homework sheet 01

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1 Assignment

For the calculation of eigenvalues and eigenvectors of the given matrix A

$$\begin{aligned} Ax &= \lambda x && (1)\\ (A-\lambda I)x &= 0 && (2)\\ det(A-\lambda I) &= 0 && (3) \end{aligned}$$

was used with x and λ being the eigenvectors and the eigenvalues of A, respectively. First the eigenvalues λ were calculated.

$$det(A-\lambda I) = 0 \tag{4}$$

$$det(\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} - \lambda I) = 0 \tag{5}$$

$$\begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0 \tag{6}$$

$$(2-\lambda)\cdot \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} - (-1)\cdot \begin{vmatrix} -1 & 0 \\ -1 & 2-\lambda \end{vmatrix} = 0 \tag{7}$$

$$(2-\lambda)\cdot((2-\lambda)^2-1) + (-1)\cdot(2-\lambda) - 0\cdot(-1) = 0 \tag{8}$$

$$(2-\lambda)\cdot((2-\lambda)^2-1) + (-1)\cdot(2-\lambda) = 0 \tag{9}$$

By division through $(2-\lambda)$ the first λ can be derived as $\lambda_1 = 2$.

$$((2-\lambda)^2-1)+(-1) = 0 \tag{10}$$

$$(2-\lambda)^2 - 2 = 0 \tag{11}$$

$$\lambda^2 - 4\lambda + 4 - 2 = 0 \tag{12}$$

$$\lambda^2 - 4\lambda + 2 = 0 \tag{13}$$

$$\lambda_{2,3} = 2 \pm \sqrt{2} \tag{14}$$

The eigenvectors of the corresponding eigenvalues can now be derived by using $(A-\lambda I)e = 0$.

For $\lambda_1 = 2$:

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \cdot x = 0 \tag{15}$$

resolves to the following system of linear equations:

$$-x_2 = 0 \tag{16}$$

$$-x_1 - x_3 = 0 \tag{17}$$

$$-x_2 = 0 \tag{18}$$

This system can be solved to

$$x_2 = 0 \tag{19}$$
$$x_1 = -x_3 \tag{20}$$

and leads to the eigenvectors

$$\begin{pmatrix} x_1 \\ 0 \\ -x_1 \end{pmatrix} \tag{21}$$

Choosing $x_1 = 1$ results in

$$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \tag{22}$$

as an eigenvector of A to the eigenvalue $\lambda_1 = 2$.

For $\lambda_2 = 2 - \sqrt{2}$:

$$\begin{pmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{pmatrix} \cdot x = 0 \tag{23}$$

resolves to the following system of linear equations:

$$\sqrt{2}x_1 - x_2 = 0 \tag{24}$$
$$-x_1 + \sqrt{2}x_2 - x_3 = 0 \tag{25}$$
$$-x_2 + \sqrt{2}x_3 = 0 \tag{26}$$

This results in

$$x_1 = x_3 \tag{27}$$
$$x_2 = \frac{2x_1}{\sqrt{2}} \tag{28}$$

and leads to the eigenvectors

$$\begin{pmatrix} x_1 \\ \sqrt{2}x_1 \\ x_1 \end{pmatrix} \tag{29}$$

Choosing $x_1 = 1$ results in

$$\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \tag{30}$$

as an eigenvector of A to the eigenvalue $\lambda_2 = 2 - \sqrt{2}$.

For $\lambda_3 = 2 + \sqrt{2}$:

$$\begin{pmatrix} -\sqrt{2} & -1 & 0 \\ -1 & -\sqrt{2} & -1 \\ 0 & -1 & -\sqrt{2} \end{pmatrix} \cdot x = 0 \tag{31}$$

resolves to the following system of linear equations:

$$-\sqrt{2}x_1 - x_2 = 0 \tag{32}$$
$$-x_1 - \sqrt{2}x_2 - x_3 = 0 \tag{33}$$
$$-x_2 - \sqrt{2}x_3 = 0 \tag{34}$$

This results in

$$x_1 = x_3 \tag{35}$$
$$x_2 = \frac{2x_1}{-\sqrt{2}} \tag{36}$$

and leads to the eigenvectors

$$\begin{pmatrix} x_1 \\ -\sqrt{2}x_1 \\ x_1 \end{pmatrix} \tag{37}$$

Choosing $x_1 = 1$ results in

$$\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix} \tag{38}$$

as an eigenvector of A to the eigenvalue $\lambda_3 = 2 + \sqrt{2}$.

The python code, solving the same problem:

```
import numpy as np
from numpy import linalg

A = np.array([(2, -1, 0), (-1, 2, -1), (0, -1, 2)])

w, v = linalg.eig(A)

print("The eigenvalues are:")
print w
print("and the eigenvectors are:")
print v
```

2 Assignment

Let $B \in \mathbb{R}^{n \times n}$ be a matrix with n linearly independent eigenvectors $x_1, x_2, ..., x_n$. U is a matrix having these eigenvectors as columns and D is a diagonal matrix with the corresponding eigenvalues λ_i in the diagonal.

To show:

$$B = UDU^{-1} \tag{39}$$

Using the rules of matrix multiplication the following transformations can be performed:

$$\begin{aligned} B &= UDU^{-1} && (40) \\ BU &= UDU^{-1}U && (41) \\ BU &= UD && (42) \end{aligned}$$

Let $j \in 1, 2, ..., n$.

The multiplication of B with U correlates with the multiplication of B with each column of U. The n resulting values written as matrix come up with the result of BU. In formulas:

$$BU = (BU_{.j})_{j=1,...,n} = (Bx_j)_{j=1,...,n} \tag{43}$$

Multiplying U and D results in a matrix where column j corresponds to the product of the eigenvector e_j i.e. column j of U and the eigenvalue λ_j. In formulas:

$$(UD_{.j})_{j=1,...,n} = (U_{.j}\lambda_j)_{j=1,...,n} = (x_j\lambda_j)_{j=1,...,n} \tag{44}$$

With 43 and 44 we can say:

$$\begin{aligned} BU &= UD && (45) \\ Bx_j &= \lambda_j x_j \quad \forall j \in 1, 2, ..., n && (46) \end{aligned}$$

which comes up with the definition of eigenvalues and eigenvectors (see 1).

3 Assignment

(1):

Let $A \in \mathbb{R}^n \times n$ be real and symmetric. Next assume

$$\lambda \in \mathbb{C} \tag{47}$$

being an eigenvalue and x its corresponding eigenvector with

$$x \in \mathbb{C} \tag{48}$$

The eigenvalue and eigenvector is defined as

$$Ax = \lambda x \tag{49}$$

and from 47 and 48 follows, that also

$$Ax \in \mathbb{C} \tag{50}$$

$$\lambda x \in \mathbb{C} \tag{51}$$

Let further $\bar{x}$ and $\bar{\lambda}$ be the complex conjugates of x and y respectively. Because of

$$(a - bi)(c - di) = \tag{52}$$

$$ac - adi - bci + bdi^2 = \tag{53}$$

$$ac - bd - (bc + ad)i = \tag{54}$$

$$ac - bd + \bar{}(bc + ad)i = \tag{55}$$

$$ac + bci + \bar{}adi + bdi^2 = \tag{56}$$

$$(a + bi)\bar{}(c + di) \tag{57}$$

the following equation is alos true

$$A\bar{x} = \bar{\lambda}\bar{x} \tag{58}$$

A left-sided multiplication of 58 and 49 with the transposed vector x or its conjugate is valid. So the following can be written:

$$\bar{x}^T Ax = \bar{x}^T \lambda x \tag{59}$$

$$x^T A\bar{x} = x^T \bar{\lambda}\bar{x} \tag{60}$$

By subtracting 60 from 59, the following equation can be stated:

$$\bar{x}^T Ax - x^T A\bar{x} = \bar{x}^T \lambda x - x^T \bar{\lambda}\bar{x} = (\lambda - \bar{\lambda}\bar{x}^T x) \tag{61}$$

As the left side of the equation equals zero, due to the symmetry property of A, also the right side has to equal zero. As $\bar{x}^T x$ cannot be zero, due to x and x^T not being the nullvector, $\lambda - \bar{\lambda}$ has to be zero:

$$\lambda - \bar{\lambda} = 0 \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R} \tag{62}$$

(2):

Assume $\lambda_i \neq \lambda_j$ The definition of eigenvalues and eigenvectors is

$$Ax_i = x_i\lambda_i \tag{63}$$

$$Ax_j = x_j\lambda_j \tag{64}$$

By left side multiplication with x_j and x_i one gets

$$x_jAx_i = x_jx_i\lambda_i \tag{65}$$

$$x_iAx_j = x_ix_j\lambda_j \tag{66}$$

Subtracting the equations leads to

$$x_jAx_i - x_iAx_j = x_jx_i\lambda_i - x_ix_j\lambda_j \tag{67}$$

Because of the symmetry of A, the left side of the equation solves to zero.

$$0 = x_jx_i\lambda_i - x_ix_j\lambda_j = (\lambda_i - \lambda_j)(x_ix_j) \tag{68}$$

As $\lambda_i - \lambda_j$ cannot be zero, due to the first assumption ($\lambda_i \neq \lambda_j$), the right part of the term (x_ix_J) has to be zero. By definition, this means, that x_i and x_j are orthogonal.

4 Assignment

$B \in \mathbb{R}^{n \times n}$ is a real symmetric matrix with distinct eigenvalues $\lambda_1, \lambda_2, \lambda_n$.

To show:

$$(1) \quad |B| = \prod_i \lambda_i \tag{69}$$

$$(2) \quad tr(B) = \sum_i \lambda_i \tag{70}$$

(1):

Since B is a real symmetric matrix with distinct eigenvalues, its eigenvectors are orthogonal (see Assignment 3). Therefore, the eigenvectors are linearly independent, which means that it is possible to diagonalize the B.

In addition, we know that the determinant of a diagonal matrix is the same as the multiplication of all values on the diagonal.

Let U and D be matrices as defined in Assignment 2.

$$\begin{aligned}
B &= UDU^{-1} && (71)\\
|B| &= |UDU^{-1}| && (72)\\
|B| &= |U||D||U^{-1}| && (73)\\
|B| &= |U||D||U|^{-1} && (74)\\
|B| &= \frac{|U|}{|U|}|D| && (75)\\
|B| &= |D| && (76)\\
|B| &= \prod_i \lambda_i && (77)
\end{aligned}$$

(2):

We use the same facts described in (1):

$$\begin{aligned}
B &= UDU^{-1} && (78)\\
tr(B) &= tr(UDU^{-1}) && (79)\\
tr(B) &= tr(U^{-1}UD) && (80)\\
tr(B) &= tr(D) && (81)\\
tr(B) &= \sum_i \lambda_i && (82)
\end{aligned}$$

5 Assignment

A hypreplane or affine set $H \in \mathbb{R}^n$ defined by the equation $h(x) \equiv w_0 + w^T x = 0$.

To show:

1. for any point $x_0 \in H, w^T x_0 = -w_0$.
2. if x_1 and x_2 lie in H, then $w^T(x_1 - x_2) = 0$.
3. $\hat{w} = w/||w||$ is the vector normal to the surface of H.
4. if $x_0 \in H$, then the sgned distance of any point x to H is given by $\hat{w}^T(x - x_0) = (w^T x + w_0)/||w||$

(1):

$$
\begin{aligned}
h(x_0) \equiv w_0 + w^T x_0 &= 0 && (83)\\
w^T x_0 &= -w_0 && (84)
\end{aligned}
$$

(2):

$$h(x_1) \equiv w_0 + w^T x_1 = 0 \tag{85}$$
$$h(x_2) \equiv w_0 + w^T x_2 = 0 \tag{86}$$

With 85 - 86:

$$w_0 + w^T x_1 - w_0 - w^T x_2 = 0 \tag{87}$$
$$w^T(x_1 - x_2) = 0 \tag{88}$$

(3):

Let $v \in H$. The scalar product of v and $\hat{w}$ has to be zero since all vectors in H have to be orthonogal to $\hat{w}$. Let $x_1, x_2 \in H$ and represent v as a linear combination of x_1 and x_2 so that $v = x_1 - x_2$.

$$< \hat{w}, v >= \hat{w}^T v = \hat{w}^T(x_1 - x_2) = 0 \tag{89}$$
$$\Leftrightarrow \frac{w^T}{||w||}(x_1 - x_2) = 0 \tag{90}$$

We know that it is equal to zero due to the knowledge of (2).

(4):

Computing the distance d of a point x to H can be computed using the projection so that $d = \frac{<\hat{w},(x-x_0)>}{||\hat{w}||}$. Since $\hat{w}$ is already normalized to a length of 1, the equation can be transformed to $\hat{w}^T(x - x_0)$.

$$
\begin{aligned}
\hat{w}^T(x - x_0) &= \frac{w^T x + w_0}{||w||} && (91)\\
\frac{w^T}{||w||}(x - x_0) &= \frac{w^T x + w_0}{||w||} && (92)\\
w^T(x - x_0) &= w^T x + w_0 && (93)\\
& && (94)
\end{aligned}
$$

with $-w_0 = w^T x_0$:

$$
\begin{aligned}
w^T(x - x_0) &= w^T x - w^T x_0 && (95)\\
w^T(x - x_0) &= w^T(x - x_0) && (96)
\end{aligned}
$$