Competitive Mathematics:
314-Problems Starter Pack

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Compiled in 2024

Contents

Preface

Background
I initially compiled these problems in early 2024 as my "problem bank" for Pintarian Mathletes and other teaching purposes, as well as for my own reference. My collection of problems eventually grew quite large, so I decided to organise it into this document that others could use as reference.

About
This compilation presents a variety of the **most common** types of problems encountered in competitive mathematics, intending to serve as a practical resource for problem-solving. This compilation is ideal for those preparing for math contests such as IMONST, SASMO, KMC, etc.

Disclaimer
Since only the most common types of problems are discussed, take note that this compilation barely even scratches the surface of competitive maths - hence the name "Starter Pack". Moreover, a solid mathematical background (especially in school maths) is assumed. Also, the included problems may not be the best selection, and there are large imbalances in the number of problems across subsections (depending on their original use, e.g., a half-hour class vs. a two-hour class).

Structure
This document contains four main sections by topic, which are Algebra, Combinatorics, Geometry and Number Theory. Within each section is several subsections that each target a specific type of problem. Subsections within a section are roughly sorted in order of decreasing importance (i.e. decreasing frequency of appearance in maths contests), and problems within a subsection are roughly sorted in order of increasing difficulty. Keep in mind the ordering is **very subjective** though, and even I was unsure how to order many parts. Additionally, each subsection includes a brief description (but very vague, with little to no explanation) of general strategies that might be insightful for those who are less familiar with the type of problem presented. However, some of these descriptions may completely spoil solutions, so read them with caution or treat them as hints if you find yourself stuck.

Feedback
If you notice any errors or have suggestions for improvement, please contact me at leiamayssa2007@gmail.com. Alternatively, if you know me personally, feel free to reach out through other platforms. :)

Closing Remarks
I hope this compilation proves beneficial. It is freely available for anyone to use, share, or refer to. Happy problem-solving!

Contest Abbreviations

Many problems are taken from the following contests:

AIME	American Invitational Mathematics Examination
AHSME	American High School Mathematics Examination
AMC	American Mathematics Competition
ARML	American Regions Mathematics League
CJR	Chen Jingrun's Cup
DOKA	Depth of Knowledge Assessment
IMONST	IMO National Selection Test
KMC	Kangaroo Math Competition
OMK	Olimpiad Matematik Kebangsaan
PML	Pintarian Mathlympics
SASMO	Singapore and Asian Schools Math Olympiad
SMO	Singapore Mathematical Olympiad

1 Algebra

1.1 Simple Sums

Strategy: Apply (and understand!) the formulas for sums of arithmetic and geometric series.

1. Evaluate $1+2+3+\cdots+2024$.
2. Find a general formula for $1+2+3+\cdots+n$.
3. Evaluate $2+4+6+...+2024$.
4. Evaluate $1+3+5+\cdots+99$.
5. Find a general formula for $1+3+5+\cdots+(2n-1)$.
6. (PML) Evaluate $1-2+3-4+5-6+\cdots+2023-2024$.
7. Search up (and memorise!) the general formula for
   (a) $1^2+2^2+3^2+\cdots+n^2$,
   (b) $1^3+2^3+3^3+\cdots+n^3$.
8. Evaluate $\frac{1^3+2^3+...2024^3}{1+2+\cdots+2024}$.
9. Show that $2^0+2^1+2^2+\cdots+2^n=2^{n+1}-1$.
10. Find a general formula for $a+ar+ar^2+\cdots+ar^n$.
11. Evaluate $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$.

1.2 Factorials

Strategy: Expand the factorials and cancel common factors.

12. Find the value of $\frac{2024!}{2023!}$.
13. (AMC 12) What is the value of $\frac{11!-10!}{9!}$?
14. Find the positive integer n such that $6!7!=n!$.
15. Evaluate
$$\left(\frac{10!+11!}{12!}\right)\left(\frac{23!}{21!+22!}\right).$$
16. (PML) Given that $\frac{(n+24)!}{(n+22)!}=2550$, find n.

1.3 Comparing Powers

Strategy: Equate the bases if possible. Otherwise, find an inequality that allows you to compare the values. For example, we find that $2^2 > 3$, which implies $2^{2024} > 3^{1012}$.

17. (PML) Determine which is larger: 2^{50} or 4^{23}.
18. (PML) Determine which is larger: 2^{3034} or 3^{2023}.
19. (AMC 8) Arrange the numbers 10^8, 5^{12} and 2^{24} in ascending order.
20. Arrange the numbers $2^{11}, 4^5, 8^4, 17^3$ and 31^2 in ascending order.
21. Among the numbers $1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}$ and $\sqrt[6]{6}$, which is the largest?

1.4 Big Numbers

Strategy: Express numbers in terms of variables and simplify your expression. Factorisations usually come in handy.

22. Evaluate $20242024^2 - (20242023)(20242025)$.
23. Evaluate $\sqrt{111113^2 - 888888}$.
24. (CJR) Find the value of $\sqrt{499^2 + 999}$.
25. (CJR) If $x^3 = 2015 \times 2017 \times 2019 + 4 \times 2017$, find x.
26. Evaluate $\sqrt[3]{2024^3 + 3(2024)^2 + 3(2024) + 1}$.
27. (OMK) Evaluate
$$\frac{66666666 \times 44444445 - 33333333 \times 88888888}{6666 \times 4445 - 3333 \times 8888}.$$
28. Evaluate
$$\frac{240^3 + 420^3}{240^3 - 180^3}.$$
29. Evaluate $\sqrt{(47)(49)(51)(53) + 16}$.
30. (SMO) Evaluate
$$\frac{(2020^2 - 20100)(20100^2 - 100^2)(2000^2 + 20100)}{2010^6 - 10^6}.$$

1.5 Infinite Nested Radicals & Fractions

> **Strategy:** Call the entire expression X, then substitute X back into the original expression.

31. Solve $\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}=3$.
32. Evaluate $10\sqrt{10\sqrt{10\sqrt{\ldots}}}$.
33. Evaluate $\sqrt{5+4\sqrt{5+4\sqrt{5+4\sqrt{\ldots}}}}$.
34. Evaluate $\frac{4}{3+\frac{4}{3+\frac{4}{3+\ldots}}}$.
35. Evaluate $1+\frac{1}{1+\frac{2}{1+\frac{1}{1+\frac{2}{1+\frac{1}{\ldots}}}}}$.
36. Evaluate $\sqrt{\frac{18}{\sqrt{\frac{18}{\sqrt{\frac{18}{\ldots}-1}}-1}}}-1$.

1.6 Rates

> **Strategy:** Use the formula Work = Rate × Time.

37. Alia can eat 30 candies in 6 minutes. How many minutes will it take her to eat 75 candies?
38. (PML) Ainul takes 5 days to solve a problem while Irfan takes 20 days to solve a problem. If they work together, how long would it take for them to solve 100 problems?
39. (CJR) A task would take 60 days to complete by 21 workers. How many days does it take to complete by 28 workers?
40. It takes 5 days for 5 workers to build 5 toys. How long does it take for 6 workers to build 6 toys?
41. (CJR) After an examination, Ms Zhang can ask her three assistants Aihui, Bilan and Cenyue to do the grading. If only Aihui and Bilan are asked to grade, it takes 20 hours to complete. If only Aihui and Cenyue are asked to grade, it takes 30 hours to complete. If only Bilan and Cenyue are asked to grade, it takes 15 hours to complete. How many hours does it take for Aihui to complete the grading alone?

1.7 Conjugate

Strategy: Eliminate surds by multiplying with its conjugate.

42. (CJR) Find the value of

$$\frac{3\sqrt{3}+5}{3\sqrt{3}-5}+\frac{3\sqrt{3}-5}{3\sqrt{3}+5}.$$

43. Given that

$$x=\frac{8}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)},$$

find the value of $(x+2)^8$.

44. Find the largest integer less than $(\sqrt{10}+3)^3$.

45. Find the largest integer less than $(\sqrt{3}+1)^6$.

46. (SMO) What is the 521st digit to the right of the decimal point in the decimal representation of

$$(1+\sqrt{2})^{2022}?$$

1.8 A Taste of Algebraic Manipulations

Strategy: Don't solve for the variables! Instead, try to directly find the value of the expression you want, by adding/subtracting/multiplying/squaring equations.

47. (CJR) If $143x-77y=451$, find the value of $299x-161y$.

48. Given that $ab=4$, $cd=6$ and $ad=8$, find the value of bc.

49. Suppose

$$\frac{x}{a-b}=\frac{y}{b-c}=\frac{z}{c-a}.$$

Find the value of $x+y+z$.

50. (PML) Given that $xy=2$ and $x+y=4$, find x^2+y^2.

51. (CJR) If a and b are positive real numbers such that $a+b=200$ and $ab=8100$, find $\sqrt{a}+\sqrt{b}$.

52. If $x^2 + y^2 + z^2 = 24$ and $x + y + z = 6$, find $xy + yz + zx$.

53. (CJR) Let x, y be two non-zero real numbers such that $x \neq y$. If $x + \frac{9}{x} = y + \frac{9}{y}$, find xy.

54. Given that $\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1$, find

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c}.$$

55. (CJR) Given that $\frac{x}{x+y+z} = \frac{1}{3}$ and $\frac{y}{x+y+z} = \frac{1}{4}$, find $\frac{24x+36y+48z}{x+y+z}$.

56. (SMO) If $x+\sqrt{xy}+y = 9$ and $x^2+xy+y^2 = 27$, find the value of $x-\sqrt{xy}+y$.

1.9 Trickier Algebraic Manipulations

Strategy: Same strategy as before.

57. (SMO) Suppose $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \sqrt{2}$ and $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0$. Find

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}.$$

58. (AMC 12) If x, y and z are positive numbers satisfying

$$x + \frac{1}{y} = 4,\ y + \frac{1}{z} = 1 \text{ and } z + \frac{1}{x} = \frac{7}{3},$$

then what is the value of xyz?

59. (CJR) Given that a, b, c are real numbers and

$$(b - 196a)^2 - 128(b - 14c)(c - 14a) = 0,$$

find the largest possible value of $\frac{b-14c}{c-14a}$.

60. (CJR) Given that x and y are real numbers such that $x > y$, $x + y = 14$ and $xy = 12$, find the value of $x^2 + \frac{168}{x}$.

61. (SMO) Let $m \neq n$ be two real numbers such that $m^2 = n + 2$ and $n^2 = m + 2$. Find the value of $4mn - m^3 - n^3$.

62. (SMO) Suppose that $a + x^2 = 2006$, $b + x^2 = 2007$ and $c + x^2 = 2008$ and $abc = 3$. Find the value of
$$\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}.$$

63. Given that $6(a + b + c) = 3(a^2 + b^2 + c^2) = 2(a^3 + b^3 + c^3) = 6$, find the value of abc.

64. (SMO) If $\sqrt{x^2 + 7x - 4} + \sqrt{x^2 - x + 4} = x - 1$, find the value of $3x^2 + 14x$.

1.10 $x \pm 1/x$

Strategy: This is a super common type of problem that uses algebraic manipulations. Apply the same strategy, i.e. adding/subtracting/multiplying/raising powers.

65. Given that $x + \frac{1}{x} = 5$, find $x^2 + \frac{1}{x^2}$.

66. Given that $x + \frac{1}{x} = 5$, find $x^3 + \frac{1}{x^3}$.

67. Given that $x + \frac{1}{x} = 5$, find $x^4 + \frac{1}{x^4}$.

68. Given that $4x^2 + \frac{1}{x^2} = 2$, find $8x^3 + \frac{1}{x^3}$.

69. x satisfies the equations $x^2 - 7x + 1 = 0$ and $x^4 - kx^2 + 1 = 0$. Find k.

70. Given that $x - \frac{1}{x} = 4$, find $x^2 + \frac{1}{x^2}$.

71. (PML) Given that $x - \frac{1}{x} = 4$, find $x^3 - \frac{1}{x^3}$.

72. (AMC 10) The real number x satisfies the equation $x + \frac{1}{x} = \sqrt{5}$. What is the value of $x^{11} - 7x^7 + x^3$?

73. Suppose $x + \frac{1}{x} = \sqrt{2}$. Find the value of $x^{37} + \frac{1}{x^{37}}$.

74. Suppose $x + \frac{1}{x} = \sqrt{3}$. Find the value of $x^{2023} + \frac{1}{x^{2023}}$.

75. Suppose $x + \frac{1}{x} = \sqrt{2 - \sqrt{2}}$. Find the value of $x^{2023} + \frac{1}{x^{2023}}$.

1.11 Surds in Surds

Strategy: If you encounter $\sqrt{a+2\sqrt{b}}$ or similar forms in a math competition, it can usually be simplified as

$$\sqrt{a+2\sqrt{b}} = \sqrt{(c+d)+2\sqrt{cd}} = \sqrt{(\sqrt{c}+\sqrt{d})^2} = \sqrt{c}+\sqrt{d}.$$

76. Simplify $\sqrt{7+2\sqrt{12}}$.
77. Simplify $\sqrt{7-2\sqrt{12}}$.
78. Simplify $\sqrt{17+12\sqrt{2}}+\sqrt{17-12\sqrt{2}}$.
79. Find the positive integers a and b such that
$$\sqrt{\sqrt{2024}+\sqrt{2025}} = \sqrt{a}+\sqrt{b}.$$

1.12 Surds in Cube Roots

Strategy: Let $\sqrt[3]{a} = x$ and $\sqrt[3]{b} = y$. Note that

$$(x+y)^3 = x^3+y^3+3xy(x+y).$$

Calculate for x^3+y^3 and xy, then solve the cubic to find $x+y$.

80. Evaluate $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$.
81. Evaluate $\sqrt[3]{20+14\sqrt{2}}+\sqrt[3]{20-14\sqrt{2}}$.

1.13 Coefficients

Strategy: Suppose

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

where a_i's represent the coefficients. Think about what values of x we could substitute to yield useful information.

82. Find the sum of coefficients of the polynomial
$$P(x) = (x+1)^5.$$

83. Find the sum of coefficients of the polynomial
$$P(x) = (x-2)^{2024}(x-3)^4.$$

84. Find the sum of coefficients of the polynomial
$$P(x) = (2024x-2023)(2023x-2022)\dots(3x-2)(2x-1).$$

85. Find the constant term in the polynomial
$$P(x) = (72x^{48} - 64x^{20} + 3)^5.$$

86. Find the constant term in the polynomial
$$P(x) = (6x-5)(5x-4)(4x-3)(3x-2).$$

87. Suppose
$$(x^4 + x^3 + 1)^{20} = a_{80}x^{80} + a_{79}x^{79} + a_{78}x^{78} + \cdots + a_2x^2 + a_1x + a_0.$$
Find the value of $a_0 + a_2 + \cdots + a_{80}$.

1.14 Trivial Inequality

Strategy: Recall that $x^2 \geq 0$ for any real number x, with equality if and only if $x = 0$.

88. Find x if $(x-2024)^2 = 0$.

89. Find $x + y + z$ if
$$22(x-22)^{22} + 24(y-24)^{24} + 26(y-26)^{26} = 0.$$

90. By completing the square, find the minimum value of
$$x^2 + 24x + 2024.$$

91. (AMC 12) What is the least possible value of
$$(xy-1)^2 + (x+y)^2$$
for real numbers x and y.

92. Find all ordered pairs of real numbers (x, y) such that
$$(4x^2 + 4x + 3)(y^2 - 6y + 13) = 8.$$

93. Find all solutions x, y, z of the equation
$$x^2 + 5y^2 + 10z^2 = 4xy + 6yx + 2z - 1.$$

1.15 Telescoping Sums & Products

Strategy: Figure out a way to keep cancelling stuff. Usually, all terms in the middle can be cancelled, leaving the first and last few terms.

94. Simplify $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times ... \times \frac{2022}{2023} \times \frac{2023}{2024}$.
95. Evaluate $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\ldots\left(1+\frac{1}{2022}\right)\left(1+\frac{1}{2023}\right)$.
96. (AMC 8) What is the value of the product
$$\left(\frac{1\times 3}{2\times 2}\right)\left(\frac{2\times 4}{3\times 3}\right)\left(\frac{3\times 5}{4\times 4}\right)\cdots\left(\frac{97\times 99}{98\times 98}\right)\left(\frac{98\times 100}{99\times 99}\right)?$$
97. Evaluate $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots$.
98. Evaluate $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots+\frac{1}{2023\times 2024}$.
99. Evaluate $\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\cdots+\frac{1}{2021\times 2023}$.
100. Evaluate $\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\cdots+\frac{1}{\sqrt{99}+\sqrt{100}}$.
101. Evaluate $1(1!)+2(2!)+3(3!)+\cdots+2024(2024!)$.
102. Evaluate $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{2024}{2025!}$.
103. (PML) Evaluate $\frac{1}{8}+\frac{1}{24}+\frac{1}{48}+\cdots+\frac{1}{1680}+\frac{1}{1848}+\frac{1}{2024}$.
104. Evaluate $\frac{2^3-1}{2^3+1}\times\frac{3^3-1}{3^3+1}\times\frac{4^3-1}{4^3+1}\times\ldots$.
105. Evaluate $\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\cdots+\sqrt{1+\frac{1}{2023^2}+\frac{1}{2024^2}}$.
106. Evaluate $\frac{1}{1^4+1^2+1}+\frac{2}{2^4+2^2+1}+\frac{3}{3^4+3^2+1}+\ldots$.
107. (SMO) Find the value of
$$\frac{1}{3+1}+\frac{2}{3^2+1}+\frac{4}{3^4+1}+\frac{8}{3^8+1}+\cdots+\frac{2^{2006}}{3^{2^{2006}}+1}.$$
108. (SMO) Calculate the following sum:
$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\cdots+\frac{10}{2^{10}}.$$

2 Combinatorics

2.1 Basic Counting

Strategy: Apply your knowledge of Combinations and Permutations (which you hopefully learnt in high school).

109. How many ways can you line up 6 people in a row?

110. How many ways can you seat 6 people around a circular table, if rotations count as one?

111. (PML) There are 24 Mathletes. Find the number of ways to elect 1 President and 2 Vice Presidents. (A Mathlete cannot be both a President and Vice President.)

112. Find the number of subsets of $\{1, 2, 3, 4, 5, 6, 7\}$ which contain exactly 3 elements.

113. Find the number of subsets (of all sizes) of $\{1, 2, 3, 4, 5, 6\}$.

114. How many ways can you seat 7 girls and 3 boys in a row such that no boy sits next to another boy?

115. (PML) Rizan, Chuah and 5 other Mathletes wish to sit at a round table. If Chuah insists he sits next to Rizan, how many ways can they be seated assuming that rotations do not count as distinct orientations?

2.2 Probabilities

Strategy: Probability = No. of desired cases / Total no. of cases.

116. (PML) Anas flips a fair coin 3 times. What is the probability that the coin lands on the same side all 3 times?

117. Suppose you roll two fair 6-sided die to obtain two numbers. What is the probability that the sum of the two numbers is prime?

118. (AMC 8) A box contains five cards, numbered 1, 2, 3, 4, and 5. Three cards are selected randomly without replacement from the box. What is the probability that 4 is the largest value selected?

2.3 Word Rearrangements

Strategy: Calculate permutations, and don't forget to account for repeated letters and extra conditions.

119. How many ways can you rearrange the letters of the word MATH?

120. (PML) How many ways can you rearrange the letters of the word MATHLETE?

121. How many ways can you rearrange the letters of the word PINTAR if the letters A and I must be next to each other?

122. How many ways can you rearrange the letters of the word MATHLETE if same letters cannot be the consecutive? (e.g. MATHL<u>EE</u>T is an invalid arrangement since two E's are next to each other)

123. How many ways are there to misspell the word MISSPELLED?

2.4 Casework

Strategy: Split into cases, and solve each case separately.

124. (AIME) How many even integers between 4000 and 7000 have four different digits?

125. (SASMO) How many 10-digit numbers only composed of 1, 2, and 3 exist, in which any two neighbouring digits differ by 1?

126. How many ways are there to place a black king and a white king on an 8×8 chessboard so that they do not attack each other?

127. (AIME) How many 4-digit numbers are there such that it begins with 1 and has exactly two identical digits? For example, 3445 is valid but 3344 is not.

2.5 Complementary Counting

Strategy: To count the cases you want, count the cases you **don't** want, then subtract that from the total number of possible cases. The phrase "at least" is usually an indicator for complementary counting.

128. Let $S = \{1, 2, 3, 4, 5\}$. How many subsets of S contain at least one prime number?

129. How many positive integers less than 240 are not divisible by 5?

130. How many four-digit positive integers contain at least one even digit?

131. (AMC 10) How many four-digit positive integers have at least one digit that is a 2 or a 3?

132. Three fair 6-sided die are rolled. Find the probability that at least one of the numbers rolled is greater than 3.

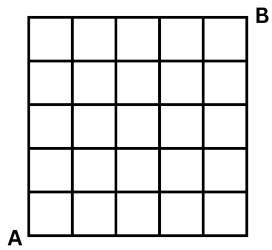
133. Chuah writes the numbers 1 to 10 in a row (in that order). Then, he wishes to colour each number either red, blue or yellow. In how many ways can he do so if at least one pair of consecutive numbers must have the same colour?

134. (AIME) Find the number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are subsets of neither $\{1, 2, 3, 4, 5\}$ nor $\{4, 5, 6, 7, 8\}$.

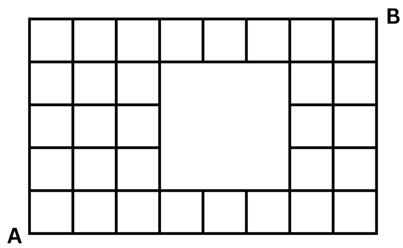
2.6 Shortest Paths

Strategy: One possible method is manually counting the number of paths to each point. Alternatively, notice that a shortest path consists of steps in one of two directions. For example, suppose we are trying to get from $(0, 0)$ to $(3, 2)$, so the direction of each step is up or right, which we can denote by U and R respectively. Some examples of shortests paths for this case are URRUR, RURUR or UURRR. Now think about how this problem has actually just transformed into a word-arrangement problem!

135. Based on the figure below, an ant has travelled from point A to point B by only walking on the gridlines of the grid. Moreover, the ant took a path with minimum length. How many possible paths could the ant have taken?

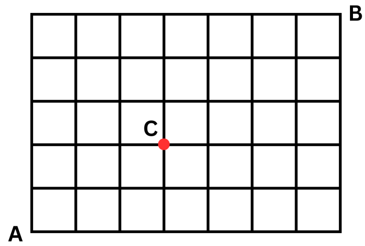


136. Based on the figure below, an ant wants to travel from point A to point B by only walking on the gridlines of the grid. However, several of the gridlines have been removed. How many shortest paths exist?



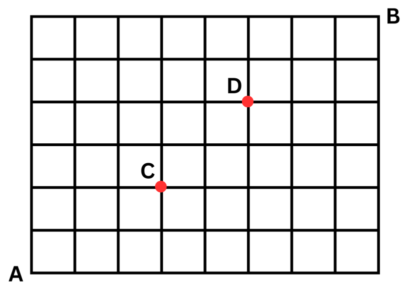
137. An ant is walking in a coordinate plane and has to get from $(3, 5)$ to $(8, 9)$. If the ant can only move one unit up, down, left or right, how many paths with minimum length can the ant take?

138. Based on the figure below, an ant wants to travel from point A to point B by only walking on the gridlines of the grid. How many shortest paths exist if the ant must visit point C?



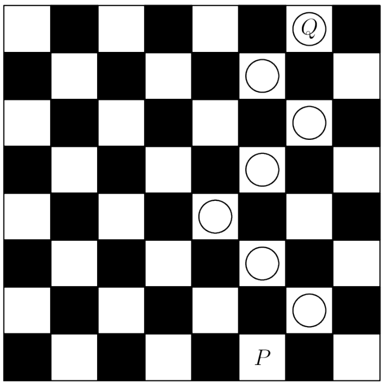
139. Same problem as above, but the ant must **not** visit point C.

140. Based on the figure below, an ant wants to travel from point A to point B by only walking on the gridlines of the grid. How many shortest paths exist if the ant must visit either point C or point D?

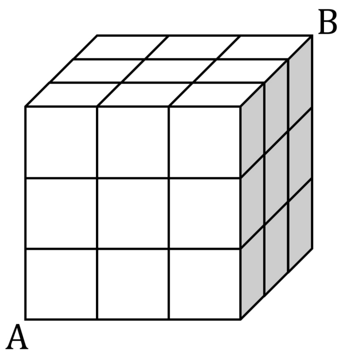


141. Same problem as above, but the ant must visit point C and must **not** visit point D.

142. (AMC 8) A game board consists of 64 squares that alternate in color between black and white. The figure below shows square P in the bottom row and square Q in the top row. A marker is placed at P. A step consists of moving the marker onto one of the adjoining white squares in the row above. How many 7-step paths are there from P to Q? (The figure shows a sample path.)



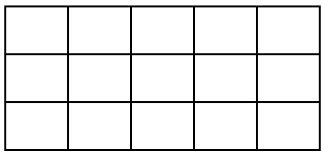
143. (PML) Obie wants to travel from A to B by only walking on the grid-lines of the 3×3 cube. Find the number of shortest paths he can walk. (Note: Obie can travel on the left, back and bottom surfaces of the cube.)



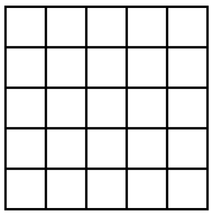
2.7 Counting Rectangles

> **Strategy:** To count rectangles that are not specifically squares, observe that every rectangle is defined by a pair of horizontal lines and a pair of vertical lines. Think about how we can count these pairs of lines.

144. How many rectangles are in the 3×5 grid below?

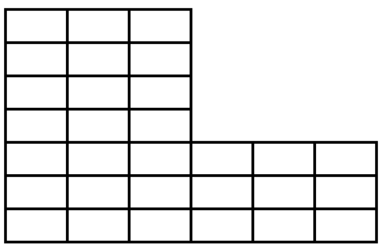


145. How many squares (of all sizes) can be found in a 5×5 grid of squares?

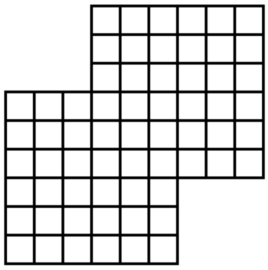


146. How many rectangles, which are not squares, can be found in a 4×6 grid of squares?

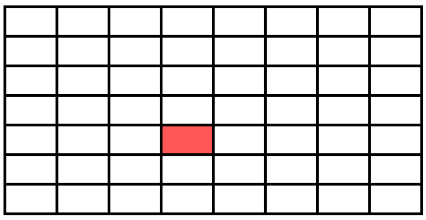
147. How many rectangles can be found in the figure below?



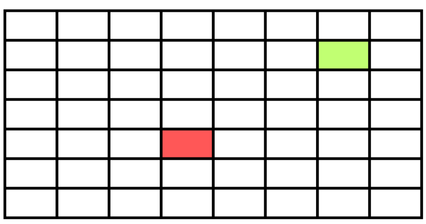
148. Given that each box is a square in the figure below, how many squares (of all sizes) can be found?



149. In the 7×8 grid below, find the number of rectangles that can be found which contain the small red rectangle.



150. In the 7×8 grid below, find the number of rectangles that can be found which contain either the small red rectangle, but **not** the small green rectangle.



2.8 Principle of Inclusion and Exclusion

Strategy: Visualise using a Venn diagram, and try filling up the numbers for each region of the Venn diagram.

151. There are a total of 40 Mathletes. 25 Mathletes like geometry, 20 Mathletes like algebra, and 12 Mathletes like both geometry and algebra. How many Mathletes do not like both geometry and algebra?

152. How many 5-digit numbers start or end with an even digit?

153. How many positive integers less than 100 are divisible by 3 or 5?

154. (AMC 10) There are 20 students participating in an after-school program offering classes in yoga, bridge, and painting. Each student must take at least one of these three classes, but may take two or all three. There are 10 students taking yoga, 13 taking bridge, and 9 taking painting. There are 9 students taking at least two classes. How many students are taking all three classes?

155. How many positive integers less than 300 are divisible by 3, 4 or 5?

156. How many positive integers less than or equal to 2024 are divisible by neither 4, 6, nor 9?

157. How many positive integers less than 180 are relatively prime to 180?

158. (SMO) 4 black balls, 4 white balls and 2 red balls are arranged in a row. Find the total number of ways this can be done if all the balls of the same colour do not appear in a consecutive block.

159. (CJR) A school has four clubs whose members are students in this school. Each club has 99 members. Every two clubs have 33 common members. Every three clubs have 11 common members. There is exactly one student that joins all four clubs. At least how many students does this school have?

2.9 Stars and Bars

Strategy: Simplify the conditions of the problem and apply the Stars and Bars technique/formula.

160. Find the number of ordered quadruples (a, b, c, d) such that

$$a + b + c + d = 12$$

where a, b, c, d are:

(a) non-negative integers
(b) positive integers
(c) odd positive integers
(d) even positive integers

161. (AMC 10) Pat is to select six cookies from a tray containing only chocolate chip, oatmeal, and peanut butter cookies. There are at least six of each of these three kinds of cookies on the tray. How many different assortments of six cookies can be selected?

162. Find the number of ordered triples of integers (a, b, c) such that

$$a + b + c = 24$$

if $a \geq 0$, $b \geq 2$ and $c \geq 4$.

163. Find the number of ways to distribute 15 stickers among 4 Mathletes if each Mathlete insists they receive at least 2 stickers.

164. Find the number of 4-digit integers whose digit sum is 8.

165. Three fair 6-sided die are rolled. Find the probability that the sum of the top faces of the three die is 6.

166. Find the number of ordered quadruples of non-negative integers (a, b, c, d) such that $0 \leq a + b + c + d \leq 24$.

167. How many terms are in the expansion of $(a + b + c)^{24}$ after simplifying?

168. (PML) Find the number of cubic polynomials $P(x)$ with non-negative integer coefficients such that $P(1) = 22$.

169. Five fair 6-sided die are rolled. Find the probability that the sum of the top faces of the five die is 24.

170. (AMC 12) The expression

$$(x+y+z)^{2006}-(x-y-z)^{2006}$$

is simplified by expanding it and combining like terms. How many terms are in the simplified expression?

171. (PML) Find the number of ordered triples of positive integers (a, b, c) such that $abc = 2024^4$.

3 Geometry

3.1 Pythagoras!

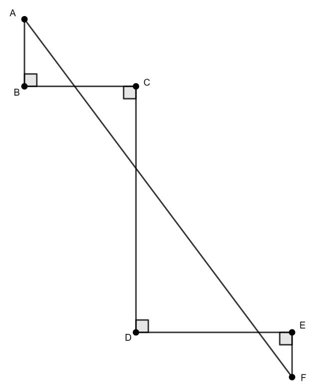
Strategy: Apply Pythagoras (usually) if there are right angles and you're trying to find a length. Sometimes, you'll need to "rearrange" line segments in order to obtain a right-angled triangle to apply Pythagoras. Pythagoras can also be used to verify that a triangle is right-angled. Also, memorise some common Pythagorean triples.

172. Find the area of a triangle with sidelengths 5, 12 and 13.

173. (AIME) In quadrilateral $ABCD$, $\angle B$ is a right angle, diagonal $\overline{AC}$ is perpendicular to $\overline{CD}$, $AB = 18$, $BC = 21$, and $CD = 14$. Find the perimeter of $ABCD$.

174. (CJR) Suppose ΔABC is a triangle with $AB = 30$, $AC = 16$ and $BC = 34$. If M is the midpoint of BC, find the length of AM.

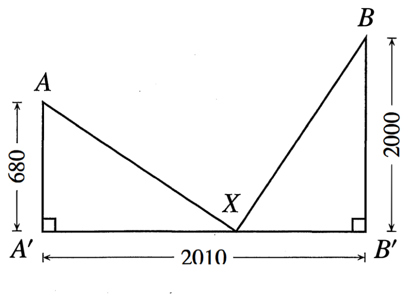
175. (PML) In the diagram below, $AB = 3$, $BC = 5$, $CD = 11$, $DE = 7$ and $EF = 2$. Find AF.



176. $ABCD$ is a square of side 12 cm. Points E, F and G lie on sides AD, BC and CD respectively. Given that $DG = 5$ cm and EF is perpendicular to AG, what is the length of EF?

177. (AMC 12) A triangle with side lengths in the ratio $3:4:5$ is inscribed in a circle with radius 3. What is the area of the triangle?

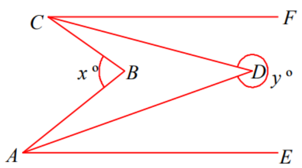
178. (SMO) Let AA' and BB' be two line segments which are perpendicular to $A'B'$. The lengths of AA', BB' and $A'B'$ are 680, 2000 and 2010 respectively. Find the minimal length of $AX + XB$ where X is a point between A' and B'.



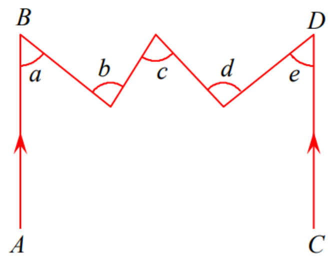
3.2 Simple Angle Chasing

Strategy: Apply basic angle rules which you learnt in school.

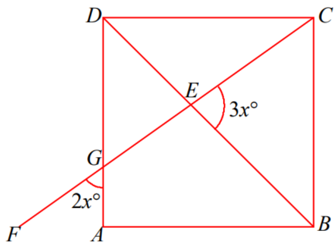
179. (CJR) In the figure below, AE is parallel to CF, AD bisects $\angle BAE$, and CD bisects $\angle BCF$. If $y = 324$, find x.



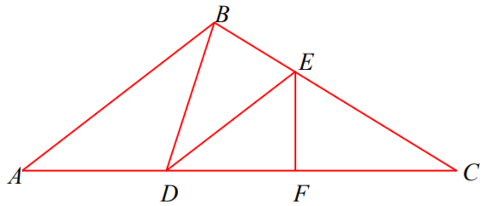
180. (CJR) In the figure below, AB is parallel to CD. If $a+b+c+d+e = 310°$ and $b + c + d = 228°$, find $\angle c$.



181. (CJR) In the figure below, $ABCD$ is a square. Find x.



182. (CJR) In the figure below, AB is parallel to DE, EF is perpendicular to AC, and DE bisects $\angle BDC$. If $\angle DBC = 71°$ and $\angle FEC = 59°$, find $\angle ABD$.



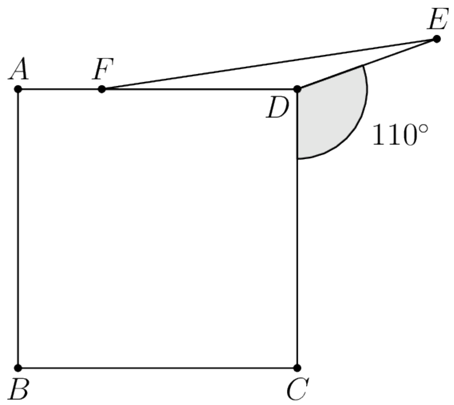
3.3 Isosceles Triangles

Strategy: Every time you encounter a triangle with equal side lengths, recall that it means there are equal angles as well, and vice versa.

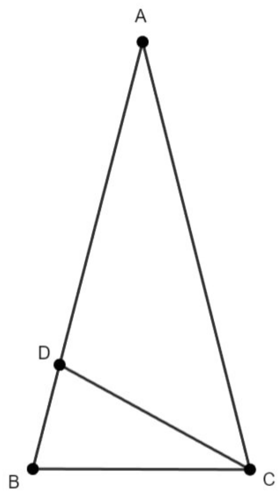
183. (AMC 12) Triangles ABC and ADC are isosceles with $AB = BC$ and $AD = DC$. Point D is inside triangle ABC, angle ABC measures 40 degrees, and angle ADC measures 140 degrees. What is the degree measure of angle BAD?

184. Suppose $AB = AC = AD$ such that $\angle ABC = 24°$ and $\angle ADC = 48°$.

185. (AMC 12) As shown in the figure below, point E lies on the opposite half-plane determined by line CD from point A so that $\angle CDE = 110°$. Point F lies on $\overline{AD}$ so that $DE = DF$, and $ABCD$ is a square. What is the degree measure of $\angle AFE$?

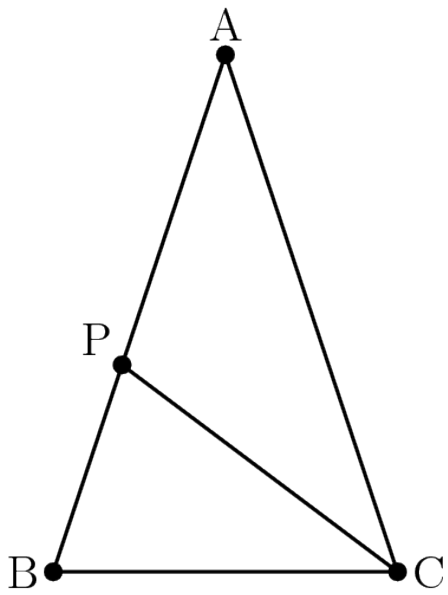


186. (PML) In the diagram below, $AB = AC$ and $BC = 4$. Given that $\angle ACD = 45°$ and $\angle DCB = 30°$, find the length of CD.

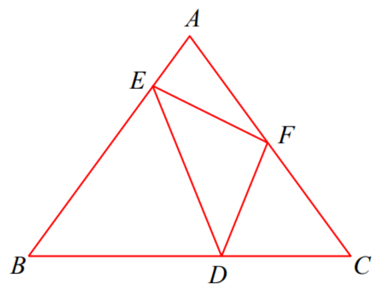


187. (CJR) Suppose $ABCD$ is a quadrilateral with $AB = BC = CD$ such that $\angle B = 84°$ and $\angle C = 60°$. Find $\angle D$.

188. (AHSME) In triangle ABC, $AB = AC$. If there is a point P strictly between A and B such that $AP = PC = CB$, then find the value of $\angle A$.



189. (CJR) In the figure below, ΔABC is an isosceles triangle with $AB = AC$. Suppose that $\angle A = 76°$, $BD = BE$ and $CD = CF$. If $\angle DEF : \angle DFE = 7 : 9$, find $\angle AEF$.



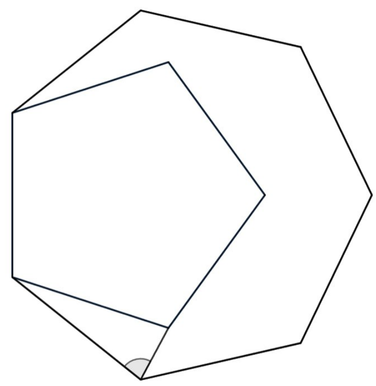
190. In ΔABC, point D is on AC such that $AB = AD$. Suppose that $\angle ACB - \angle ABC = 30°$. Find $\angle CBD$.

3.4 Regular Polygons

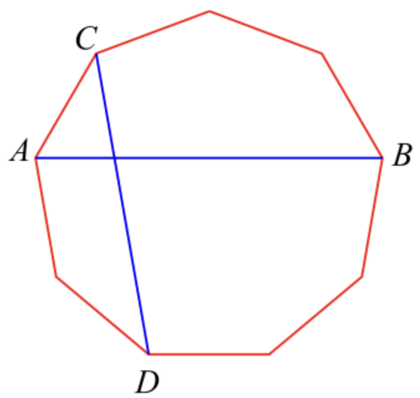
Strategy: Learn how to find the sum of interior angles for an n-sided polygon. Now, recall that regular polygons have equal interior angles and equal side lengths. (This also means lots of isosceles triangles.)

191. (CJR) Let $ABCDE$ be a regular pentagon. Find $\angle CAD$.

192. (PML) The diagram below shows a regular pentagon and a regular heptagon. Find the marked angle.



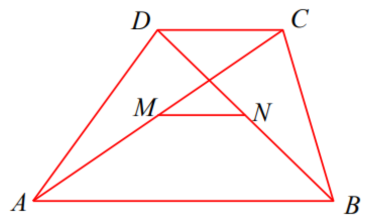
193. (CJR) The figure shown below is a regular 9-gon. Find the value of the acute angle between the diagonals AB and CD.



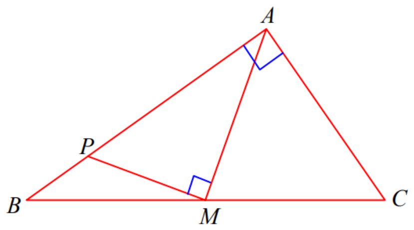
3.5 Similar Triangles

Strategy: After noticing a pair of similar triangles, you can either find side lengths (by considering ratios), or deduce angles.

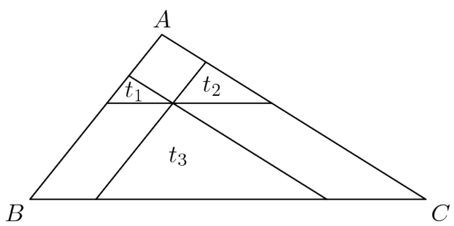
194. (CJR) In the figure shown below, $AB > CD$ and AB is parallel to CD. M and N are respectively the midpoints of the line segments AC and BD. If $AB = 1024$ and $MN = 124$, find the length of CD.



195. (CJR) In the figure below, ΔABC is a right-angled triangle with $\angle BAC = 90°$. M is the midpoint of BC, and P is a point on AB such that PM is perpendicular to AM. Given that $AB = 96$, $AC = 72$, find the length of BP.



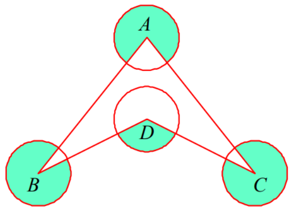
196. (AIME) A point P is chosen in the interior of $\triangle ABC$ such that when lines are drawn through P parallel to the sides of $\triangle ABC$, the resulting smaller triangles t_1, t_2, and t_3 in the figure, have areas 4, 9, and 49, respectively. Find the area of $\triangle ABC$.



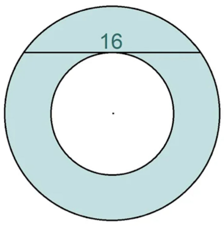
3.6 Circle Areas

Strategy: Cleverly apply the usual formula.

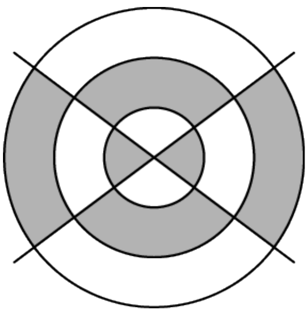
197. (CJR) In the figure shown below, the centers of the four identical circles are on the vertices of quadrilateral $ABCD$. If the area of each circle is 32, find the sum of the areas of the four shaded regions.



198. Consider 2 concentric circles as shown in the diagram below. Suppose that the chord of the outer circle of length 16 is tangent to the inner circle. Find the area of the shaded region.



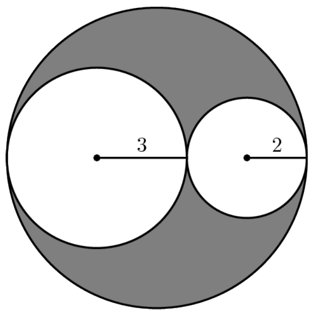
199. (AMC 10) Two distinct lines pass through the center of three concentric circles of radii 3, 2, and 1. The area of the shaded region in the diagram is $\frac{8}{13}$ of the area of the unshaded region. What is the radian measure of the acute angle formed by the two lines?



3.7 Tangency

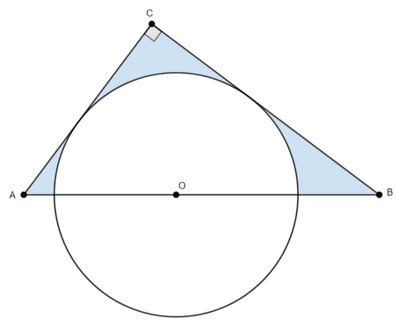
Strategy: If two circles are tangent to each other, then take note that their respective centers and the point of tangency are collinear. Also, if a circle is tangent to the line, then the angle betwewn its center, the point of tangency, and the line is 90°. Drawing these two things in your diagram is useful.

200. (AMC 10) Circles of radius 2 and 3 are externally tangent and are circumscribed by a third circle, as shown in the figure. Find the area of the shaded region.

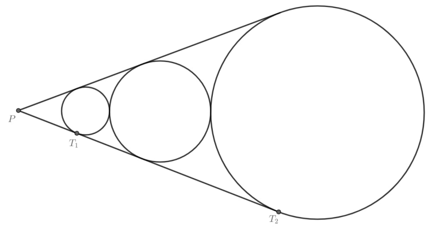


201. Externally tangent circles with centers at points A and B have radii of lengths 5 and 3, respectively. A line externally tangent to both circles intersects ray AB at point C. What is BC?

202. (PML) In the diagram below, $AB = 35$ and $AC = 21$. Given that the circle is tangent to sides AC and BC, and O is the center of the circle, find the area of the shaded region in terms of π.



203. In the diagram below, circles ω_1, ω_2 and ω_3 (with radii r_1, r_2, r_3 respectively such that $r_1 < r_2 < r_3$) share two common tangents that meet at a point P. T_1 and T_2 are the tangency points of ω_1 and ω_3 with the tangent. Given that $r_1 + r_2 = 8$ and $r_2 + r_3 = 18$, find the length of T_1T_2.



204. (AMC 12) Three circles of radius s are drawn in the first quadrant of the xy-plane. The first circle is tangent to both axes, the second is tangent to the first circle and the x-axis, and the third is tangent to the first circle and the y-axis. A circle of radius $r > s$ is tangent to both axes and to the second and third circles. What is r/s?

3.8 30-60-90 Triangle

Strategy: Apply this fact whenever you can:
Given a triangle ABC with $\angle ABC = 90°$, $\angle BAC = 30°$, $\angle ACB = 60°$, it is known that $AB : BC : AC = \sqrt{3} : 1 : 2$.

205. (AMC 10) Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on side AB so that $\angle AMD = \angle CMD$. What is the degree measure of $\angle AMD$?

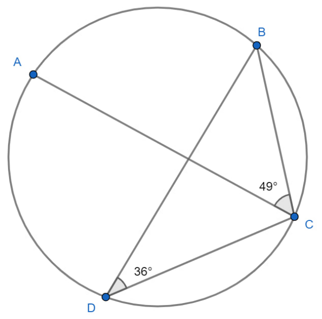
206. (AMC 10) Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on side AB so that $\angle AMD = \angle CMD$. What is the degree measure of $\angle AMD$?

207. (AMC 12) Points A and B lie on a circle centered at O, and $\angle AOB = 60°$. A second circle is internally tangent to the first and tangent to both $\overline{OA}$ and $\overline{OB}$. What is the ratio of the area of the smaller circle to that of the larger circle?

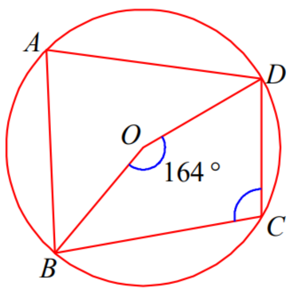
3.9 Angle Chasing in Circles

Strategy: Apply these rules, and also take note of the many isosceles triangles that arise (since the length of a radius within a circle is equal).

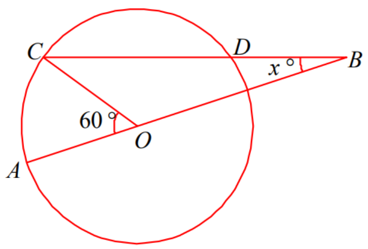
208. (PML) Based on the diagram below, find $\angle ABC$.



209. (CJR) In the diagram below, find $\angle BCD$.

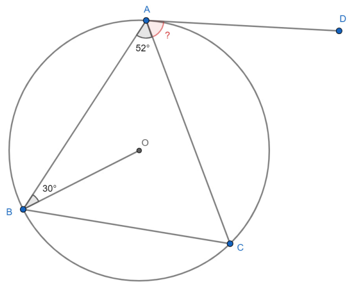


210. (CJR) In the diagram below, find x if BD has the same length as the radius of the circle.

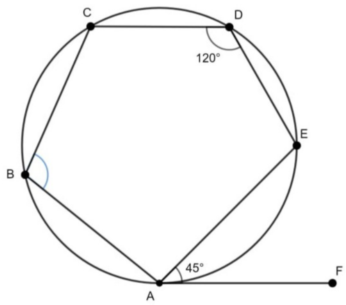


211. (PML) Consider points A, B, C, D, E, F that lie on a circle in that order. Given that $\angle ABC = 92°$ and $\angle CDE = 111°$, find $\angle EFA$.

212. (PML) In the diagram below, AD is tangent to (ABC) and O is the center of (ABC). Find $\angle CAD$.



213. (PML) In the diagram below, AF is tangent to the circle passing through points A, B, C, D and E. Given $\angle FAE = 45°$ and $\angle CDE = 120°$, find $\angle ABC$.



3.10 Triangle Inequality

Strategy: Obeserve that the sum of lengths of any two sides of a (non-degenerate) triangle must be at least the length of the third side.

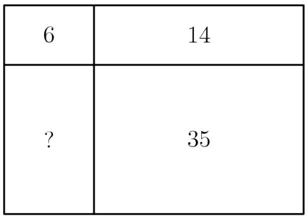
214. (CJR) Given that ΔABC is an acute-angled triangle with $AB = 15$ and $BC = 8$, find the largest possible length of AC if the length AC is an integer.

215. (AMC 10) In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 15. What is the greatest possible perimeter of the triangle?

3.11 Divided Areas

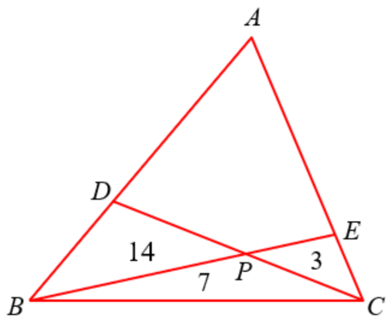
Strategy: Apply the usual area formulas and be big brain. :D

216. (AHSME) A large rectangle is partitioned into four rectangles by two segments parallel to its sides. The areas of three of the resulting rectangles are shown. What is the area of the fourth rectangle?



217. Consider a trapezium $ABCD$ with AB parallel to CD, and suppose the diagonals AC and BD intersect at E. Given that the area of ΔABE is 16 and the area of ΔBEC is 40, find the area of trapezium $ABCD$.

218. (CJR) In the figure below, D and E are points on AB and AC respectively. The lines CD and BE intersect at the point P. If the areas of the triangles ΔPBC, ΔPCE and ΔPBD are 7, 3 and 14 respectively, find the area of the quadrilateral $ADPE$.

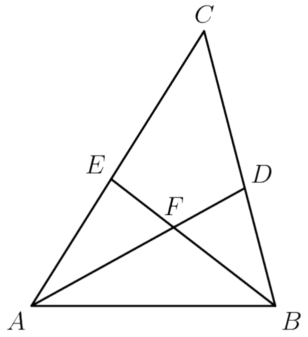


3.12 Angle Bisectors

Strategy: Apply the angle bisector theorem.

219. Suppose ΔABC has $AB = 21$ and $AC = 28$. Let D be the point on BC such that AD bisects $\angle BAC$. Given that $DB = 15$, find $\angle BAC$.

220. (AMC 12) In $\triangle ABC$, $AB = 6$, $BC = 7$, and $CA = 8$. Point D lies on $\overline{BC}$, and $\overline{AD}$ bisects $\angle BAC$. Point E lies on $\overline{AC}$, and $\overline{BE}$ bisects $\angle ABC$. The bisectors intersect at F. What is the ratio $AF : FD$?

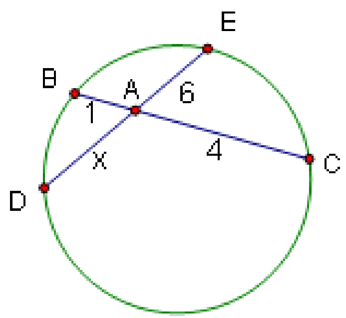


221. (AMC 10) Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

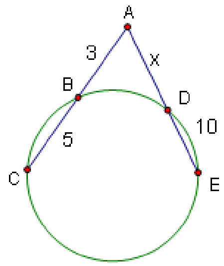
3.13 Power of a Point

Strategy: Apply Power of a Point formulas.

222. Find the value of x in the following diagram:



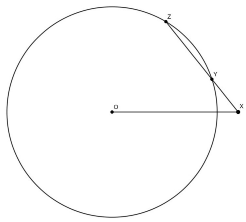
223. Find the value of x in the following diagram:



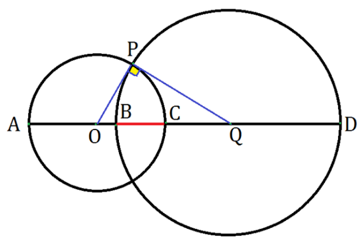
224. (AMC 12) In unit square $ABCD$, the inscribed circle ω intersects $\overline{CD}$ at M, and $\overline{AM}$ intersects ω at a point P different from M. What is AP?

225. (ARML) In a circle, chords AB and CD intersect at R. If $AR : BR = 1 : 4$ and $CR : DR = 4 : 9$, find the ratio $AB : CD$.

226. (PML) In the diagram below, O is the center of the circle. If $OX = 12$, $XY = 4$ and $YZ = 7$, find the radius of the circle.



227. In the diagram below, O and Q are the centers of the circles. P is one of the intersection points of the two circles, and $\angle OPQ = 90°$. If $AB = 10$ and $CD = 24$, find the length of BC.



228. (AMC 12) Consider two concentric circles of radius 17 and 19. The larger circle has a chord, half of which lies inside the smaller circle. What is the length of the chord in the larger circle?

4 Number Theory

4.1 Last Digits

Strategy: To find last digits, we only need to consider the last digits when performing operations. In many cases, finding patterns is useful. Additionally, some problems which ask for 2 or more last digits may require the use of the binomial theorem.

229. Find the last digit of 20^{2024}.

230. Find the last digit of 9^{2024}.

231. Find the last digit of $(1+5^{2024})^{2024}$.

232. (PML) Find the last digit of $2024^{2023^{2022}}$.

233. Find the last digit of $21^{21} \times 22^{22} \times 23^{23} \times 24^{24} \times 25^{25}$.

234. Find the last digit of 7^{2021}, 7^{2022}, 7^{2023} and 7^{2024}.

235. Find the last digit of $2022^{2023} + 2023^{2022}$.

236. Find the last digit of $2024!$.

237. (PML) Find the last 2 digits of $20! \times 24!$.

238. Find the last digit of $1 \times 3 \times 5 \times \cdots \times 999$.

239. Find the last 3 digits of $2 \times 4 \times 6 \times \cdots \times 2022 \times 2024$.

240. Find the last digit of $1^{2024} + 2^{2024} + 3^{2024} + \cdots + 99^{2024} + 100^{2024}$.

241. (AMC 10) Find the last 2 digits of $2015^{2016} - 2017$.

242. Find the last 2 digits of 9^{2024}.

243. Find the last 2 digits of 4321^{1234}.

4.2 Number of Factors

Strategy: Find the prime factorisation of the number, then apply (and understand!) the formula for number of factors.

244. Find the number of positive factors of 2400.

245. Find the number of positive odd factors of 2400.

246. Find the number of positive factors of 2400 which are multiples of 3.

247. Find the number of positive factors of 9!

248. (PML) Find the number of positive odd divisors of 12!

249. (AMC 10) How many positive cubes divide $3! \times 5! \times 7!$?

250. (AMC 10) How many positive integer divisors of 201^9 are perfect squares or perfect cubes (or both)?

251. (AIME) Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2 are less than n but do not divide n?

4.3 Divisibility Rules

Strategy: Exploit divisibility rules to obtain information about the digits of a number. (Search up the rules yourself.)

252. (CJR) If the 3-digit number $2a7$ is divisible by 11, find the value of a.

253. (CJR) Given that the 5-digit number $2a5a6$ is divisible by 36, find the digit a.

254. (CJR) The 5-digit number $a789b$ is a multiple of 12. Find the number of possible solutions for (a, b).

255. (SMO) What is the smallest 5-digit integer of the form $5x20y$ that is divisible by 33?

256. (AMC 12) The six-digit number $20210A$ is prime for only one digit A. What is A?

257. (IMONST) What is the smallest positive multiple of 24 that can be written using digits 4 and 5 only?

4.4 Trailing Zeroes

Strategy: Notice that every factor of 10 adds a trailing zero.

258. Find the number of trailing zeroes of 10^{24}.
259. Find the number of trailing zeroes of $15^{20} \times 20^{15}$.
260. Find the number of trailing zeroes of 30!
261. Find the number of trailing zeroes of $20! + 40! + 60! + \cdots + 980! + 1000!$
262. Find the number of trailing zeroes of 2024!
263. (AIME) Find the number of trailing zeroes of $1!2!3!\ldots99!100!$

4.5 Number of Digits

Strategy: Rearrange factors to obtain the form

$$\text{some number} \times 10^{\text{some number}}.$$

264. Find the number of digits of the number 24×10^{24}.
265. (CJR) How many digits are in the number $5^{20} \times 4^{17}$.
266. (PML) Find the number of digits of the number

$$20^{24} \times 24^4 \times 25^{18}.$$

267. (AMC 8) How many digits are in the base-ten representation of

$$8^5 \times 5^{10} \times 15^5?$$

4.6 Remainders!

Strategy: For problems with a^b, find the remainders for several smaller values of b and look out for a pattern that repeats. Harder problems will require the use of modular arithmetic.

268. Today is Monday. What day is it in 2024 days?

269. Consider the following string of letters:

$$MATHLETEMATHLETEMATHLETE\ldots$$

What is the 2024th letter in the sequence?

270. (CJR) Find the remainder when 5^{61} is divided by 7.

271. Find the remainder when 7^{2023} is divided by $2, 3, 4, \ldots, 9, 10$ respectively.

272. (CJR) Find the remainder when $2^{1234567} - 1$ is divided by 7.

273. (SMO) What is the remainder when $7^{2008} + 9^{2008}$ is divided by 64?

274. (AMC 10) What is the remainder when $3^0 + 3^1 + 3^2 + \cdots + 3^{2009}$ is divided by 8.

275. (AMC 10) What is the remainder when $2^{202} + 202$ is divided by $2^{101} + 2^{51} + 1$?

276. (AMC 12) Let $N = 123456789101112\ldots4344$ be the 79-digit number that is formed by writing the integers from 1 to 44 in order, one after the other. What is the remainder when N is divided by 45?

277. (AMC 8) How many positive three-digit integers have a remainder of 2 when divided by 6, a remainder of 5 when divided by 9, and a remainder of 7 when divided by 11?

278. (CJR) A bag contains not more than 300 sweets. If the sweets are taken out 3 at a time, or 5 at a time, or 7 at a time, there are always 2 sweets left in the bag. What is the largest possible number of sweets in the bag?

4.7 Fraction = Integer

Strategy: If x/y is an integer, rewrite x/y to obtain

$$\frac{x}{y} = \text{some integer} + \frac{\text{known integer}}{y}.$$

Thus, $\frac{\text{known integer}}{y}$ is an integer, so y must be a factor of the known integer. By considering the possible values of y, we can solve for the desired answer. Polynomial long division can be useful for harder problems.

279. Find all positive integers n such that $\frac{6}{n}$ is an integer.

280. Find all positive integers n such that $n + 20$ is a multiple of n.

281. (CJR) If $2x$ and $\frac{55}{x}$ are both positive integers, how many possible values can x assume?

282. (PML) Find all positive integers n such that $\frac{n^2+23n}{n+10}$ is an integer.

283. (SMO) There are a few integer values of a such that

$$\frac{a^2 - 3a - 3}{a - 2}$$

is an integer. Find the sum of all these integer values of a.

284. (CJR) How many integers n are there such that

$$\frac{2n^4 + 6n^3 - 3n^2 - 108n + 3}{n + 3}$$

is also an integer?

285. (AIME) What is the largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?

286. (CJR) If n is a positive integer such that

$$\frac{140}{n-1} - \frac{140}{n+1}$$

is also an integer, find the largest possible value of n.

4.8 Expression = Prime

Strategy: Factorise the expression. Recall that the only factors of a prime are 1 and itself.

287. Find all positive integers a such that $a^2 + a$ is prime.

288. Find all positive integers a such that $a^2 - 1$ is prime.

289. Find all positive integers n such that $n^2 - 10n + 24$ is prime.

290. (PML) Find all positive integers n such that $n^4 - 24n^2 + 36$ is prime.

4.9 Equations in Integers

Strategy: Factorise to obtain

$$\text{some factorisation} = \text{some integer}.$$

Then, consider the factors of the integer.

291. Find all ordered pairs of positive integers (x, y) such that

$$x^2 = y^2 + 5.$$

292. Find all ordered pairs of positive integers (x, y) such that

$$x + y = 50 - xy.$$

293. Find all ordered pairs of positive integers (x, y) such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{9}.$$

294. Find all ordered pairs of positive integers (x, y) such that

$$x^2 + 2x = y^2 + 4y + 24.$$

295. Find all ordered pairs of positive integers (x, y) such that

$$x^3 = y^3 + 37.$$

296. (AIME) Find $3x^2y^2$ if x and y are integers such that

$$y^2 + 3x^2y^2 = 30x^2 + 517.$$

4.10 An Even Prime???

Strategy: Consider parity and recall that 2 is the only even prime.

297. Find all pairs (p, q) such that $p + q = 103$ where p and q are primes.

298. (PML) Let p and q be primes. Find all pairs (p, q) such that

$$p + 6q = 2024.$$

299. (AMC 8) The sum of two prime numbers is 85. What is the product of these two prime numbers?

300. (Classic) Find all positive integers n such that $3n - 4$, $4n - 5$ and $5n - 3$ are all prime.

301. (AHSME) If p and q are primes and $x^2 - px + q = 0$ has distinct positive integral roots, find p and q.

302. (SMO) Suppose that p and q are prime numbers and they are roots of the equation $x^2 - 99x + m = 0$ for some m. What is the value of $\frac{p}{q} + \frac{q}{p}$?

4.11 Something × ? = Perfect Power

Strategy: Consider prime factorisations. Apply the fact that if x is a power of k, then the exponent of every prime in the prime factorisation of x is a multiple of k.

303. Determine the smallest positive integer m such that $2m$ is a perfect square and $3m$ is a perfect cube.

304. (CJR) If m and n are positive integers such that $n^5 = 18000m$, find the smallest possible value of m.

305. (DOKA) Given that p, q, r are prime numbers, find the value of pqr such that $104104 \times p^2q^2r$ is a perfect cube.

4.12 Expression $= x^2$

Strategy: Apply the same technique with equations in integers, but specifically, exploit the difference of squares factorisation.

306. (CJR) Let n be an integer such that $n+100$ and $n-24$ are perfect squares. Find the smallest possible value of n.

307. (SMO) Let n be a positive integer such that $n^2+19n+48$ is a perfect square. Find the value of n.

4.13 Highest Powers

Strategy: Factorise!

308. (AMC 8) What is the largest power of 2 that is a divisor of 13^4-11^4?

309. (AMC 10) What is the greatest power of 2 that is a factor of $10^{1002}-5^{501}$?

310. (KMC) What is the highest power of 3 dividing $7!+8!+9!$?

4.14 GCD and LCM

Strategy: Learn the definitions and how to calculate GCD and LCM. An important fact to keep in mind is that

$$\gcd(a,b)\times \operatorname{lcm}(a,b)=ab.$$

311. Find $\gcd(270,144)$ and $\operatorname{lcm}(270,144)$.

312. The least common multiple of a positive integer n and 18 is 180, and the greatest common divisor of n and 45 is 15. What is n?

313. (CJR) Given that the product of two positive integers is 4320, and their greatest common divisor is 12. Find the least common multiple of these two numbers.

314. (AMC 10) How many ordered pairs (a,b) of positive integers satisfy the equation

$$ab+63=20\operatorname{lcm}(a,b)+12\gcd(a,b).$$

5 Appendix

5.1 Factorisations and Expansions

- $(x+y)^2 = x^2 + 2xy + y^2 = (x-y)^2 + 4xy$
- $(x-y)^2 = x^2 - 2xy + y^2 = (x+y)^2 - 4xy$
- $(x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + xz)$
- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + 3xy(x+y) + y^3$
- $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3 = x^3 - 3xy(x-y) - y^3$
- $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - xz)$
- Difference of squares: $x^2 - y^2 = (x-y)(x+y)$
- Difference of cubes: $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$
- Sum of cubes: $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$
- SFFT: $xy + kx + jy + jk = (x+j)(y+k)$

5.2 Angles in Circles