Supplementary Materials

Appendix A
Physical Interpretation of Fuel-Rate-Speed Function

A truck running on a road with grade/slope θ (positive if moving up and negative if moving down) faces three resistances: aerodynamic (air) resistance, rolling resistance and grade resistance [62]. The air resistance is the friction of air, which is modeled as

$$F_a(v) = \frac{1}{2}\rho A_f c_d v^2, \tag{18}$$

where ρ is the air density and A_f the frontal area of the truck and c_d is drag coefficient of the truck (see Tab. III for c_d and A_f) and v is the speed of the truck. The rolling resistance is the friction between the tires and the ground, which is modeled as

$$F_r = c_r mg\cos\theta, \tag{19}$$

where c_r is the coefficient of rolling resistance (friction coefficient) between the tires and the ground, m is the truck mass and g is gravitational acceleration. The grade resistance is the force of the gravity on the opposite direction of truck movement, i.e.,

$$F_g = mg\sin\theta. \tag{20}$$

Then the tractive force is

$$F_t(v) = F_a(v) + F_r + F_g, \tag{21}$$

which yields to the power consumption

$$P_f(v) = F_t(v)\cdot v = \frac{1}{2}\rho A_f c_d v^3 + (c_r\cos\theta + \sin\theta)mgv. \tag{22}$$

We can regard $P_f(v)$ as the power demand to move the truck on the road with constant speed v. To provision such power demand, the internal combustion engine (ICE) needs to convert fuel into mechanical energy. There are a substantial number of models for ICE [49]. For the purpose of this physical interpretation, we use the following relationship (see [49, Equation 10]),

$$P_f = f(v)\cdot \mathsf{LHV}\cdot\eta, \tag{23}$$

where $f(v)$ is the fuel rate consumption (unit: gallon per hour), LHV is the lower heating value of the fuel (unit: KJ per gallon), and η is the fuel efficiency[8]. Eq. (23) gives the fuel-rate-speed function $f(v)$ as follows,

$$f(v) = \frac{P_f}{\mathsf{LHV}\cdot\eta} = \frac{\frac{1}{2}\rho A_f c_d v^3 + (c_r\cos\theta+\sin\theta)mgv}{\mathsf{LHV}\cdot\eta}, \tag{24}$$

which shows that the fuel-rate-speed function is polynomial with speed v and also strictly convex.

Therefore, such physical interpretation justifies our assumption for the fuel-rate-speed function in Sec. II-A.

[8]The unit of power demand P_f would be KW. We can appropriately make all units consistent.

Appendix B
Proof of Lemma 1

We can prove this lemma by using the continuous Jensen's inequality. For any speed profile $v : [0, t_e] \to \mathbb{R}^+$ over road/edge e, the incurred fuel consumption is $\int_0^{t_e} f_e(v(t))dt$, and the travelled distance is $\int_0^{t_e} v(t)dt$. As we require that the truck must pass edge e with exactly t_e hours, we must have

$$\int_0^{t_e} v(t)dt = D_e. \tag{25}$$

Since $f_e(\cdot)$ is convex, according to the continuous Jensen's inequality [63, Ch. 12.411], we have

$$\frac{\int_0^{t_e} f_e(v(t))dt}{t_e} \geq f_e\left(\frac{\int_0^{t_e} v(t)dt}{t_e}\right) = f_e\left(\frac{D_e}{t_e}\right), \tag{26}$$

which means

$$\int_0^{t_e} f_e(v(t))dt \geq t_e f_e\left(\frac{D_e}{t_e}\right), \tag{27}$$

with equality when $v(t) = \frac{D_e}{t_e}$ for all $t \in [0, t_e]$.

The proof is completed.

Appendix C
Proof of Lemma 2

Since the fuel-rate-speed function $f_e(v)$ is a polynomial function (and thus twice differentiable) with respect to v, we can thus obtain the first and second-order derivative of $c_e(t_e) = t_e f_e\left(\frac{D_e}{t_e}\right)$ with respect to t_e, i.e.,

$$c_e'(t_e) = f_e(\frac{D_e}{t_e}) - \frac{D_e}{t_e}f_e'(\frac{D_e}{t_e}), \tag{28}$$

and

$$\begin{aligned} c_e''(t_e) &= f_e'(\frac{D_e}{t_e})(-\frac{D_e}{t_e^2}) \\ &\quad -[(-\frac{D_e}{t_e^2})f_e'(\frac{D_e}{t_e}) + \frac{D_e}{t_e}f_e''(\frac{D_e}{t_e})(-\frac{D_e}{t_e^2})] \\ &= \frac{D_e^2}{t_e^3}f_e''(\frac{D_e}{t_e}). \end{aligned} \tag{29}$$

Since $f_e(\cdot)$ is strictly convex over the speed limit region, we have $f_e''(\frac{D_e}{t_e}) > 0$ and thus we conclude that

$$c_e''(t_e) > 0, \tag{30}$$

which proves that $c_e(\cdot)$ is strictly convex with respect to t_e over $[t_e^{\mathsf{lb}}, t_e^{\mathsf{ub}}]$.

For the second part of this lemma, we first observe that $c_e'(t_e)$ is a differentiable (and thus continuous) and strictly increasing function. Thus we will consider the following three cases.

Case 1 $0 \leq c_e'(t_e^{\mathsf{lb}})$**:** In this case, we know that $c_e(t_e)$ is strictly increasing over $[t_e^{\mathsf{lb}}, t_e^{\mathsf{ub}}]$ and we can set $\hat{t}_e = t_e^{\mathsf{lb}}$.

Case 2 $0 \in (c_e'(t_e^{\mathsf{lb}}), c_e'(t_e^{\mathsf{ub}}))$**:** In this case, we can find a $\hat{t}_e \in (c_e'(t_e^{\mathsf{lb}}), c_e'(t_e^{\mathsf{ub}}))$ such that $c_e'(\hat{t}_e) = 0$ due to the continuity of $c_e'(t_e)$.

Case 3 $0 \geq c_e'(t_e^{\mathsf{ub}})$**:** In this case, we know that $c_e(t_e)$ is strictly decreasing over $[t_e^{\mathsf{lb}}, t_e^{\mathsf{ub}}]$ and we can set $\hat{t}_e = t_e^{\mathsf{ub}}$.

In all three cases, we obtain that $c_e(t_e)$ is first strictly decreasing over $[t_e^{\mathsf{lb}}, \hat{t}_e]$ and then strictly increasing over $[\hat{t}_e, t_e^{\mathsf{ub}}]$. Note that $\hat{t}_e$ could be on the boundary of $[t_e^{\mathsf{lb}}, t_e^{\mathsf{ub}}]$, as shown in **Case 1** and **Case 3**.

The proof is completed.

APPENDIX D
PROOF OF LEMMA 3

First, since p and $\boldsymbol{t}_p$ is a feasible solution to PASO, we have $\mathsf{OPT} \leq c(p, \boldsymbol{t}_p)$.

Second, since Algorithm 2 returns in line 13, the path cost will be no greater than some $c \leq N$, thus we have

$$\tilde{c}(p, \boldsymbol{t}_p) \triangleq \sum_{e\in p} \tilde{c}_e(t_e) = \sum_{e\in p} \min\{\lfloor \frac{c_e(t_e)}{V} \rfloor + 1, N+1\} \leq N,$$

which clearly implies that

$$\tilde{c}_e(t_e) = \lfloor \frac{c_e(t_e)}{V} \rfloor + 1, \forall e \in p.$$

Then we have

$$\begin{aligned} \tilde{c}(p, \boldsymbol{t}_p) &= \sum_{e\in p} \tilde{c}_e(t_e) = \sum_{e\in p} \left[\lfloor \frac{c_e(t_e)}{V} \rfloor + 1 \right] \\ &\geq \sum_{e\in p} \frac{c_e(t_e)}{V} = \frac{c(p, t_p)}{V}, \end{aligned}$$

which yields to

$$\begin{aligned} c(p, \boldsymbol{t}_p) &\leq \tilde{c}(p, \boldsymbol{t}_p)V \leq NV = \left(\lfloor \frac{U}{V} \rfloor + n + 1 \right) V \\ &\leq \left(\frac{U}{V} + n + 1 \right) V = U + (n+1)V = U + L\delta. \end{aligned}$$

The proof is completed.

APPENDIX E
PROOF OF LEMMA 4

For PASO, let us denote $(p^*, \boldsymbol{t}_{p^*})$ as an optimal solution. Namely, p^* is an optimal path and $\boldsymbol{t}_{p^*}$ is the corresponding optimal travel time set. For each edge $e \in p^*$, we must have

$$\min\{\lfloor \frac{c_e(t_e)}{V} \rfloor + 1, N+1\} = \lfloor \frac{c_e(t_e)}{V} \rfloor + 1.$$

Suppose not. Then

$$\lfloor \frac{c_e(t_e)}{V} \rfloor + 1 > N + 1,$$

which means

$$c_e(t_e) \geq V \lfloor \frac{c_e(t_e)}{V} \rfloor > VN = V(\lfloor \frac{U}{V} \rfloor + n + 1) > U \geq \mathsf{OPT}.$$

This is a contradiction to $c_e(t_e) \leq \sum_{e\in p^*} c_e(t_e) = \mathsf{OPT}$.

Then we have

$$\begin{aligned} &\tilde{c}(p^*, \boldsymbol{t}_{p^*}) = \sum_{e\in p^*} \tilde{c}_e(t_e) = \sum_{e\in p^*} \min\{\lfloor \frac{c_e(t_e)}{V} \rfloor + 1, N+1\} \\ &= \sum_{e\in p^*} \left[\lfloor \frac{c_e(t_e)}{V} \rfloor + 1 \right] \leq \sum_{e\in p^*} \left[\frac{c_e(t_e)}{V} + 1 \right] \\ &\leq \frac{\mathsf{OPT}}{V} + n \leq \frac{U}{V} + n \leq (\lfloor \frac{U}{V} \rfloor + 1) + n = N. \end{aligned} \tag{31}$$

Here is a critical step which is different from Lemma 3 in [51] for RSP problem. For each edge $e \in p^*$, t_e may not be a representative point in vector $\boldsymbol{\tau}_e$. However, we can consider the representative point $\tilde{t}_e = \tau_e^i$ where $i \triangleq \tilde{c}_e(t_e)$, which incurs the same fuel cost, i.e., $\tilde{c}_e(t_e) = \tilde{c}_e(\tilde{t}_e)$. Clearly, we also have $\tilde{c}(p^*, \tilde{\boldsymbol{t}}_{p^*}) \leq N$ and $\tilde{t}_e \leq t_e$ where $\tilde{\boldsymbol{t}}_{p^*} \triangleq \{\tilde{t}_e : e \in p^*\}$.

Therefore path p^* and travel time $\tilde{\boldsymbol{t}}_{p^*}$ must be examined by Algorithm 2, which completes the proof of the first part, i.e., Algorithm 2 must return a feasible path p and travel time $\boldsymbol{t}_p$. Moreover, we have

$$\tilde{c}(p, \boldsymbol{t}_p) \leq \tilde{c}(p^*, \tilde{\boldsymbol{t}}_{p^*}) = \tilde{c}(p^*, \boldsymbol{t}_{p^*}). \tag{32}$$

From (31), we first note that

$$\tilde{c}(p^*, \boldsymbol{t}_{p^*}) \leq \frac{\mathsf{OPT}}{V} + n. \tag{33}$$

Second, since Algorithm 2 returns in line 13, we must have

$$\tilde{c}(p, \boldsymbol{t}_p) \triangleq \sum_{e\in p} \tilde{c}_e(t_e) = \sum_{e\in p} \min\{\lfloor \frac{c_e(t_e)}{V} \rfloor + 1, N+1\} \leq N,$$

which clearly implies that

$$\tilde{c}_e(t_e) = \lfloor \frac{c_e(t_e)}{V} \rfloor + 1, \forall e \in p.$$

We then note that

$$\begin{aligned} \tilde{c}(p, \boldsymbol{t}_p) &= \sum_{e\in p} c_e(t_e) = \sum_{e\in p} \min\{\lfloor \frac{c_e(t_e)}{V} \rfloor + 1, N+1\} \\ &= \sum_{e\in p} \left(\lfloor \frac{c_e(t_e)}{V} \rfloor + 1 \right) \\ &\geq \sum_{e\in p} \left(\frac{c_e(t_e)}{V} \right) \\ &= \frac{c(p, \boldsymbol{t}_p)}{V}. \end{aligned} \tag{34}$$

Inserting inequalities (33) and (34) into (32), we obtain

$$\frac{c(p, \boldsymbol{t}_p)}{V} \leq \frac{\mathsf{OPT}}{V} + n,$$

which means

$$c(p, \boldsymbol{t}_p) \leq \mathsf{OPT} + nV \leq \mathsf{OPT} + L\delta.$$

The proof is competed.

APPENDIX F
PROOF OF THEOREM 2

The first part of this theorem directly follow the analysis of *Steps 1-3* in Sec. III-C. Namely, Algorithm 3 returns a $(1+\epsilon)$-approximate solution for PASO in time

$$O((mn\log\xi + mn^2)\log\log\frac{\mathsf{UB}}{\mathsf{LB}} + \frac{mn\log\xi}{\epsilon} + \frac{mn^2}{\epsilon^2}). \tag{35}$$

Now we prove the second part of this theorem. Namely, if we use $\mathsf{LB} = \mathsf{C}^{\mathsf{lb}}$ and $\mathsf{UB} = n\mathsf{C}^{\mathsf{ub}}$ where $\mathsf{C}^{\mathsf{lb}} \triangleq \min_{e\in\mathcal{E}} c_e(t_e^{\mathsf{ub}})$ and $\mathsf{C}^{\mathsf{ub}} \triangleq \max_{e\in\mathcal{E}} c_e(t_e^{\mathsf{lb}})$, Algorithm 3 has time complexity polynomial in the input size of the problem PASO and therefore is an FPTAS. According to (35), we only need to show $\log\log\frac{\mathsf{UB}}{\mathsf{LB}} = \log\log\frac{n\mathsf{C}^{\mathsf{ub}}}{\mathsf{C}^{\mathsf{lb}}}$ is polynomial in the input size.

Suppose that $\mathsf{C}^{\mathsf{ub}} \triangleq \max_{e\in\mathcal{E}} c_e(t_e^{\mathsf{lb}}) = c_{e_1}(t_{e_1}^{\mathsf{lb}})$. For edge e_1, we should input all its properties, i.e., $\{D_{e_1}, R_{e_1}^{\mathsf{lb}}, R_{e_1}^{\mathsf{ub}}, f_{e_1}\}$ where f_{e_1} is a polynomial function. Suppose that

$$f_{e_1}(x) = a_1 x^{k_1} + a_2 x^{k_2} + \cdots + a_q x^{k_q}.$$

Then to input fuel-rate-speed function f_{e_1}, we only need to input $a_1, k_1, a_2, k_2, \cdots, a_q, k_q$. Therefore, for edge e_1, we should input the following real numbers,

$$\{D_{e_1}, R_{e_1}^{\mathsf{lb}}, R_{e_1}^{\mathsf{ub}}, a_1, k_1, a_2, k_2, \cdots, a_q, k_q\}.$$

The input size for edge e_1 is

$$I_{e_1} \geq \log\left(\frac{D_{e_1} + R_{e_1}^{\mathsf{lb}} + R_{e_1}^{\mathsf{ub}} + a_1 + k_1 + a_2 + k_2 + \cdots + a_q + k_q}{\mathsf{eps}}\right),$$

where $\mathsf{eps} \ll 1$ is the machine epsilon, i.e., the maximum relative error of for rounding a real number to the nearest floating point number that can be represented by a digital machine. Now let us show that $\log\log\frac{\mathsf{C}^{\mathsf{ub}}}{\mathsf{eps}}$ is polynomial in I_{e_1}.

According to the definition of the fuel-time function $c_{e_1}(\cdot)$ in (2), we get

$$\begin{aligned}
&\log\log\left(\frac{\mathsf{C}^{\mathsf{ub}}}{\mathsf{eps}}\right) = \log\log\left(\frac{c_{e_1}(t_{e_1}^{\mathsf{lb}})}{\mathsf{eps}}\right)\\
&= \log\log\left(\frac{t_{e_1}^{\mathsf{lb}} \cdot f_{e_1}(\frac{D_{e_1}}{t_{e_1}^{\mathsf{lb}}})}{\mathsf{eps}}\right) = \log\log\left(\frac{\frac{D_{e_1}}{R_{e_1}^{\mathsf{ub}}} \cdot f_{e_1}(R_{e_1}^{\mathsf{ub}})}{\mathsf{eps}}\right)\\
&= \log\left[\log\left(\frac{D_{e_1}}{R_{e_1}^{\mathsf{ub}}}\right) + \log\left(\frac{f_{e_1}(R_{e_1}^{\mathsf{ub}})}{\mathsf{eps}}\right)\right]\\
&= \log\left[\log\left(\frac{D_{e_1}}{\mathsf{eps}}\right) - \log\left(\frac{R_{e_1}^{\mathsf{ub}}}{\mathsf{eps}}\right) + \log\left(\frac{f_{e_1}(R_{e_1}^{\mathsf{ub}})}{\mathsf{eps}}\right)\right]\\
&\leq \log\left[I_{e_1} + \log\left(\frac{f_{e_1}(R_{e_1}^{\mathsf{ub}})}{\mathsf{eps}}\right)\right]\\
&\qquad \text{(Since } R_{e_1}^{\mathsf{ub}} > 0 \text{ and thus } R_{e_1}^{\mathsf{ub}} \geq \mathsf{eps})\\
&= \log\left[I_{e_1} + \log\left(\frac{a_1(R_{e_1}^{\mathsf{ub}})^{k_1} + \cdots + a_q(R_{e_1}^{\mathsf{ub}})^{k_q}}{\mathsf{eps}}\right)\right]\\
&\leq \log\left[I_{e_1} + \log\left(\frac{q a_i (R_{e_1}^{\mathsf{ub}})^{k_i}}{\mathsf{eps}}\right)\right]\\
&\qquad \left(\text{Define } i \in \arg\max_{j\in[1,q]} a_j (R_{e_1}^{\mathsf{ub}})^{k_j}\right)\\
&= \log\left[I_{e_1} + \log q + \log\left(\frac{a_i}{\mathsf{eps}}\right) + \log\left(\frac{(R_{e_1}^{\mathsf{ub}})^{k_i}}{\mathsf{eps}^{k_i}} \cdot \mathsf{eps}^{k_i}\right)\right]\\
&\leq \log\left[I_{e_1} + I_{e_1} + I_{e_1} + k_i \log\left(\frac{R_{e_1}^{\mathsf{ub}}}{\mathsf{eps}}\right)\right]\\
&\qquad \text{(Since } \log\mathsf{eps} < 0)\\
&\leq \log\left[I_{e_1} + I_{e_1} + I_{e_1} + \frac{k_i}{\mathsf{eps}}\log\left(\frac{R_{e_1}^{\mathsf{ub}}}{\mathsf{eps}}\right)\right]\\
&\qquad \text{(Since } \mathsf{eps} < 1)\\
&\leq \log\left[I_{e_1} + I_{e_1} + I_{e_1} + \frac{k_i}{\mathsf{eps}} \cdot I_{e_1}\right]\\
&\leq \log\left[I_{e_1} + I_{e_1} + I_{e_1} + 2^{I_{e_1}} \cdot I_{e_1}\right]\\
&= \log I_{e_1} + \log(3 + 2^{I_{e_1}})\\
&\leq \log I_{e_1} + \log(3 \cdot 2^{I_{e_1}} + 2^{I_{e_1}})\\
&= \log I_{e_1} + I_{e_1} + 2 = O(I_{e_1}),
\end{aligned}$$

which is thus polynomial in I_{e_1}.

Then

$$\begin{aligned}
&\log\log\frac{n\mathsf{C}^{\mathsf{ub}}}{\mathsf{C}^{\mathsf{lb}}} = \log\log\frac{\frac{n\mathsf{C}^{\mathsf{ub}}}{\mathsf{eps}}}{\frac{\mathsf{C}^{\mathsf{lb}}}{\mathsf{eps}}} \leq \log\log\frac{n\mathsf{C}^{\mathsf{ub}}}{\mathsf{eps}}\\
&= \log\left(\log n + \log\frac{\mathsf{C}^{\mathsf{ub}}}{\mathsf{eps}}\right)\\
&\leq 2\max\left\{\log\log n, \log\log\frac{\mathsf{C}^{\mathsf{ub}}}{\mathsf{eps}}\right\}\\
&= \max\{O(\log\log n), O(I_{e_1})\},
\end{aligned} \tag{36}$$

which is polynomial in the input size of PASO because both $O(\log\log n)$ and $O(I_{e_1})$ are polynomial in the input size of PASO. We thus prove the second part of this theorem.

The proof is completed.

Appendix G
Proof of Lemma 6

Define function $h(t_e) = c_e(t_e) + \lambda t_e$. Then we can get the first derivative as

$$h'(t_e) = c_e'(t_e) + \lambda. \tag{37}$$

Since $c_e(t_e)$ is a strictly convex and strict decreasing function, we know that $c_e'(t_e)$ (and also $h'(t_e)$) is a strictly increasing function and $c_e'(t_e) < 0$ at interval $[t_e^{\mathsf{lb}}, t_e^{\mathsf{ub}}]$. We then consider the following three cases.

Case 1: If $0 \leq \lambda < -c_e'(t_e^{\mathsf{ub}})$, we get that $c_e'(t_e^{\mathsf{ub}}) + \lambda < 0$ and thus

$$h'(t_e) \leq h'(t_e^{\mathsf{ub}}) < 0, \forall, t \in [t_e^{\mathsf{lb}}, t_e^{\mathsf{ub}}]. \tag{38}$$

This shows that $h(t_e)$ is strictly decreasing at $[t_e^{\mathsf{lb}}, t_e^{\mathsf{ub}}]$ and the minimal value is attained at $t_e^*(\lambda) = t_e^{\mathsf{ub}}$.

Case 2: If $-c_e'(t_e^{\mathsf{ub}}) \leq \lambda \leq -c_e'(t_e^{\mathsf{lb}})$, then we can get that $c_e'^{-1}(-\lambda) \in [t_e^{\mathsf{lb}}, t_e^{\mathsf{ub}}]$. Clearly, the monotonic increasing property of $h'(t_e)$ implies that $h'(t_e) < 0$ at $[t_e^{\mathsf{lb}}, c_e'^{-1}(-\lambda))$ and $h'(t_e) > 0$ at $(c_e'^{-1}(-\lambda), t_e^{\mathsf{lb}}]$. This means that the minimal value is attained at $t_e^*(\lambda) = c_e'^{-1}(-\lambda)$.

Case 3: If $\lambda > -c_e'(t_e^{\mathsf{lb}})$, we get that $c_e'(t_e^{\mathsf{lb}}) + \lambda > 0$ and thus

$$h'(t_e) \geq h'(t_e^{\mathsf{lb}}) > 0, \forall, t \in [t_e^{\mathsf{lb}}, t_e^{\mathsf{ub}}]. \tag{39}$$

This shows that $h(t_e)$ is strictly increasing at $[t_e^{\mathsf{lb}}, t_e^{\mathsf{ub}}]$ and the minimal value is attained at $t_e^*(\lambda) = t_e^{\mathsf{lb}}$.

The proof is completed.

Appendix H
Proof of Theorem 3

Let us consider any two λ_1, λ_2 with $0 \leq \lambda_1 < \lambda_2$. We need to prove $\delta(\lambda_1) \geq \delta(\lambda_2)$. Suppose that the optimal path at λ_1 is $p^*(\lambda_1) = p_1$ and the optimal path at λ_2 is $p^*(\lambda_2) = p_2$[9].

For any path p and any $\lambda \geq 0$, we denote its (optimal) generalized path cost as

$$W_p(\lambda) \triangleq \sum_{e\in p} w_e(\lambda) = \sum_{e\in p} \left[c_e(t_e^*(\lambda)) + \lambda t_e^*(\lambda)\right], \quad (40)$$

and denote its corresponding path fuel cost as

$$C_p(\lambda) \triangleq \sum_{e\in p} c_e(t_e^*(\lambda)). \quad (41)$$

and denote its corresponding path delay

$$T_p(\lambda) \triangleq \sum_{e\in p} t_e^*(\lambda). \quad (42)$$

Clearly, we have $W_p(\lambda) = C_p(\lambda) + \lambda T_p(\lambda)$.

Based on such notations, we have $\delta(\lambda_1) = T_{p_1}(\lambda_1)$ and $\delta(\lambda_2) = T_{p_2}(\lambda_2)$, and we need to prove $T_{p_1}(\lambda_1) \geq T_{p_2}(\lambda_2)$.

When $\lambda = \lambda_1$, the optimal path is p_1, which means that

$$\begin{aligned} W_{p_1}(\lambda_1) &= C_{p_1}(\lambda_1) + \lambda_1 T_{p_1}(\lambda_1) \\ &\leq W_{p_2}(\lambda_1) = C_{p_2}(\lambda_1) + \lambda_1 T_{p_2}(\lambda_1) \end{aligned} \quad (43)$$

Similarly, when $\lambda = \lambda_2$, the optimal path is p_2, which means that

$$\begin{aligned} W_{p_2}(\lambda_2) &= C_{p_2}(\lambda_2) + \lambda_2 T_{p_2}(\lambda_2) \\ &\leq W_{p_1}(\lambda_2) = C_{p_1}(\lambda_2) + \lambda_2 T_{p_1}(\lambda_2) \end{aligned} \quad (44)$$

Now we will use the fact that $t_e^*(\lambda)$ minimizes $w_e(\lambda)$, as defined in (13). Since both $t_e^*(\lambda_1)$ and $t_e^*(\lambda_2)$ are feasible, i.e., in the interval $[t_e^{\mathsf{lb}}, t_e^{\mathsf{ub}}]$, we get that

$$\begin{aligned} W_{p_2}(\lambda_1) &= C_{p_2}(\lambda_1) + \lambda_1 T_{p_2}(\lambda_1) \\ &= \sum_{e\in p_2} \left(c_e(t_e^*(\lambda_1)) + \lambda_1 t_e^*(\lambda_1)\right) \\ &= \sum_{e\in p_2} \min_{t_e^{\mathsf{lb}} \leq t_e \leq t_e^{\mathsf{ub}}} \left(c_e(t_e) + \lambda_1 t_e\right) \\ &\leq \sum_{e\in p_2} \left(c_e(t_e^*(\lambda_2)) + \lambda_1 t_e^*(\lambda_2)\right) \\ &= C_{p_2}(\lambda_2) + \lambda_1 T_{p_2}(\lambda_2). \end{aligned} \quad (45)$$

Similarly, we have

$$W_{p_1}(\lambda_2) = C_{p_1}(\lambda_2) + \lambda_2 T_{p_1}(\lambda_2) \leq C_{p_1}(\lambda_1) + \lambda_2 T_{p_1}(\lambda_1). \quad (46)$$

Inserting (45) into (43), we get that

$$C_{p_1}(\lambda_1) + \lambda_1 T_{p_1}(\lambda_1) \leq C_{p_2}(\lambda_2) + \lambda_1 T_{p_2}(\lambda_2),$$

which implies that

$$\lambda_1 \left[T_{p_1}(\lambda_1) - T_{p_2}(\lambda_2)\right] \leq C_{p_2}(\lambda_2) - C_{p_1}(\lambda_1). \quad (47)$$

Similarly, inserting (46) into (44), we get that

$$C_{p_2}(\lambda_2) + \lambda_2 T_{p_2}(\lambda_2) \leq C_{p_1}(\lambda_1) + \lambda_2 T_{p_1}(\lambda_1),$$

[9]Paths p_1 and p_2 could be the same.

which implies that

$$-\lambda_2 \left[T_{p_1}(\lambda_1) - T_{p_2}(\lambda_2)\right] \leq C_{p_1}(\lambda_1) - C_{p_2}(\lambda_2). \quad (48)$$

Summing (47) and (48), we get that

$$(\lambda_1 - \lambda_2)\left[T_{p_1}(\lambda_1) - T_{p_2}(\lambda_2)\right] \leq 0. \quad (49)$$

Since we assume that $\lambda_1 < \lambda_2$, we must have

$$T_{p_1}(\lambda_1) \geq T_{p_2}(\lambda_2). \quad (50)$$

The proof is completed.

Appendix I
Proof of Theorem 4

At the point λ_0, the dual function has value

$$\begin{aligned} D(\lambda_0) &= -\lambda_0 T + \min_{x\in\mathcal{X}} \sum_{e\in\mathcal{E}} x_e \cdot \min_{t_e^{\mathsf{lb}} \leq t_e \leq t_e^{\mathsf{ub}}} (c_e(t_e) + \lambda_0 t_e) \\ &= -\lambda_0 T + \min_{x\in\mathcal{X}} \sum_{e\in\mathcal{E}} x_e \cdot (c_e(t_e^*(\lambda_0)) + \lambda_0 t_e^*(\lambda_0)) \\ &= -\lambda_0 T + \sum_{e\in p^*(\lambda_0)} \left[c_e(t_e^*(\lambda_0)) + \lambda_0 t_e^*(\lambda_0)\right] \\ &= -\lambda_0 T + \sum_{e\in p^*(\lambda_0)} c_e(t_e^*(\lambda_0)) + \lambda_0 \sum_{e\in p^*(\lambda_0)} t_e^*(\lambda_0) \\ &= -\lambda_0 T + \lambda_0 \delta(\lambda_0) + \sum_{e\in p^*(\lambda_0)} c_e(t_e^*(\lambda_0)) \\ &= -\lambda_0 T + \lambda_0 T + \sum_{e\in p^*(\lambda_0)} c_e(t_e^*(\lambda_0)) \\ &= \sum_{e\in p^*(\lambda_0)} c_e(t_e^*(\lambda_0)). \end{aligned} \quad (51)$$

On one hand, we know that any dual function value will be a lower bound of OPT according to the weak duality. Thus,

$$D(\lambda_0) \leq \mathsf{OPT}. \quad (52)$$

On the other hand, we know that $p^*(\lambda_0)$ is a feasible path and $\{t_e^*(\lambda_0), e \in p^*(\lambda_0)\}$ satisfies

$$\sum_{e\in p^*(\lambda_0)} t_e^*(\lambda_0) = T. \quad (53)$$

Here $p^*(\lambda_0)$ and $\{t_e^*(\lambda_0), e \in p^*(\lambda_0)\}$ is a feasible solution to PASO with the objective value $\sum_{e\in p^*(\lambda_0)} c_e(t_e^*(\lambda_0)) = D(\lambda_0)$, which is an upper bound of OPT, i.e.,

$$D(\lambda_0) \geq \mathsf{OPT}. \quad (54)$$

Eq. (52) and (54) conclude that $D(\lambda_0) = \mathsf{OPT}$, and $p^*(\lambda_0)$ and $\{t_e^*(\lambda_0), e \in p^*(\lambda_0)\}$ is an optimal solution to PASO.

The proof is completed.

Appendix J
Proof of Theorem 5

First, if we let total travel delay be $T' = \sum_{e\in p^*(\lambda_L)} t_e^*(\lambda_L) > T$, we get a relaxed version of PASO. According to Theorem 4, we know that $\mathsf{LB} = \sum_{e\in p^*(\lambda_L)} c_e(t_e^*(\lambda_L))$ is the optimal solution of the relaxed version, and thus we have $\mathsf{LB} \leq \mathsf{OPT}$.

Second, since $\sum_{e \in p^*(\lambda_U)} t_e^*(\lambda_U) < T$, we know that $p^*(\lambda_U)$ and $\{t_e^*(\lambda_U) : e \in p^*(\lambda_U)\}$ is a feasible solution to PASO. Thus, $\mathsf{UB} = \sum_{e \in p^*(\lambda_L)} c_e(t_e^*(\lambda_U)) \geq \mathsf{OPT}$.

The proof is completed.