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1 Initial Smoothed Gresho Vortex

[1] **Gresho P.M., Chan S.T**., *On The Theory Of Semi-Implicit Projection Methodes For Viscous Incompressible Flow And Its Implementation Via A Finite Element Method That Also Introduces A Nearly Consistent Mass Matrix. Part 2: Implementation*, International Journal For Numerical Methods in Fluids, Vol. 11, 621-659 (1990)

[2] **Kardioglu S.Y., Klein R., Minion M.L.**., *A Fourth-Order Auxiliary Variable Projection Method For Zero-Mach Numer Gas Dynamics*, Journal of Computational Physics, Vol. 227, 2012-2043 (2008)

The following gives in detail the 2D initial conditions for density, velocity and pressure for a smoothed vortex according to Gresho. The original is found in [1], the smoothed version in [2].
The relation between the radial coordinate r and the Cartesian coordinates x and y used in the following is given by

$$r = \frac{\sqrt{(x - x_c(t))^2 + (y - y_c(t))^2}}{R} \tag{1}$$

where R is the radius of the vortex and set to $R = 0.4$ and

$$x_c(t) = x_0 + t\, u_c \tag{2}$$
$$y_c(t) = y_0 + t\, v_c \tag{3}$$

and the initial position of the vortex being $(x_c(0), y_c(0)) = (x_0, y_0)$.

1.1 Density

The dimensionless density

$$\rho(x, y, t) = \begin{cases} \rho_c + \frac{1}{2}(1 - r^2)^6 & if \quad r < 1 \\ \rho_c & otherwise \end{cases} \tag{4}$$

with $\rho_c = \frac{1}{2}$ given in [2] is due to a scaling of the density bump according to the value of the undisturbed density more general in the form

$$\rho(x, y, t) = \begin{cases} \rho_c \left(1 + (1 - r^2)^6\right) & if \quad r < 1 \\ \rho_c & otherwise \end{cases} \tag{5}$$

and boils down to the first density equation in case of $\rho_c = \frac{1}{2}$. Multiplication with the reference value $\hat{\rho}_{ref} = 1\frac{kg}{m^3}$ yields

$$\hat{\rho}(x, y, t) = \begin{cases} \hat{\rho}_c \left(1 + (1 - r^2)^6\right) & if \quad r < 1 \\ \hat{\rho}_c & otherwise \end{cases} \tag{6}$$

with

$$\hat{\rho}_c = \rho_c \, \hat{\rho}_{ref} \tag{7}$$

$$\hat{\rho} = \rho \, \hat{\rho}_{ref} \tag{8}$$

where $\hat{\rho}_c$ is what we specify in the input file as rho_ref.

1.2 Velocity

The velocity components given in [2] are

$$u(x,y,t) = \begin{cases} u_c - 1024 sin(\theta) \, (1-r)^6 \, r^6 & if \quad r < 1 \\ u_c & otherwise \end{cases} \tag{9}$$

$$v(x,y,t) = \begin{cases} v_c + 1024 sin(\theta) \, (1-r)^6 \, r^6 & if \quad r < 1 \\ v_c & otherwise \end{cases} \tag{10}$$

with $\theta = atan\left(\frac{y-y_c(t)}{x-x_c(t)}\right)$ and $u_c = v_c = 1$. However, to achieve the behavior of an counterclockwise rotating vortex, the velocity components must be

$$u(x,y,t) = \begin{cases} u_c - \varphi \frac{x-x_c(t)}{|x-x_c(t)|} sin(\theta) \, (1-r)^6 \, r^6 & if \quad r < 1 \\ u_c & otherwise \end{cases} \tag{11}$$

$$v(x,y,t) = \begin{cases} v_c + \varphi \frac{x-x_c(t)}{|x-x_c(t)|} cos(\theta) \, (1-r)^6 \, r^6 & if \quad r < 1 \\ v_c & otherwise \end{cases} \tag{12}$$

with $\theta = atan\left(\frac{y-y_c(t)}{x-x_c(t)}\right)$ and $\varphi = 1024$. Multiplication with a reference velocity component $\hat{u}_{ref} = 1\frac{m}{s}$ or $\hat{v}_{ref} = 1\frac{m}{s}$ respectively, yields

$$\hat{u}(x,y,t) = \begin{cases} \hat{u}_c - \varphi \hat{u}_{ref} \frac{x-x_c(t)}{|x-x_c(t)|} sin(\theta) \, (1-r)^6 \, r^6 & if \quad r < 1 \\ \hat{u}_c & otherwise \end{cases} \tag{13}$$

$$\hat{v}(x,y,t) = \begin{cases} \hat{v}_c + \varphi \hat{v}_{ref} \frac{x-x_c(t)}{|x-x_c(t)|} cos(\theta) \, (1-r)^6 \, r^6 & if \quad r < 1 \\ \hat{v}_c & otherwise \end{cases} \tag{14}$$

where $\hat{u}_c$ and $\hat{v}_c$ is what we specify in the input file as components of $velocity\_ref$.
$\hat{u}_{ref} = 1\frac{m}{s}$ and $\hat{v}_{ref} = 1\frac{m}{s}$ are constants just as $\hat{\rho}_{ref}$ and do not need to be touched. However, $\hat{u}_{ref} = \hat{v}_{ref} = 1\frac{m}{s}$ can be expressed as $\hat{w}_{ref} = 1\frac{m}{s}$ an thus

$$\hat{u}(x,y,t) = \begin{cases} \hat{u}_c - \varphi \hat{w}_{ref} \frac{x-x_c(t)}{|x-x_c(t)|} sin(\theta) \, (1-r)^6 \, r^6 & if \quad r < 1 \\ \hat{u}_c & otherwise \end{cases} \tag{15}$$

$$\hat{v}(x,y,t) = \begin{cases} \hat{v}_c + \varphi \hat{w}_{ref} \frac{x-x_c(t)}{|x-x_c(t)|} cos(\theta) \, (1-r)^6 \, r^6 & if \quad r < 1 \\ \hat{v}_c & otherwise \end{cases} \tag{16}$$

In the code φ is chosen to be $4 \cdot 1024$ for the smoothed Gresho vortex initial conditions which has the effect that the maximum rotation velocity has the same size as the advection velocity if $velocity\_ref$ (for the rotation) and $velocity\_inflow$ (for the advection) are chosen to be of the same size (e.g. if all components of $velocity\_ref$ and $velocity_inflow$ are set to 1 the max. rotation velocity is 1).

1.3 Pressure

To obtain the pressure one can look at a 1 dimensional path between the vortex center and the outer boundary of the vortex which is represented by $0 \leq r \leq 1$. Due to the non-constant density along this path and the rotation velocity, in a coordinates system which moves with the velocity $(\hat{u}_c, \hat{v}_c)$ centrifugal forces try to pull mass away from the vortex center and so the closer one gets to the vortex center, the stronger a pressure force has to be which keeps the vortex in place, yielding the following equilibrium

$$\underbrace{\int_{dS} \hat{p}(x,y,t) dS}_{=F_p} = \underbrace{\int_{dV} \frac{\hat{\rho}(r)\hat{\mathbf{v}}_r^2(r)}{r} \, dV}_{=F_z} = \int_{dV} \frac{\hat{\rho}(r)\left(\hat{u}_r^2(r) + \hat{v}_r^2(r)\right)}{r} \, dV \tag{17}$$

where $\hat{\mathbf{v}}_r$ is only the rotational part of the original velocity field and thus $\hat{\mathbf{v}}_r^2 = \left[\hat{u}_r^2(r) + \hat{v}_r^2(r)\right]$ and

$$\hat{u}_r(x,y,t) = \begin{cases} -\varphi \hat{w}_{ref} \frac{x - x_c(t)}{|x - x_c(t)|} sin(\theta) \left(1 - r\right)^6 r^6 & if \quad r < 1 \\ 0 & otherwise \end{cases} \tag{18}$$

$$\hat{v}_r(x,y,t) = \begin{cases} \varphi \hat{w}_{ref} \frac{x - x_c(t)}{|x - x_c(t)|} cos(\theta) \left(1 - r\right)^6 r^6 & if \quad r < 1 \\ 0 & otherwise \end{cases} \tag{19}$$

Along the 1 dimensional path the pressure equation is

$$\hat{p}(x,y,t) = \int_1^{\tilde{r}} \frac{\hat{\rho}(r)\hat{\mathbf{v}}_r^2(r)}{r} \, dr = \int_1^{\tilde{r}} \frac{\hat{\rho}(r)\left(\hat{u}_r^2(r) + \hat{v}_r^2(r)\right)}{r} \, dr \tag{20}$$

This can also be written as

$$\left(\hat{\rho}_{ref}\hat{w}_{ref}^2\right) p(x,y,t) = \left(\hat{\rho}_{ref}\hat{w}_{ref}^2\right) \int_1^{\tilde{r}} \frac{\rho(r)\left(u_r^2(r) + v_r^2(r)\right)}{r} \, dr \tag{21}$$

and so the dimensionless equation

$$p(x,y,t) = \int_1^{\tilde{r}} \frac{\rho(r)\left(u_r^2(r) + v_r^2(r)\right)}{r} \, dr \tag{22}$$

remains for the moment. Choosing the integration path starting at the left outer boundary of the vortex approaching the vortex center from the right which is possible due to the rotational symmetry of the problem, the following things simplify:

- $y - y_c(0) = 0$
- $\theta = 0$
- $sin(\theta) = 0$
- $cos(\theta) = 1$
- $\frac{x - x_c(t)}{|x - x_c(t)|} = 1$

Therefore

$$
\begin{aligned}
u_r(x,y,t) &= 0 \qquad (23)\\
v_r(x,y,t) &= \begin{cases} \varphi\,(1-r)^6\,r^6 & if \quad r<1 \\ 0 & otherwise \end{cases} \qquad (24)
\end{aligned}
$$

and (as above)

$$
\rho(x,y,t) = \begin{cases} \rho_c\left(1+(1-r^2)^6\right) & if \quad r<1 \\ \rho_c & otherwise \end{cases} \qquad (25)
$$

The square of the velocity vector finally is

$$
\mathbf{v}_r^2(x,y,t) = \begin{cases} \varphi^2\,(1-r)^{12}\,r^{12} & if \quad r<1 \\ 0 & otherwise \end{cases} \qquad (26)
$$

and the pressure integral yields

$$
p(r) = \int \frac{\rho(r)\mathbf{v}^2(r)}{r}\,dr = \begin{cases} \varphi^2\rho_c\left(\sum_{k=12}^{36}\left(2\,a_k\,r^k\right)\right)+C & if \quad r<1 \\ 0 & otherwise \end{cases} \qquad (27)
$$

with coefficients a_k

$$
\begin{array}{ccccc}
a_{12} = \frac{1}{12} & a_{13} = -\frac{12}{13} & a_{14} = \frac{9}{2} & a_{15} = -\frac{184}{15} & a_{16} = \frac{609}{32} \\
a_{17} = -\frac{222}{17} & a_{18} = -\frac{38}{9} & a_{19} = \frac{54}{19} & a_{20} = \frac{783}{20} & a_{21} = -\frac{558}{7} \\
a_{22} = \frac{1053}{22} & a_{23} = \frac{1014}{23} & a_{24} = -\frac{1473}{16} & a_{25} = \frac{204}{5} & a_{26} = \frac{510}{13} \\
a_{27} = -\frac{1564}{27} & a_{28} = \frac{153}{8} & a_{29} = \frac{450}{29} & a_{30} = -\frac{269}{15} & a_{31} = \frac{174}{31} \\
a_{32} = \frac{57}{32} & a_{33} = -\frac{74}{33} & a_{34} = \frac{15}{17} & a_{35} = -\frac{6}{35} & a_{36} = \frac{1}{72}
\end{array}
\qquad (28)
$$

Comparing to the results in [2] it can be seen that there is a typo for the coefficient in front of r^{30} in [2].

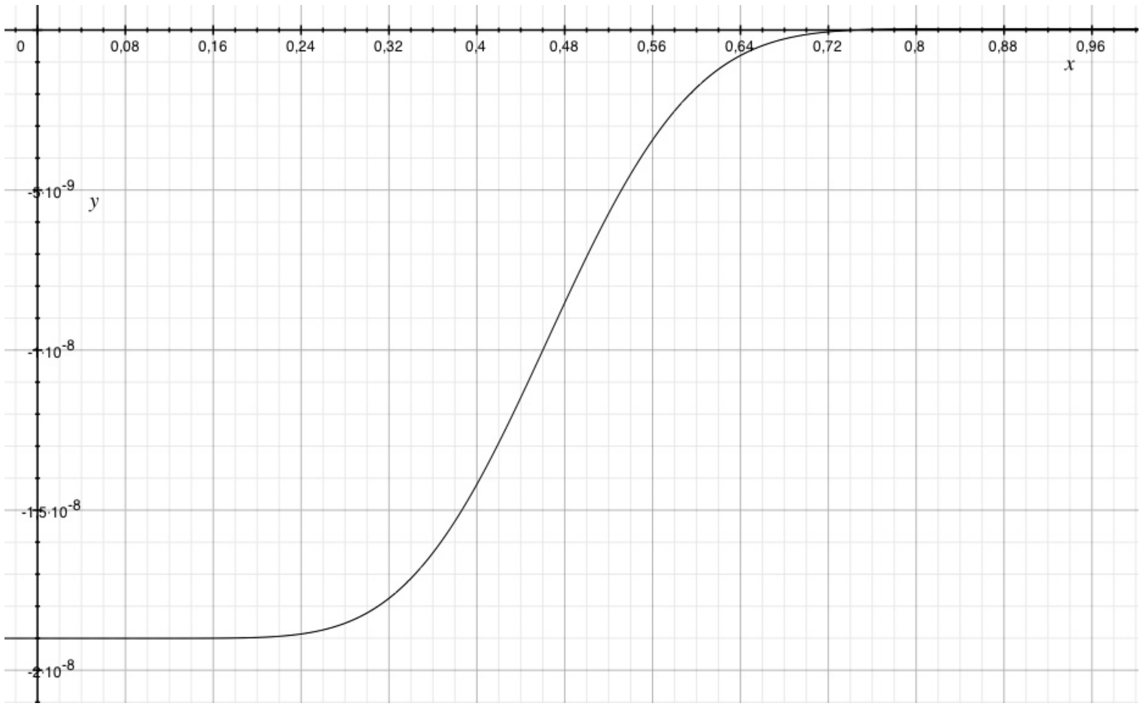

Figure 1: 1-D pressure distribution for $u_c = v_c = 0$ and $\rho_c = \frac{1}{2}$, symmetric with respect to rotation around y-axis

The (dimensionless) 1 dimensional integral from 1 (outer vortex boundary) to $\tilde{r}$ (arbitrary location towards the vortex center) finally is

$$p(x, y, t) = \begin{cases} p(\tilde{r}) - p(1) & if \quad r < 1 \\ 0 & otherwise \end{cases} \tag{29}$$

which is with dimensions

$$\hat{p}(x, y, t) = \begin{cases} \hat{\rho}_{ref}\hat{w}_{ref}^2 \left[p(\tilde{r}) - p(1)\right] & if \quad r < 1 \\ 0 & otherwise \end{cases} \tag{30}$$

where $\hat{\rho}_{ref}\hat{w}_{ref}^2 = 1\frac{N}{m^2}$.