homework sheet 02

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1 Assignment: Learning by doing

Problem 1 and 2

Python code required for the solution:

```
''' node.py '''

import numpy as np

class Node(object):
    '''
    class represents a node in a tree having children and a decision
    '''
    def __init__(self, features, values, current_depth, max_depth,
        classes):

        self.features = features
        self.values = values
        self.classes = classes

        # further slipping only if maximum depth is not reached and
            the data does not belong only to one class
        if current_depth < max_depth and np.unique(self.values[:,
            features.size-1]).size != 1:
            self.split_feature, self.split_value = self.find_decision
                ()
            data_leq, data_g = self.split_data(self.split_feature,
                self.split_value)
            self.child_leq = Node(self.features, data_leq,
                current_depth+1, max_depth, self.classes)    # node
                creates its children on its own
            self.child_g = Node(self.features, data_g, current_depth
                +1, max_depth, self.classes)

    # determining the feature and its value for splitting optimised by
         the gini index
    def find_decision(self):
        bins = np.append(self.classes, [self.classes.size])
        split_feature = ''
        split_value = -100000000000000
        optimal_cost = 1000000000000000

        # try each feature and each value within
        for feature in range(0, self.features.size-1):
            for i in np.sort(self.values[:,feature]):
                current_data_leq, current_data_g = self.split_data(
                    feature, i)
```

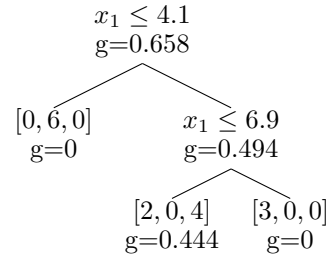
```
            distribution_leq = np.histogram(current_data_leq[:,
                self.classes.size], bins)[0]
            distribution_g = np.histogram(current_data_g[:, self.
                classes.size], bins)[0]
            cost_leq = self.gini(distribution_leq)
            cost_g = self.gini(distribution_g)
            total = self.values[:,0].size
            current_cost = cost_leq*np.sum(distribution_leq)/total
                 + cost_g*np.sum(distribution_g)/total
            if current_cost < optimal_cost:
                optimal_cost = current_cost
                split_feature = feature
                split_value = i

    return split_feature, split_value


# split the data corresponding to a feature and its value
def split_data(self, split_feature, split_value):

    data_leq = self.values[self.values[:, split_feature]<=
        split_value]
    data_g = self.values[self.values[:, split_feature]>split_value
        ]

    return data_leq, data_g


# computes the gini index of a given distribution
def gini(self,v):
    sum_all = np.sum(v)
    sum_square = np.sum(np.square(v))
    if np.sum(v)==0:
        return 1.0
    else:
        return 1.0-(sum_square/((sum_all)**2.0))


# classifies a new data point
def classify(self, x):
    if hasattr(self, 'split_feature'):
        if x[self.split_feature] <= self.split_value:
            self.child_leq.classify(x)
        else:
            self.child_g.classify(x)
    else:
        bins = np.append(self.classes, [self.classes.size])
        distribution = np.histogram(self.values[:,self.classes.
            size], bins)[0]
        maximum = max(distribution)
        c = np.where(distribution == maximum)
        p = 1.0*maximum / np.sum(distribution)
        print(str(x) + ' belongs to class ' + str(c[0]) + ' with a
             probability of ' + str(p))


# visualizes the decision tree
def visualize(self, current_depth):
    if hasattr(self, 'split_feature'):
        bins = np.append(self.classes, [self.classes.size])
        distribution = np.histogram(self.values[:,self.classes.
            size], bins)[0]
        print (str(current_depth)  + '\t' + str(distribution)+ '\t
            ' + self.features[self.split_feature] + '<=' + str(
            self.split_value) +'\t'+ str(self.gini(distribution)))
```

```
            self.child_leq.visualize(current_depth+1)
            self.child_g.visualize(current_depth+1)
        else:
            bins = np.append(self.classes, [self.classes.size])
            distribution = np.histogram(self.values[:,self.classes.
                size], bins)[0]
            print (str(current_depth) + '\t'+str(distribution)+'\t\t'+
                str(self.gini(distribution)))
```

```
''' main.py '''

import sys
import numpy as np
import csv

import node


def main(argv):
    # read csv file
    csv_file = argv[1]
    with open(csv_file, 'r') as csvfile:
            reader = csv.reader(csvfile, delimiter=',')
            features = np.array(reader.next())
            values = np.array([row for row in reader], dtype = 'float'
                )

    classes = np.unique(values[:, features.size-1])     # vector with
        the specific classes

    root = node.Node(features, values, 0, 2, classes)   # create the
        root node, which creates its children on its own

    root.visualize(0)   # visualize the tree

    # Problem 2
    x1 = np.array([4.1, -0.1, 2.2])
    root.classify(x1)

    x2 = np.array([6.1, 0.4, 1.3])
    root.classify(x2)


if __name__ == '__main__':
    main(sys.argv)
```

Results:

1. Decision tree:



2. Classification:

[4.1, -0.1, 2.2] belongs to class [1] with a probability of 1.0.

[6.1, 0.4, 1.3] belongs to class [2] with a probability of $\frac{2}{3}$.

Problem 3 and 4

Python code required for the solution:

```
''' kNN.py '''

import numpy as np

class KNN(object):
    '''
    classdocs
    '''

    def __init__(self, values):
        self.values = values

    # classifies a new data point
    def classify(self, x, k, classes):
        sorted_dist = self.find_rows_with_nn(x, k)
        rows = sorted_dist[0:k,0].astype(np.int32)

        cl = self.values[rows,self.values.shape[1]-1]

        bins = np.append(classes, [classes.size])
        distribution = np.histogram(cl, bins)[0]
        maximum = max(distribution)
        c = np.where(distribution == maximum)   #class
        p = 1.0*maximum / np.sum(distribution)  #probability

        return c[0], p

    # determines the class by regression
    def regress(self, x, k):
        sorted_dist = self.find_rows_with_nn(x, k)
        summe = 0
        normalization = 0
        for i in range(0, k):
            summe += 1/sorted_dist[i, 1]*self.values[sorted_dist[i,0],
                self.values.shape[1]-1]
            normalization += 1/sorted_dist[i, 1]
        return summe/normalization

    # determines the rows in the values correpsonding to the k nearest
        neighbors
    def find_rows_with_nn(self, x, k):
        distances = np.zeros([self.values.shape[0],2])
        for i in range(0, self.values.shape[0]):
            j = self.values.shape[1]-1
            distances[i,1] = np.linalg.norm(x-self.values[i,0:j])
            distances[i,0] = i

        return np.sort(distances.view('i8,i8'), order=['f1'], axis=0).
            view(np.float)

''' main.py '''
import sys
import csv
import numpy as np

import kNN

def main(argv):
    # read csv file
    csv_file = argv[1]
    with open(csv_file, 'r') as csvfile:
        reader = csv.reader(csvfile, delimiter=',')
```

```
            features = np.array(reader.next())
            values = np.array([row for row in reader], dtype = 'float')

        classes = np.unique(values[:, features.size-1])     # vector with
            the specific classes

        knn = kNN.KNN(values)
        k = 3

        # Problem 3
        x1 = np.array([4.1, -0.1, 2.2])
        c1, p1 = knn.classify(x1, k, classes)

        x2 = np.array([6.1, 0.4, 1.3])
        c2, p2 = knn.classify(x2, k, classes)

        print(str(x1) + ' belongs to class ' + str(c1) + ' with a
            probability of ' + str(p1))
        print(str(x2) + ' belongs to class ' + str(c2) + ' with a
            probability of ' + str(p2))


        # Problem 4
        reg1 = knn.regress(x1, k)
        reg2 = knn.regress(x2, k)

        print('Regression for '+ str(x1) + ' results in ' + str(reg1))
        print('Regression for '+ str(x2) + ' results in ' + str(reg2))


    if __name__ == '__main__':
        main(sys.argv)
```

Results:

1. KNN:

[4.1, -0.1, 2.2] belongs to class [0, 1, 2] with a probability of $\frac{1}{3}$. Therefore, pick randomly class 1.

[6.1, 0.4, 1.3] belongs to class [2] with a probability of $\frac{2}{3}$.

2. KNN Regression:

Regression for [4.1, -0.1, 2.2] results in 0.561.

Regression for [6.1, 0.4, 1.3] results in 1.396.

Problem 5

The values of the second column are much smaller than the values of the first and third column. Scaling the data could compensate this problem affecting the euclidian distance (standardization). Another solution could be using another distance measurement.

This problem does not arise when training a decision tree since only one feature is considered for splitting the set at a time. Therefore, only the relative differences within each column are important.

2 Assignment: Probabilistic kNN

Problem 6

To show:

$$\frac{p(c=0|x^*)}{p(c=1|x^*)} \approx \frac{e^{-\frac{\|x^*-x_0\|^2}{2\sigma^2}}}{e^{-\frac{\|x^*-x_1\|^2}{2\sigma^2}}} \tag{1}$$

We know from the lecture:

$$p(c=b|x^*) = \frac{p(x^*|c=b)p(c=b)}{\sum_{i\in classes} p(x^*|c=i)p(c=i)} \tag{2}$$

and

$$\begin{aligned} p(x|c=b) &= \frac{1}{N_b} \sum_{n\in classb} \mathcal{N}(x|x_n, \sigma^2 I) & (3)\\ &= \frac{1}{N_b} \frac{1}{(2\pi\sigma^2)^{D/2}} \sum_{n\in classb} e^{-\frac{\|x-x_n\|^2}{2\sigma^2}} & (4) \end{aligned}$$

Here we have the two classes 0 and 1 with N_0 and N_1 elements, respectively. It holds:

$$\begin{aligned} p(c=0) &= \frac{N_0}{N_0+N_1} & (5)\\ p(c=0) &= \frac{N_1}{N_0+N_1} & (6) \end{aligned}$$

$$\begin{aligned} \Rightarrow p(c=0|x^*) &= \frac{p(x^*|c=0)p(c=0)}{p(x^*|c=0)p(c=0)+p(x^*|c=1)p(c=1)} & (7)\\ &= \frac{\frac{1}{N_0}\frac{1}{(2\pi\sigma^2)^{D/2}}\sum_{n\in class0} e^{-\frac{\|x-x_n\|^2}{2\sigma^2}} \frac{N_0}{N_0+N_1}}{p(x^*|c=0)\frac{N_0}{N_0+N_1}+p(x^*|c=1)\frac{N_1}{N_0+N_1}} & (8)\\ \Rightarrow p(c=1|x^*) &= \frac{p(x^*|c=1)p(c=1)}{p(x^*|c=0)p(c=0)+p(x^*|c=1)p(c=1)} & (9)\\ &= \frac{\frac{1}{N_1}\frac{1}{(2\pi\sigma^2)^{D/2}}\sum_{n\in class1} e^{-\frac{\|x-x_n\|^2}{2\sigma^2}} \frac{N_1}{N_0+N_1}}{p(x^*|c=0)\frac{N_0}{N_0+N_1}+p(x^*|c=1)\frac{N_1}{N_0+N_1}} & (10) \end{aligned}$$

This leads to the following transformations:

$$\begin{aligned} \frac{p(c=0|x^*)}{p(c=1|x^*)} &= \frac{\frac{1}{N_0}\frac{1}{(2\pi\sigma^2)^{D/2}}\sum_{n\in class0} e^{-\frac{\|x-x_n\|^2}{2\sigma^2}} \frac{N_0}{N_0+N_1}}{\frac{1}{N_1}\frac{1}{(2\pi\sigma^2)^{D/2}}\sum_{n\in class1} e^{-\frac{\|x-x_n\|^2}{2\sigma^2}} \frac{N_1}{N_0+N_1}} & (11)\\ &= \frac{\sum_{n\in class0} e^{-\frac{\|x-x_n\|^2}{2\sigma^2}}}{\sum_{n\in class1} e^{-\frac{\|x-x_n\|^2}{2\sigma^2}}} & (12)\\ &= \frac{e^{-\frac{\|x-x_0\|^2}{2\sigma^2}} + \sum_{n\in class0\setminus x_0} e^{-\frac{\|x-x_n\|^2}{2\sigma^2}}}{e^{-\frac{\|x-x_1\|^2}{2\sigma^2}} + \sum_{n\in class1\setminus x_1} e^{-\frac{\|x-x_n\|^2}{2\sigma^2}}} & (13) \end{aligned}$$

Since σ^2 is very small the last part of the sums is very small in comparison to the first part and therefore can be neglected. This results in the approximation that was meant to be shown:

$$\frac{p(c=0|x^*)}{p(c=1|x^*)} \approx \frac{e^{-\frac{\|x^*-x_0\|^2}{2\sigma^2}}}{e^{-\frac{\|x^*-x_1\|^2}{2\sigma^2}}} \tag{14}$$

Problem 7

Since $\sigma \to 0$, we know from problem 6 that

$$\frac{p(c=0|x^*)}{p(c=1|x^*)} \approx \frac{e^{-\frac{\|x^*-x_0\|^2}{2\sigma^2}}}{e^{-\frac{\|x^*-x_1\|^2}{2\sigma^2}}} = e^{-\frac{\|x^*-x_0\|^2}{2\sigma^2}+\frac{\|x^*-x_1\|^2}{2\sigma^2}} \tag{15}$$

$$= e^{\frac{-\|x^*-x_0\|^2+\|x^*-x_1\|^2}{2\sigma^2}} \tag{16}$$

If

$$0 < \frac{p(c=0|x^*)}{p(c=1|x^*)} < 1 \tag{17}$$

$$\Leftrightarrow p(c=0|x^*) < p(c=1|x^*) \tag{18}$$

and therefore, class 1 is chosen for x^*.

If

$$\frac{p(c=0|x^*)}{p(c=1|x^*)} > 1 \tag{19}$$

$$\Leftrightarrow p(c=0|x^*) > p(c=1|x^*) \tag{20}$$

and therefore, class 0 is chosen for x^*.

For e^x: If $x < 0$, then $0 < e^x < 1$. If $x > 0$, then $e^x > 1$.

Let $x = \frac{-\|x^*-x_0\|^2+\|x^*-x_1\|^2}{2\sigma^2}$.

If x^* is closer to x_0 than to x_1, $\|x^*-x_0\|^2 < \|x^*-x_1\|^2$. This can further be transformed:

$$\Leftrightarrow \|x^*-x_0\|^2 - \|x^*-x_1\|^2 < 0 \tag{21}$$

$$\Leftrightarrow \frac{-\|x^*-x_0\|^2 + \|x^*-x_1\|^2}{2\sigma^2} > 0 \tag{22}$$

$$\Leftrightarrow x > 0 \tag{23}$$

For $x > 0$ we know that $e^x > 1$ and therefore $e^{\frac{-\|x^*-x_0\|^2+\|x^*-x_1\|^2}{2\sigma^2}} > 1$. Since Equation 15, we know $\frac{p(c=0|x^*)}{p(c=1|x^*)} > 1$, which let us come to the conclusion that class 0 is chosen if x^* is closer to x_0 than to x_1 (see Equation 19).

Problem 8

The higher the variance, the more neighbours should be considered since it is more probable that there is an element from another class in between the elements of the depicted class due to the high variance. The smaller the variance, the smaller the number of considered neighbours. Due to the small variance the probability of finding an element from another class in the area of the depicted class is small.

Additionally, the approximation of Problem 6 indicates this correlation. A very small variance is considered which allows the disregard of all neighbours except the nearest neighbour.

3 Assignment: Neighbourhood Component Analysis

Problem 9

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