A Unified Energy Efficiency and Spectral Efficiency Tradeoff Metric in Wireless Networks

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***Abstract*—In this letter, we propose a unified metric for energy efficiency (EE) and spectral efficiency (SE) tradeoff design in wireless networks. Different from previous EE-based or SE-based design, we simultaneously optimize both EE and SE. First, we formulate our design problem as a multi-object optimization (MOO) problem, in which the Pareto optimal set is characterized. After normalizing EE and SE to make them comparable, we then convert the MOO problem into a single-object optimization (SOO) problem by the weighted product scalarization method. We further show that the objective function of this SOO problem, i.e., our proposed EE and SE tradeoff (EST) metric, is quasi-concave with the transmit power and a unique globally optimal solution is derived. Numerical results validate the effectiveness of the proposed unified EST metric.**

***Index Terms*—Energy efficiency (EE), spectral efficiency (SE), EE and SE tradeoff (EST), multi-object optimization (MOO).**

I. Introduction

SPECTRAL efficiency (SE), defined as the throughput per unit of bandwidth, is one of the most important metrics for wireless network design. With the increasing attention on energy saving and the development of green radio techniques [1], [2], energy efficiency (EE), defined as the number of bits that can be transmitted per unit of energy consumption [3], is also becoming important for wireless communications. Ideally, it is desirable to maximize EE and SE simultaneously. However, maximizing one metric (EE or SE) does not mean that the other one is also maximized. In fact, the optimal EE performance often leads to low SE performance and vice verse [1]. Therefore, it is often imperative to make a tradeoff between EE and SE.

Existing studies on EST can be divided into two categories: one is to characterize the relationship between EE and SE as accurately as possible [4], [5]; the other is to maximize EE with an SE requirement [6], [7]. More specifically, in [4] and [5], a closed-form approximation for EE-SE relationship is derived for different communication systems. In [6], the EST optimization problem in downlink orthogonal frequency division multiple access (OFDMA) networks is formulated to maximize EE with a minimal SE requirement. More recently, the EST optimization problem in downlink distributed antenna systems is investigated in [7] to maximize EE while keeping SE at a certain level. There are certain limitations for these previous works. For the first category, due to lack of a unique solution for global maximization, the EE-SE relationship cannot give direct guidance for the system operator; for the second category, maximizing EE while regarding SE as a constraint is inflexible and it restricts the SE performance.

To overcome the limitations of these previous design criteria, we propose a unified EST metric that can be used to optimize both EE and SE simultaneously. We first show that the optimization of both EE and SE can be formulated as a MOO problem, for which the Pareto optimal set will be characterized. To find a unique globally optimal solution, this MOO problem is then transformed into a SOO problem via the weighted product scalarization method, which yields our proposed unified EST metric. We prove that the proposed EST metric is quasi-concave with the transmit power. Numerical results demonstrate that the proposed EST metric covers both the SE-based and EE-based designs, and it is flexible to make tradeoffs between EE and SE for different preferences.

II. System Model

Consider a point-to-point communication link, where both the transmitter and receiver are equipped with only one antenna. Denote P as the transmit power, G as the channel gain between the transmitter and the receiver, and σ^2 as the variance of the additive white Gaussian noise at the receiver. Correspondingly, the signal-to-noise ratio (SNR) at the receiver can be defined as

$$\Gamma = \frac{GP}{\sigma^2} = \rho P, \tag{1}$$

where $\rho = G/\sigma^2$. SE is defined as the transmission rate per unit of bandwidth, which is given by

$$f_{SE}(P) = \log(1+\Gamma). \tag{2}$$

EE, defined as the ratio of SE to the total power consumption[1], can be shown as

$$f_{EE}(P) = \frac{f_{SE}(P)}{P+P_c} = \frac{\log(1+\Gamma)}{P+P_c}, \tag{3}$$

where the denominator is the total power consumption and P_c represents the average power consumed in the relevant electronic devices.

[1]Note that here EE is defined as the bits per unit of energy consumption per unit of bandwidth, which is different from [3]. But it does not affect our results due to the constant bandwidth factor.

Manuscript received August 7, 2012. The associate editor coordinating the review of this letter and approving it for publication was K. K. Wong.

This work was supported by the National 863 Program of China under Grant No.2011AA01A105, and the Key Deployment Project of the Chinese Academic of Sciences (KGZD-EW-103).

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Digital Object Identifier 10.1109/LCOMM.2012.112812.121744

Our goal is to maximize EE and SE simultaneously. Thus, we obtain an EE-SE MOO problem as

$$\mathbf{P(1)} \quad \max_{P} \{f_{SE}(P), f_{EE}(P)\} \quad s.t. \quad P \in \mathcal{P}$$

where $\mathcal{P} = \{P | 0 \leq P \leq P_{\max}\}$ is the transmit power constraint with $P_{\max}$ denoting the maximal transmit power.

III. EE-SE Tradeoff and a New EST Metric

A. The Pareto Optimal Set

Pareto optimality is a basic concept in MOO, which is stated below.

Definition 1: A point, $P_0 \in \mathcal{P}$ is *Pareto optimal* if and only if there does not exist any other point, $P' \in \mathcal{P}$, such that $f_{EE}(P') \geq f_{EE}(P_0)$, $f_{SE}(P') \geq f_{SE}(P_0)$ and at least one (f_{EE} or f_{SE}) has been strictly improved.

Definition 2: The *Pareto optimal set* is the set of all Pareto optimal points.

In other words, a point is Pareto optimal if there is no other point that can improve both EE and SE simultaneously. To solve an MOO, the Pareto optimal set needs to be characterized, in which we can find the globally optimal solution. To get the Pareto optimal set of $\mathbf{P(1)}$, we first summarize the properties of EE and SE as shown in the following lemmas:

Lemma 1: $f_{SE}(P)$ is strictly increasing and concave with P.

Lemma 2: There exists one and only one point $P^* \in [0, \infty)$ that maximizes $f_{EE}(P)$. $f_{EE}(P)$ is strictly increasing and concave at $P \in [0, P^*)$ while strictly decreasing and neither concave nor convex at $P \in [P^*, \infty)$.

Proof: We get the first derivative of $f_{EE}(P)$ as

$$f'_{EE}(P) = \frac{\alpha(P)}{(1 + \rho P)(P + P_c)^2}, \tag{4}$$

where

$$\alpha(P) = \rho(P + P_c) - (1 + \rho P)\log(1 + \rho P). \tag{5}$$

Since

$$\alpha'(P) = -\rho \log(1 + \rho P) < 0, \tag{6}$$

$\alpha(P)$ is decreasing with P. Denote P^* as $f'_{EE}(P^*) = 0$, i.e., $\alpha(P^*) = 0$. Then Lemma 2 holds according to the proof of Lemma 1 in [8]. ■

Next the Pareto optimal set of $\mathbf{P(1)}$, denoted as $\mathcal{P}^{POS}$, is characterized in the following proposition.

Proposition 1: The Pareto optimal set of $\mathbf{P(1)}$ is

$$\mathcal{P}^{POS} = \begin{cases} \{P | P^* \leq P \leq P_{\max}\} & \text{if} \quad P^* < P_{\max} \\ \{P | P = P_{\max}\} & \text{if} \quad P^* \geq P_{\max} \end{cases} \tag{7}$$

Proof: See Appendix A. ■

From Proposition 1, in the case of $P^* \geq P_{\max}$, $\mathcal{P}^{POS}$ contains a single point, i.e., $P_{\max}$, which means that the globally optimal solution for $\mathbf{P(1)}$ is unique. So we will only analyze the case of $P^* < P_{\max}$ in the rest of this letter. Since the globally optimal solution to $\mathbf{P(1)}$ must be Pareto optimal, we can equivalently transform the MOO problem $\mathbf{P(1)}$ to $\mathbf{P(2)}$,

$$\mathbf{P(2)} \quad \max_{P} \{f_{SE}(P), f_{EE}(P)\} \quad s.t. \quad P \in \mathcal{P}^{POS}$$

where $\mathcal{P}^{POS} = \{P^* \leq P \leq P_{\max}\}$.

B. The Unified EST Metric

Though all points in $\mathcal{P}^{POS}$ is Pareto optimal, we should try to find a unique global solution to facilitate system design. An efficient solution to distinguish a unique point in the Pareto optimal set is the scalarization methods, especially with a priori articulation of preferences. In this letter, we assume that the priori articulation of preferences for EE and SE is provided by the system operator. Then we will apply scalarization methods to transform the MOO problem $\mathbf{P(2)}$ to a single-object optimization (SOO) problem. To apply the scalarization methods, it is important to make all object functions comparable [9]. We then will take a function normalization process to regard EE and SE as quantities without an associated physical unit so that EE and SE are comparable to be integrated in a utility function.

Denote $f_{EE}^{\max}$ and $f_{SE}^{\max}$ as the maximal EE and SE in $\mathcal{P}^{POS}$ respectively. Then we have $f_{SE}^{\max} = f_{SE}(P_{\max})$ according to Lemma 1 and $f_{EE}^{\max} = f_{EE}(P^*)$ according to Lemma 2. As such, we will make EE and SE dimensionless via the following function normalization process [9],

$$f_{SE}^{norm}(P) = \frac{f_{SE}(P)}{f_{SE}^{\max}}, \tag{8}$$

$$f_{EE}^{norm}(P) = \frac{f_{EE}(P)}{f_{EE}^{\max}}. \tag{9}$$

Clearly, now we have $f_{SE}^{norm}(P) \in (0, 1]$ and $f_{EE}^{norm}(P) \in (0, 1]$, respectively. Then the MOO problem $\mathbf{P(2)}$ is equivalent to $\mathbf{P(3)}$,

$$\mathbf{P(3)} \quad \max_{P} \quad \{f_{SE}^{norm}(P), f_{EE}^{norm}(P)\} \quad s.t. \quad P \in \mathcal{P}^{POS}$$

Next we will propose our unified EST metric with any given preference configuration. The weighted product scalarization method [9] is used since it is similar to the widely-used Cobb-Douglas production function in economics [10]. The Cobb-Douglas production function matches well with history data, so it is reasonable for us to choose its expression where we can regard the utility as the production, and regard SE, EE as labor, capital respectively.

Denoting $w \in [0, 1]$ as the preference for SE [2], we define a new EST metric as

$$U(P) = [f_{SE}^{norm}(P)]^{w} \times [f_{EE}^{norm}(P)]^{1-w}. \tag{10}$$

Note that $U(P)$ is also referred as the utility function considering both EE and SE with a given preference configuration.

Thus $\mathbf{P(3)}$ can be transformed to the following SOO problem,

$$\mathbf{P(4)} \quad \max_{P} U(P) \quad s.t. \quad P \in \mathcal{P}^{POS}$$

To solve $\mathbf{P(4)}$, we first prove the following lemma,

Lemma 3: $U(P)$ is strictly quasi-concave in $\mathcal{P}^{POS}$.

Proof: First, we do the following utility transformation,

$$\begin{aligned} V(P) &= \log U(P) \\ &= w \log f_{SE}^{norm}(P) + (1 - w) \log f_{EE}^{norm}(P) \\ &= \log f_{SE}(P) - (1 - w)\log(P + P_c) - \gamma, \end{aligned} \tag{11}$$

[2] w for SE means $(1 - w)$ for EE, so $(w, 1 - w)$ is a given preference configuration for SE and EE.

where $\gamma = w \log f_{SE}^{\max} + (1-w)\log f_{EE}^{\max}$ is a constant. Then the first derivation of $V(P)$ is

$$\begin{aligned} V'(P) &= \frac{f'_{SE}(P)}{f_{SE}(P)} - \frac{1-w}{P+P_c} \\ &= \frac{\rho(P+P_c) - (1-w)(1+\rho P)\log(1+\rho P)}{(1+\rho P)(P+P_c)\log(1+\rho P)} \\ &= \frac{\beta(P) - (1-w)}{P+P_c}, \end{aligned} \tag{12}$$

where

$$\beta(P) = \frac{\rho(P+P_c)}{(1+\rho P)\log(1+\rho P)}. \tag{13}$$

Let $V'(P) = 0$, yielding

$$\beta(P) = 1 - w. \tag{14}$$

According to Appendix B, we can get that $\beta(P)$ is strictly decreasing with P. Then in $\mathcal{P}^{POS}$, we have

$$0 < \beta(P_{\max}) \le \beta(P) \le \beta(P^*) = 1, \tag{15}$$

where $\beta(P^*) = 1$ can be easily derived from $\alpha(P^*) = 0$.

Since $1 - w \in [0, 1]$, we have the following two cases.

Case 1: $w > 1 - \beta(P_{\max})$

In this case, we have $1-w < \beta(P_{\max}) \le \beta(P)$. Thus, (14) has no solution, and $V'(P) > 0$. Therefore, $V(P)$ is strictly increasing in $\mathcal{P}^{POS}$. Since all strictly monotonic functions are strictly quasi-concave [11], $V(P)$ is strictly quasi-concave in $\mathcal{P}^{POS}$.

Case 2: $w \le 1 - \beta(P_{\max})$

In this case, we have $\beta(P_{\max}) \le 1 - w \le 1$. So (14) has one and only one solution due to $\beta(P)$'s strictly decreasing. Denote the unique solution as P^{**}. In $[P^*, P^{**})$, since $1-w < \beta(P)$, we have $V'(P) > 0$. And in $(P^{**}, P_{\max}]$, since $1-w > \beta(P)$, we have $V'(P) < 0$. Therefore, $V(P)$ is first strictly increasing and then strictly decreasing in $\mathcal{P}^{POS}$, which means $V(P)$ is strictly quasi-concave in $\mathcal{P}^{POS}$. Since $U(P) = e^{V(P)}$ is increasing with $V(P)$, $U(P)$ is strictly quasi-concave in $\mathcal{P}^{POS}$ [11]. ∎

Furthermore, with the proof of Lemma 3, we also get the optimal solution for $\mathbf{P(4)}$, as shown in the following proposition.

Proposition 2: The optimal solution for $\mathbf{P(4)}$ is

$$P^{opt} = \begin{cases} P_{\max} & \text{if} \quad w > 1 - \beta(P_{\max}) \\ P^{**} & \text{if} \quad w \le 1 - \beta(P_{\max}) \end{cases} \tag{16}$$

Proof: The analysis of two cases in the proof of Lemma 3 completes the proof. ∎

C. Some Discussions

Based on Proposition 2, we can make some discussions to provide insights to the EST. First, the preference for SE (i.e., w) can be divided into the following three cases. (1) $w = 1$, i.e., only SE is considered. Thus $w > 1 - \beta(P_{\max})$ and the optimal power is $P^{opt} = P_{\max}$, which is in accordance with the SE-oriented analysis. (2) $w = 0$, i.e., only EE is considered. Then $w \le 1 - \beta(P_{\max})$ and the optimal power is $P^{opt} = P^{**}$. In addition, P^{**} satisfies (14), i.e., $\beta(P^{**}) = 1 - w = 1$. Due to the monotonicity of $\beta(P)$, we can get $P^{**} = P^*$ [3]. So $P^{opt} = P^*$, which is in accordance with the EE-oriented analysis. (3) $0 < w < 1$, i.e., both EE and SE are considered. Then we can adjust w for different preference configurations, and then obtain the optimal system performance to maximize the unified EST metric, i.e., $U(P)$.

In addition, we have the following properties for the impact of preference for SE, i.e., w.

Property 1: The optimal power P^{opt} is non-decreasing with the preference for SE, i.e., w. Specifically, when $w = 0$, the optimal power $P^{opt} = P^*$; when $0 \le w \le 1 - \beta(P_{\max})$, P^{opt} is strictly increasing with w; when $w > 1 - \beta(P_{\max})$, P^{opt} will remain to be $P^{\max}$.

Proof: Note that 1-$\beta(P)$ is strictly increasing with P, then Property 1 holds according to Proposition 2. ∎

Property 2: $f_{SE}^{norm}(P^{opt})$ is non-decreasing with the preference for SE, i.e., w, while $f_{EE}^{norm}(P^{opt})$ is non-increasing with w. In addition, there exists one and only one $w^0 \in (0, 1 - \beta(P_{\max}))$ such that $f_{SE}^{norm}(P^{opt}) = f_{EE}^{norm}(P^{opt}) = U(P^{opt})$.

Proof: See Appendix C. ∎

IV. Numerical Results

In this section, numerical results are provided to validate the effectiveness of our proposed unified EST metric. We consider a static channel gain $G = 1$, the noise variance $\sigma^2 = 0.01$W, and the device power consumption $P_c = 0.1$W. Moreover, $P_{\max}$ is assumed to be 1W in our simulation. Fig. 1 shows the optimal transmit power with our proposed EST metric, which is in line with Proposition 2 and Property 1. Fig. 2 shows the normalized SE, EE and the utility achieved with the optimal transmit power P^{opt}. It verifies Property 2. It is clear that the normalized SE will be non-decreasing while the normalized EE will be non-increasing with w, and the intersection of the three lines is shown in Fig. 2. Furthermore, the normalized SE and EE will stay to be constants when w is close to 1, which matches the observation in Fig. 1. This phenomenon can simplify the selection of the preference w. To better understand our proposed EST metric, we illustrate the EE-SE relationship [4] in Fig. 3. It can be seen that the optimal solution leads to maximal EE with $w = 0$ while to maximal SE with $w = 1$. Meanwhile, we can make a tradeoff between EE and SE for different preferences based on the EST metric. For example, we can choose w to be 0.5 if we equally prefer EE and SE. And the designer can regard w as the guideline for wireless communication system design. Rather than only considering the point-to-point communication, we will study the unified EST metric for more complicated networks in the future.

Appendix A
Proof of Proposition 1

Proof: First, $f_{SE}(P)$ is increasing at $[0, P_{\max}]$ according to Lemma 1. If $P^* \ge P_{\max}$, $f_{EE}(P)$ is increasing at $[0, P_{\max}]$ according to Lemma 2. Thus, $\forall P \in [0, P_{\max})$, we have $f_{SE}(P_{\max}) > f_{SE}(P)$, and $f_{EE}(P_{\max}) > f_{EE}(P)$, which results in $\mathcal{P}^{POS} = \{P | P = P_{\max}\}$. If $P^* < P_{\max}$, $f_{EE}(P)$ is increasing at $[0, P^*)$ while decreasing at $[P^*, P_{\max}]$ due to Lemma 2. $\forall P \in [0, P^*)$, $f_{SE}(P^*) > f_{SE}(P)$, and

[3]Since $\beta(P) = \rho(P+P_c)/[(1+\rho P)\log(1+\rho P)] = 1$ is equal to $\alpha(P) = \rho(P+P_c) - (1+\rho P)\log(1+\rho P) = 0$ and $\alpha(P^*) = 0$.

[4]Note that in Fig. 3, P^* and $P_{\max}$ are not values for x-axis, but the corresponding transmit power for related SE.

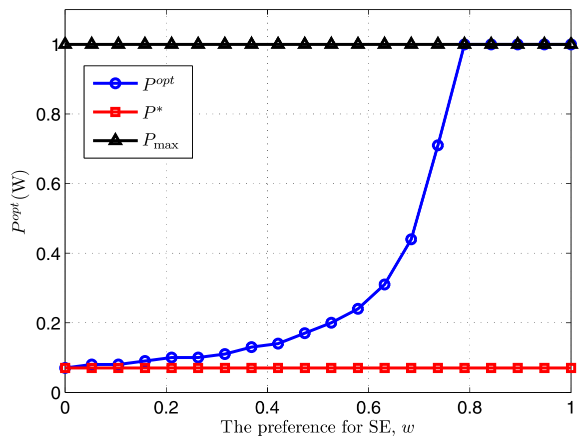

Fig. 1. The optimal transmit power with different w.

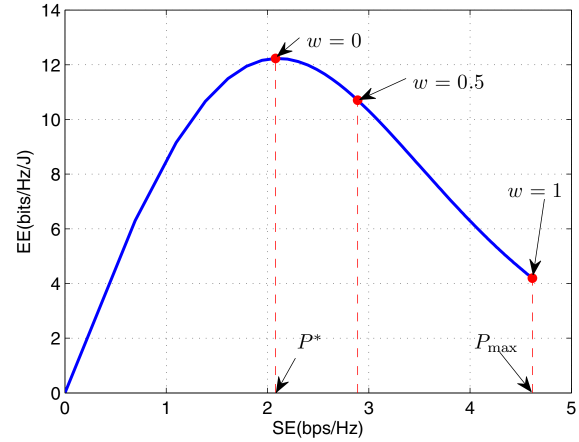

Fig. 3. The EE-SE relationship.

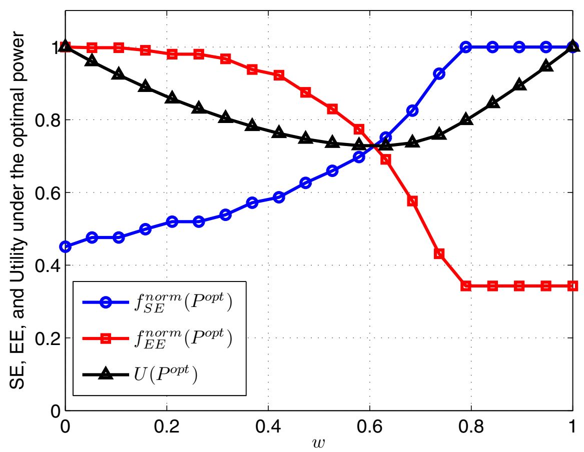

Fig. 2. $f_{SE}^{norm}(P)$, $f_{EE}^{norm}(P)$ and $U(P)$ under the optimal transmit power with different w.

$f_{EE}(P^*) > f_{EE}(P)$, which means $[0, P^*) \nsubseteq \mathcal{P}^{POS}$. However, $\forall P \in [P^*, P_{\max}]$, there does not exist any other point P' such that both $f_{SE}(P') \geq f_{SE}(P)$, and $f_{EE}(P') \geq f_{EE}(P)$. Thus, $\mathcal{P}^{POS} = \{P | P^* \leq P \leq P_{\max}\}$. ■

Appendix B
Proof of Monotonicity of $\beta(P)$

Proof: First, we have

$$\beta(P) = \frac{\rho P}{(1+\rho P)\log(1+\rho P)} + \frac{\rho P_c}{(1+\rho P)\log(1+\rho P)}. \tag{17}$$

Then, we get the first derivative of the first item in (17) as

$$\left[\frac{\rho P}{(1+\rho P)\log(1+\rho P)}\right]' = \frac{\rho P[\log(1+\rho P) - \rho P]}{[(1+\rho P)\log(1+\rho P)]^2} \leq 0. \tag{18}$$

Since the second item in (17) is strictly decreasing with P, we achieve that $\beta(P)$ is strictly decreasing with P. ■

Appendix C
Proof of Property 2

Proof: Since P^{opt} is non-decreasing with w as shown in Property 1, $f_{SE}^{norm}(P^{opt}) = \frac{f_{SE}(P^{opt})}{f_{SE}^{\max}}$ is non-decreasing with w according to Lemma 1. Similarly, with Property 1 and Lemma 2, we obtain that $f_{EE}^{norm}(P^{opt})$ is non-increasing with w. Specifically, as shown in Proposition 1, P^{opt} is strictly increasing with w when $w \in [0, 1-\beta(P_{\max})]$ and remains to be $P^{\max}$ when $w \in (1-\beta(P_{\max}), 1]$. Thus, we can obtain that $f_{SE}^{norm}(P^{opt})$ ($f_{EE}^{norm}(P^{opt})$) is strictly increasing (decreasing) with w when $w \in [0, 1-\beta(P_{\max})]$ and keeps to be a constant when $w \in (1-\beta(P_{\max}), 1]$. In addition, when $w = 0$, we have $P^{opt} = P^*$, and thus $f_{EE}^{norm}(P^{opt}) = 1 > f_{SE}^{norm}(P^{opt})$. When $w \in (1-\beta(P_{\max}), 1]$, we have $P^{opt} = P^{\max}$, and thus $f_{SE}^{norm}(P^{opt}) = 1 > f_{EE}^{norm}(P^{opt})$. So there exists one and only one $w^0 \in (0, 1-\beta(P_{\max}))$ such that $f_{SE}^{norm}(P^{opt}) = f_{EE}^{norm}(P^{opt})$. Then $U(P^{opt}) = [f_{SE}^{norm}(P^{opt})]^w \times [f_{EE}^{norm}(P^{opt})]^{1-w} = f_{SE}^{norm}(P^{opt}) = f_{EE}^{norm}(P^{opt})$. ■

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