IMONST-MRSM

Thursday, 13 June, 2024 7:56 AM

- expansion → $(a+b)(c+d) = ac+ad+bc+bd$
- factorisation → opposite

$a^2+2ab+b^2 = (a+b)^2$ ✓

$= (a+b)(a+b)$

$a^2 - b^2 \cdot (a+b)(a-b)$ → diff of squares

$n^2 - 1 = (n+1)(n-1)$

TOPIC 1: BIG NUMBERS

1) $\sqrt{2024^2 - (2023)(2025)}$ ($2024-1$, $2024+1$)

$2024 = m$

$m^2 - (m-1)(m+1)$

$= m^2 - [m^2-1]$

$= m^2 - m^2 + 1$

$= 1$

2) $\sqrt{111113^2 - 888888}$

$111111 = m$

$111113 = m+2$

$888\,888 = 8m$

$= \sqrt{(m+2)^2 - 8m}$

$= \sqrt{m^2+2m+2m+4-8m}$ ($4m$)

$= \sqrt{m^2-4m+4}$

$= \sqrt{(m-2)^2}$

$= m-2$

$= 111\,111 - 2$

$= 111\,109$

$(m+2)(m-2) = m^2 - 2^2 = m^2-4$

3) $\sqrt{499^2 + 999}$

- 500
- 499

$m = 500$

$499 = m-1$

$999 = 2m-1$

$\sqrt{(m-1)^2 + (2m-1)}$

$= \sqrt{m^2 - 2m + 1 + 2m - 1}$

$= \sqrt{m^2}$

$= m$

$= 500$

$m = 499$

$499 = m$

$999 = 2m+1$

$\sqrt{m^2 + (2m+1)}$

$= \sqrt{(m+1)^2}$

$= m+1$

$= 500$

$m^2 + 2m + 1$

$= (m+1)^2$

$= m^2 + m + m + 1$

$= m^2 + 2m + 1$

4) $\sqrt[3]{2022 \times 2024 \times 2026 + 4 \times 2024}$

5) OMK

$6666\,6666 \times 4444\,4445 - 3333\,3333 \times 8888\,8888$ (6m, 4m+1, 3m, 8m)

$\rightarrow 6666 \times 4445 - 3333 \times 8888$ (6m, 4m+1, 3m, 8m)

$m = 1111\,1111$ $\quad n = 1111$

$6m(4m+1) - 3m(8m)$

$= 24m^2 + 6m - 24m^2$

$= 6m$

$6n(4n+1) - 3n(8n)$

$= 24n^2 + 6n - 24n^2$

$= 6n$

$\frac{6m}{6n} = \frac{m}{n} = \frac{1111\,1111}{1111} = 10001$

3/4 var. ✓

m n p q

TOPIC 2 : SUMS n PRODUCTS

1) $1+2+3+\cdots+2023+2024$

$1+2024 = 2025$

$2+2023 = 2025$

$3+2022 = 2025$

$\cdots$

$2023+2 = 2025$

$2024+1 = 2025$

$2024 \times 2025 = 2S$

$S = \frac{2024 \times 2025}{2}$

$S = 1012 \times 2025$

general: $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

$2+4+6+\cdots+2024 = 2[1+2+3+\cdots+1012]$

$= 2\left(\frac{1012(1012+1)}{2}\right)$

note: arithmetic progression (+/−) $\frac{n(a+e)}{2}$ (no. terms, first term, last term)

2, 5, 8, 11 (+3, +3, +3)

3, 8, 13, (+5, +5, +5)

2) $1+3+5+\cdots+(2n-1) = \frac{n(1+(2n-1))}{2} = \frac{n(2n)}{2} = n^2$

$2(1)-1, 2(2)-1, 2(3)-1, \ldots, 2n-1$ (n terms)

$1 = 1^2$

$1+3 = 4 = 2^2$

$1+3+5 = 9 = 3^2$

$\vdots$

★ $1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

★ $1^3+2^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

3) $(1-2)+(3-4)+(5-6)+\cdots+(2023-2024)$ (−1, −1, −1, −1)

$\frac{2024}{2} = 1012$ pair

each pair $= -1$

$1012 \times (-1) = -1012$

4) geometric prog. (×/÷)

$1 \xrightarrow{\times 3} 3 \xrightarrow{\times 3} 9 \xrightarrow{\times 3} 27 \xrightarrow{\times 3} 81 \ldots$

$1+2+4+8+\cdots+2^n = S$

$2(1+2+4+8+\cdots+2^n) = 2S$

$= 2+4+8+16+\cdots+2^{n+1}$ ←

$2 \times 2 \times \cdots \times 2 \times 2$ (n 2's) $= (n+1)$ 2's $= 2^{n+1}$

$S = 2S - S$

$= (2+4+8+16+\cdots+2^{n+1}) - (1+2+4+8+\cdots+2^n)$

$\therefore S = 2^{n+1} - 1$

5) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots = S$

$2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right) = 2S$

$= 1+\frac{2}{4}+\frac{2}{8}+\frac{2}{16}+\frac{2}{32}+\cdots$

$= 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots = 2S$

$\frac{1}{2^n} \times 2 = \frac{2}{2^n} = \frac{2^n \times 2}{2^n \times 2^n} = \frac{2^n}{2^{2n}}$

$S = 2S - S$

$= \left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right) - \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\right)$

$= 1$

$\therefore \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots = 1$

6) $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{2024}\right)$

$= \left(\frac{3}{2}\right)\left(\frac{4}{3}\right)\left(\frac{5}{4}\right)\left(\frac{6}{5}\right)\cdots\left(\frac{2024}{2023}\right)\left(\frac{2025}{2024}\right)$

$= \frac{2025}{2}$

$1+\frac{1}{n} = \frac{n}{n}+\frac{1}{n} = \frac{n+1}{n}$

TELESCOPING SUM / PRODUCTS

7) $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{5}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}$

$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a-b$ (conjugate)

$\frac{1}{\sqrt{1}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{1}} = \frac{\sqrt{2}-\sqrt{1}}{(\sqrt{2}-\sqrt{1})(\sqrt{2}+\sqrt{1})} = \frac{\sqrt{2}-\sqrt{1}}{1} = \sqrt{2}-\sqrt{1}$

$= (\sqrt{2}-\sqrt{1}) + (\sqrt{3}-\sqrt{2}) + (\sqrt{4}-\sqrt{3}) + (\sqrt{5}-\sqrt{4}) + \cdots + (\sqrt{100}-\sqrt{99})$

$= \sqrt{100} - \sqrt{1}$

$= 10 - 1$

$= 9$

8) $\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \cdots + \frac{1}{2022\times2023} + \frac{1}{2023\times2024}$

$\frac{1}{2} \quad \frac{1}{6} \quad \frac{1}{12}$

$\frac{1}{\square} = \frac{1}{\square+1} - \frac{1}{\square}$

$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ ✓

$= \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)}$

$= \frac{1}{n(n+1)}$

$\frac{1}{1\times2} = \frac{1}{1} - \frac{1}{2}$

$\frac{1}{2\times3} = \frac{1}{2} - \frac{1}{3}$

$\frac{1}{3\times4} = \frac{1}{3} - \frac{1}{4}$

$\frac{1}{4\times5} = \frac{1}{4} - \frac{1}{5}$

$\vdots$

$\left(\frac{1}{1}-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{5}\right) + \cdots + \left(\frac{1}{2022}-\frac{1}{2023}\right) + \left(\frac{1}{2023}-\frac{1}{2024}\right)$

$= \frac{1}{1} - \frac{1}{2024}$

$= \frac{2024}{2024} - \frac{1}{2024}$

$= \frac{2023}{2024}$

TOPIC 3 : ALGEBRAIC MANIPULATION

→ add / subtract
→ multiply
→ raise power (e.g. square)

e.g. $\frac{a}{b}=2$, $\frac{b}{c}=4$, $\frac{a}{c}=?$

$\frac{a}{c}=\frac{a}{b}\times\frac{b}{c}=2\times4=8$

1) $xy=2$, $x+y=4$

$x^3+y^3=?$

$(x+y)^2=4^2=16$

$x^2+y^2+2xy=16$

$x^2+y^2=16-2xy$
$=16-2(2)$
$=16-4$
$=12$

2) $\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=1$

Find $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}$.

$(\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c})+(\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c})$

$=\frac{a+1}{1+a}+\frac{b+1}{1+b}+\frac{c+1}{1+c}$

$=1+1+1$

$=3$

$(+\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=3$

$\therefore \frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}=2$

3) $143x-77y=451$, find $299x-161y$.

$\frac{143x-77y}{11}=\frac{451}{11}$

$\therefore 13x-7y=41$

$299x-161y=23(13x-7y)$
$=23(41)$
$=943$

4) $x+\frac{1}{x}=5$

a) $x^2+\frac{1}{x^2}$

$(x+\frac{1}{x})^2=5^2=25$

$x^2+2x(\frac{1}{x})+(\frac{1}{x})^2=25$

$x^2+2+\frac{1}{x^2}=25$

$x^2+\frac{1}{x^2}=23$.

$x^4+2x^2(\frac{1}{x^2})+\frac{1}{x^4}=23^2$

$x^4+\frac{1}{x^4}=23^2-2$

b) $x^3+\frac{1}{x^3}$

$(x+\frac{1}{x})^3=5^3=125$

$x^3+3x^2(\frac{1}{x})+3x(\frac{1}{x})^2+\frac{1}{x^3}=125$

$x^3+3x+3(\frac{1}{x})+\frac{1}{x^3}=125$

$x^3+3[x+\frac{1}{x}]+\frac{1}{x^3}=125$

$x^3+3(5)+\frac{1}{x^3}=125$

$x^3+\frac{1}{x^3}=110$

5) $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$

x, $x^3=2+\sqrt{5}$; y, $y^3=2-\sqrt{5}$??

$x^3+y^3=4$

$xy=\sqrt[3]{2+\sqrt{5}}\times\sqrt[3]{2-\sqrt{5}}$
$=\sqrt[3]{(2+\sqrt{5})(2-\sqrt{5})}$
$=\sqrt[3]{2^2-5}$
$=\sqrt[3]{-1}$
$=-1$

$x^3+y^3=4$, $xy=-1$

$x+y$

$(x+y)^3=x^3+3x^2y+3xy^2+y^3$
$=x^3+y^3+3x^2y+3xy^2$
$=4+3xy(x+y)$
$=4+3(-1)(x+y)$
$=4-3(x+y)$

$z=x+y$

$(x+y)^3+3(x+y)-4=0$

$z^3+3z-4=0$

3 solution, 1 real solution

$z=1$ ($z-1$)

$z-1\overline{)z^3+3z-4}$ → quadratic

z^3-z^2

z^2+z-4

$\Delta=b^2-4ac$

$\rightarrow x+y=1$

$x+y=1$

$1+3(1)-4=0$

6) $x>y$

$x+y=14$, $xy=12$

$x^2+\frac{168}{x}=?$

What is $\frac{168}{x}$?
$=14\times12$

$x^2+\frac{168}{x}=x^2+\frac{(x+y)(xy)}{x}$
$=x^2+(x+y)(y)$
$=x^2+xy+y^2$
$=x^2+y^2+12$
$=172+12$
$=184$

$(x+y)^2=14^2$

$x^2+2xy+y^2=196$

$x^2+2(12)+y^2=196$

$x^2+y^2=196-24=172$

7) $x\neq y$

$x+\frac{9}{x}=y+\frac{9}{y}$

Find xy.

$x^2+9=xy+\frac{9x}{y}$ xy

$x^2y+9y=xy^2+9x$
$xy(x)$ $xy(y)$

$x^2y-xy^2=9x-9y$

$xy(x-y)=9(x-y)$

$xy=9$

$x=y$

$x-y=0$ ← × boleh potong

$5(0)=7(0)$

$\Rightarrow 5=7$

$x-y\neq0$

TOPIC 4 : TRIVIAL INEQUALITY

$x^2=0$ ✓ → $x=0$

$x^2=5$ ✓ → $x=\sqrt{5}\approx2.2$, $-\sqrt{5}$

$x^2=-5$ ✗

x real number

1.5492

$x^3=4$

$x=2,-2$

$x^2>0$
↳ 2 sol.

$+\times+=+$
$-\times-=+$
$0\times0=0$

$x^2\geq0$

$x^2<0$ ✗

1) $22(x-22)^{23}+24(y-24)^{24}+26(z-26)^{26}=0$

≥0 ≥0 ≥0

$=0$ $=0$ $=0$

$22(x-22)^{23}=0$
$(x-22)^{23}=0$
$\Rightarrow x-22=0$
$x=22$

$y=24$
$z=26$

PROBLEM - SOLVING Session

1) (AMC) $(\frac{1\times3}{2\times2})(\frac{2\times4}{3\times3})(\frac{3\times5}{4\times4})\cdots(\frac{97\times99}{98\times98})(\frac{98\times100}{99})$

$(\frac{1\times\cancel{3}}{2\times\cancel{2}})(\frac{\cancel{2}\times\cancel{4}}{\cancel{3}\times\cancel{3}})(\frac{\cancel{3}\times5}{\cancel{4}\times\cancel{4}})\cdots$

$(\frac{1}{2})(\frac{100}{99})=\frac{50}{99}$

2) complete the square
minimum value

x^2+2x+4
$=x^2+2x+1+3$
$=(x+1)^2+3$
≥0

A^2+B
≥0 ↓ min

$(x+1)^2=0$
$\Rightarrow x+1$ $\Rightarrow x=-1$

3) Find all x,y,z s.t.

$x^2+5y^2+10z^2=4xy+6yz+2z-1$

$(z-1)^2=z^2-2z+1$

$x^2+5y^2+10z^2-4xy-6yz-2z+1=0$

$x^2+5y^2+z^2+9z^2-4xy-6yz-2z+1=0$

$x^2+5y^2+9z^2-4xy-6yz+z^2-2z+1=0$

$x^2+5y^2+9z^2-4xy-6yz+(z-1)^2=0$

y^2+4y^2

$(x-2y)^2+y^2+9z^2-6yz+(z-1)^2=0$

$(x-2y)^2+(y-3z)^2+(z-1)^2=0$

$z-1=0$ $y-3z=0$ $x-2y=0$
$z=1$ $y=3z=3$ $x=2y=6$

$z=1$, $y=3$, $x=6$