# project 2: *tic tac toe* by the numbers

### task 2.1: the *tic tac toe* game tree

Implement a program that builds / computes the *complete* game tree for *tic tac toe*. Once that tree is available, determine the average branching factor, the state space complexity, and the game tree size for *tic tac toe*.

### task 2.2: probabilities of contributing to a winning pattern

Previously, in task 1.2, you were asked to implement a function for making *tic tac toe* moves based on statistical considerations.
Recall that you had to host a tournament with randomly moving players and use data gathered during the tournament in order to determine auspicious positions on the *tic tac toe* board. In other words, you were asked to empirically estimate which fields contribute more or less frequently to a winning game state.
To be precise, the tournament was supposed to last for $N$ rounds and, after each round with a winner, you had to determine which fields the winner occupied in order to, in the end, estimate how likely each field contributes to winning the game.
The following tables show corresponding results obtained from four tournaments of length $N \in \{100, 1000, 10000, 100000\}$

| | | |
|---|---|---|
| 0.10 | 0.09 | 0.12 |
| 0.08 | 0.14 | 0.11 |
| 0.12 | 0.11 | 0.13 |

$N = 100$

| | | |
|---|---|---|
| 0.12 | 0.10 | 0.13 |
| 0.09 | 0.15 | 0.10 |
| 0.12 | 0.09 | 0.11 |

$N = 1000$

| | | |
|---|---|---|
| 0.12 | 0.09 | 0.12 |
| 0.09 | 0.15 | 0.09 |
| 0.12 | 0.09 | 0.12 |

$N = 10000$

| | | |
|---|---|---|
| 0.12 | 0.09 | 0.12 |
| 0.09 | 0.15 | 0.09 |
| 0.12 | 0.09 | 0.12 |

$N = 100000$

For a growing number of rounds, the probabilities appear to converge to certain values (note that if the values in a table do not add to $1$, this is due to rounding errors during printing). Yet, to know for sure, we would have to consider the case where $N \to \infty$.
Since playing such a tournament is impossible, your task is to answer the following questions:
How could / would you estimate the probability for a field to contribute to a win *without* having to run a tournament? I.e. can you think of a theoretical rather than an empirical approach?

If cannot come up with an answer, have a look at the following page . . .

### solution 2.2

On the $3 \times 3$ *tic tac toe* board there are eight ways of placing the number $1$ three times in a row, namely

$$
\begin{array}{c|c|c} 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array} \quad
\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \end{array} \quad
\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 1 & 1 & 1 \end{array} \quad
\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline 1 & 0 & 0 \end{array} \quad
\begin{array}{c|c|c} 0 & 1 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 1 & 0 \end{array} \quad
\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & 0 & 1 \\ \hline 0 & 0 & 1 \end{array} \quad
\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array} \quad
\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & 1 & 0 \\ \hline 1 & 0 & 0 \end{array}
$$

Hence, to count how often a each single field appears in these patterns, we simply add the above matrices and obtain

$$
\begin{array}{c|c|c} 3 & 2 & 3 \\ \hline 2 & 4 & 2 \\ \hline 3 & 2 & 3 \end{array}
$$

Since we have eight patterns of three in a row, the entries in this table should add to $8 \cdot 3 = 24$ and indeed they do. To determine the probability for each field to contribute to such a pattern, we therefore compute

$$
\begin{array}{c|c|c} 3/24 & 2/24 & 3/24 \\ \hline 2/24 & 4/24 & 2/24 \\ \hline 3/24 & 2/24 & 3/24 \end{array}
=
\begin{array}{c|c|c} 0.125 & 0.08\bar{3} & 0.125 \\ \hline 0.08\bar{3} & 0.1\bar{6} & 0.08\bar{3} \\ \hline 0.125 & 0.08\bar{3} & 0.125 \end{array}
$$

If we could run a tournament with infinitely many rounds, these theoretical estimates would also result empirically.