Machine Learning Worksheet 1

Linear Algebra – Refresher

Problem 1: Compute the eigenvalues and eigenvectors of the following matrix $\boldsymbol{A}$ by hand and using python/numpy. Hand in both your derivation and your code.

$$\boldsymbol{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Problem 2: Suppose some matrix $\boldsymbol{B} \in \mathbb{R}^{n \times n}$ has n linearly independent eigenvectors $\boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_n}$. Define the matrix $\boldsymbol{U}$ having these vectors as columns and the diagonal matrix $\boldsymbol{D}$ with the eigenvalues λ_i in the diagonal. Show that

$$\boldsymbol{B} = \boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{-1}$$

Problem 3: For a real symmetric matrix $\boldsymbol{B} \in \mathbb{R}^{n \times n}$, show that

- $\boldsymbol{B}$ has real eigenvalues.
- its eigenvectors are orthogonal if the eigenvalues are pairwise distinct.

Problem 4: For a real symmetric matrix $\boldsymbol{B} \in \mathbb{R}^{n \times n}$ with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, show that for $|\cdot|$ (*Determinant*) and $tr(\cdot)$ (*Trace*) the following holds respectively:

$$|\boldsymbol{B}| = \prod_i \lambda_i, \qquad tr(\boldsymbol{B}) = \sum_i \lambda_i.$$

Problem 5: A *hyperplane* or *affine set* H in $\mathbb{R}^n$ defined by the equation $h(\boldsymbol{x}) \equiv w_0 + \boldsymbol{w}^T\boldsymbol{x} = 0$ (in $\mathbb{R}^2$ this is a line). Show that

- for any point $\boldsymbol{x_0} \in H$, $\boldsymbol{w}^T\boldsymbol{x_0} = -w_0$.
- if $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ lie in H, then $\boldsymbol{w}^T(\boldsymbol{x_1} - \boldsymbol{x_2}) = 0$.
- $\hat{\boldsymbol{w}} = \boldsymbol{w}/\|\boldsymbol{w}\|$ is the vector normal to the surface of H.
- if $\boldsymbol{x_0} \in H$, then the signed distance of any point $\boldsymbol{x}$ to H is given by $\hat{\boldsymbol{w}}^T(\boldsymbol{x} - \boldsymbol{x_0}) = (\boldsymbol{w}^T\boldsymbol{x} + w_0)/\|\boldsymbol{w}\|$.