homework sheet 03

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1 Assignment: Basic Probability

Problem 1

$T \hat{=}$ terrorist and $\neg T \hat{=}$ not a terrorist
$S \hat{=}$ positive scan as terrorist and $\neg S \hat{=}$ negative scan as terrorist

The following information about probabilities are included in the text:

$p(T) = 0.01$, $p(\neg T) = 0.99$
$p(S|T) = 0.95$, $p(\neg S|T) = 0.05$
$p(\neg S|\neg T) = 0.95$, $p(S|\neg T) = 0.05$

To find: $p(T|S)$

$$\begin{aligned} p(T|S) &= \frac{p(S|T)p(T)}{p(S)} && (1) \\ &= \frac{p(S|T)p(T)}{p(S|T)p(T) + p(S|\neg T)p(\neg T)} && (2) \\ &= \frac{0.95 \cdot 0.01}{0.95 \cdot 0.01 + 0.05 \cdot 0.99} && (3) \\ &\approx 0.161 && (4) \end{aligned}$$

Problem 2

$B \hat{=}$ Box and $D \hat{=}$ Drawn
$r \hat{=}$ red and $w \hat{=}$ white

To find: $p(B = 2red|D = 3red)$

$$\begin{aligned} p(B = 2r|D = 3r) &= \frac{p(D = 3r|B = 2r)p(B = 2r)}{p(D = 3r)} && (5) \\ &= \frac{p(D = 3r|B = 2r)p(B = 2r)}{p(D = 3r|B = 2r)p(B = 2r) + p(D = 3r|B = 2w)p(B = 2w) + p(D = 3r|B = 1r1w)p(B = 1r1w)} && (6) \\ &= \frac{1 \cdot (0.5 \cdot 0.5)}{1 \cdot (0.5 \cdot 0.5) + 0 \cdot (0.5 \cdot 0.5) + (0.5 \cdot 0.5 \cdot 0.5) \cdot (2 \cdot 0.5 \cdot 0.5)} && (7) \\ &= \frac{0.25}{0.25 + 0.125 \cdot 0.5} && (8) \\ &= \frac{0.25}{0.3125} && (9) \\ &= 0.8 && (10) \end{aligned}$$

Problem 3

Mean:

$$\begin{aligned} E[X] &= \int_{-\infty}^{\infty} x \cdot p(x) dx && (11) \\ &= \int_0^1 x \cdot 1 && (12) \\ &= \left[\frac{1}{2}x^2\right]_0^1 && (13) \\ &= \frac{1}{2} - 0 && (14) \\ &= \frac{1}{2} && (15) \end{aligned}$$

Variance:

$$\begin{aligned} Var[X] &= E[X^2] - E[X]^2 && (16) \\ &= \int_{-\infty}^{\infty} x^2 \cdot p(x) dx - \left(\frac{1}{2}\right)^2 && (17) \\ &= \left[\frac{1}{3}x^3\right]_0^1 - \frac{1}{4} && (18) \\ &= \frac{1}{3} - 0 - \frac{1}{4} && (19) \\ &= \frac{1}{12} && (20) \end{aligned}$$

Problem 4

2 Assignment: Probability Inequalities

2.1 Markov Inequality

Problem 5

To show:

$$P(X > c) \leq \frac{E[X]}{c} \quad (21)$$

$$\begin{aligned} P(X > c) &\leq \frac{E[X]}{c} && (22) \\ \sum_{x>c} p(x) &\leq \frac{\sum_x x \cdot p(x)}{c} && (23) \\ \sum_{x>c} p(x) &\leq \frac{\sum_{x\leq c} x \cdot p(x) + \sum_{x>c} x \cdot p(x)}{c} && (24) \\ 0 &\leq \frac{\sum_{x\leq c} x \cdot p(x) + \sum_{x>c} x \cdot p(x)}{c} - \sum_{x>c} p(x) && (25) \\ 0 &\leq \sum_{x\leq c} x \cdot p(x) + \sum_{x>c} x \cdot p(x) - \sum_{x>c} c \cdot p(x) && (26) \end{aligned}$$

The first sum in the right term of inequality 26 is greater 0 since the random variable is non-negative and probabilities are always greater than 0.

For the second and third sum the following holds:

$$\sum_{x>c} c \cdot p(x) \quad \leq \quad \sum_{x>c} x \cdot p(x) \tag{27}$$

since each element of the right sum is greater than the corresponding element of the left due to the relation $x > c$. Therefore,

$$0 \leq \sum_{x>c} x \cdot p(x) - \sum_{x>c} c \cdot p(x) \tag{28}$$

This proves that the relation in inequality 26 is correct. Therefore, the Markov inequality is proved.

Application of the Markov inequality:

Let X be a random variable for drawing a head by flipping a fair coin.

$$P(X > \frac{3}{4}n) \leq \frac{E[X]}{\frac{3}{4}n} \tag{29}$$

With $E[X] = \frac{1}{2}n$ we get:

$$P(X > \frac{3}{4}n) \leq \frac{\frac{1}{2}n}{\frac{3}{4}n} = \frac{2}{3} \tag{30}$$

2.2 Chebyshev Inequality

Problem 6

2.3 Jensen's Inequality

Problem 7