0, 2);
}

model {
  vector[2] v[nTrials+1];
  real pe[nTrials];

  v[1] = initV;

  for (t in 1:nTrials) {
    choice[t] ~ categorical_logit( tau * v[t] );

    pe[t] = reward[t] - v[t,choice[t]];

    v[t+1] = v[t];
    v[t+1, choice[t]] = v[t, choice[t]] + lr * pe[t];
  }
}
```

RL – Implementation

cognitive model

statistics

computing

```
model {
  vector[2] v[nTrials+1];
  real pe[nTrials];

  v[1] = initV;

  for (t in 1:nTrials) {
    choice[t] ~ categorical_logit( tau * v[t] );
    pe[t] = reward[t] - v[t,choice[t]];

    v[t+1] = v[t];
    v[t+1, choice[t]] = v[t, choice[t]] + lr * pe[t];
  }
}
```

```
model {
  vector[2] v;
  real pe;

  v = initV;

  for (t in 1:nTrials) {
    choice[t] ~ categorical_logit( tau * v );
    pe = reward[t] - v[choice[t]];

    v[choice[t]] = v[choice[t]] + lr * pe;
  }
}
```

RL – Fitting with Stan

cognitive model
statistics
computing

```
…/BayesCog/06.reinforcement_learning/_scripts/reinforcement_learning_single_parm_main.R
```

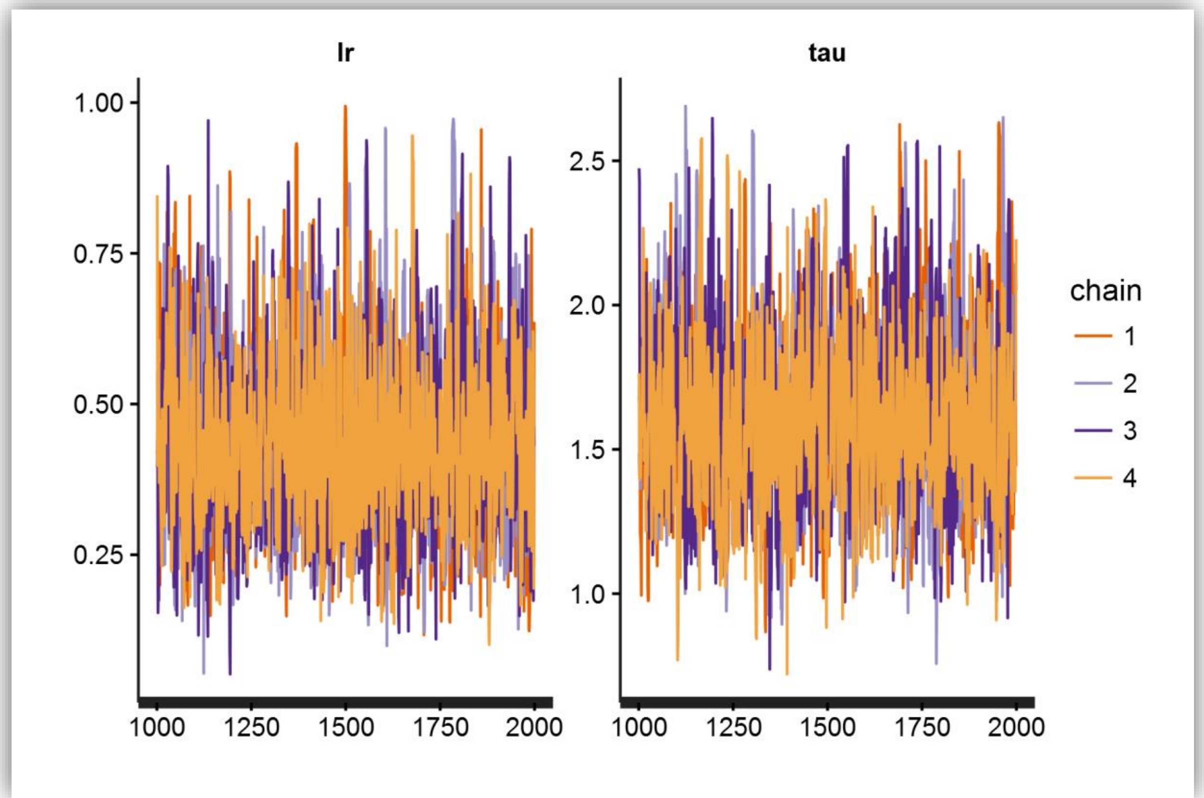
TASK: fit the model for single participants

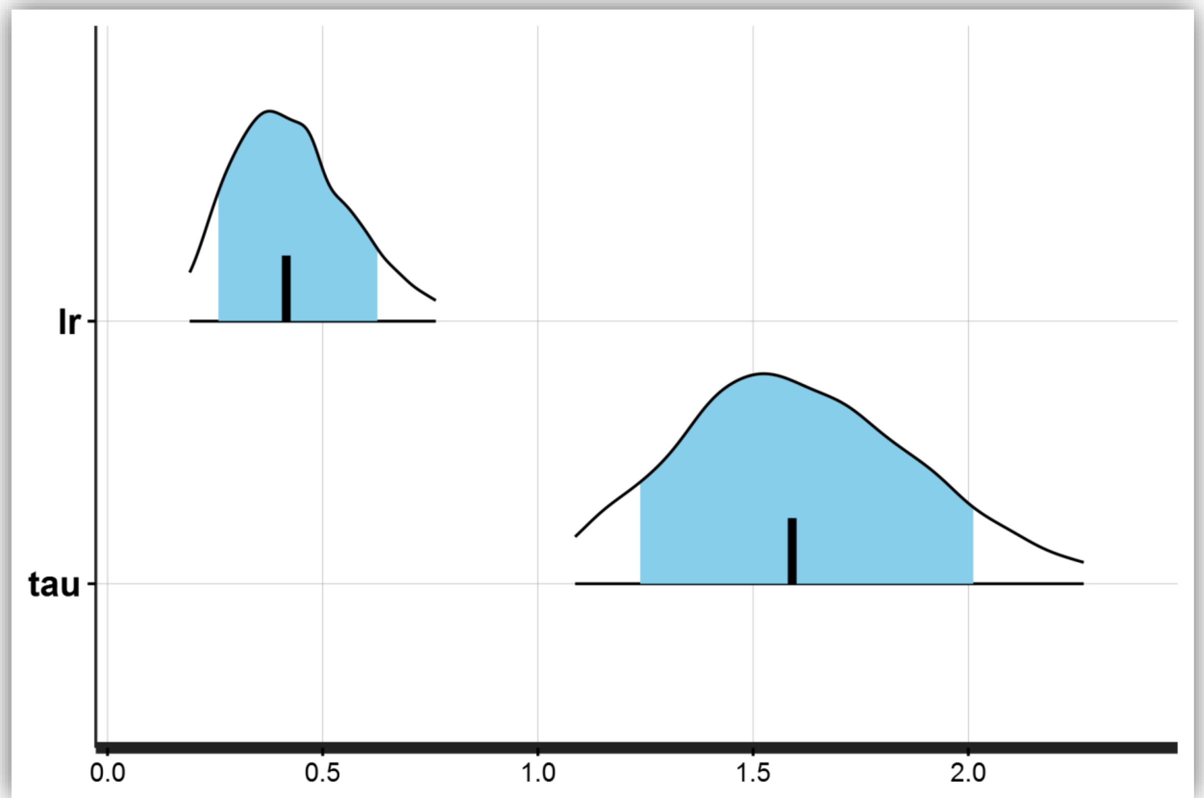
```
> source('_scripts/reinforcement_learning_single_parm_main.R’) # a function

> fit_rl1 <- run_rl_sp(multiSubj = FALSE)
```

```
> load('_data/rl_sp_ss.RData')
> head(rl_ss)
     [,1] [,2]
[1,]    2   -1
[2,]    1    1
[3,]    1    1
[4,]    1    1
[5,]    2   -1
[6,]    1    1
```

RL – MCMC Output

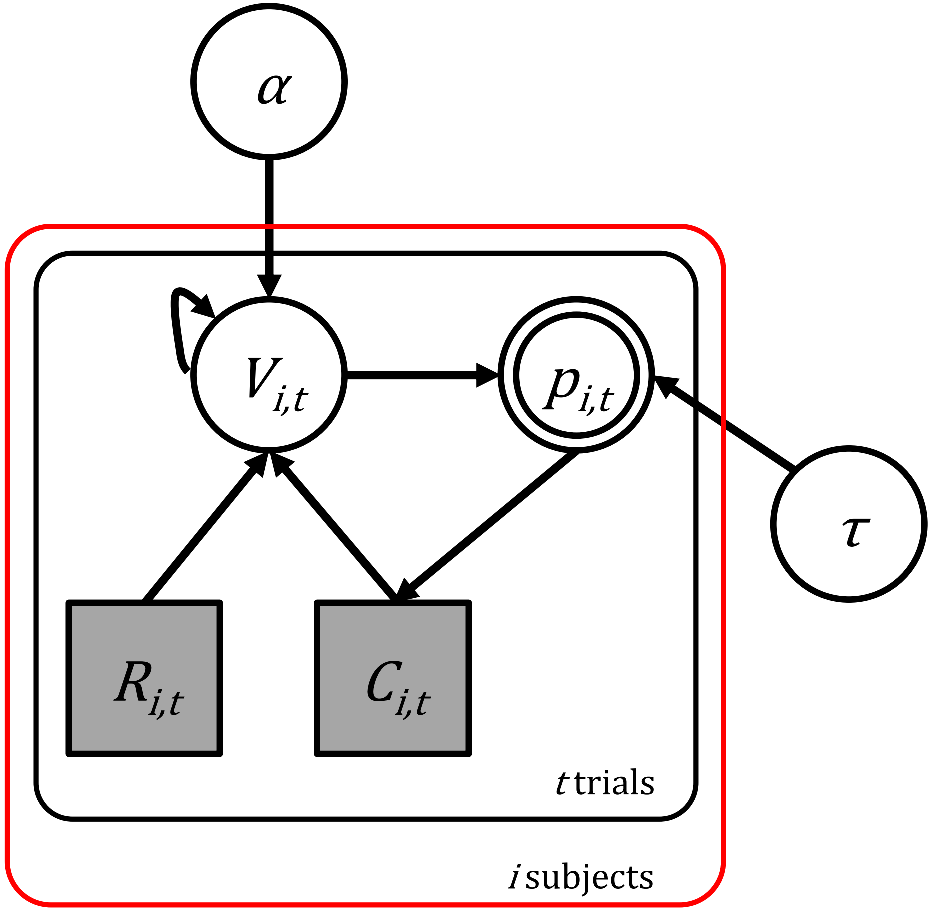
cognitive model
statistics
computing





Fitting Multiple Participants as ONE

cognitive model
statistics
computing




```
model {
  for (s in 1:nSubjects) {
    vector[2] v;
    real pe;
    v = initV;

    for (t in 1:nTrials) {
      choice[s,t] ~ categorical_logit( tau * v );
      pe = reward[s,t] - v[choice[s,t]];
      v[choice[s,t]] = v[choice[s,t]] + lr * pe;
    }
  }
}
```

Exercise IV

cognitive model
statistics
computing

```
…/BayesCog/06.reinforcement_learning/_scripts/reinforcement_learning_single_parm_main.R
```

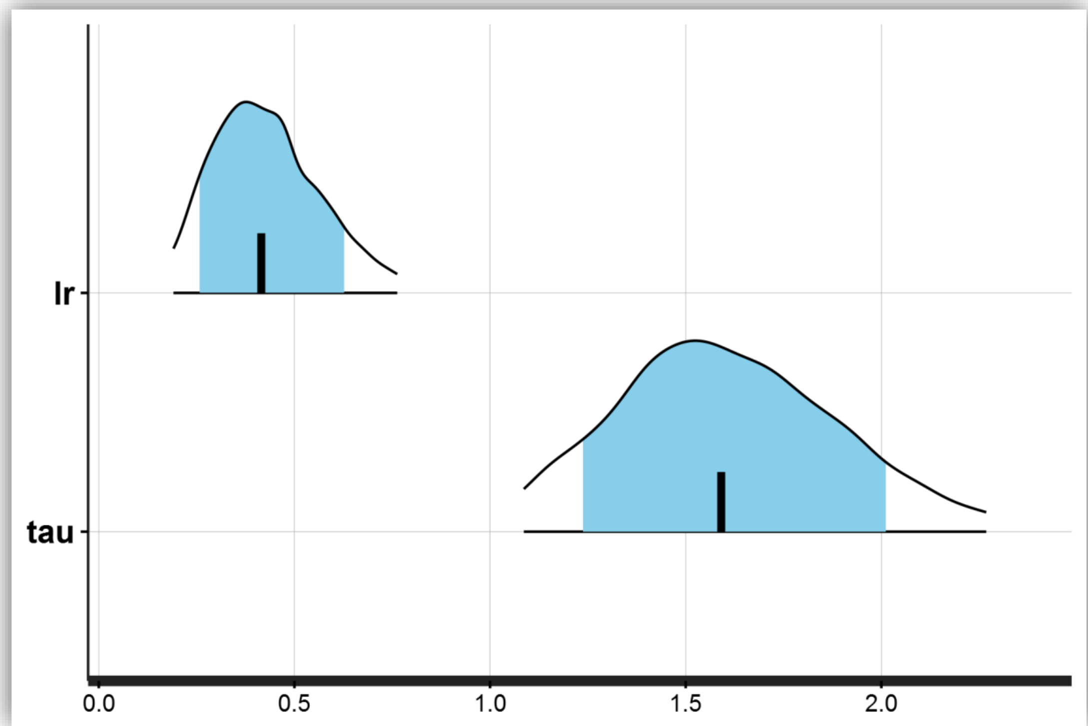
TASK: fit the model for multiple participants (assuming same parameters)

```
> source('_scripts/reinforcement_learning_single_parm_main.R')

> fit_rl2 <- run_rl_sp(multiSubj = TRUE)
```

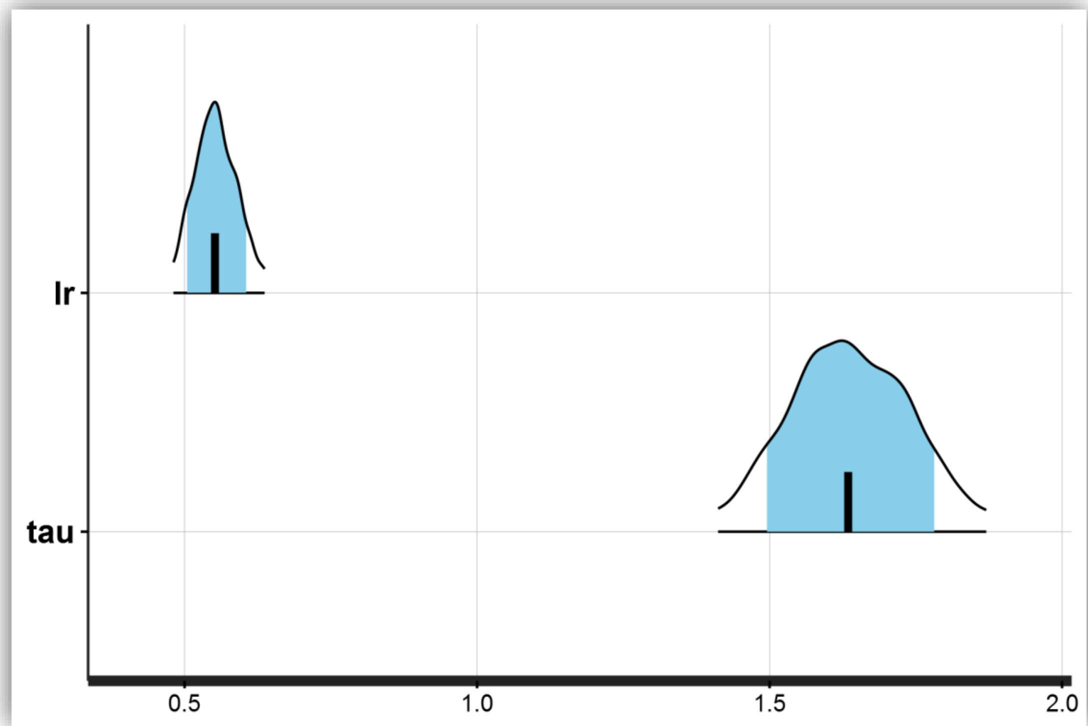
Exercise IV

cognitive model
statistics
computing

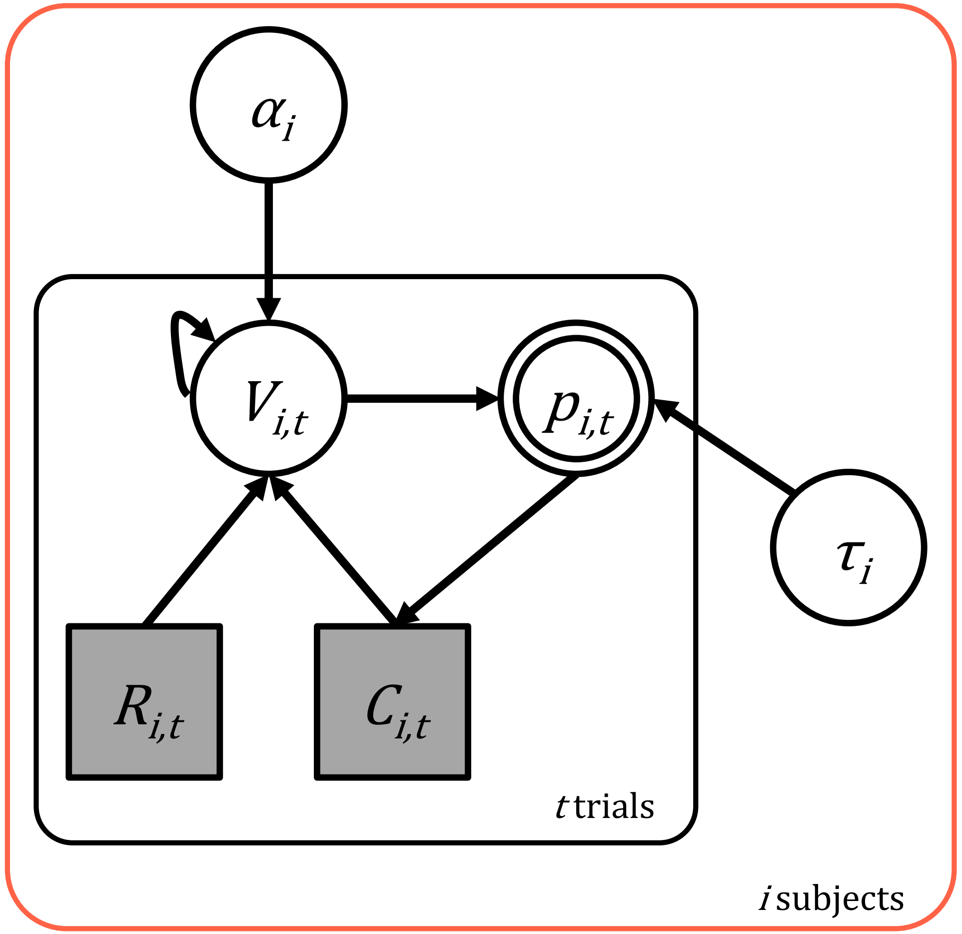
N = 1



N = 10



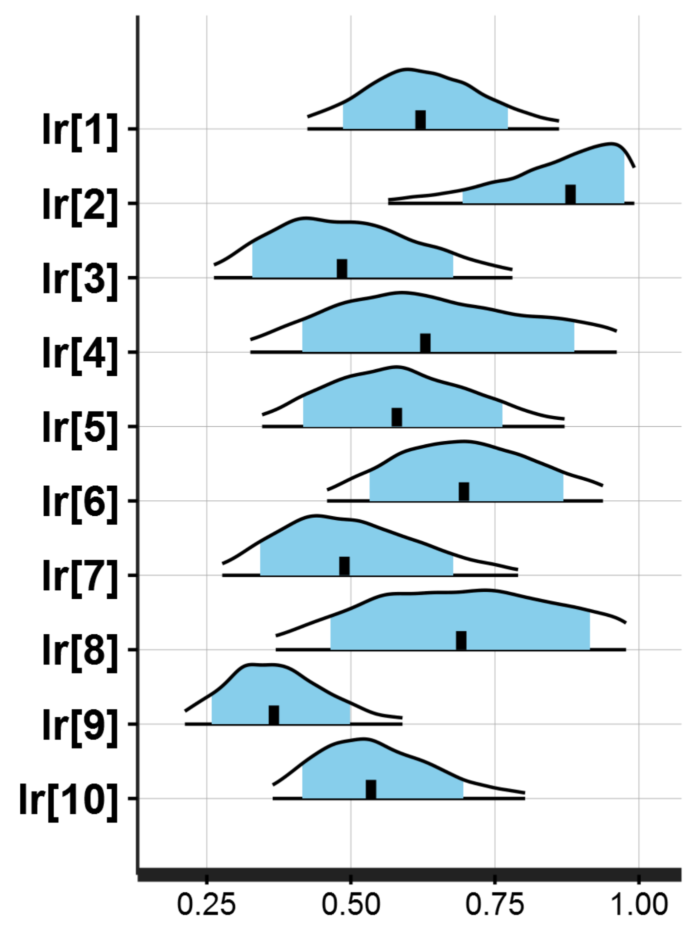
Fitting Multiple Participants Independently

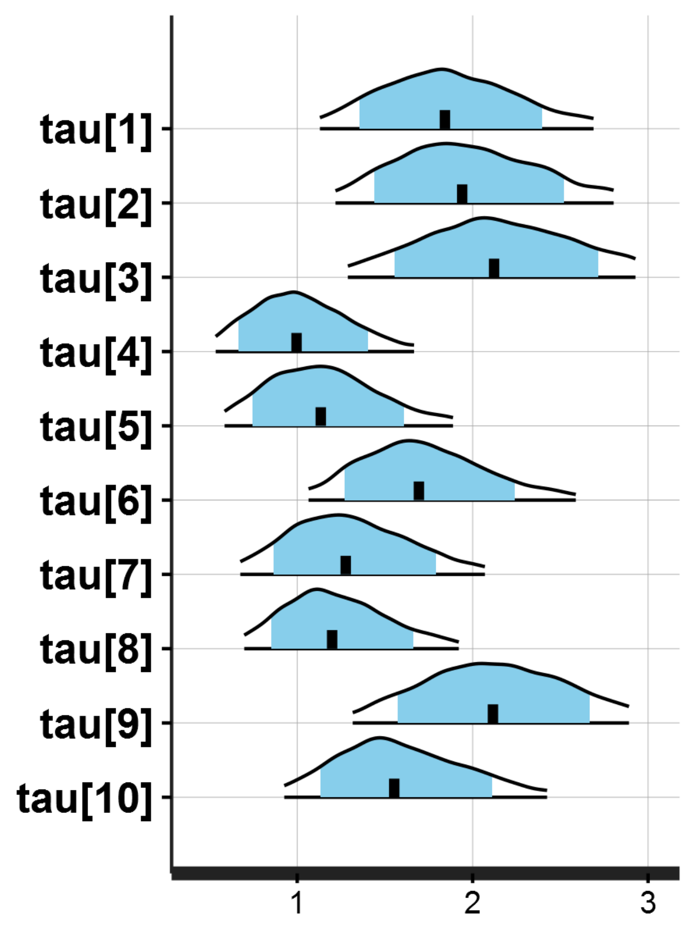


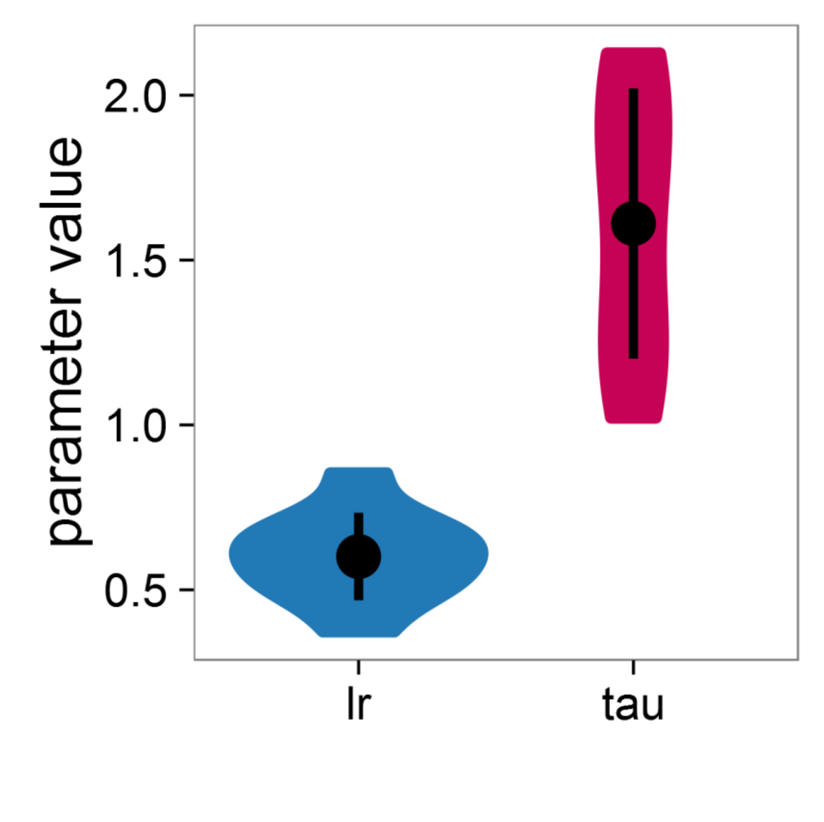

```
model {
  for (s in 1:nSubjects) {
    vector[2] v;
    real pe;
    v = initV;

    for (t in 1:nTrials) {
      choice[s,t] ~ categorical_logit( tau[s] * v );
      pe = reward[s,t] - v[choice[s,t]];
      v[choice[s,t]] = v[choice[s,t]] + lr[s] * pe;
    }
  }
}
```

Individual Fitting

cognitive model
statistics
computing



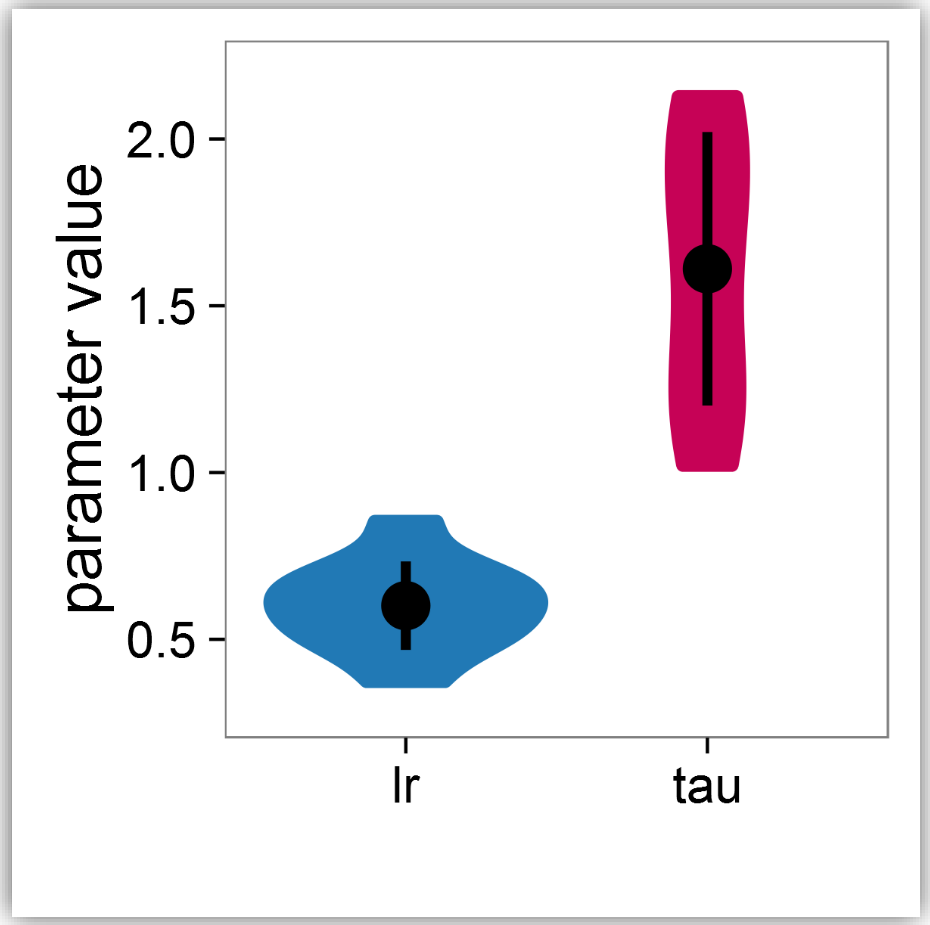




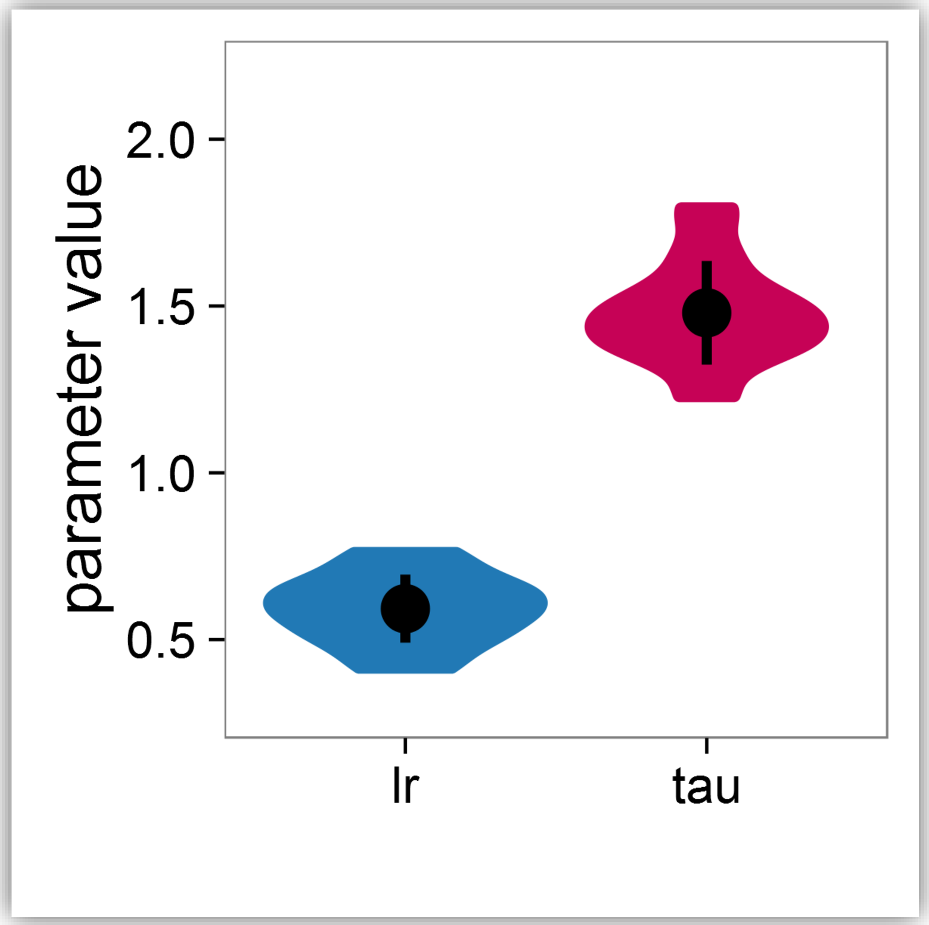
Comparing with True Parameters

cognitive model
statistics
computing

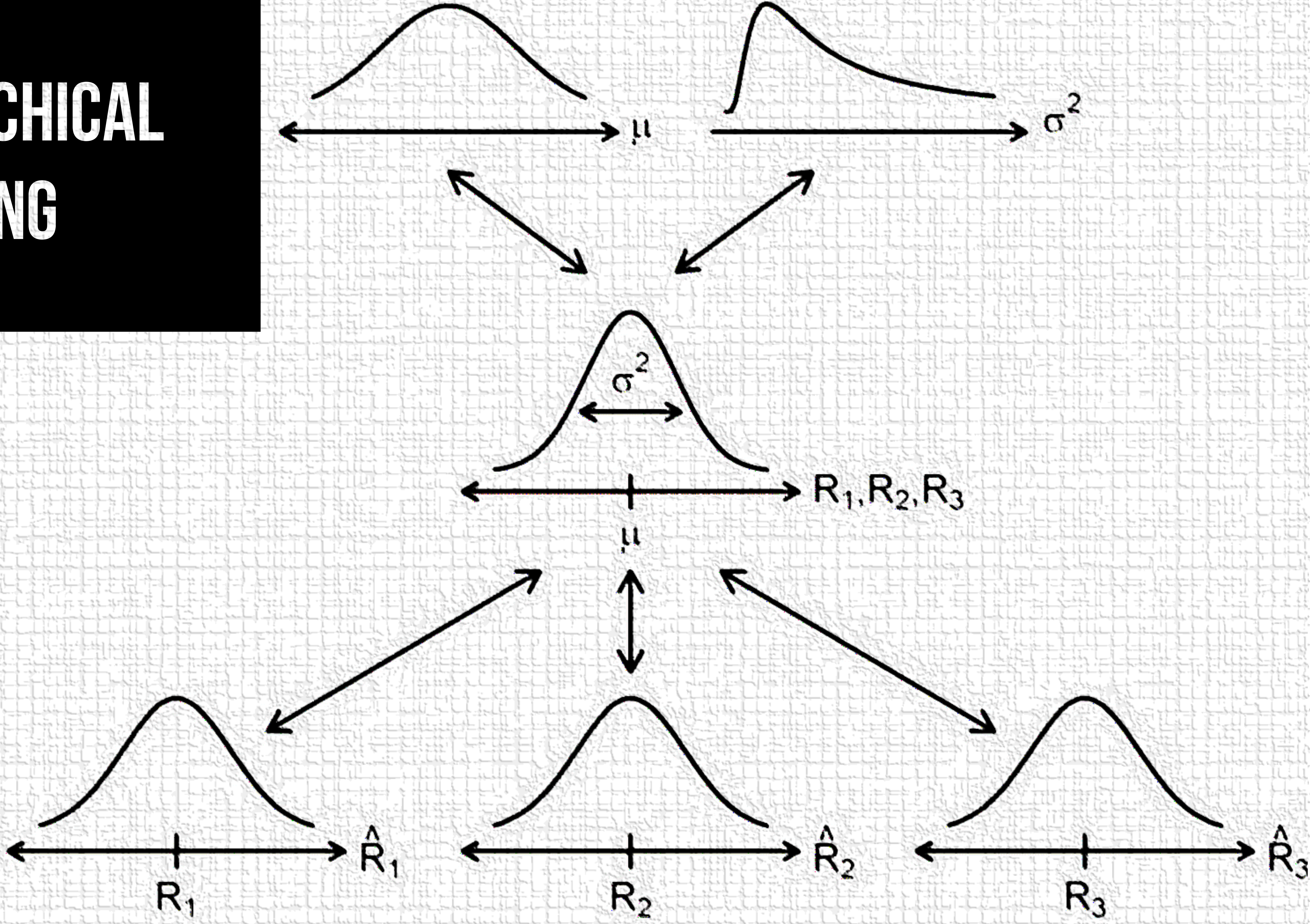
Posterior Means

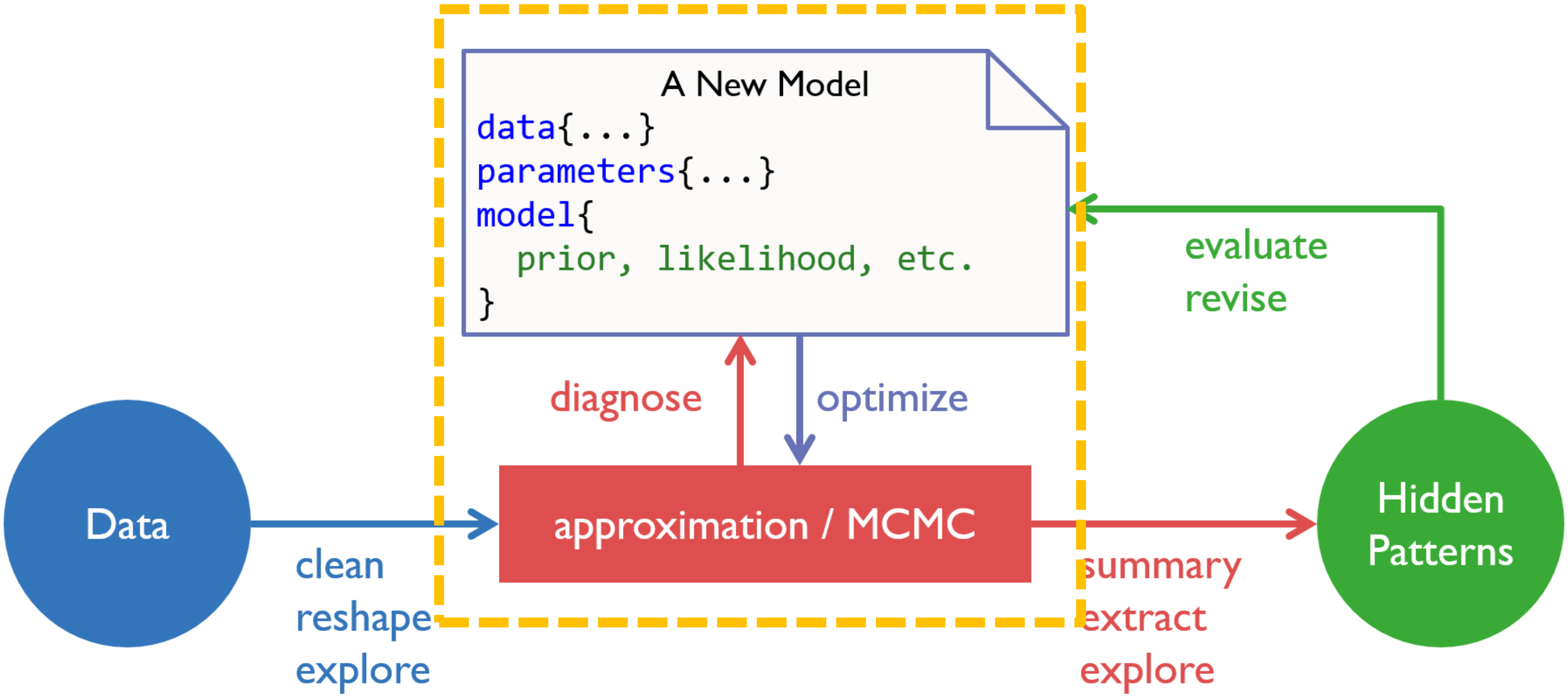


True Parameters



HIERARCHICAL MODELING



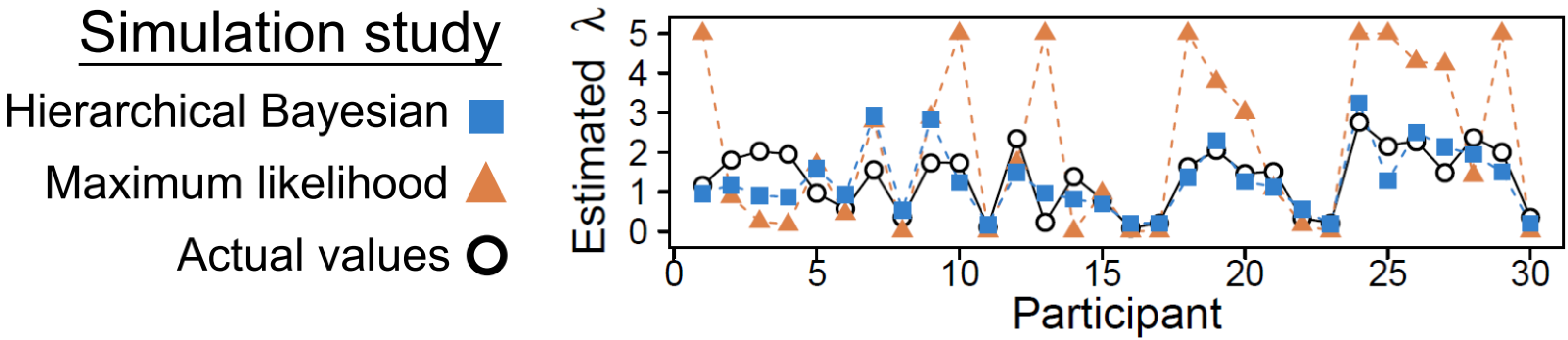

A New Model
data{...}
parameters{...}
model{
prior, likelihood, etc.
}
evaluate
revise
diagnose
optimize
Data
approximation / MCMC
Hidden Patterns
clean
reshape
explore
summary
extract
explore

Why Hierarchical Bayesian Cognitive Modeling?

cognitive model

statistics

computing



Ahn et al. (2011)

Why Hierarchical Bayesian Cognitive Modeling?

Fixed effects

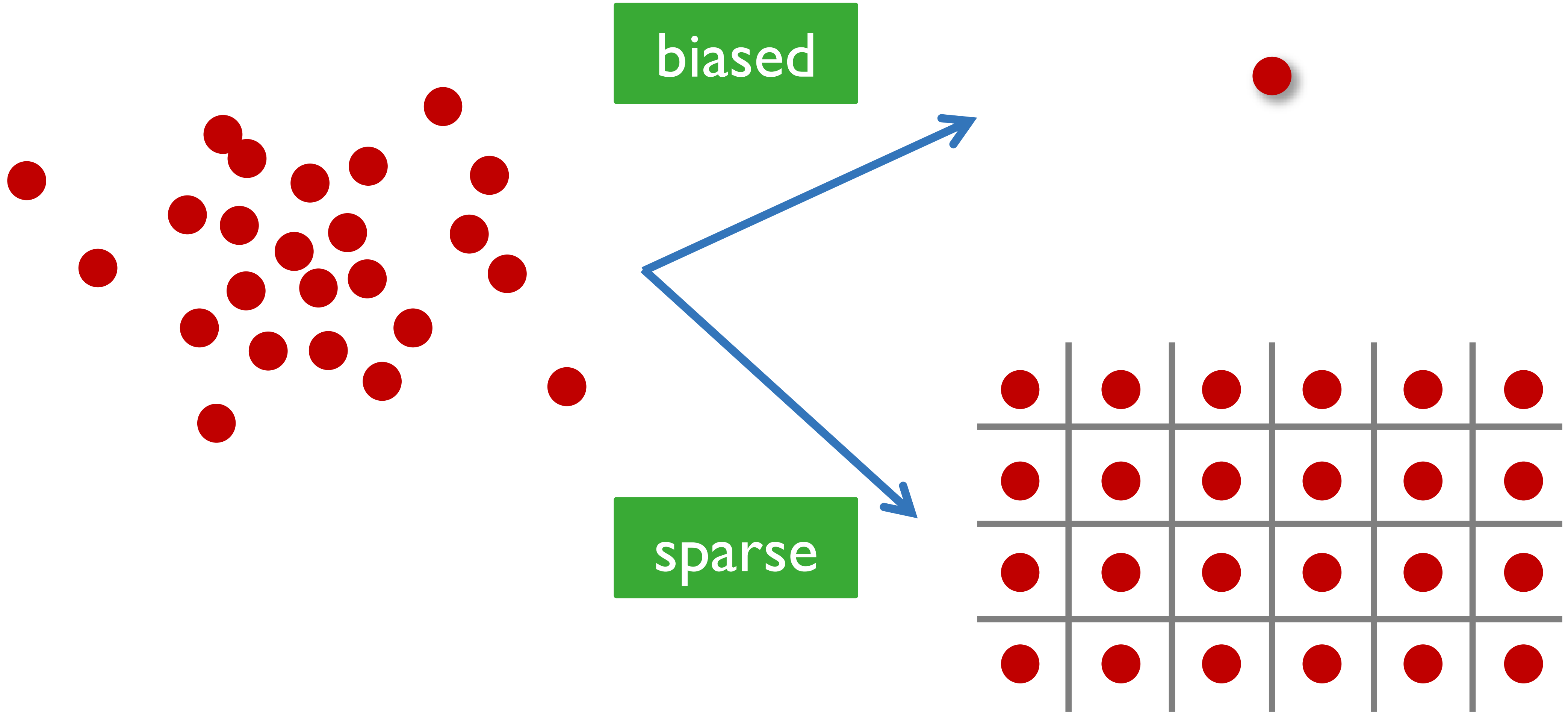
- all subjects are fitted with the same set of parameters
- worse model fit than "random effects"

Random effects

- each subject is fitted independently of the others
- best model fit for each subject
- parameter estimates can be noisy

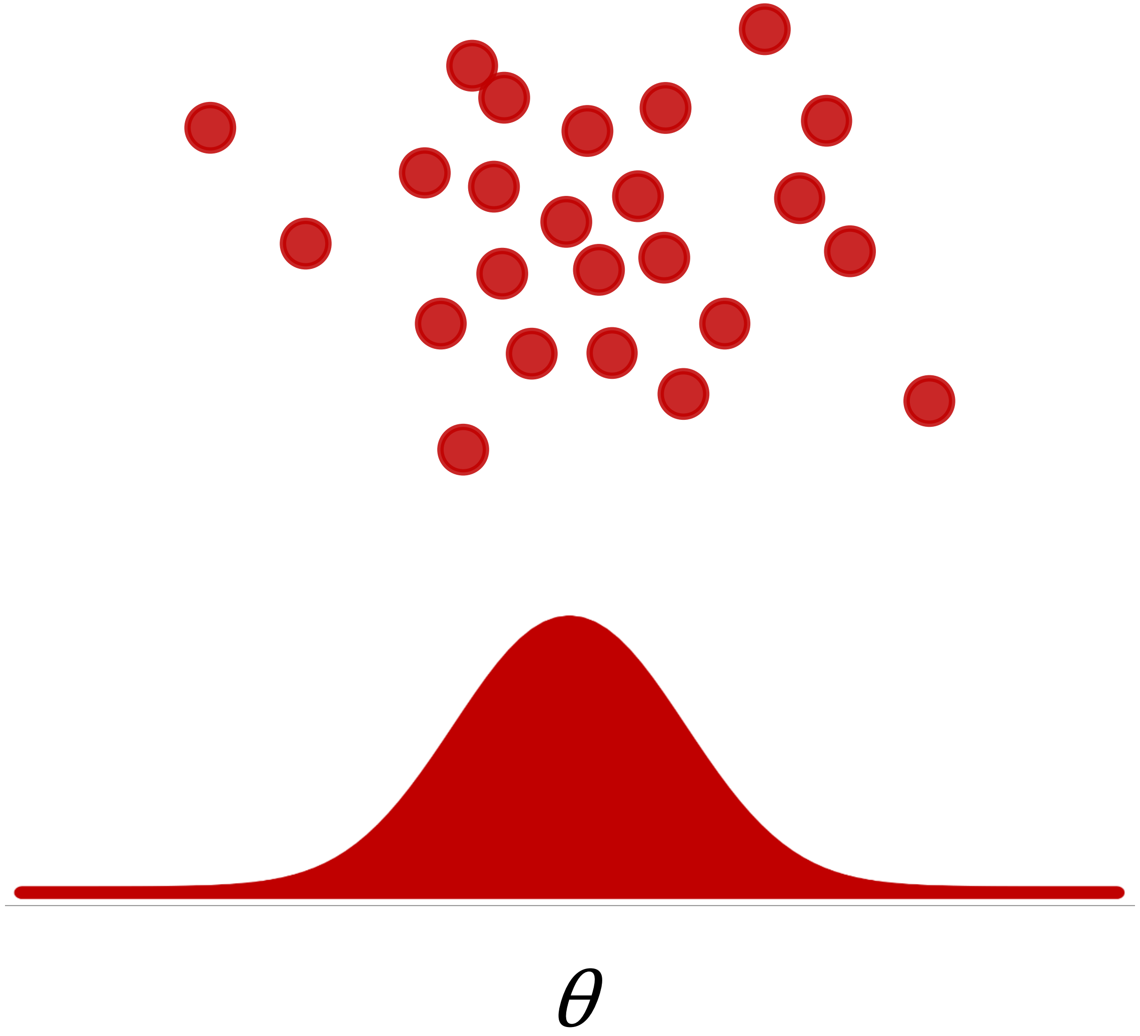
Adapted from Jan Gläscher's workshop

Fitting Multiple Participants



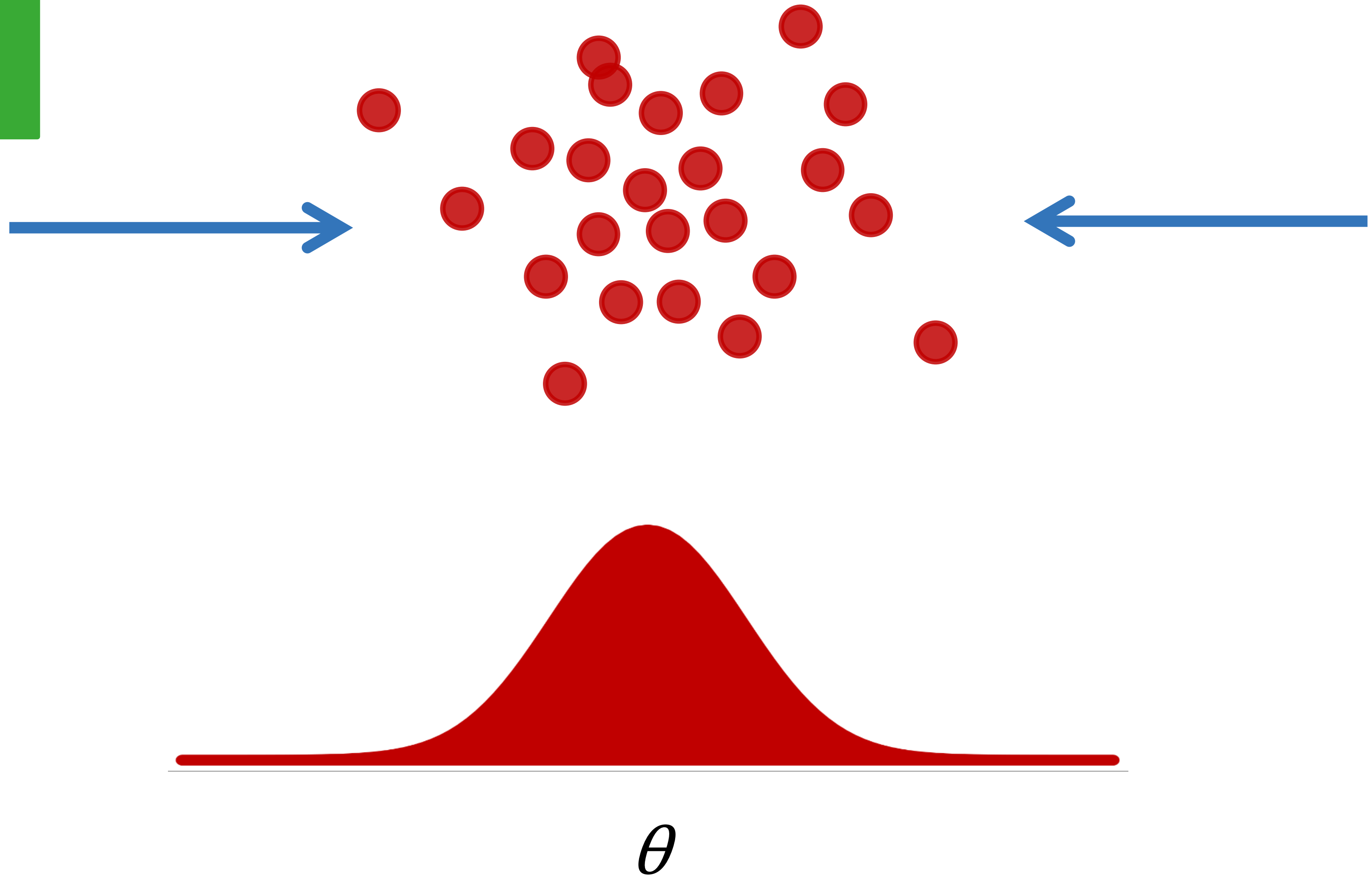
Fitting Multiple Participants

cognitive model
statistics
computing



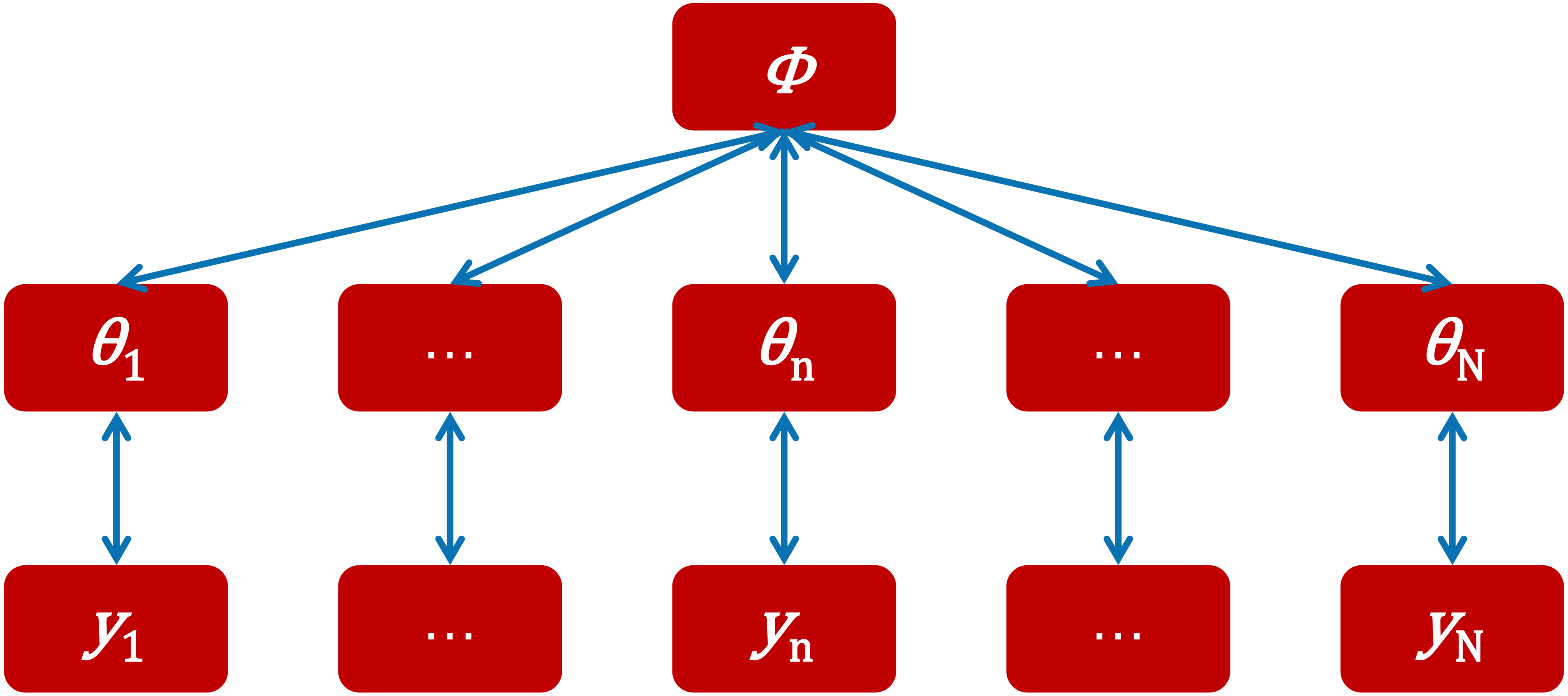
Fitting Multiple Participants

shrinkage effect



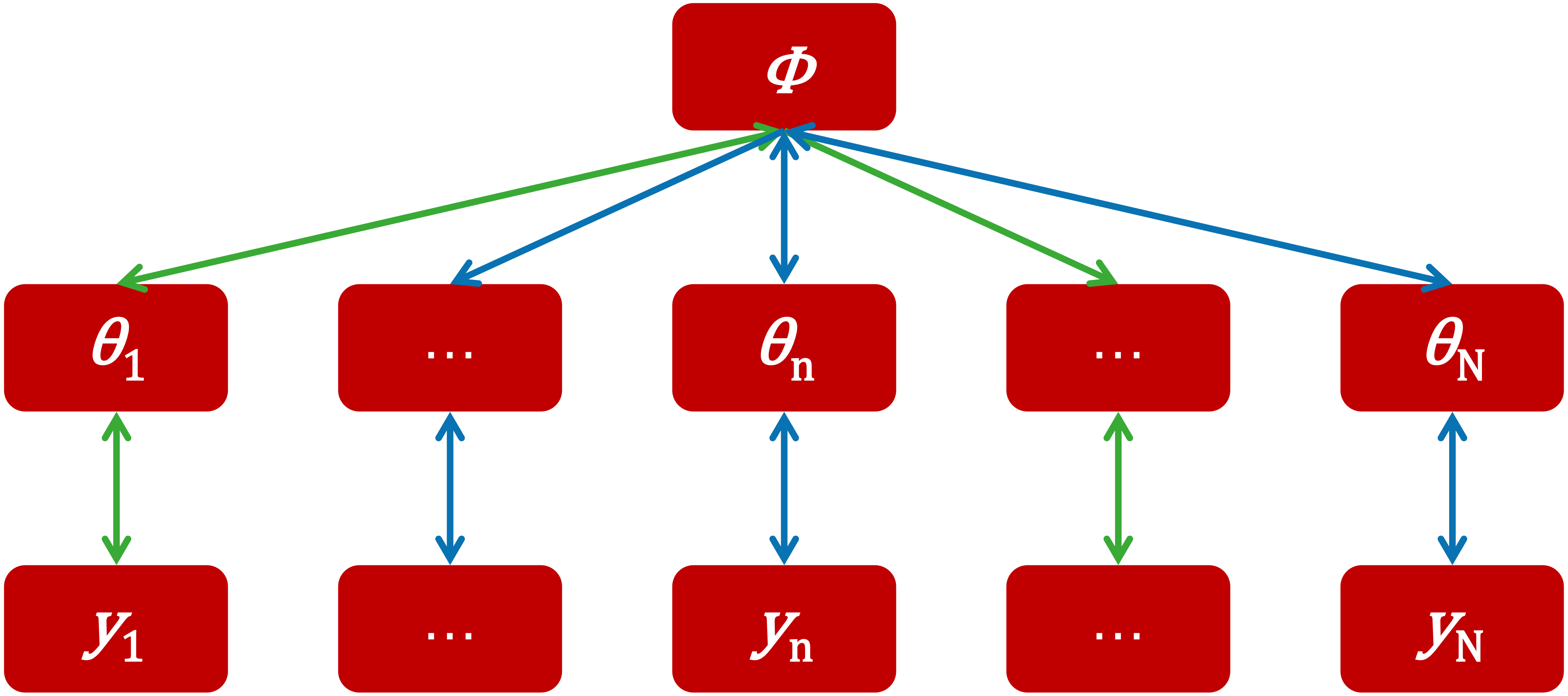
Hierarchical Structure

cognitive model
statistics
computing



Hierarchical Structure

cognitive model
statistics
computing

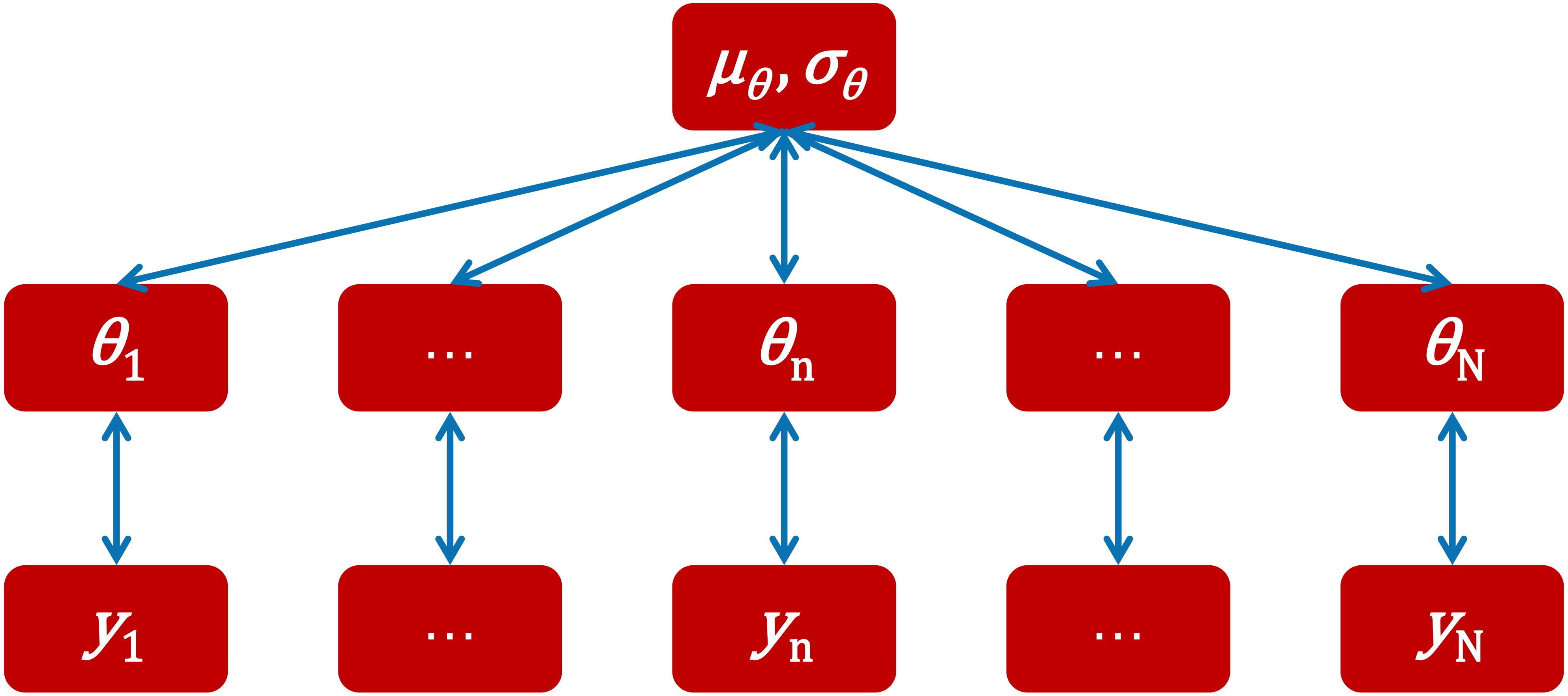


Hierarchical Structure

cognitive model

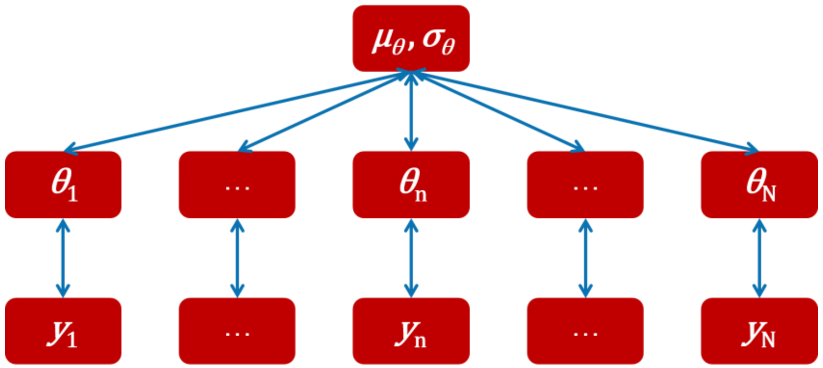
statistics

computing

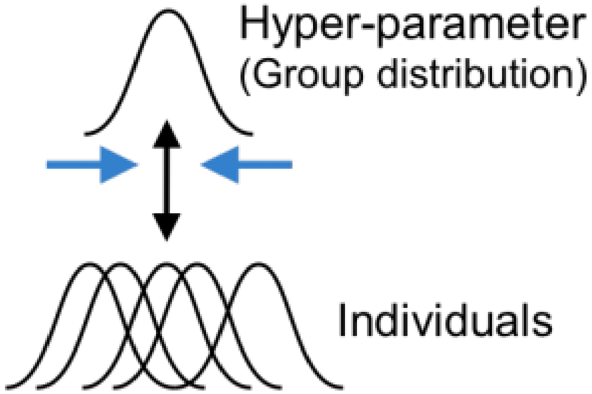


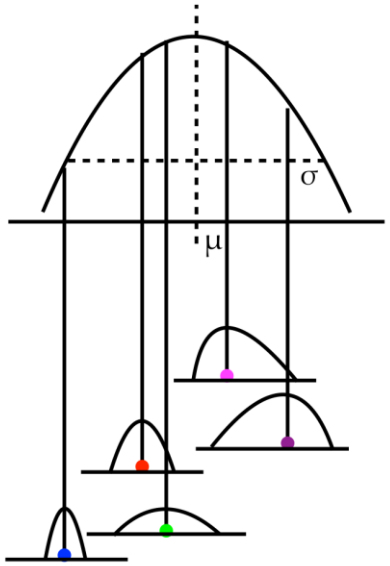
Hierarchical Structure

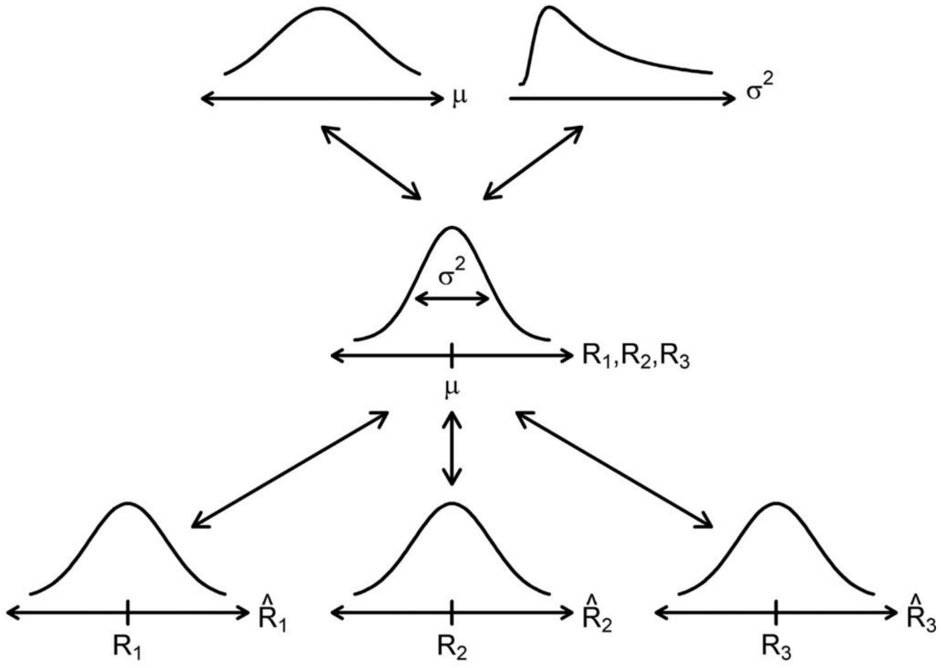
cognitive model
statistics
computing



$$P(\Theta,\Phi \mid D) = \frac{P(D \mid \Theta,\Phi)P(\Theta,\Phi)}{P(D)} \propto P(D \mid \Theta)P(\Theta \mid \Phi)P(\Phi)$$

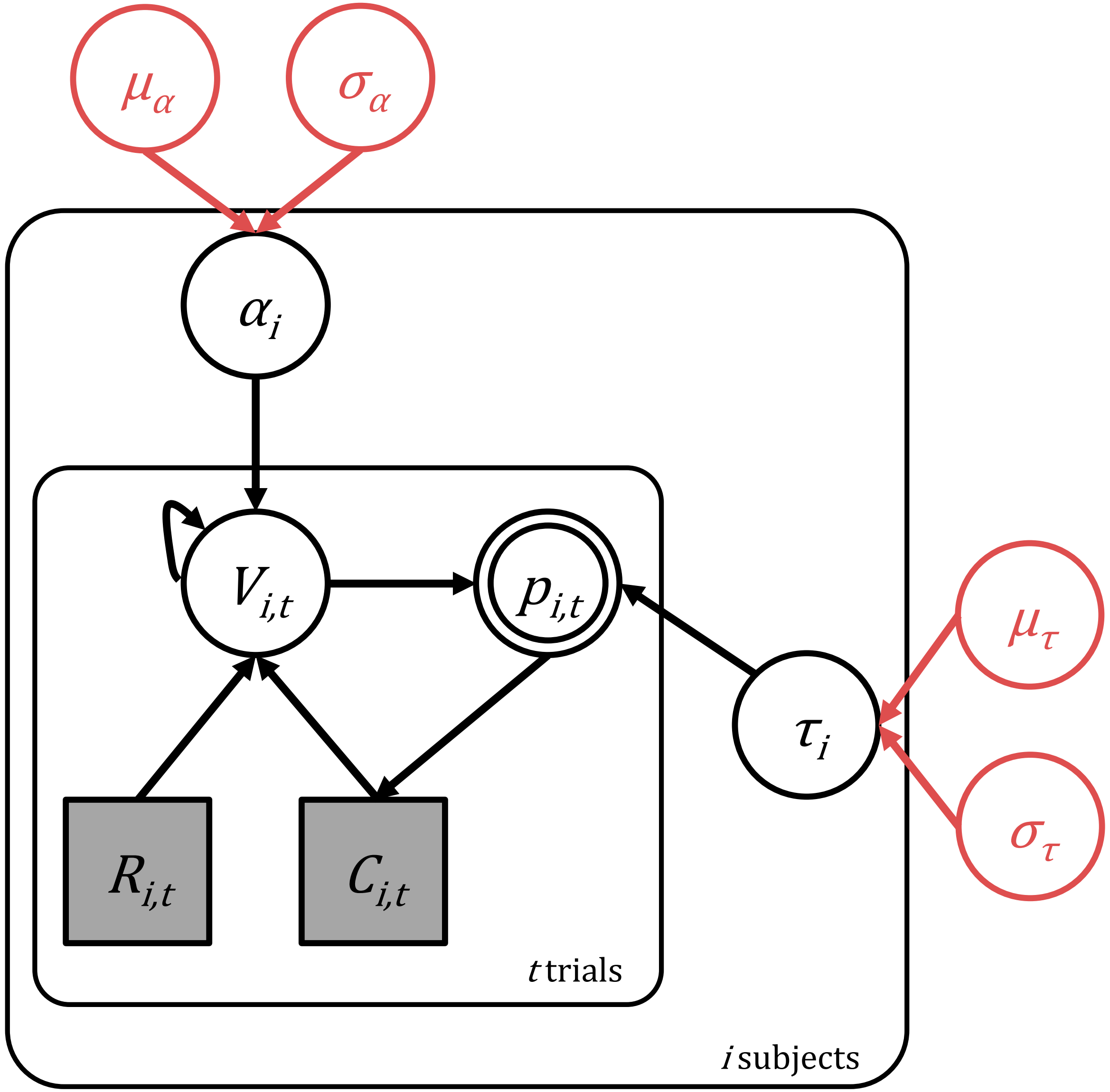






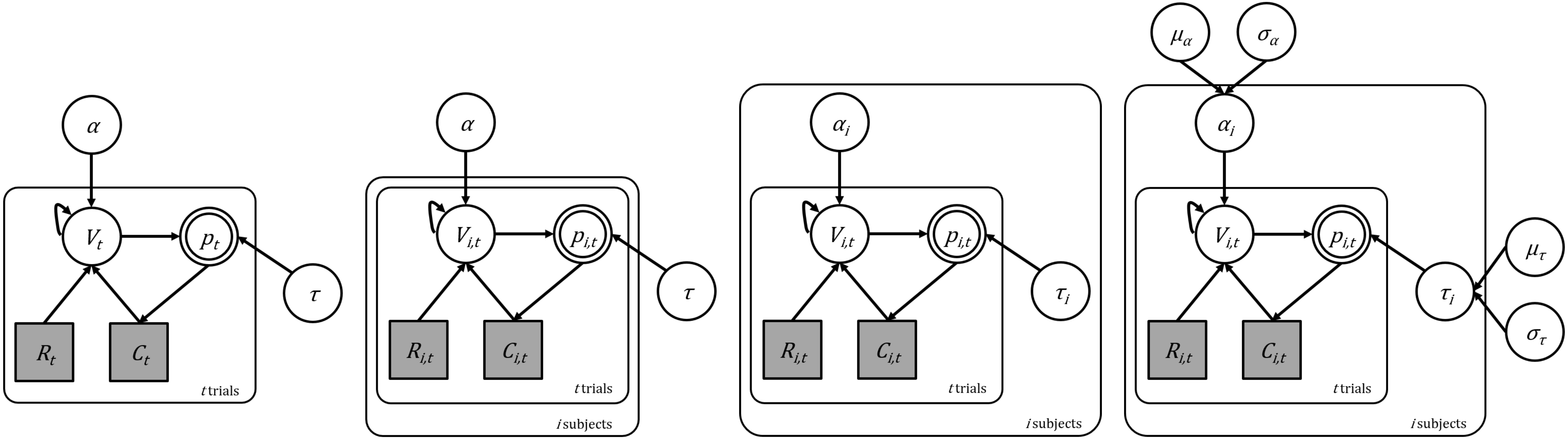
Hierarchical RL Model

cognitive model
statistics
computing



HOW DID WE GET HERE?

cognitive model
statistics
computing



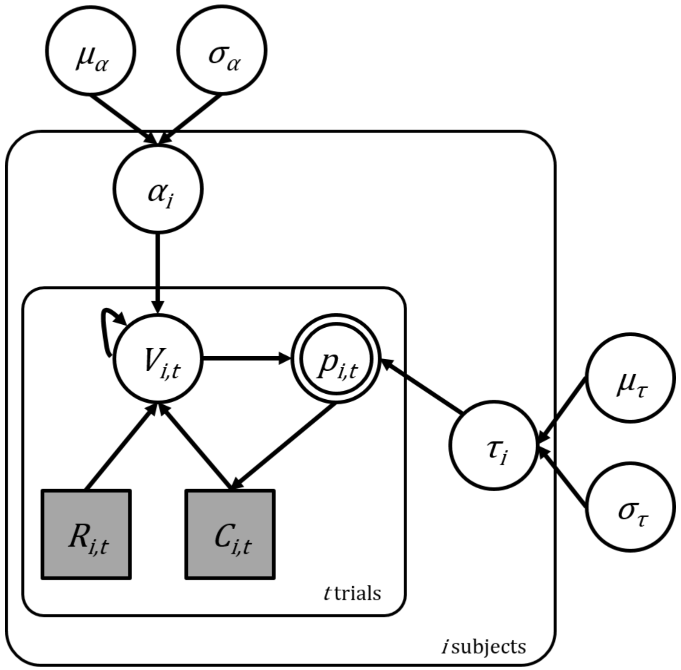
The cognitive model *per se* is the same!

Implementing Hierarchical RL Model

cognitive model

statistics

computing



$$\mu_\alpha \sim Uniform(0,1)$$
$$\sigma_\alpha \sim halfCauchy(0,1)$$
$$\mu_\tau \sim Uniform(0,3)$$
$$\sigma_\tau \sim halfCauchy(0,3)$$
$$\alpha_i \sim Normal(\mu_\alpha, \sigma_\alpha)_{\mathcal{T}(0,1)}$$
$$\tau_i \sim Normal(\mu_\tau, \sigma_\tau)_{\mathcal{T}(0,3)}$$

$$p_{i,t}(C=A) = \frac{1}{1+e^{\tau_i(V_{i,t}(B)-V_{i,t}(A))}}$$

$$V^c_{i,t+1} = V^C_{i,t} + \alpha_i(R_{i,t} - V^C_{i,t})$$

```
parameters {
  real<lower=0,upper=1> lr_mu;
  real<lower=0,upper=3> tau_mu;

  real<lower=0> lr_sd;
  real<lower=0> tau_sd;

  real<lower=0,upper=1> lr[nSubjects];
  real<lower=0,upper=3> tau[nSubjects];
}

model {
  lr_sd  ~ cauchy(0,1);
  tau_sd ~ cauchy(0,3);
  lr     ~ normal(lr_mu, lr_sd) ;
  tau    ~ normal(tau_mu, tau_sd) ;

  for (s in 1:nSubjects) {
    vector[2] v;
    real pe;
    v = initV;

    for (t in 1:nTrials) {
      choice[s,t] ~ categorical_logit( tau[s] * v );
      pe = reward[s,t] - v[choice[s,t]];
      v[choice[s,t]] = v[choice[s,t]] + lr[s] * pe;
    }
  }
}
```

Exercise V

cognitive model
statistics
computing

```
…/BayesCog/06.reinforcement_learning/_scripts/reinforcement_learning_multi_parm_main.R
```

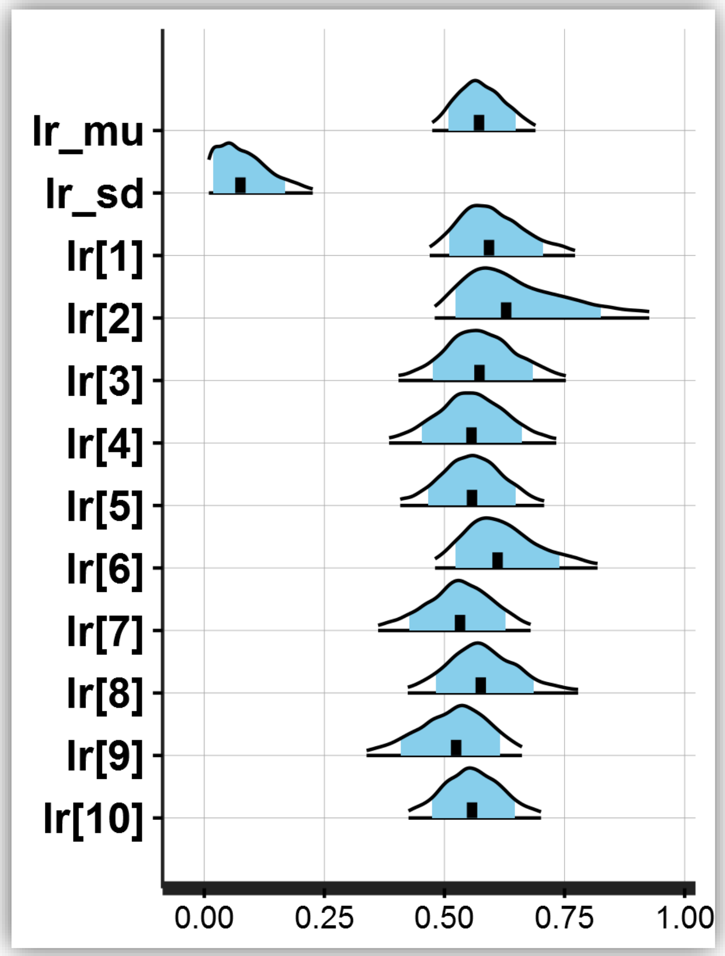
TASK: fit the hierarchical RL model

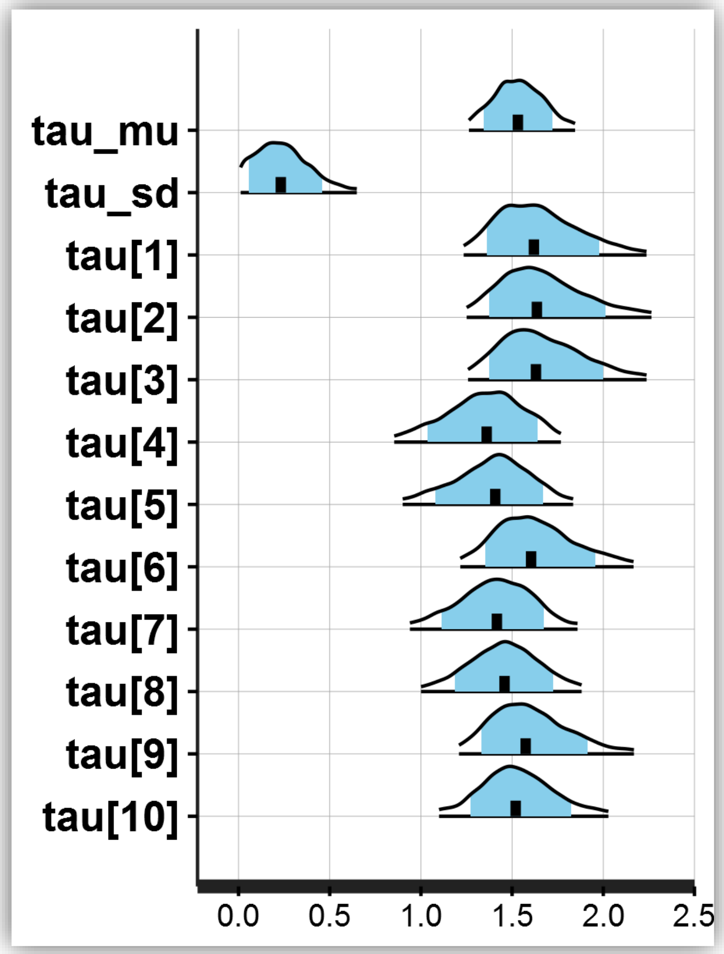
```
> source('_scripts/reinforcement_learning_multi_parm_main.R')

> fit_rl3 <- run_rl_mp( modelType ='hrch' )
```

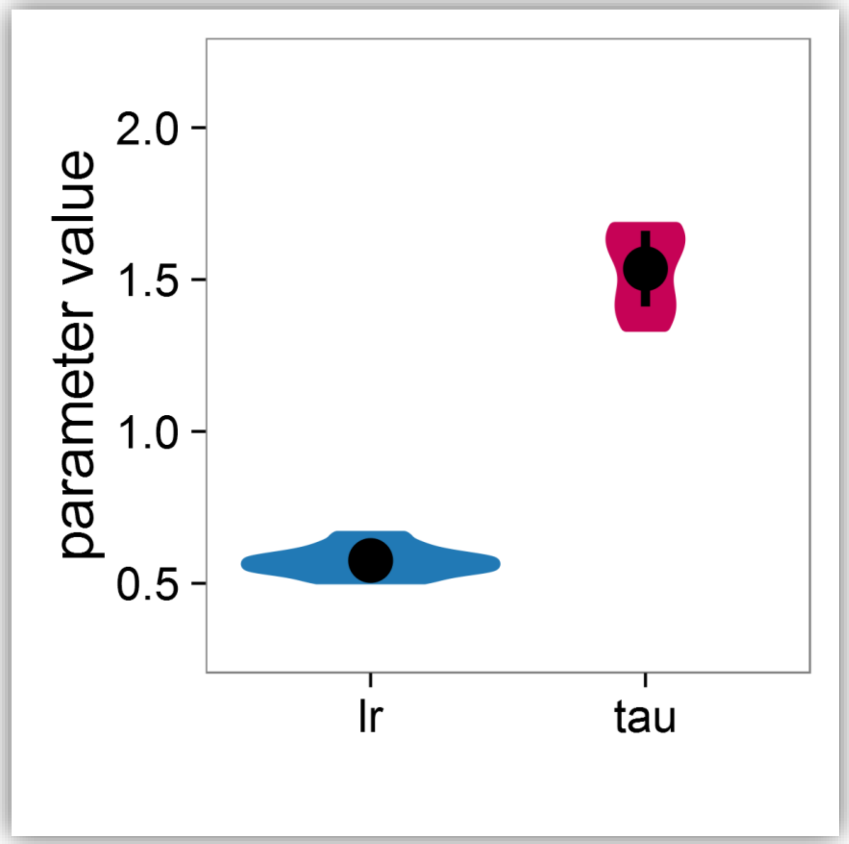
```
In addition: Warning messages:
1: There were 97 divergent transitions after warmup. Increasing adapt_delta above 0.8 may help. See
http://mc-stan.org/misc/warnings.html#divergent-transitions-after-warmup
2: Examine the pairs() plot to diagnose sampling problems
```

cognitive model
statistics
computing

Hierarchical Fitting*



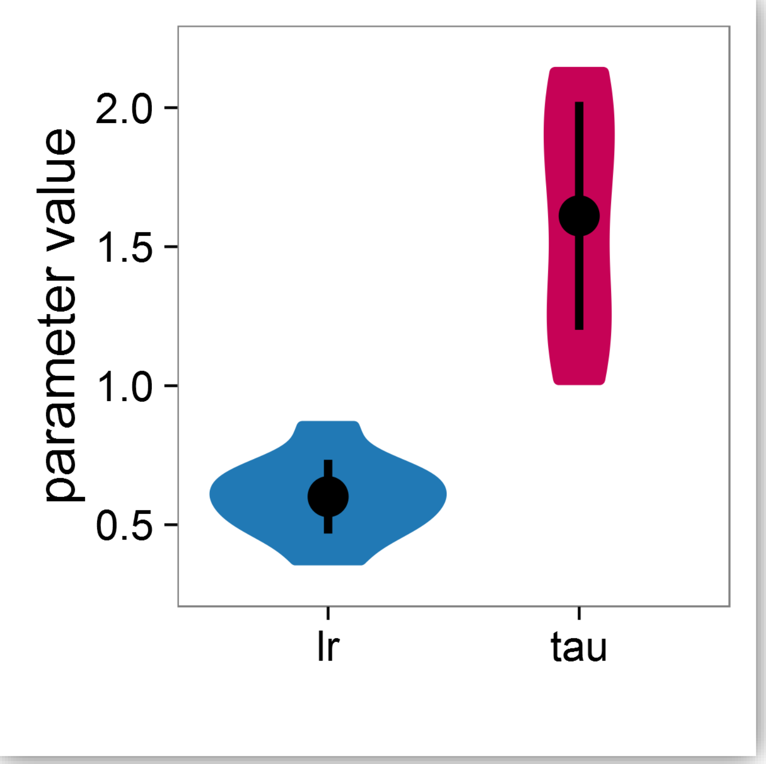




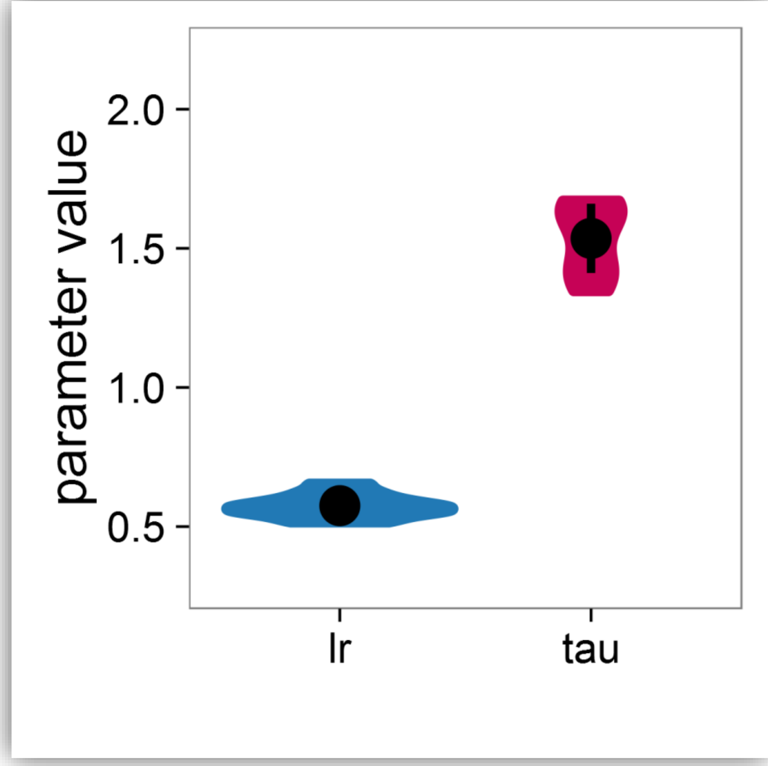
*: adapt_delta=0.999, max_treedepth=100

Comparing with True Parameters

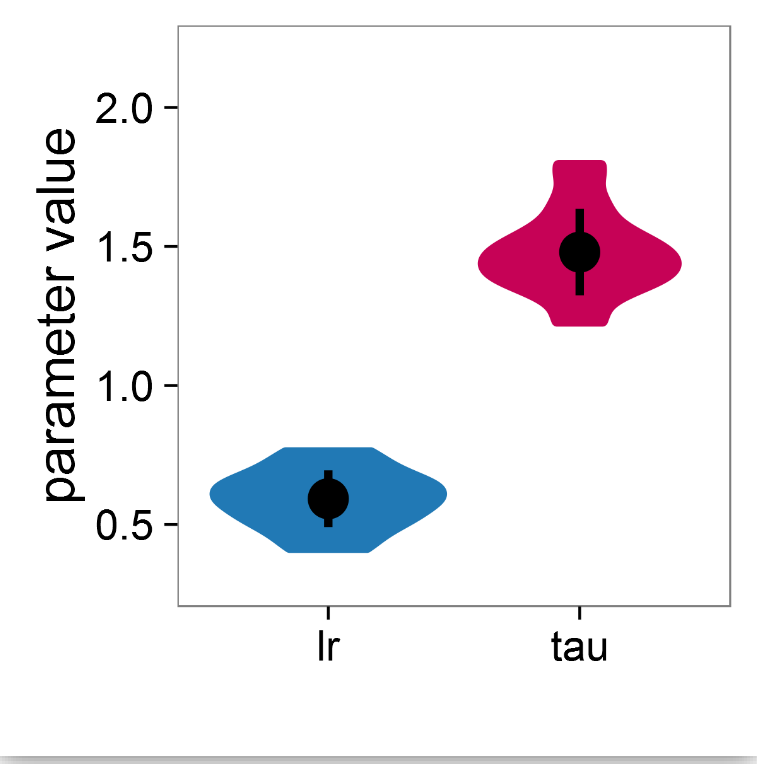
Posterior Means (indv)



Posterior Means (hrch)*



True Parameters



*: adapt_delta=0.999, max_treedepth=100

Group-level Parameters

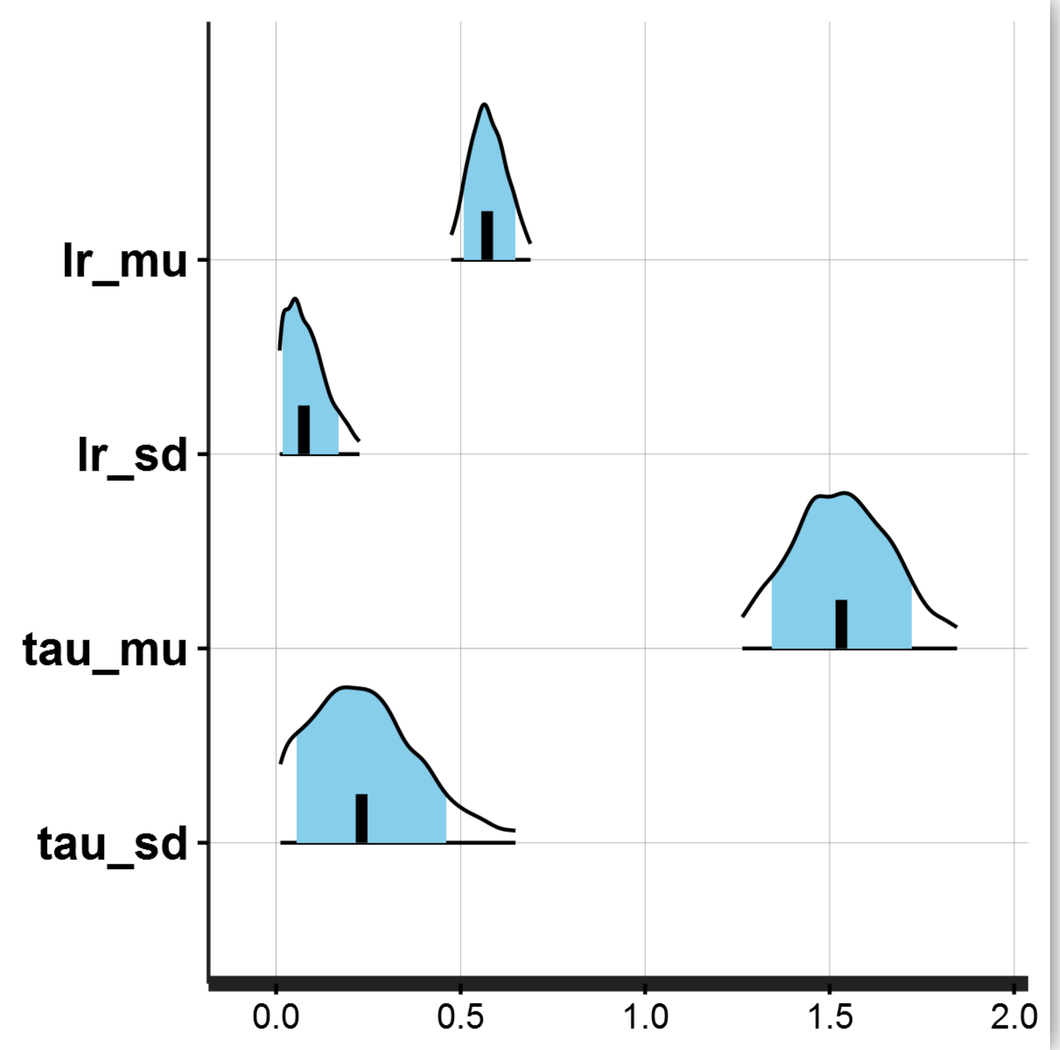
cognitive model
statistics
computing

True group parameters

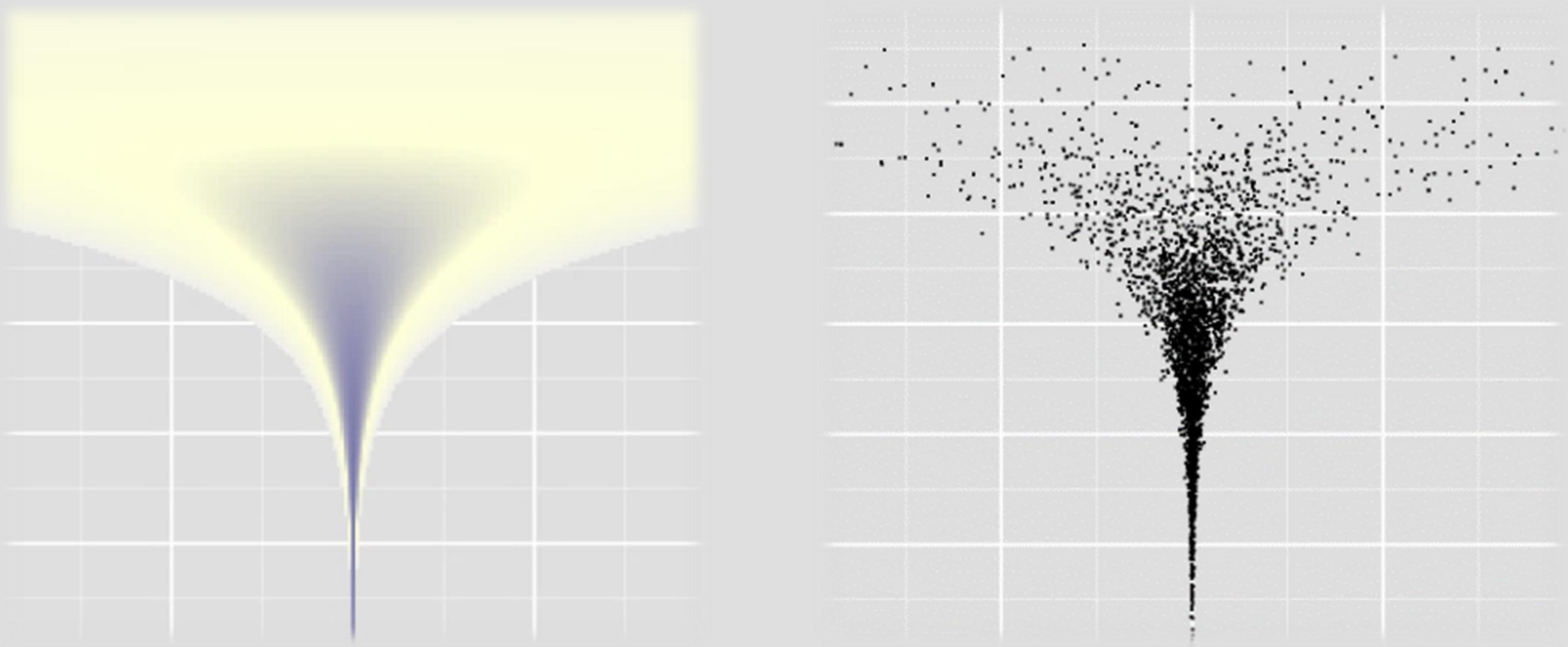
```
lr  = rnorm(10, mean=0.6, sd=0.12)
tau = rnorm(10, mean=1.5, sd=0.2)
```

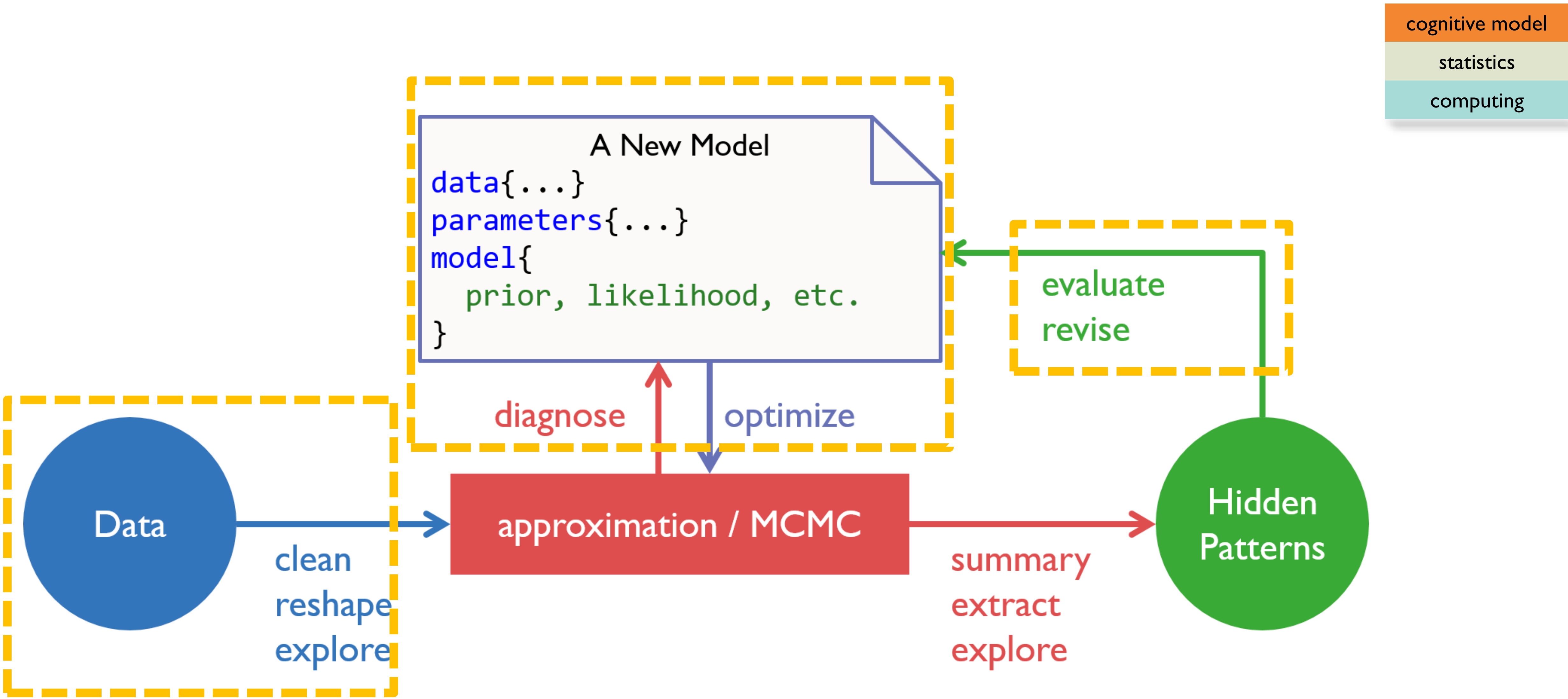
Estimated group parameters

	mean	2.5%	25%	50%	75%	97.5%
lr_mu	0.58	0.47	0.54	0.57	0.61	0.69
lr_sd	0.09	0.01	0.04	0.08	0.12	0.23
tau_mu	1.54	1.26	1.43	1.53	1.63	1.85
tau_sd	0.25	0.01	0.13	0.23	0.34	0.65



OPTIMIZING STAN CODES




cognitive model
statistics
computing
A New Model
data{...}
parameters{...}
model{
prior, likelihood, etc.
}
evaluate
revise
diagnose
optimize
Data
clean
reshape
explore
approximation / MCMC
summary
extract
explore
Hidden Patterns

Optimizing Stan Code

cognitive model
statistics
computing

Preprocess data

run as many calculations as you can outside Stan

Specify a proper model

follow literature, supervision, experience, etc.

Vectorizing

vectorize Stan code whenever you can

Reparameterizing

reparameterize target parameter to simple distributions

Preprocess Data

cognitive model

statistics

computing

$$\overline{\text{height}} = \alpha + \beta 1 * \text{weight} + \beta 2 * \text{weight}^2$$

```
d$weight_sq <- d$weight^2
```

```
data {
  int<lower=0> N;
  vector<lower=0>[N] height;
  vector<lower=0>[N] weight;
  vector<lower=0>[N] weight_sq;
}
```

Specify a Proper Model

cognitive model
statistics
computing

- Follow Literatures
- Start from simple and then build complexities
- Simulate data and run model recovery

A New Model

```
data{...}
parameters{...}
model{
  prior, likelihood, etc.
}
```

Vectorization

cognitive model
statistics
computing

```
model {
  for (n in 1:N) {
    flip[n] ~ bernoulli(theta);
  }
}
```

```
model {
  flip ~ bernoulli(theta);
}
```

```
parameters {
  ...
  real<lower=0,upper=1> lr[nSubjects];
  real<lower=0,upper=3> tau[nSubjects];
}

model {
  ...
  lr      ~ normal(lr_mu, lr_sd) ;
  tau     ~ normal(tau_mu, tau_sd) ;
  ...
}
```

```
model {
  vector[N] mu;
  for (i in 1:N) {
    mu[i] = alpha + beta * weight[i];
    height[i] ~ normal(mu[i], sigma)
  }
}
```

```
model {
  vector[N] mu;
  mu = alpha + beta * weight;
  height ~ normal(mu, sigma);
}
```

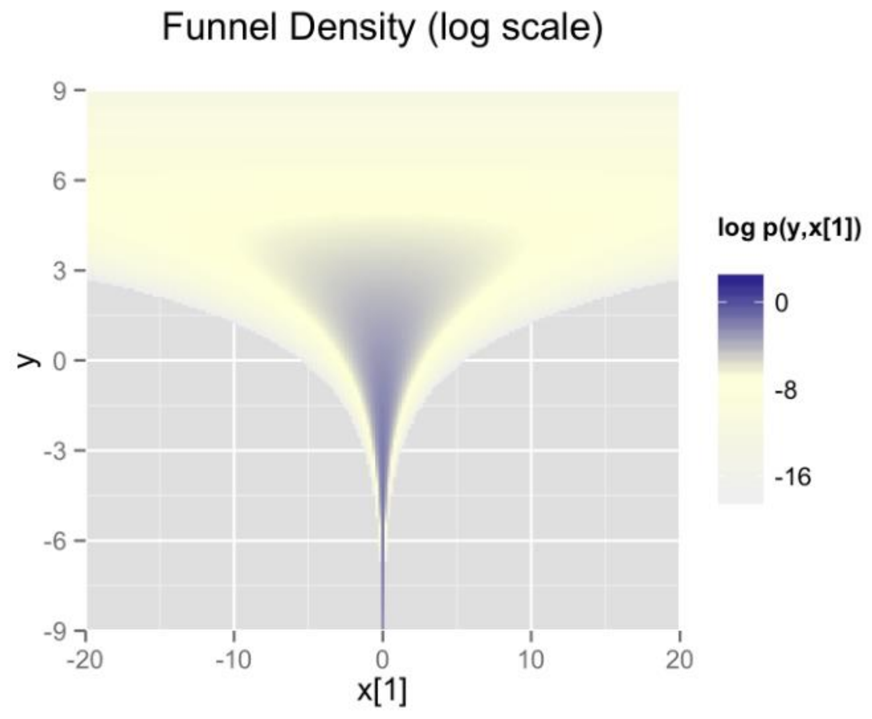
```
model {
  height ~ normal(alpha + beta * weight, sigma);
}
```

Reparameterization

Neal's Funnel

$$p(y, x) = \text{Normal}(y|0, 3) \times \prod_{n=1}^{9} \text{Normal}(x_n|0, \exp(y/2))$$

```
parameters {
  real y;
  vector[9] x;
}
model {
  y ~ normal(0,3);
  x ~ normal(0,exp(y/2));
}
```



Non-centered Reparameterization

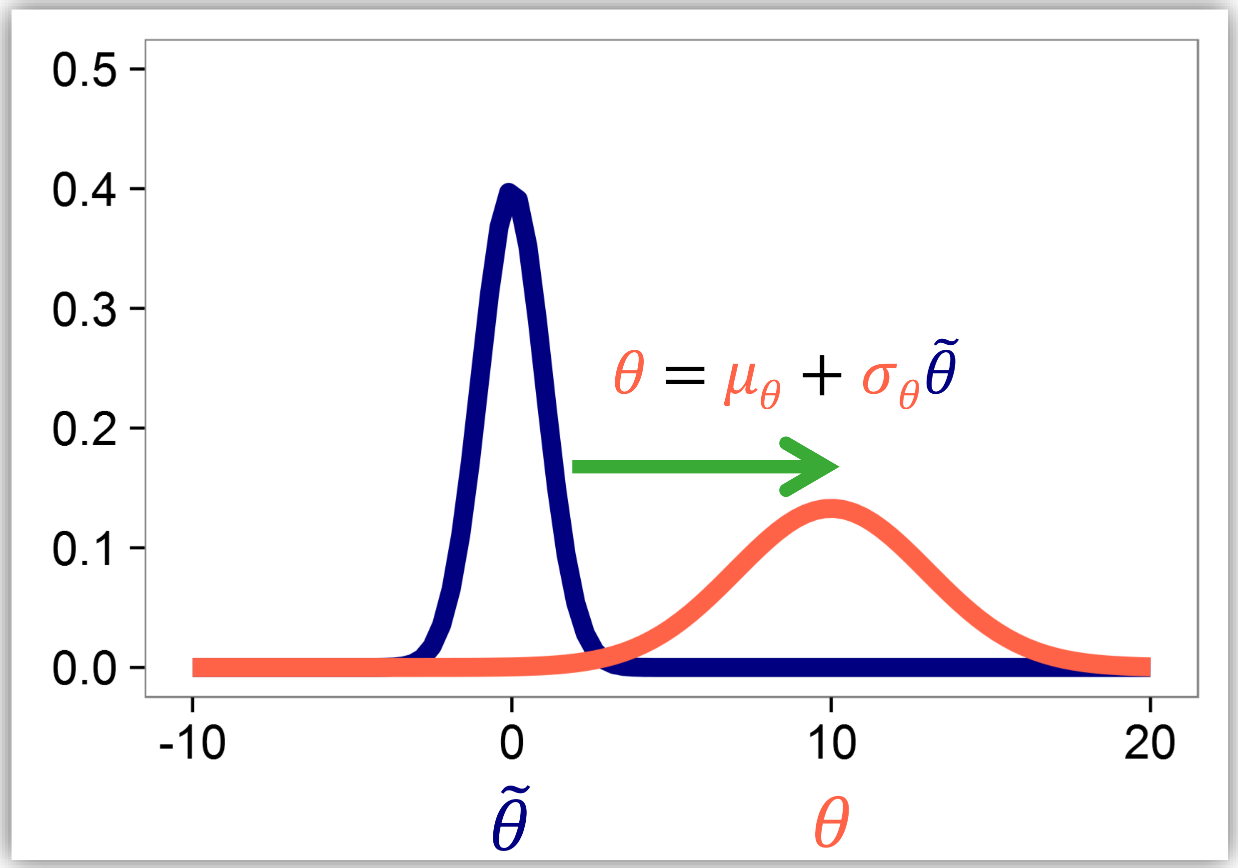
cognitive model
statistics
computing

$$\theta \sim Normal(\mu_\theta, \sigma_\theta)$$

$$\tilde{\theta} \sim Normal(0, 1)$$

$$\theta = \mu_\theta + \sigma_\theta \tilde{\theta}$$

Stan likes **simple** distributions!



Stan Development Team (2016)

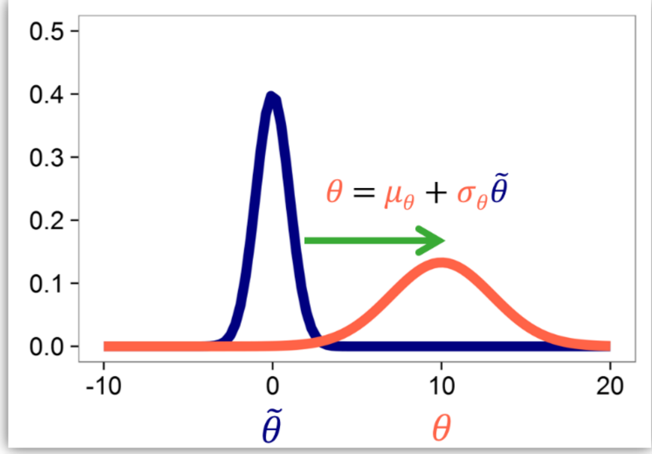
Reparameterization

cognitive model
statistics
computing

Neal's Funnel

$$p(y,x) = \text{Normal}(y|0,3) \times \prod_{n=1}^{9} \text{Normal}(x_n|0,\exp(y/2))$$

```
parameters {
  real y;
  vector[9] x;
}
model {
  y ~ normal(0,3);
  x ~ normal(0,exp(y/2));
}
```




```
parameters {
  real y_raw;
  vector[9] x_raw;
}
transformed parameters {
  real y;
  vector[9] x;

  y = 3.0 * y_raw;
  x = exp(y/2) * x_raw;
}
model {
  y_raw ~ normal(0,1);
  x_raw ~ normal(0,1);
}
```

Stan Sampling Parameters

parameter	description	constraint	default
iterations	number of MCMC samples (per chain)	int, > 0	2000
delta: δ	target Metropolis acceptance rate	$\delta \in [0,1]$	0.80
stepsize: ε	initial HMC step size	real, $\varepsilon > 0$	2.0
max_treedepth: L	maximum HMC steps per iteration	int, $L > 0$	10

Typical adjustments

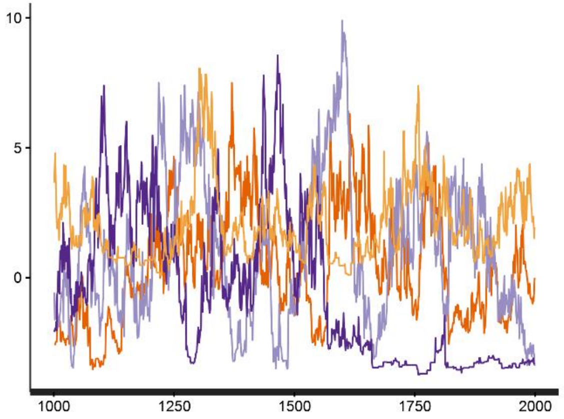
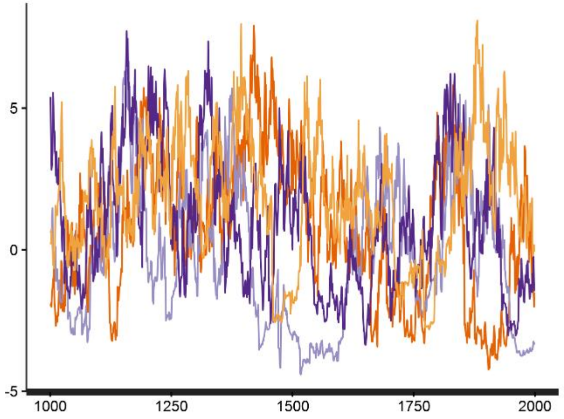
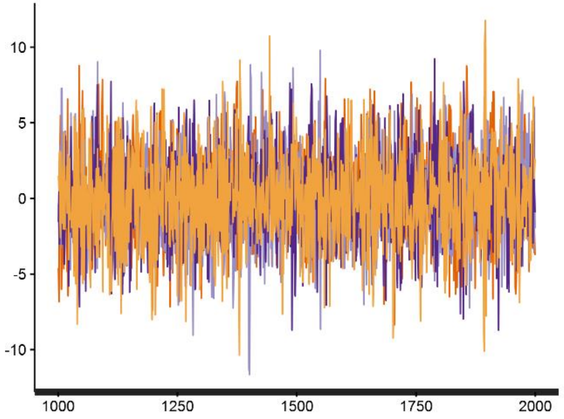
- Increase iterations
- Increase delta
- Decrease stepsize
- Might have to increase max_treedepth

```
funnel_fit2 <- stan("_scripts/funnel.stan",
        iter = 4000,
        control = list(adapt_delta = 0.999,
                       stepsize = 1.0,
                       max_treedepth = 20))
```

Stan Development Team (2016)

Neal's Funnel: Comparing Performance

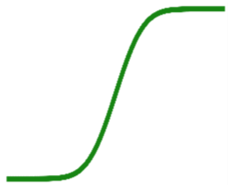
cognitive model
statistics
computing

	direct model	adjusted direct model	reparameterized model
Rhat (*y*)	1.22	1.1	1.0
n_eff (*y*)	18	42	3886
runtime*	48.50 sec	50.76 sec	50.12 sec
n_eff (*y*) / runtime	0.37 / sec	0.82 / sec	77.53 / sec
n_divergent	53	0	0
traceplot (*y*)			

*: 2 cores in parallel, including compiling time

How about Bounded Parameters?

$$\tilde{\theta} \sim Normal(0,1)$$
$$\theta = \mu_\theta + \sigma_\theta \tilde{\theta}$$
$$\theta \in (-\infty, +\infty)$$



$$\tilde{\theta} \sim Normal(0,1)$$
$$\theta = Probit^{-1}(\mu_\theta + \sigma_\theta \tilde{\theta})$$
$$\theta \in [0,1]$$

constraint	**reparameterization**
$\theta \in (-\infty, +\infty)$	$\theta = \mu_\theta + \sigma_\theta \tilde{\theta}$
$\theta \in [0, N]$	$\theta = Probit^{-1}(\mu_\theta + \sigma_\theta \tilde{\theta}) \times N$
$\theta \in [M, N]$	$\theta = Probit^{-1}(\mu_\theta + \sigma_\theta \tilde{\theta}) \times (N\text{-}M) + M$
$\theta \in (0, +\infty)$	$\theta = exp(\mu_\theta + \sigma_\theta \tilde{\theta})$

Apply to Our Hierarchical RL Model

cognitive model

statistics

computing

```
parameters {
  real<lower=0,upper=1> lr_mu;
  real<lower=0,upper=3> tau_mu;

  real<lower=0> lr_sd;
  real<lower=0> tau_sd;

  real<lower=0,upper=1> lr[nSubjects];
  real<lower=0,upper=3> tau[nSubjects];
}
```

```
parameters {
  # group-level parameters
  real lr_mu_raw;
  real tau_mu_raw;
  real<lower=0> lr_sd_raw;
  real<lower=0> tau_sd_raw;

  # subject-level raw parameters
  vector[nSubjects] lr_raw;
  vector[nSubjects] tau_raw;
}

transformed parameters {
  vector<lower=0,upper=1>[nSubjects] lr;
  vector<lower=0,upper=3>[nSubjects] tau;

  for (s in 1:nSubjects) {
    lr[s]  = Phi_approx( lr_mu_raw  + lr_sd_raw * lr_raw[s] );
    tau[s] = Phi_approx( tau_mu_raw + tau_sd_raw * tau_raw[s] ) * 3;
  }
}
```

Apply to Our Hierarchical RL Model

cognitive model

statistics

computing

```
model {
  lr_sd  ~ cauchy(0,1);
  tau_sd ~ cauchy(0,3);
  lr     ~ normal(lr_mu, lr_sd) ;
  tau    ~ normal(tau_mu, tau_sd) ;

  for (s in 1:nSubjects) {
    vector[2] v;
    real pe;
    v = initV;

    for (t in 1:nTrials) {
      choice[s,t] ~ categorical_logit( tau[s] * v );
      pe = reward[s,t] - v[choice[s,t]];
      v[choice[s,t]] = v[choice[s,t]] + lr[s] * pe;
    }
  }
}
```

```
model {
  lr_mu_raw  ~ normal(0,1);
  tau_mu_raw ~ normal(0,1);
  lr_sd_raw  ~ cauchy(0,3);
  tau_sd_raw ~ cauchy(0,3);

  lr_raw  ~ normal(0,1);
  tau_raw ~ normal(0,1);

  for (s in 1:nSubjects) {
    ...
```

```
generated quantities {
  real<lower=0,upper=1> lr_mu;
  real<lower=0,upper=3> tau_mu;

  lr_mu  = Phi_approx(lr_mu_raw);
  tau_mu = Phi_approx(tau_mu_raw) * 3;
}
```

Exercise VI

cognitive model
statistics
computing

```
…/BayesCog/07.optm_rl/_scripts/reinforcement_learning_hrch_main.R
```

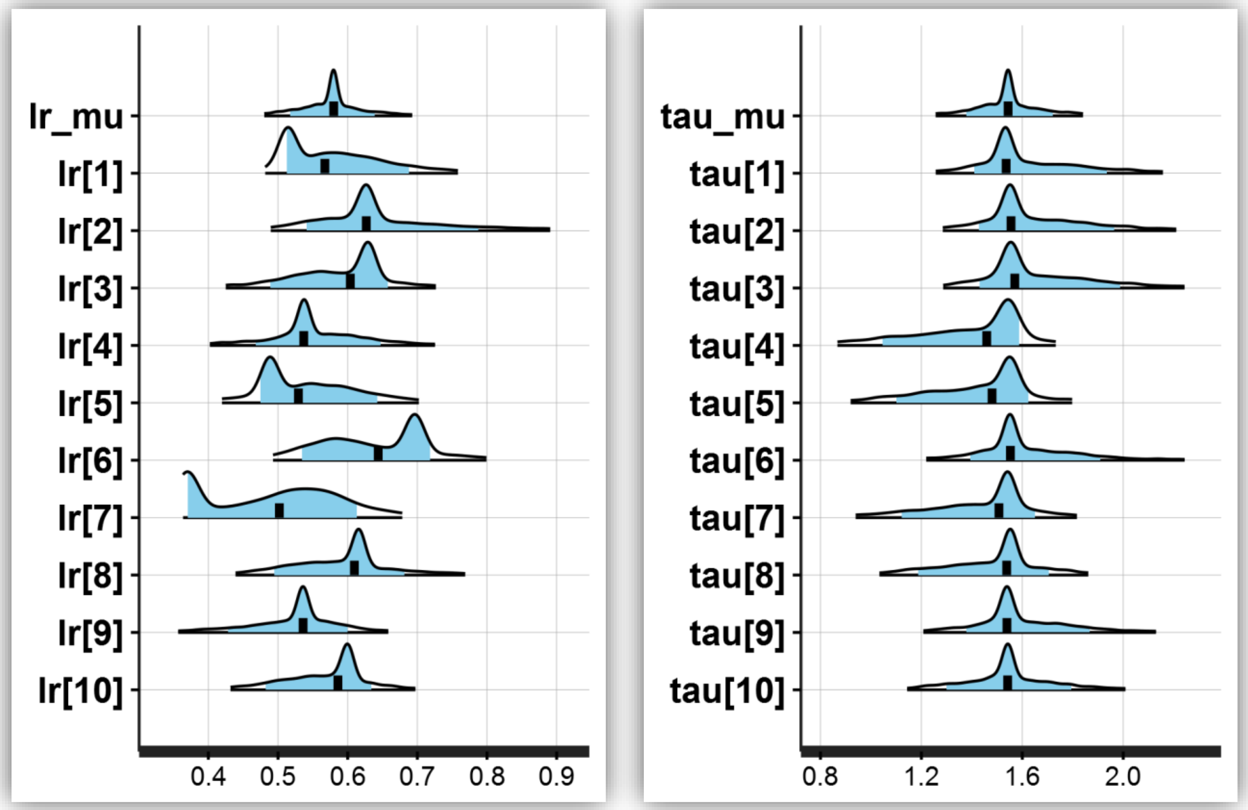
TASK: fit the optimized hierarchical RL model

```
> source('_scripts/reinforcement_learning_hrch_main.R')

> fit_rl4 <- run_rl_mp2(optimized = TRUE)
```

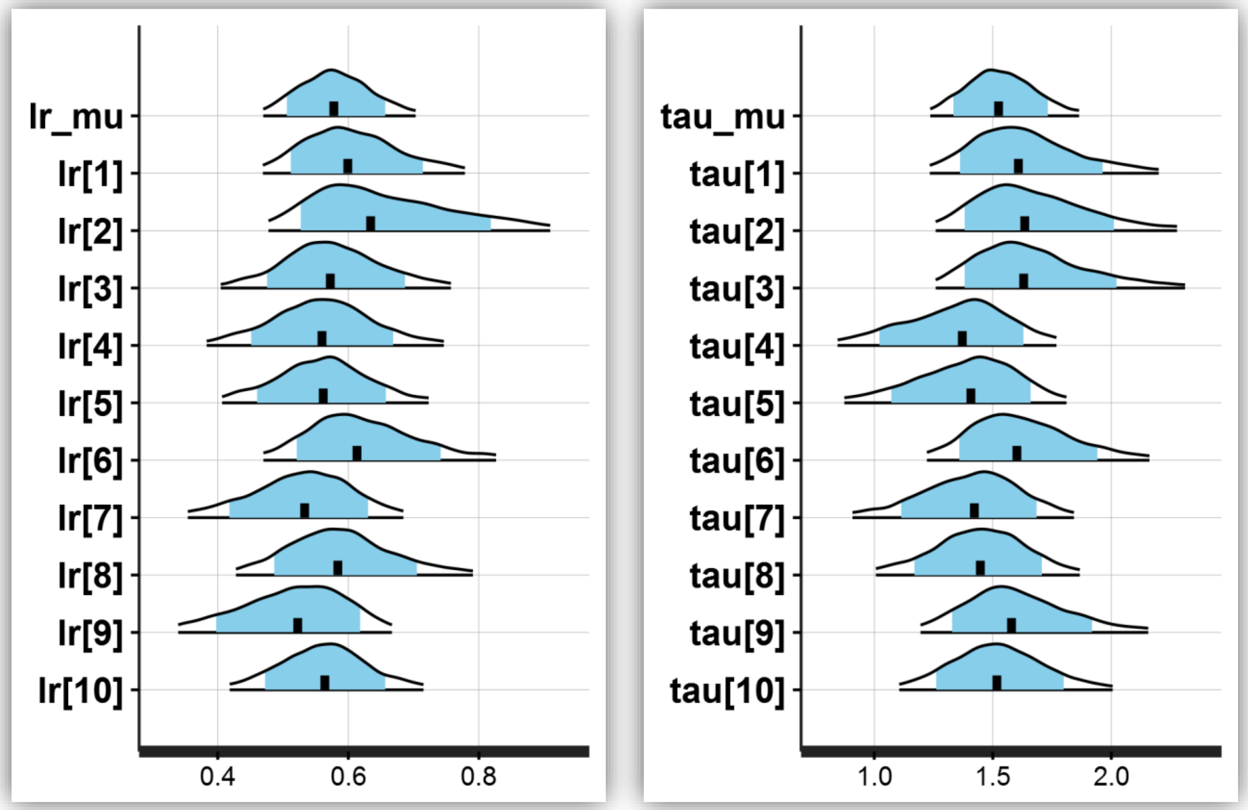
Hierarchical Fitting – Optimized

cognitive model
statistics
computing

Posterior Means (hrch)



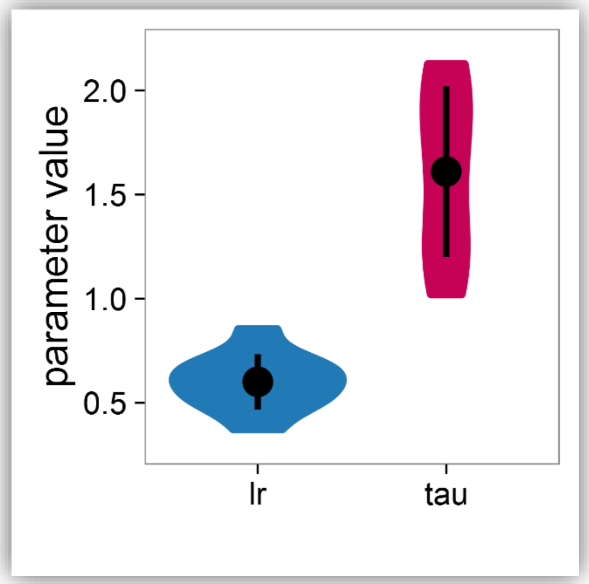
Posterior Means (hrch + optm)



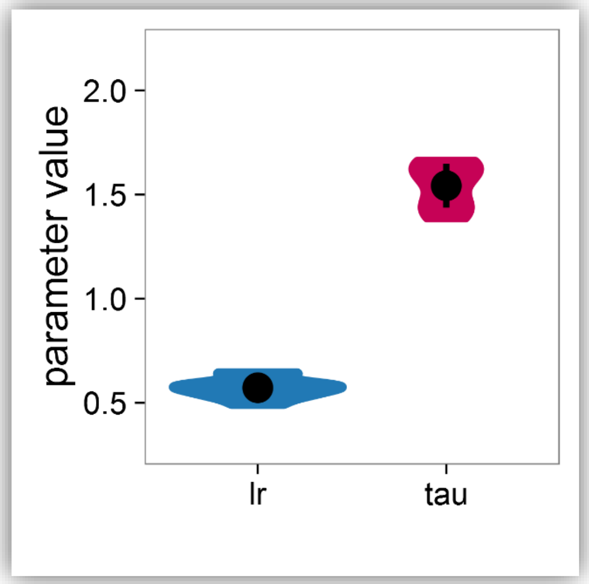
Comparing with True Parameters

cognitive model
statistics
computing

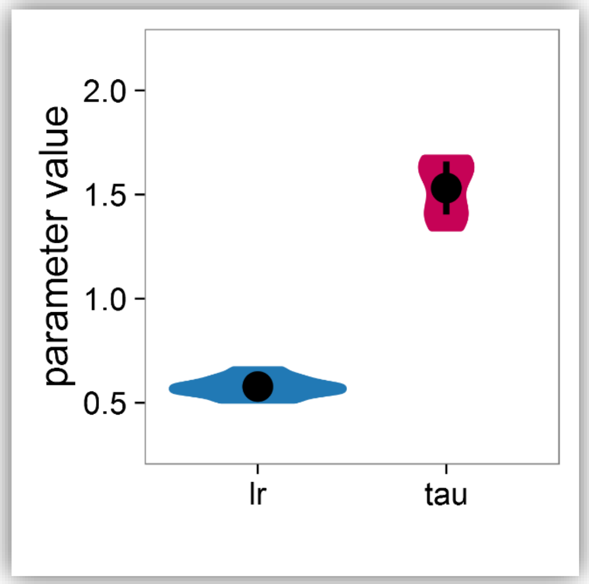
Posterior Means (indv)



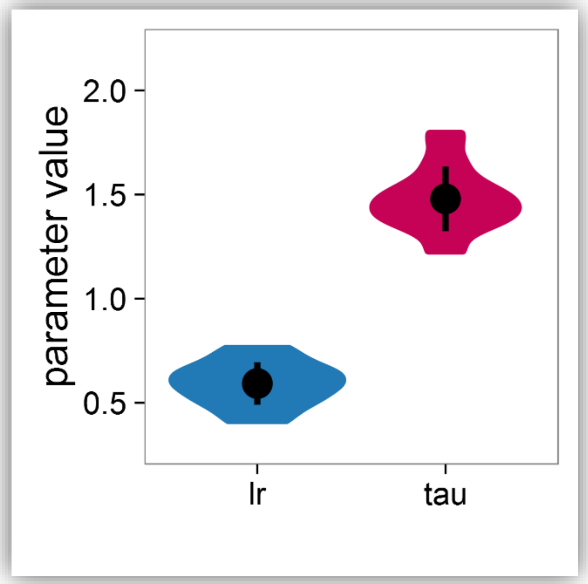
Posterior Means (hrch)

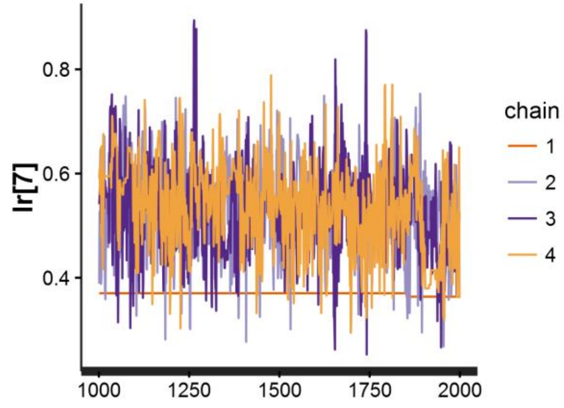


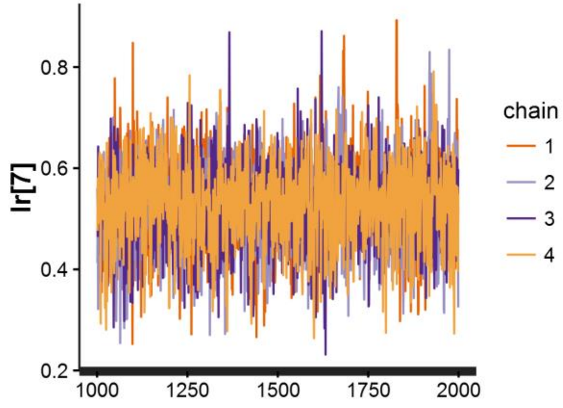
Posterior Means (hrch+optm)



True Parameters

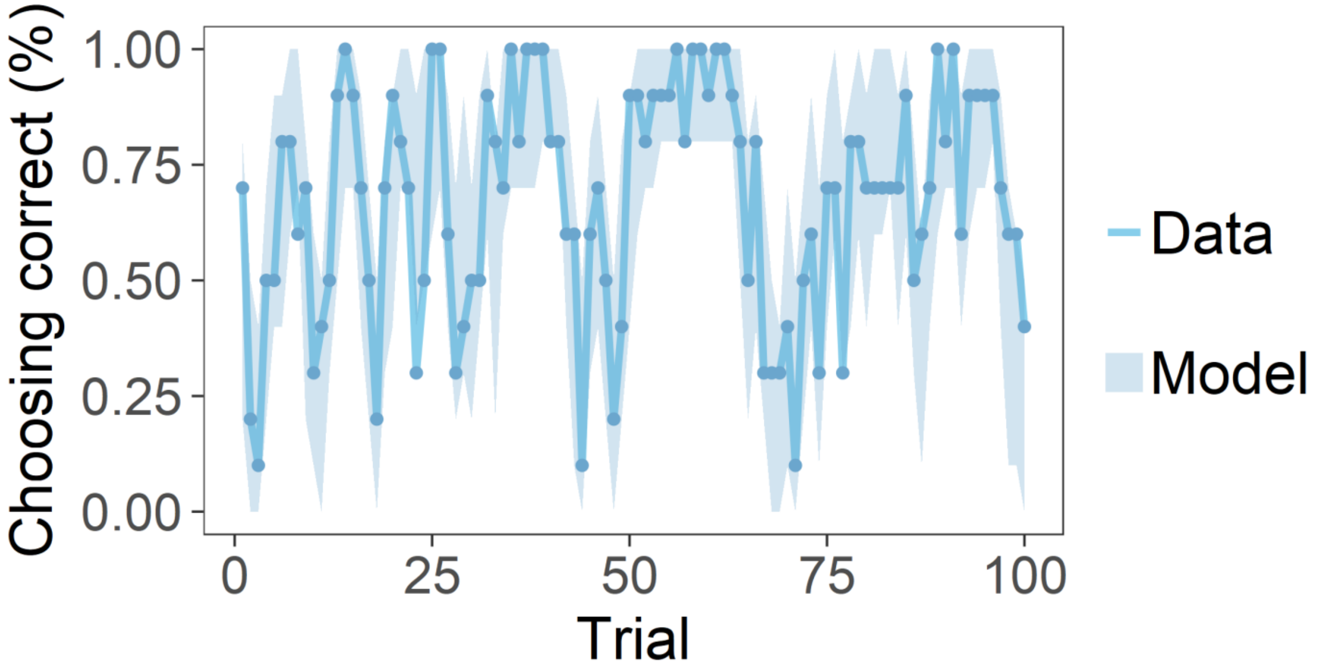


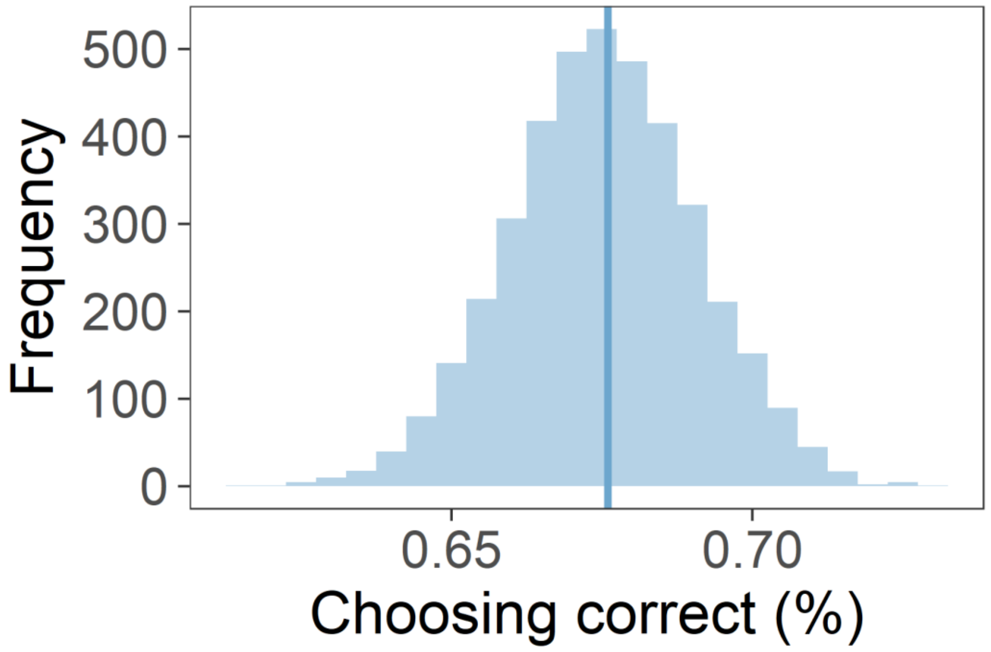




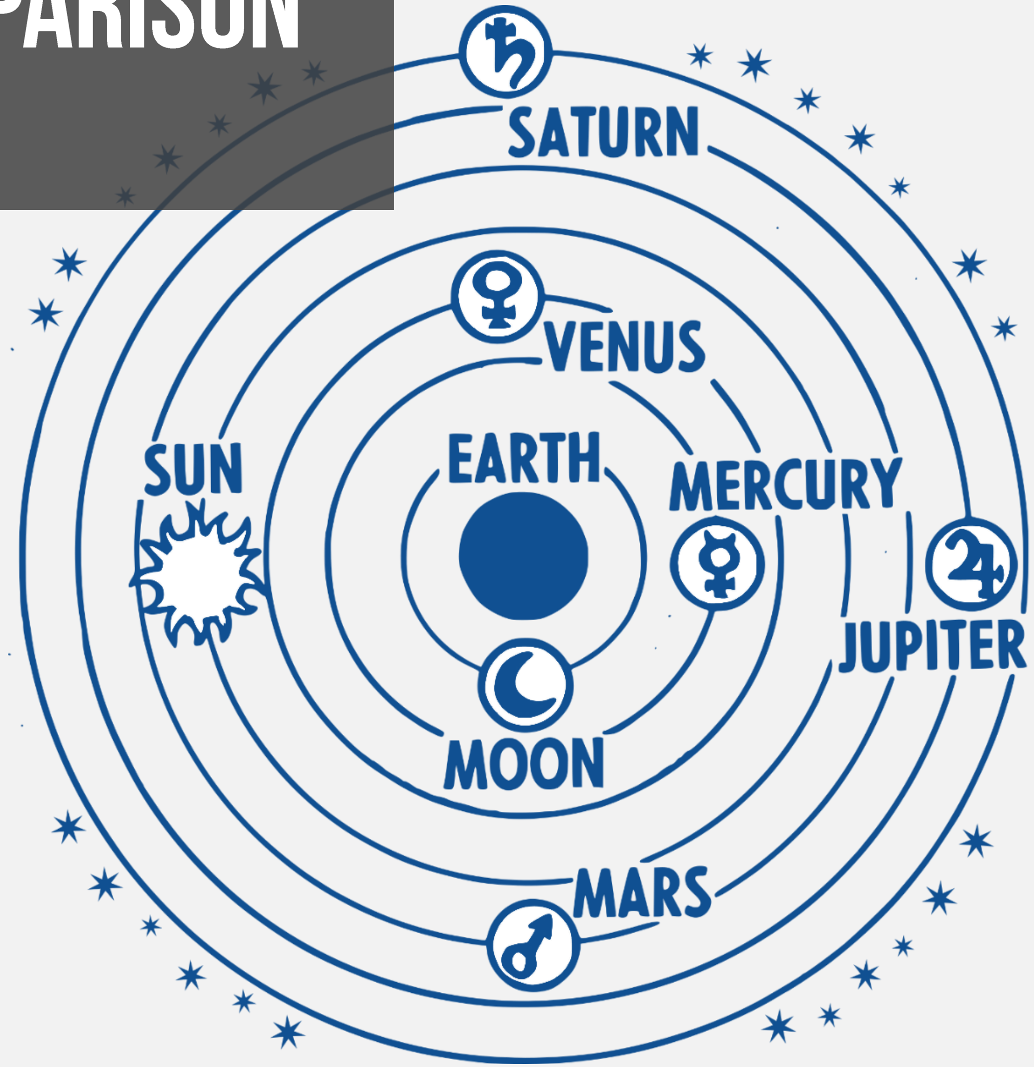
Posterior Predictive Check

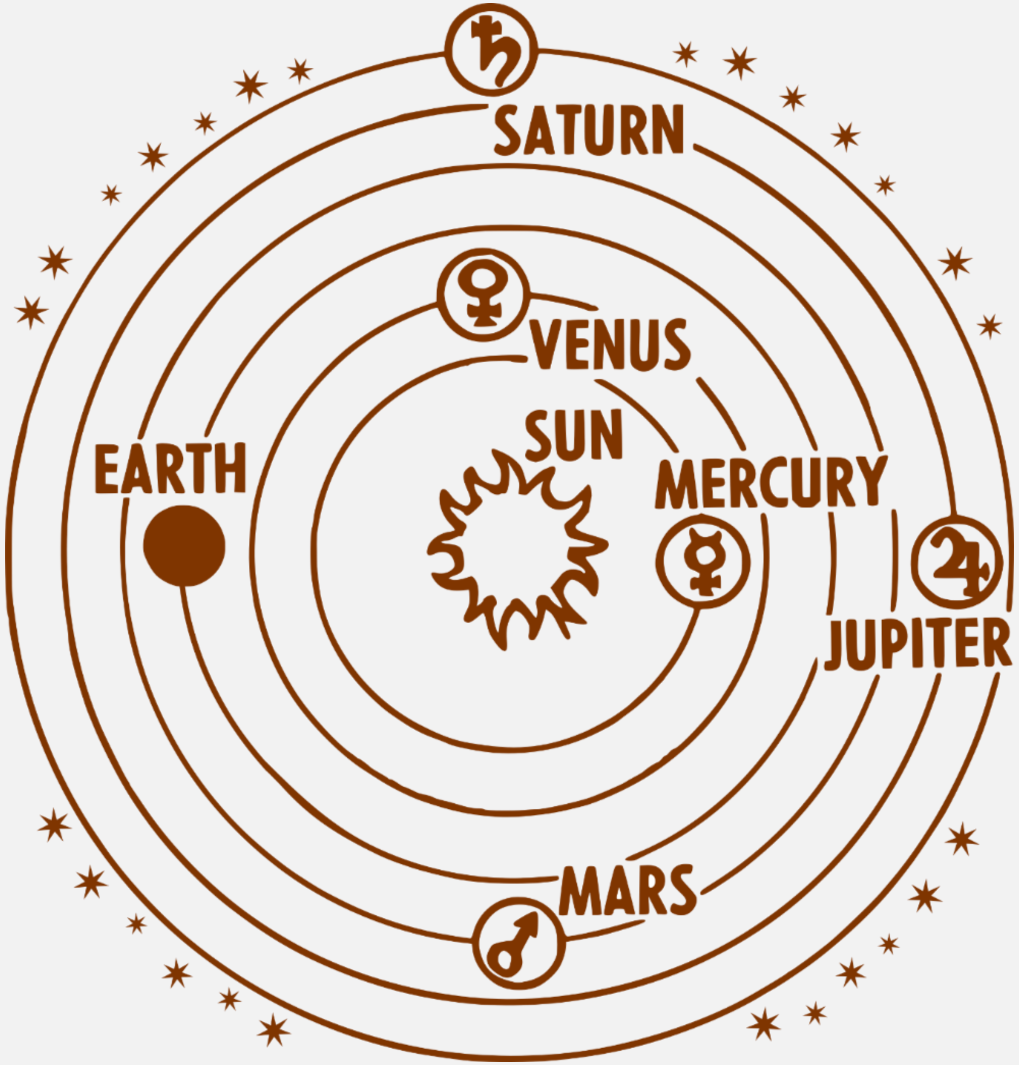
cognitive model
statistics
computing





MODEL COMPARISON





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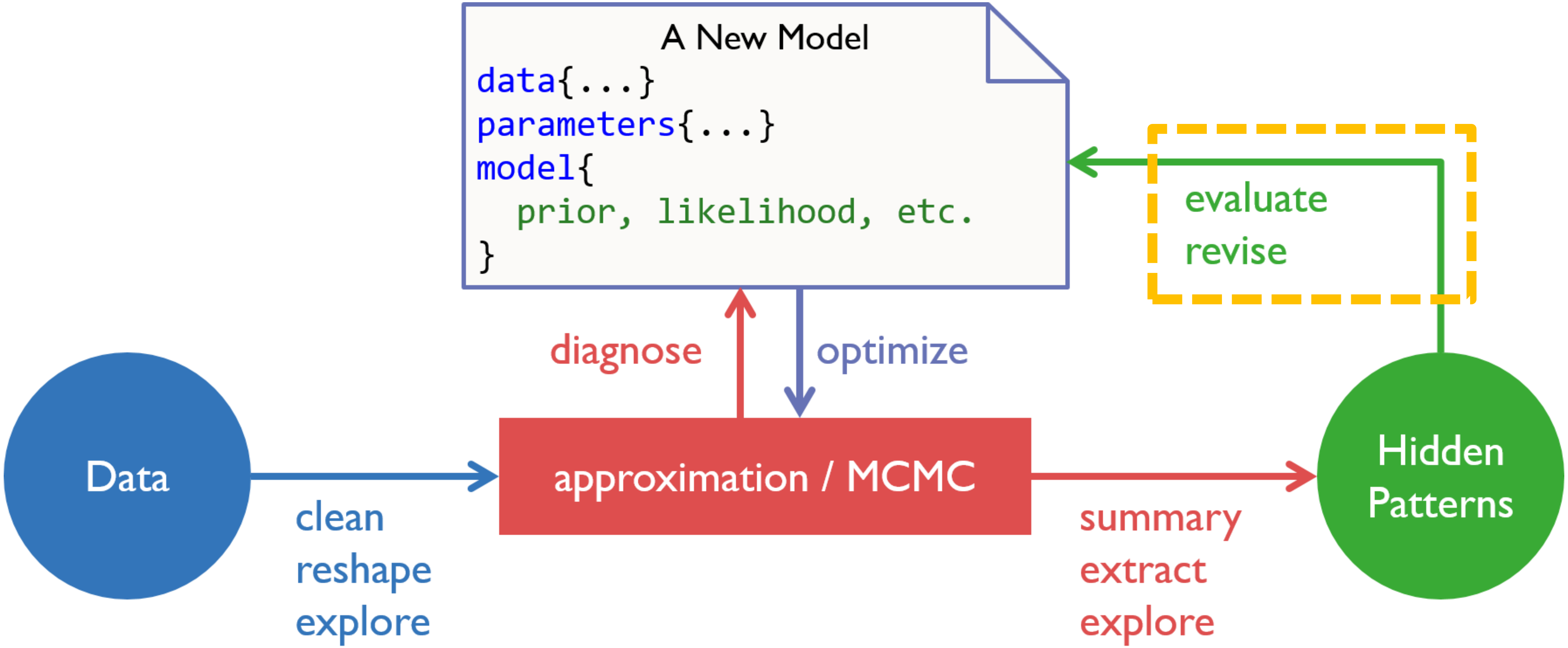
statistics
computing

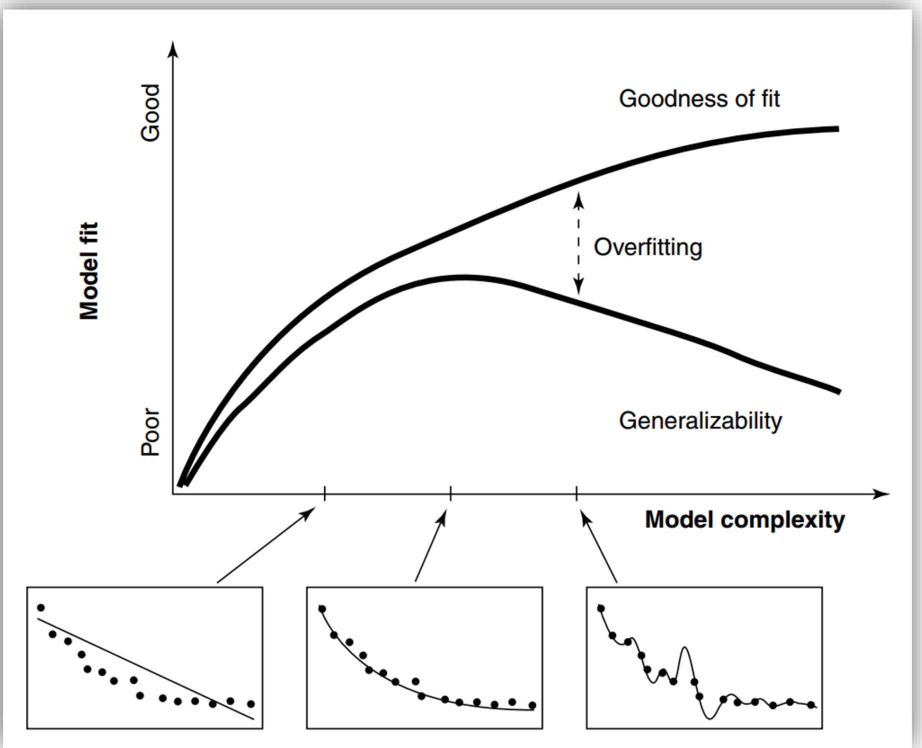


Model Comparison

Which model provides the best fit?

Which model represents the best balance between model fit and model complexity?

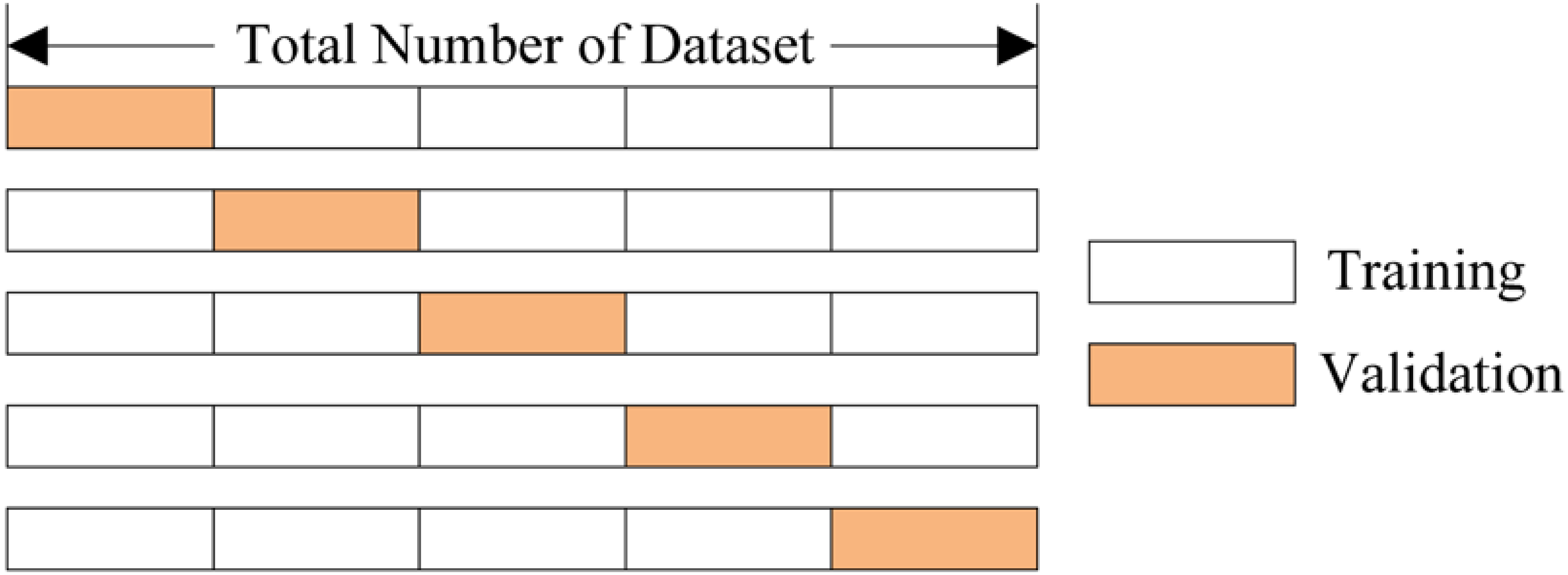
Ockham's razor:
Models with fewer assumptions are to be preferred



- overfitting: learn too much from the data
- underfitting: learn too little from the data

Pitt & Miyung (2002)

Focusing on Predictive Accuracy



- Nothing prevents you from doing that in a Bayesian context but holding out data makes your posterior distribution more diffuse
- Bayesians usually condition on *all* the data and evaluate how well a model is *expected* to predict out of sample using "information criteria": model with the highest expected log predictive density (ELPD) for new data

Information Criteria

AIC – Akaike information criterion

DIC – Deviance Information Criterion

WAIC – Widely Applicable Information Criterion

finding the model that has the highest out-of-sample predictive accuracy

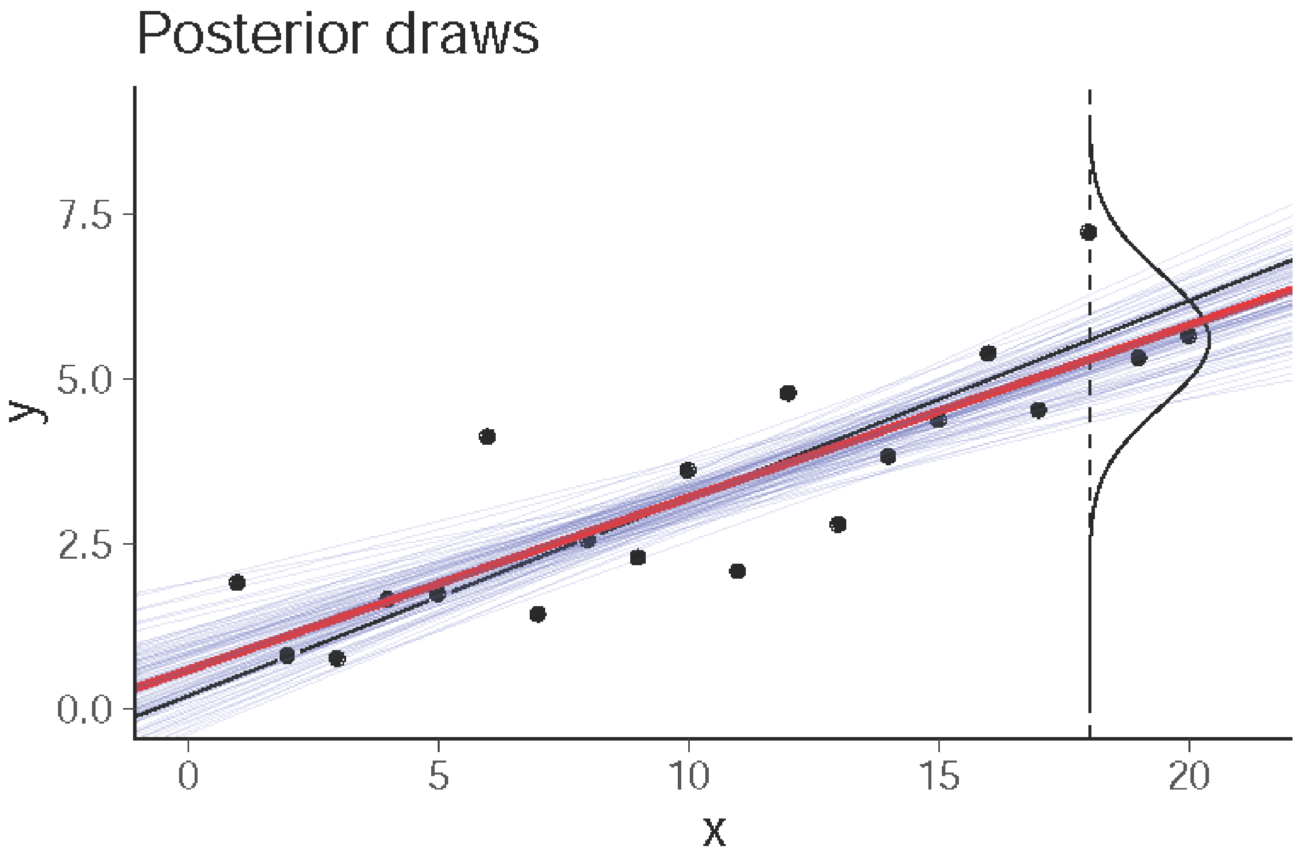
approximation to LOO

BIC – Bayesian Information Criterion

finding the "true" model

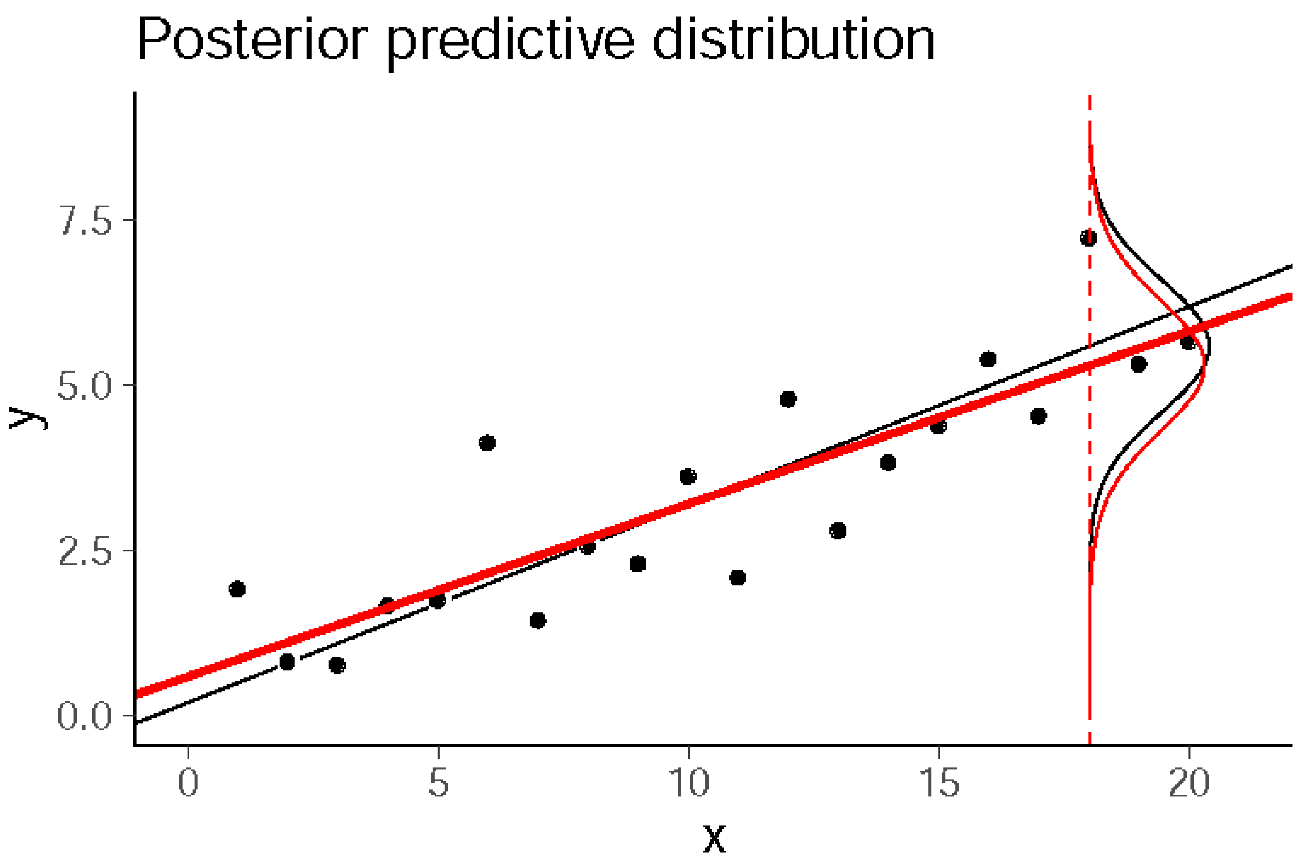
Understand model prediction

cognitive model
statistics
computing



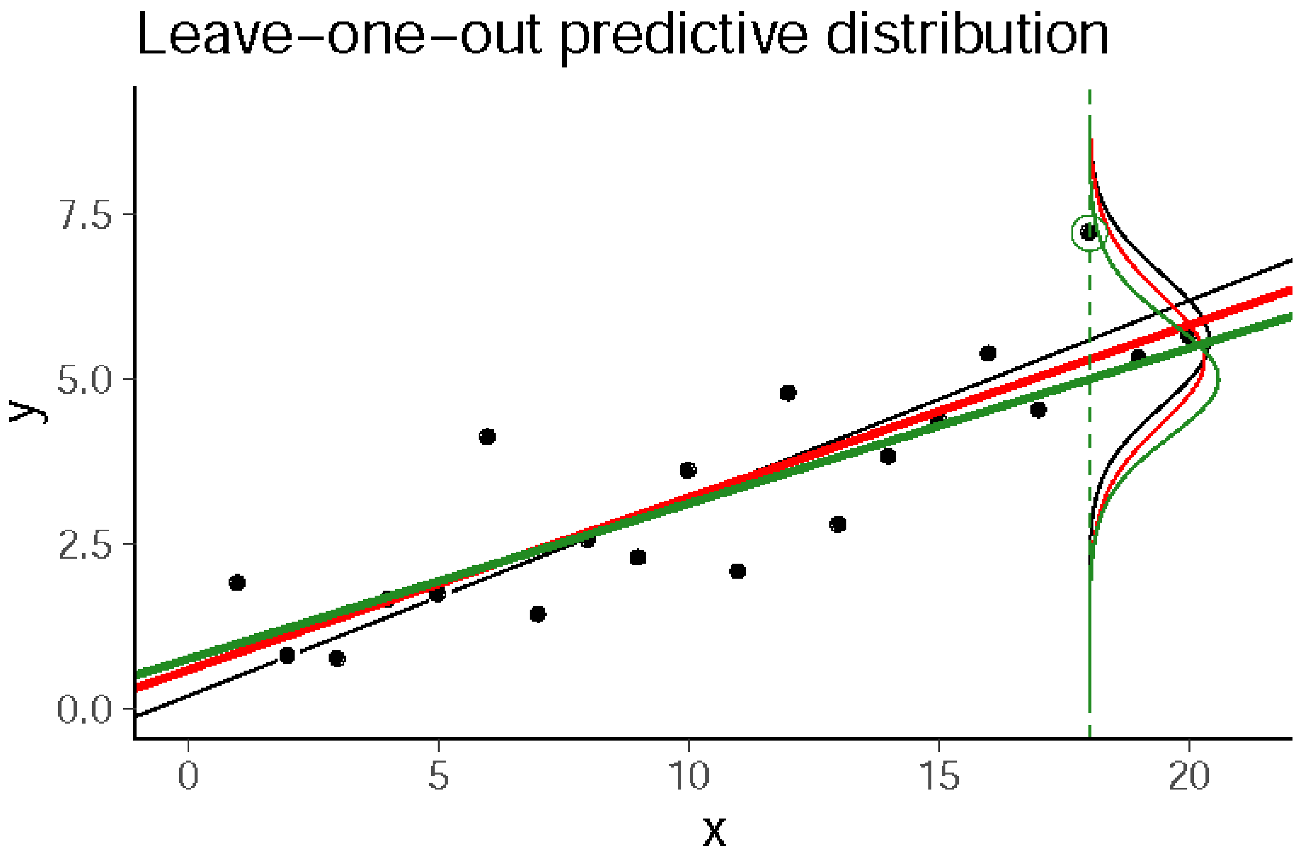
Adapted from Aki Vehtari's workshop

Understand model prediction



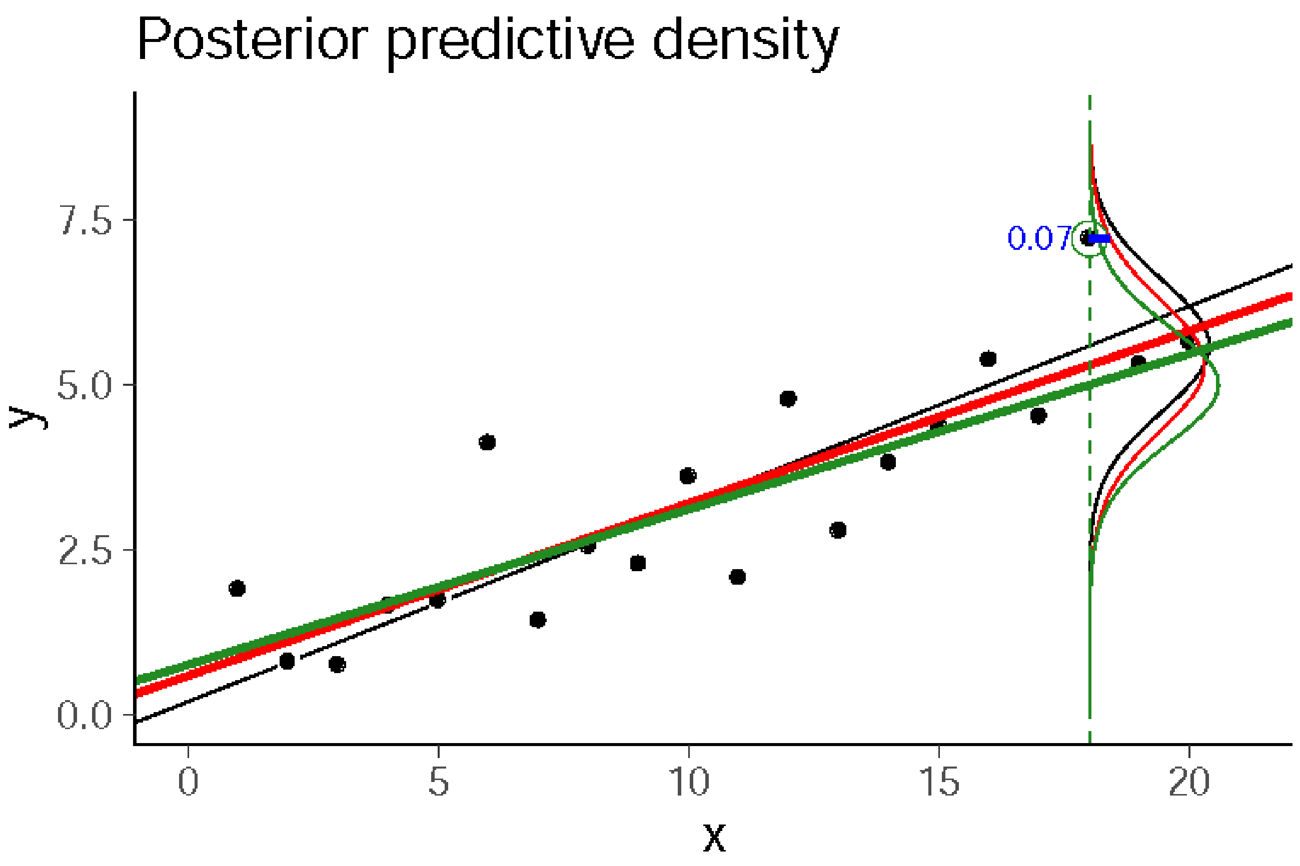
$$p(\tilde{y}|\tilde{x} = 18, x, y) = \int p(\tilde{y}|\tilde{x} = 18, \theta)p(\theta|x, y)d\theta$$

Understand model prediction



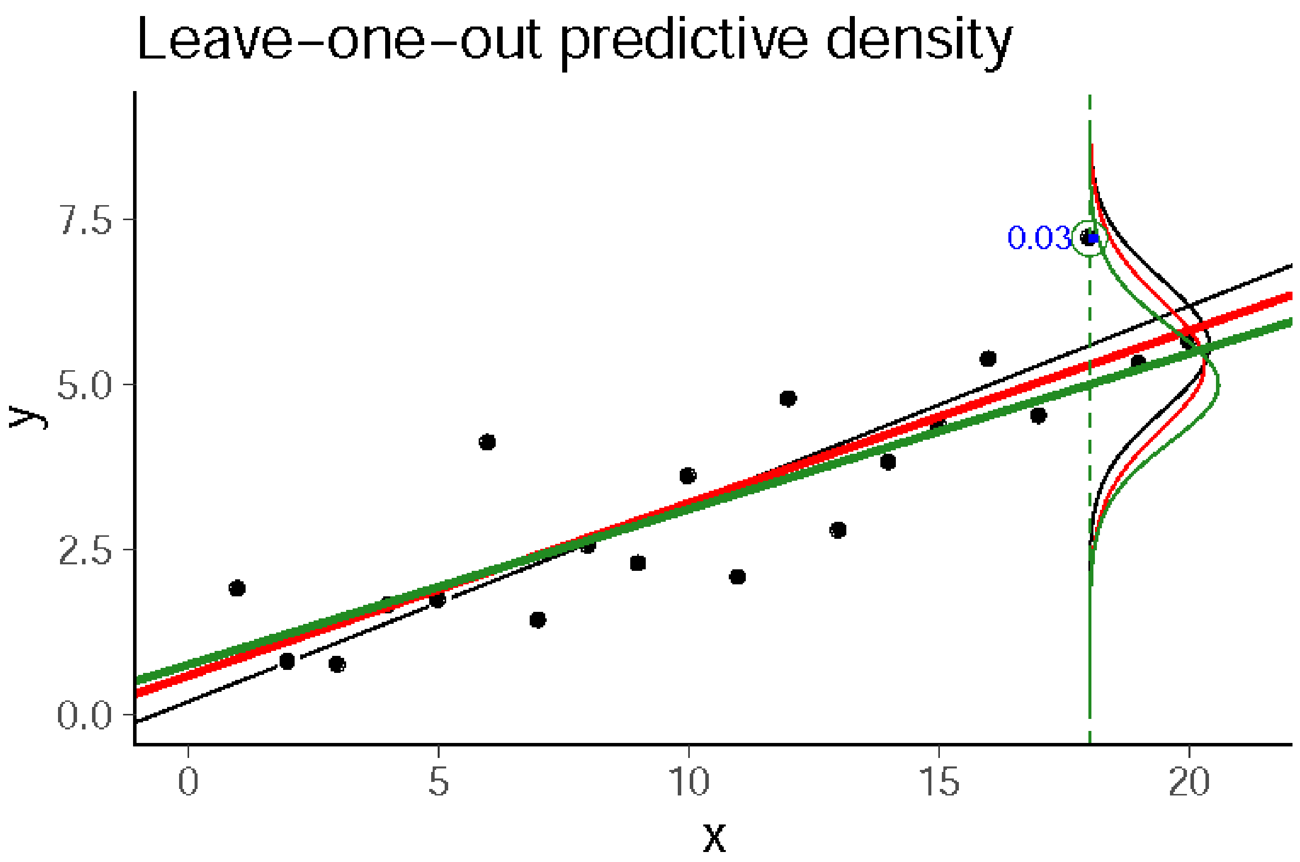
$$p(\tilde{y}|\tilde{x} = 18, x_{-18}, y_{-18}) = \int p(\tilde{y}|\tilde{x} = 18, \theta) \boxed{p(\theta|x_{-18}, y_{-18})} d\theta$$

Understand model prediction



$p(\tilde{y} = y_{18} | \tilde{x} = 18, x, y) \approx 0.07$

Understand model prediction

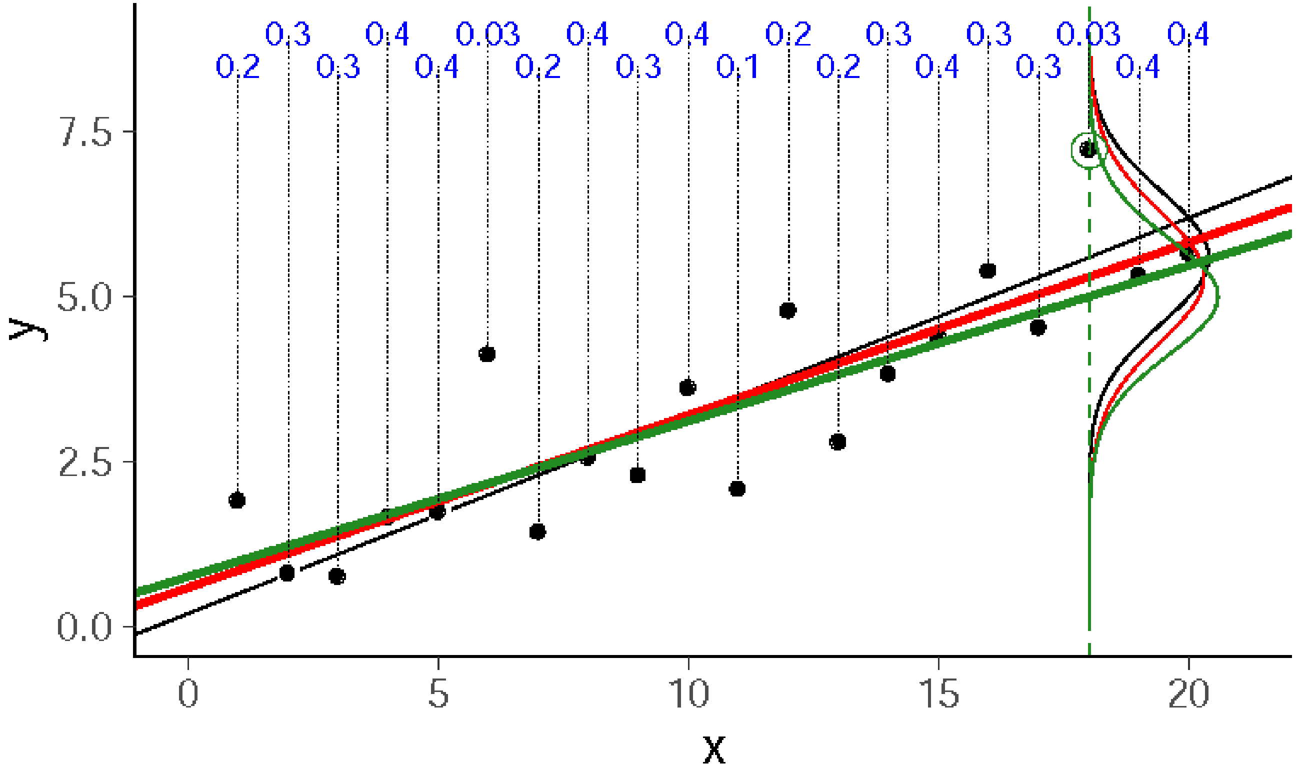


$p(\tilde{y} = y_{18} | \tilde{x} = 18, x, y) \approx 0.07$

$p(\tilde{y} = y_{18} | \tilde{x} = 18, x_{-18}, y_{-18}) \approx 0.03$

Understand model prediction

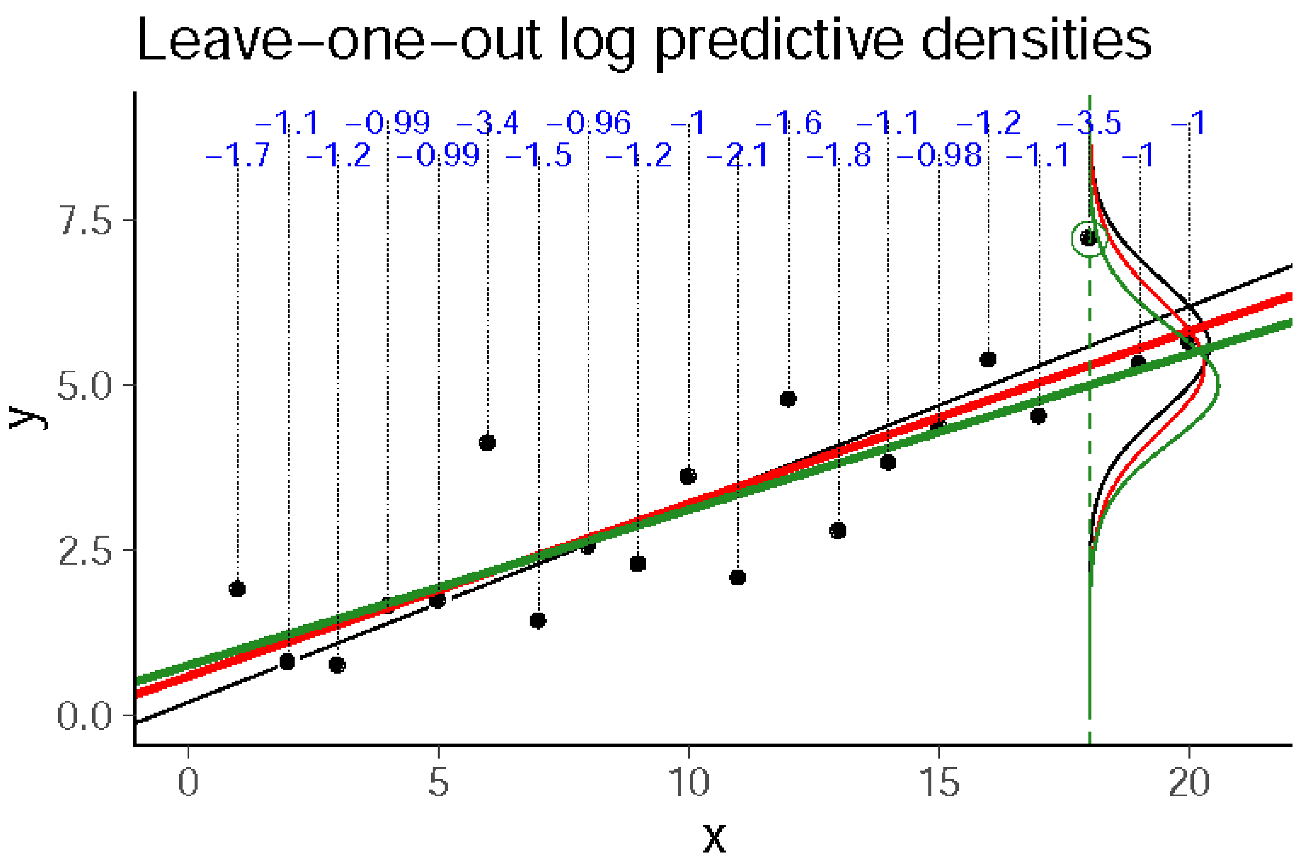
Leave−one−out predictive densities



$p(y_i|x_i, x_{-i}, y_{-i}), \quad i = 1, \ldots, 20$

Understand model prediction

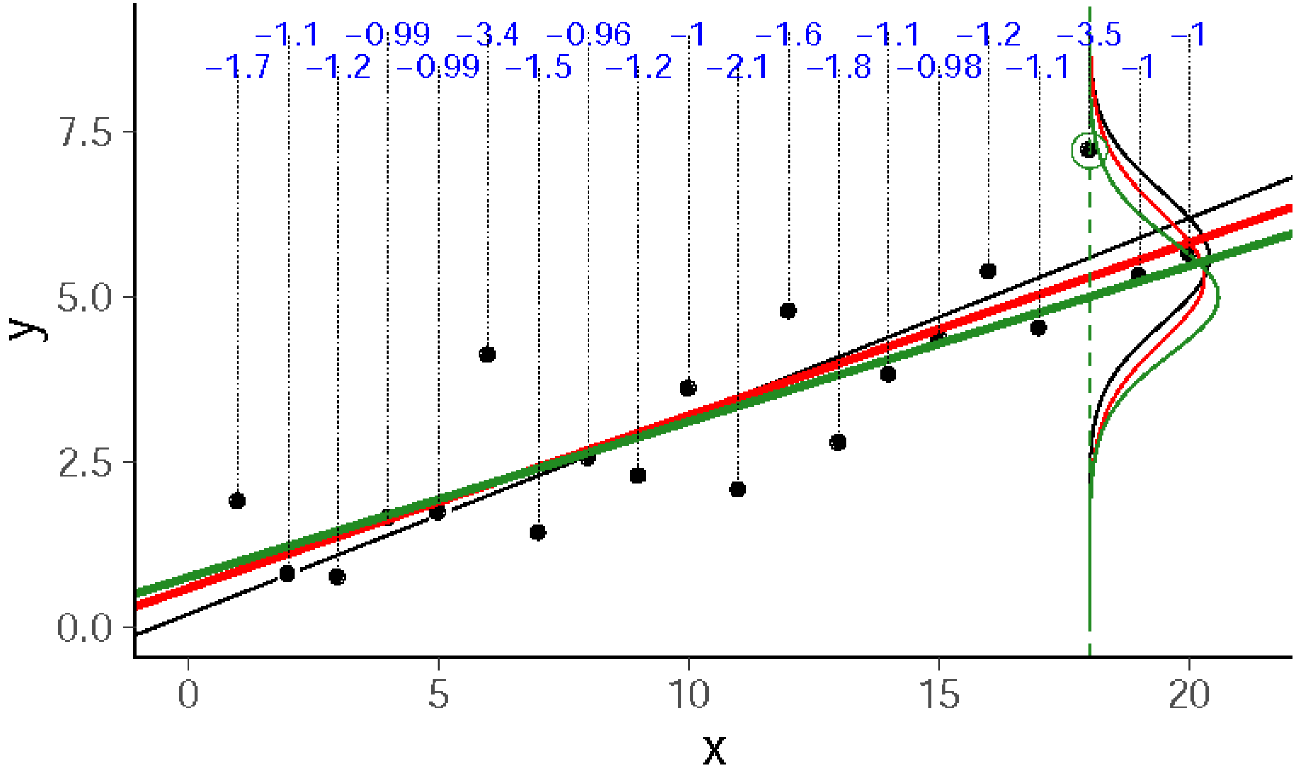
cognitive model
statistics
computing



$\log p(y_i | x_i, x_{-i}, y_{-i}), \quad i = 1, \ldots, 20$

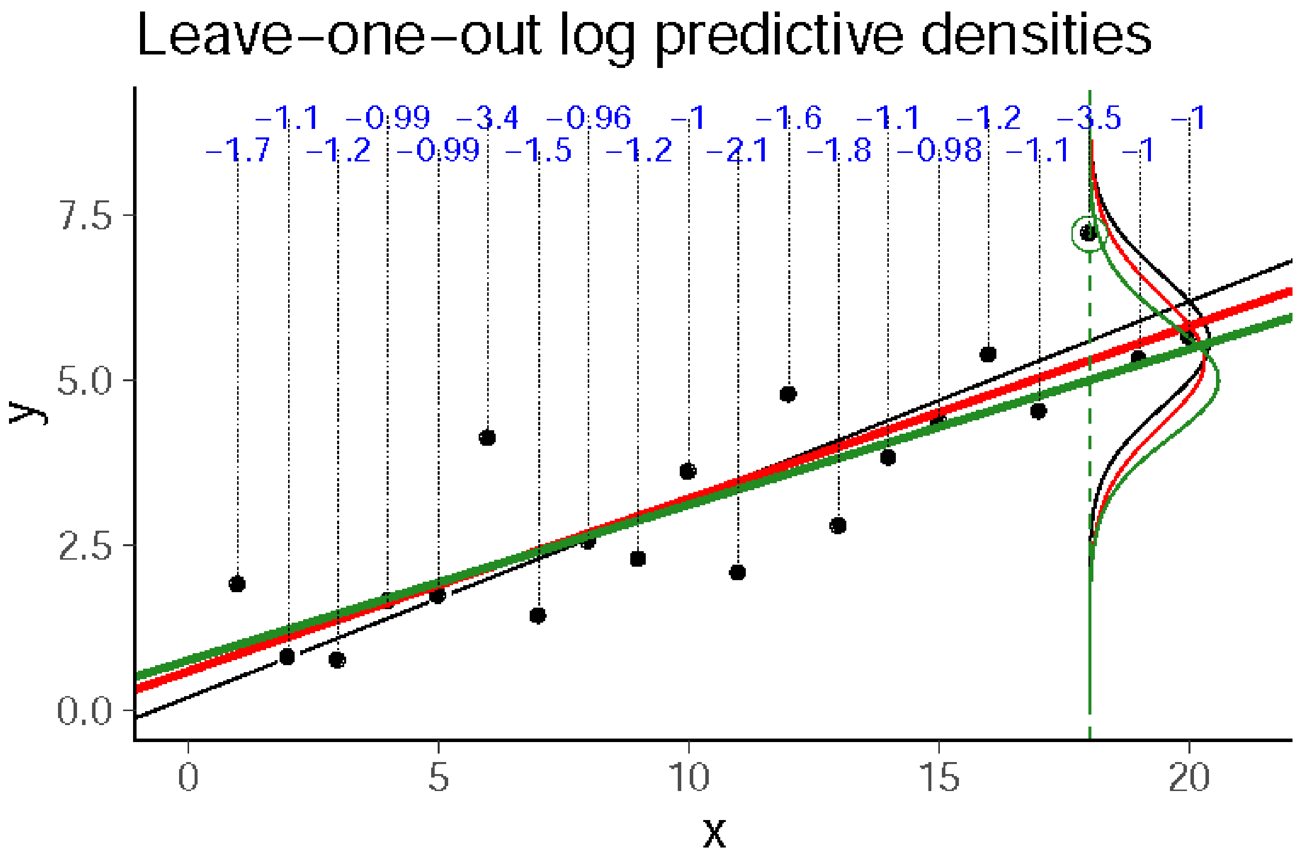
Understand model prediction

Leave−one−out log predictive densities



$\sum_{i=1}^{20} \log p(y_i | x_i, x_{-i}, y_{-i}) \approx -29.5$

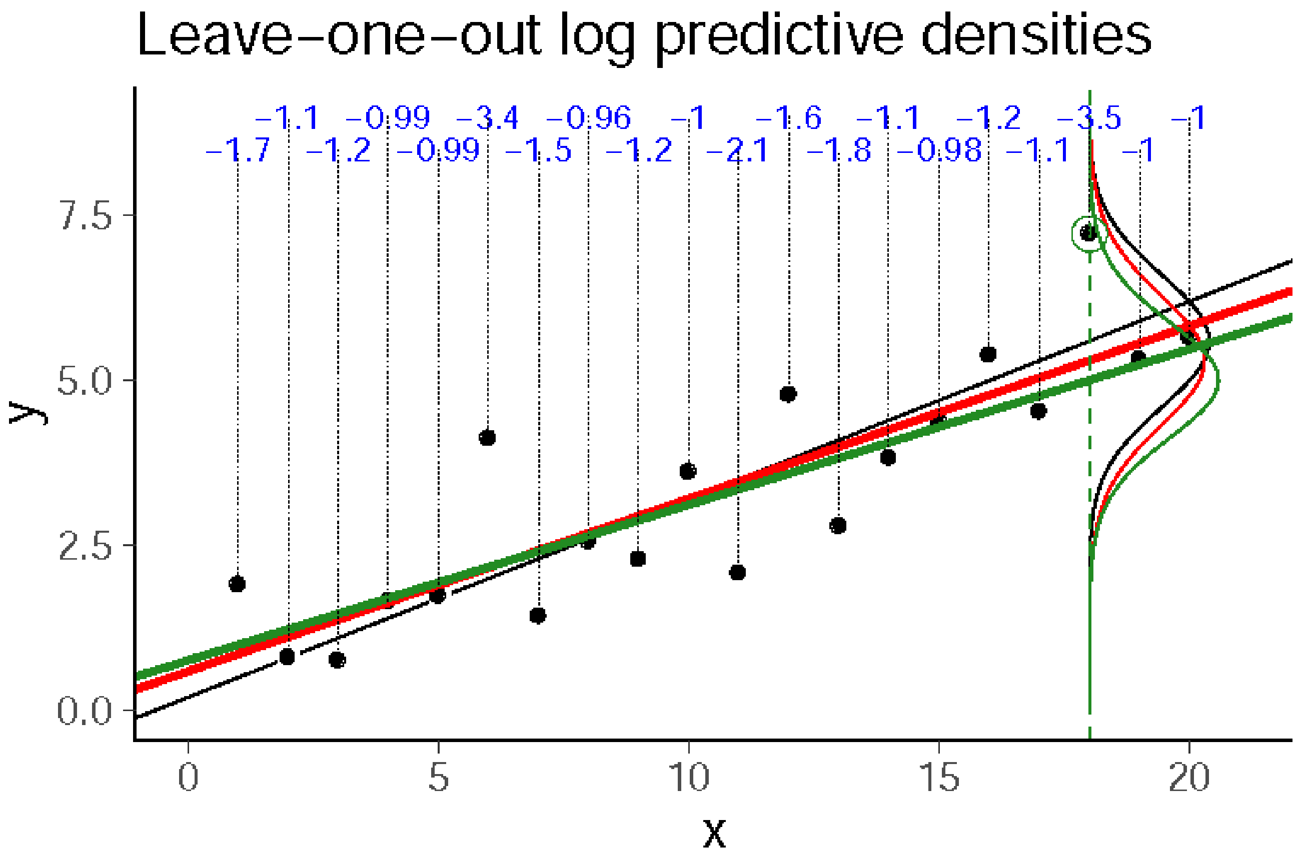
Understand model prediction



$\text{elpd\_loo} = \sum_{i=1}^{20} \log p(y_i | x_i, x_{-i}, y_{-i}) \approx -29.5$

unbiased estimate of log posterior pred. density for new data

Understand model prediction



$\text{elpd\_loo} = \sum_{i=1}^{20} \log p(y_i | x_i, x_{-i}, y_{-i}) \approx -29.5$

$\text{lpd} = \sum_{i=1}^{20} \log p(y_i | x_i, x, y) \approx -26.8$

$\text{p\_loo} = \text{lpd} - \text{elpd\_loo} \approx 2.7$

Compute WAIC from Likelihood

$$\mathrm{WAIC} = -2\,\widehat{\mathrm{elpd}}_{\mathrm{waic}}$$

expected log pointwise predictive density

$$\widehat{\mathrm{elpd}}_{\mathrm{waic}} = \widehat{\mathrm{lpd}} - \widehat{p}_{\mathrm{waic}}$$

$$\begin{aligned}\widehat{\mathrm{lpd}} &= \text{computed log pointwise predictive density}\\ &= \sum_{i=1}^{n} \log\left(\frac{1}{S}\sum_{s=1}^{S} p(y_i|\theta^s)\right).\end{aligned}$$

estimated effective number of parameters

$$\widehat{p}_{\mathrm{waic}} = \sum_{i=1}^{n} V_{s=1}^{S}\left(\log p(y_i|\theta^s)\right)$$

```
lpd <- log(colMeans(exp(log_lik)))
```

```
p_waic <- colVars(log_lik)
```

Vehtari & Gelman (2014)

*IC comparisons

		No pooling ($\tau = \infty$)	Complete pooling ($\tau = 0$)	Hierarchical model (τ estimated)
AIC	$-2\,\text{lpd} = -2\log p(y\|\hat{\theta}_{\text{mle}})$	54.6	59.4	
	k	8.0	1.0	
	$\text{AIC} = -2\,\widehat{\text{elpd}}_{\text{AIC}}$	70.6	61.4	
DIC	$-2\,\text{lpd} = -2\log p(y\|\hat{\theta}_{\text{Bayes}})$	54.6	59.4	57.4
	p_{DIC}	8.0	1.0	2.8
	$\text{DIC} = -2\,\widehat{\text{elpd}}_{\text{DIC}}$	70.6	61.4	63.0
WAIC	$-2\,\text{lppd} = -2\sum_i \log p_{\text{post}}(y_i)$	60.2	59.8	59.2
	$p_{\text{WAIC}\,1}$	2.5	0.6	1.0
	$p_{\text{WAIC}\,2}$	4.0	0.7	1.3
	$\text{WAIC} = -2\,\widehat{\text{elppd}}_{\text{WAIC}\,2}$	68.2	61.2	61.8
LOO-CV	$-2\,\text{lppd}$		59.8	59.2
	$p_{\text{loo-cv}}$		0.5	1.8
	$-2\,\text{lppd}_{\text{loo-cv}}$		60.8	62.8

Gelman et al. (2013)

Recording the Log-Likelihood in Stan


```
generated quantities {
  ...
  real log_lik[nSubjects];
  ...

  { # local section, this saves time and space
    for (s in 1:nSubjects) {
      vector[2] v;
      real pe;

      log_lik[s] = 0;
      v = initV;

      for (t in 1:nTrials) {
        log_lik[s] = log_lik[s] + categorical_logit_lpmf(choice[s,t] | tau[s] * v);

        pe = reward[s,t] - v[choice[s,t]];
        v[choice[s,t]] = v[choice[s,t]] + lr[s] * pe;
```

The {loo} Package

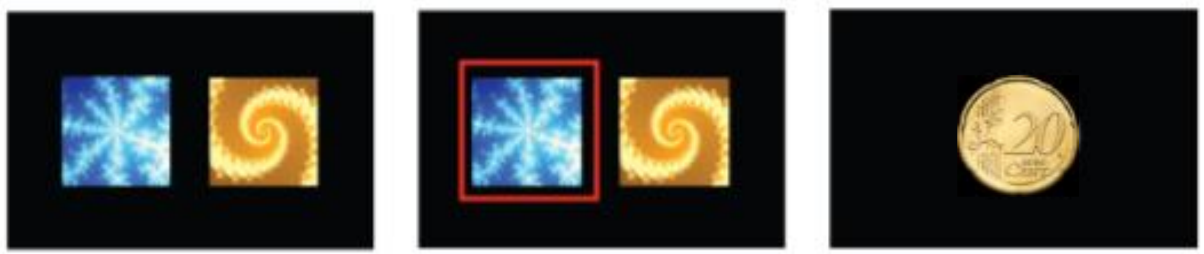

```
> library(loo)
> LL1   <- extract_log_lik(stanfit)
> loo1  <- loo(LL1)    # PSIS leave-one-out
> waic1 <- waic(LL1)   # WAIC
```

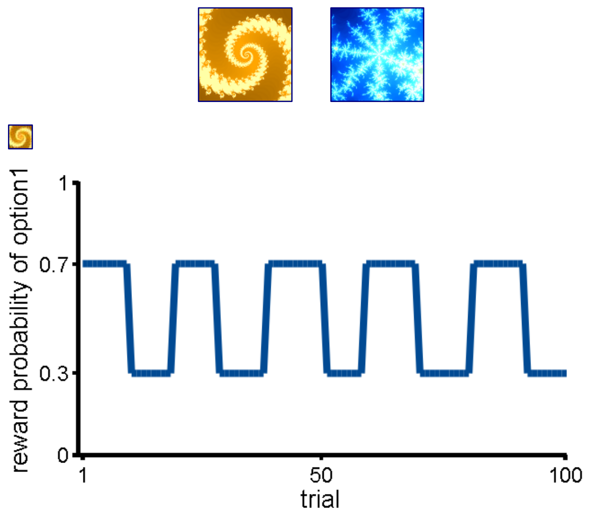
Pareto Smoothed Importance Sampling

```
Computed from 4000 by 20 log-likelihood matrix

          Estimate  SE
elpd_loo     -29.5 3.3
p_loo          2.7 1.0
looic         58.9 6.7
```

Vehtari et al. (2015)

Reversal Learning Task





fictitious RL model

Value update:

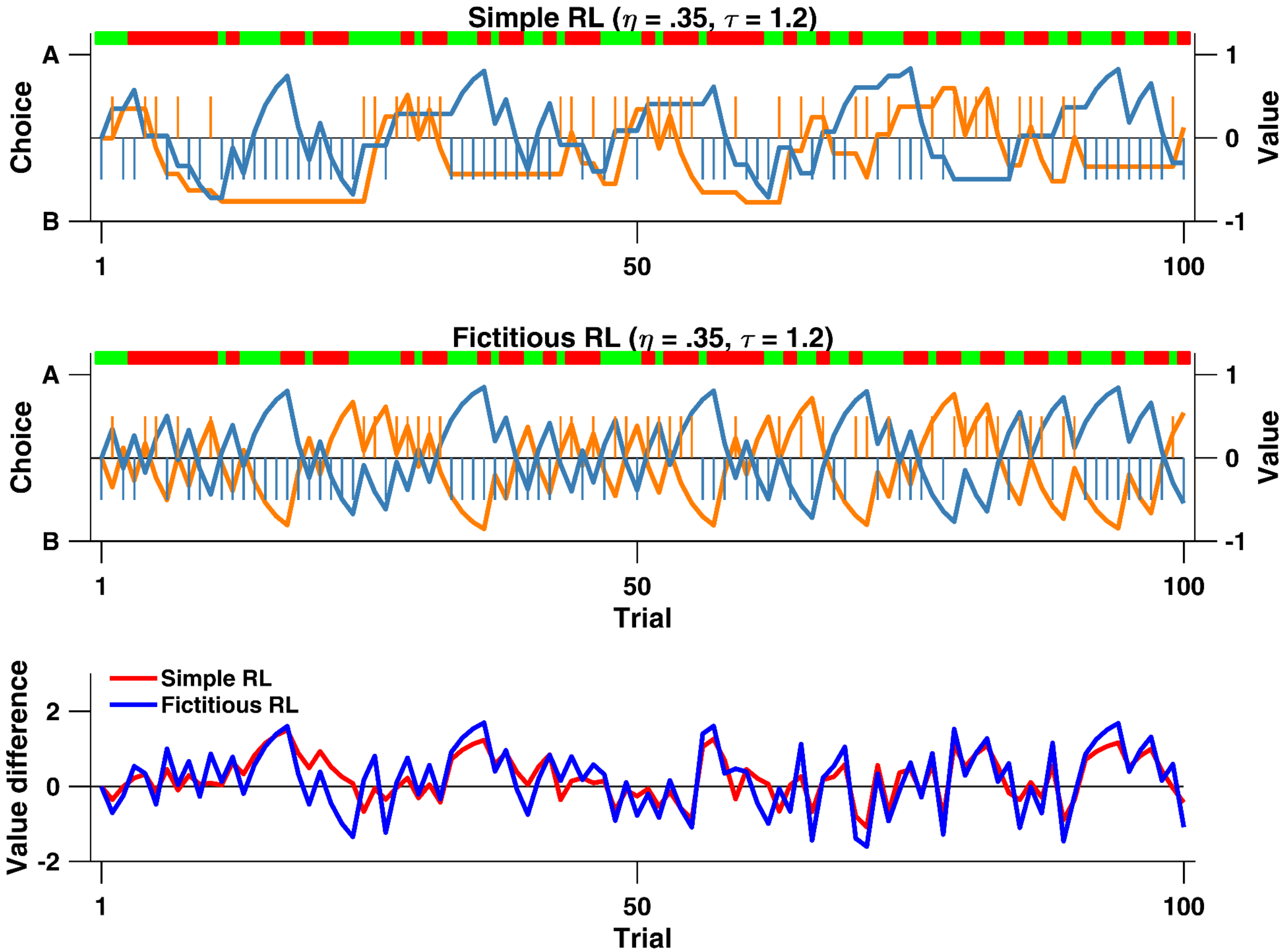
$$V_{t+1}^{c} = V_t^{c} + lr * PE$$
$$V_{t+1}^{nc} = V_t^{nc} + lr * PEnc$$

Prediction error:

$$PE = R_t - V_t^{c}$$
$$PEnc = -R_t - V_t^{nc}$$

Hampton et al. (2007)
Gläscher et al. (2009)

More on Fictitious RL



Exercise VII


```
…/BayesCog/08.compare_models/_scripts/compare_models_main.R
```

TASK: complete the fictitious RL model (model2)

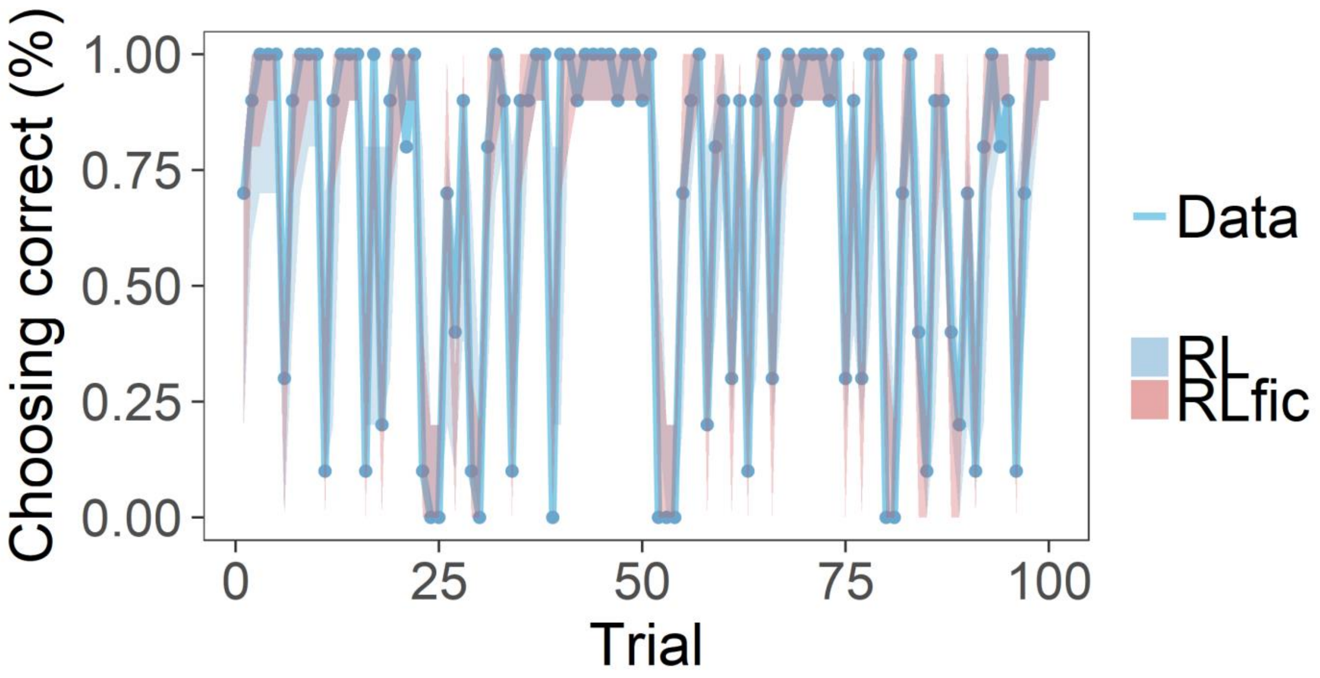
fit and compare the 2 models

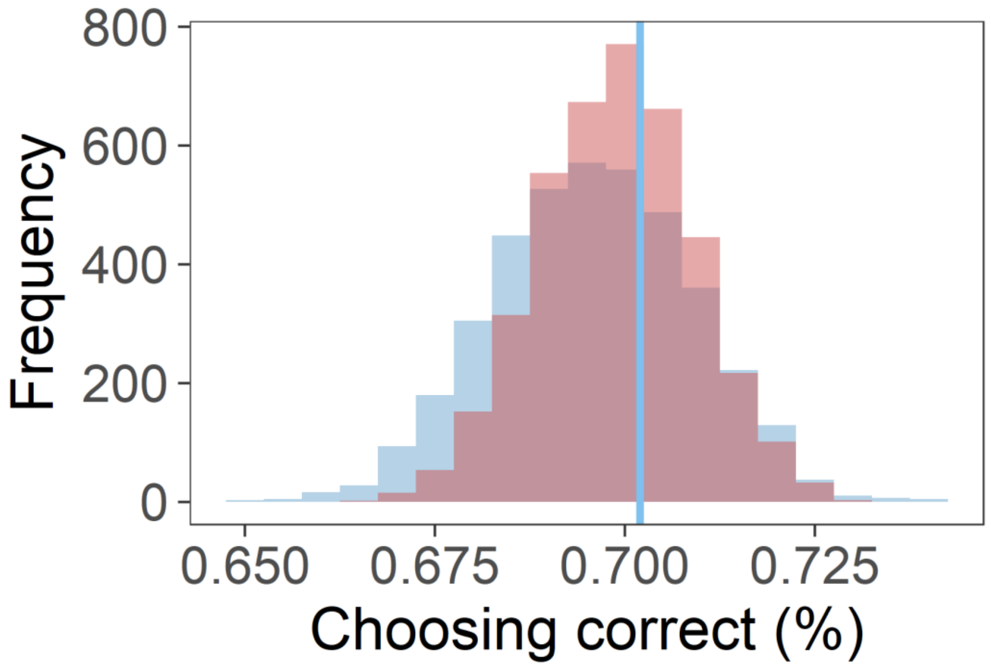
Exercise VII – output

```
> LL1 <- extract_log_lik(fit_rl1)
> ( loo1 <- loo(LL1) )
Computed from 4000 by 10 log-likelihood matrix

         Estimate   SE
elpd_loo   -389.8 15.4
p_loo         3.8  0.8
looic       779.5 30.9

> ( loo2 <- loo(LL2) )
Computed from 4000 by 10 log-likelihood matrix

         Estimate   SE
elpd_loo   -281.3 17.5
p_loo         3.4  0.5
looic       562.6 35.0
```

Posterior Predictive Check

cognitive model
statistics
computing






ANY
QUESTIONS
?

Stay tuned and
bis morgen!