homework sheet 03

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1 Assignment: Basic Probability

Problem 1

$T \mathrel{\hat{=}}$ terrorist and $\neg T \mathrel{\hat{=}}$ not a terrorist
$S \mathrel{\hat{=}}$ positive scan as terrorist and $\neg S \mathrel{\hat{=}}$ negative scan as terrorist

The following information about probabilities are included in the text:

$p(T) = 0.01$, $p(\neg T) = 0.99$
$p(S|T) = 0.95$, $p(\neg S|T) = 0.05$
$p(\neg S|\neg T) = 0.95$, $p(S|\neg T) = 0.05$

To find: $p(T|S)$

$$\begin{aligned} p(T|S) &= \frac{p(S|T)p(T)}{p(S)} && (1)\\ &= \frac{p(S|T)p(T)}{p(S|T)p(T) + p(S|\neg T)p(\neg T)} && (2)\\ &= \frac{0.95 \cdot 0.01}{0.95 \cdot 0.01 + 0.05 \cdot 0.99} && (3)\\ &\approx 0.161 && (4) \end{aligned}$$

Problem 2

$B \mathrel{\hat{=}}$ Box and $D \mathrel{\hat{=}}$ Drawn
$r \mathrel{\hat{=}}$ red and $w \mathrel{\hat{=}}$ white

To find: $p(B = 2red|D = 3red)$

$$\begin{aligned} p(B = 2r|D = 3r) &= \frac{p(D = 3r|B = 2r)p(B = 2r)}{p(D = 3r)} && (5)\\ &= \frac{p(D = 3r|B = 2r)p(B = 2r)}{p(D = 3r|B = 2r)p(B = 2r) + p(D = 3r|B = 2w)p(B = 2w) + p(D = 3r|B = 1r1w)p(B = 1r1w)} && (6)\\ &= \frac{1 \cdot (0.5 \cdot 0.5)}{1 \cdot (0.5 \cdot 0.5) + 0 \cdot (0.5 \cdot 0.5) + (0.5 \cdot 0.5 \cdot 0.5) \cdot (2 \cdot 0.5 \cdot 0.5)} && (7)\\ &= \frac{0.25}{0.25 + 0.125 \cdot 0.5} && (8)\\ &= \frac{0.25}{0.3125} && (9)\\ &= 0.8 && (10) \end{aligned}$$

Problem 3

Mean:

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x \cdot p(x)dx && (11)\\
&= \int_{0}^{1} x \cdot 1 && (12)\\
&= \left[\frac{1}{2}x^2\right]_0^1 && (13)\\
&= \frac{1}{2} - 0 && (14)\\
&= \frac{1}{2} && (15)
\end{aligned}$$

Variance:

$$\begin{aligned}
Var[X] &= E[X^2] - E[X]^2 && (16)\\
&= \int_{-\infty}^{\infty} x^2 \cdot p(x)dx - \left(\frac{1}{2}\right)^2 && (17)\\
&= \left[\frac{1}{3}x^3\right]_0^1 - \frac{1}{4} && (18)\\
&= \frac{1}{3} - 0 - \frac{1}{4} && (19)\\
&= \frac{1}{12} && (20)
\end{aligned}$$

Problem 4

Equation 1:

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x \cdot p(x)dx && (21)\\
&= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} p_{X,Y}(x,y)dydx && (22)\\
&= \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} p_Y(y) \cdot p_{X|Y}(x|y)dydx && (23)\\
&= \int_{-\infty}^{\infty} p_Y(y) \int_{-\infty}^{\infty} x \cdot p_{X|Y}(x|y)dxdy && (24)\\
&= \int_{-\infty}^{\infty} p_Y(y)E_{X|Y}[X]dy && (25)\\
&= E_Y[E_{X|Y}[X]] && (26)
\end{aligned}$$

Equation 2:

With usage of equation1:

$$\begin{aligned}
E_Y[Var_{X|Y}(X)] + Var_Y[E_{X|Y}(X)] &= && (27)\\
E_Y[E_{X|Y}(X^2) - E_{X|Y}(X)^2] + E_Y[E_{X|Y}(X)^2] - E_Y[E_{X|Y}(X)]^2 &= && (28)\\
E_Y[E_{X|Y}(X^2)] - E_Y[E_{X|Y}(X)^2] + E_Y[E_{X|Y}(X)^2] - E_Y[E_{X|Y}(X)]^2 &= && (29)\\
E_Y[E_{X|Y}(X^2)] - E_Y[E_{X|Y}(X)]^2 &= && (30)\\
E[X^2] - E[X]^2 &= && (31)\\
Var[X] & && (32)
\end{aligned}$$

2 Assignment: Probability Inequalities

2.1 Markov Inequality

Problem 5

To show:

$$P(X > c) \leq \frac{E[X]}{c} \tag{33}$$

$$\begin{aligned}
P(X > c) &\leq \frac{E[X]}{c} &(34)\\
\sum_{x>c} p(x) &\leq \frac{\sum_x x \cdot p(x)}{c} &(35)\\
\sum_{x>c} p(x) &\leq \frac{\sum_{x\leq c} x \cdot p(x) + \sum_{x>c} x \cdot p(x)}{c} &(36)\\
0 &\leq \frac{\sum_{x\leq c} x \cdot p(x) + \sum_{x>c} x \cdot p(x)}{c} - \sum_{x>c} p(x) &(37)\\
0 &\leq \sum_{x\leq c} x \cdot p(x) + \sum_{x>c} x \cdot p(x) - \sum_{x>c} c \cdot p(x) &(38)
\end{aligned}$$

The first sum in the right term of inequality 38 is greater 0 since the random variable is non-negative and probabilities are always greater than 0.

For the second and third sum the following holds:

$$\sum_{x>c} c \cdot p(x) \quad \leq \quad \sum_{x>c} x \cdot p(x) \tag{39}$$

since each element of the right sum is greater than the corresponding element of the left due to the relation $x > c$. Therefore,

$$0 \leq \sum_{x>c} x \cdot p(x) - \sum_{x>c} c \cdot p(x) \tag{40}$$

This proves that the relation in inequality 38 is correct. Therefore, the Markov inequality is proved.

Application of the Markov inequality:

Let X be a random variable for drawing a head by flipping a fair coin n times.

$$P(X > \frac{3}{4}n) \leq \frac{E[X]}{\frac{3}{4}n} \tag{41}$$

With $E[X] = \frac{1}{2}n$ we get:

$$P(X > \frac{3}{4}n) \leq \frac{\frac{1}{2}n}{\frac{3}{4}n} = \frac{2}{3} \tag{42}$$

2.2 Chebyshev Inequality

Problem 6

To show:

$$P(|X - E[X]| > a) \leq \frac{Var[X]}{a^2} \tag{43}$$

From the Markov inequality we know:

$$P(X > a) \leq \frac{E[X]}{a} \tag{44}$$

Computing $P(|X - E[X]| > a) = P((X - E[X])^2 > a^2)$ using the Markov inequality leads to the following:

$$\begin{aligned} P(X > a) &\leq \frac{E[X]}{a} && (45) \\ \Leftrightarrow P((X - E[X])^2 > a^2) &\leq \frac{E[(X - E[X])^2]}{a^2} && (46) \\ \Leftrightarrow P(|X - E[X]| > a) &\leq \frac{Var[X]}{a^2} && (47) \end{aligned}$$

Application of the Chebyshev inequality:

Let X be a random variable for drawing a head by flipping a fair coin n times.

$$E[X] = \frac{1}{2}n \tag{48}$$

$$Var[X] = \frac{1}{4}n \tag{49}$$

With this we get:

$$P(X > \frac{3}{4}n) = P\left(\left|X - \frac{1}{2}n\right| > \frac{1}{4}n\right) \leq \frac{\frac{1}{4}n}{\left(\frac{1}{4}n\right)^2} = \frac{4}{n} \tag{50}$$

2.3 Jensen's Inequality

Problem 7