Sheet 8 - PCA and SVD

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Assignment 1) Eigendecomposition, SVD and PCA

a)

(i) wrong. The notion of eigenvalues λ_i and eigenvectors $v_i \in \mathbb{R}^n$ is only defined for quadratic matrices. This is easy to verify if we take a look at the formula $Av_i = \lambda_i v_i$. The LHS of this equation has to be a vector of the same size as v_i for the equation to be well-defined. Therefore A has to be an endomorphism of $\mathbb{R}^n$, i.e. $A \in \mathbb{R}^{n\times n}$.

(ii) true. This follows directly from the spectral theorem for symmetric matrices:

> If $A \in \mathbb{R}^{n\times n}$ is symmetric, there exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors v_i of A and the corresponding eigenvalues λ_i are real.

This means we can choose $\Phi = (v_1|...|v_n)$ and $\Lambda = \text{diag}(\lambda_1, ..., \lambda_n)$ with $\lambda_1 \geq ... \geq \lambda_n$. Φ is by definition an orthogonal matrix. For $A = U\Sigma V^T$ with $U = V = \Phi$ to be a SVD, the singular values have to be non-negative. This is satisfied if we require Λ to be positive-semidefinite.

(iii) wrong. Given a basis $v_1, ..., v_n \in \mathbb{R}^n \setminus \{0\}$ of real eigenvectors of A, we know that an eigendecomposition of A exists. It suffices to show that the eigenvalues are real. Assume there exists a non-real eigenvalue $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$ corresponding to the eigenvector v_i. First, observe that

$$a \in \mathbb{C} \setminus \mathbb{R}, \quad b \in \mathbb{R} \setminus \{0\} \implies a \cdot b \in \mathbb{C} \setminus \mathbb{R}.$$

Using this, we get

$$Av_i \in \mathbb{R}^n \text{ and } \lambda_i v_i \in \mathbb{C}^n \setminus \mathbb{R}^n.$$

But then $Av_i = \lambda_i v_i$ can not be true, as $\mathbb{R}^n \cap (\mathbb{C}^n \setminus \mathbb{R}^n) = \emptyset$. Therefore λ_i has to be real.

b) Choosing $V = \Phi, \Lambda = \Sigma^2$, we get

$$\begin{aligned}
A^T A &= (U\Sigma V^T)^T U\Sigma V^T \\
&= V\Sigma^T U^T U\Sigma V^T \\
&\overset{(*)}{=} V\Sigma^2 V^T \\
&\overset{(*)}{=} V\Sigma^2 V^{-1} \\
&= \Phi\Lambda\Phi^{-1}.
\end{aligned}$$

$(*)$ use unitary property of U and V, respectively.

To get U, we rearrange the definition of the SVD:

$$A = U\Sigma V^T \iff AV\Sigma^{-1} = U.$$

Assignment 2) Singular Value Decomposition by hand

a) First we calculate B:

$$B = \frac{2}{3\sqrt{10}} \begin{pmatrix} 5 & 3 \\ 0 & 4 \end{pmatrix} \frac{2}{3\sqrt{10}} \begin{pmatrix} 5 & 0 \\ 3 & 4 \end{pmatrix} = \frac{4}{45} \begin{pmatrix} 17 & 6 \\ 6 & 8 \end{pmatrix}.$$

Determine the characteristic polynomial:

$$\begin{aligned}
p(\lambda) &= \det(B - \lambda \cdot I) \\
&= \det \begin{pmatrix} \frac{4\cdot 17}{45} - \lambda & \frac{4\cdot 6}{45} \\ \frac{4\cdot 6}{45} & \frac{4\cdot 8}{45} - \lambda \end{pmatrix} \\
&= \det \begin{pmatrix} \frac{68}{45} - \lambda & \frac{24}{45} \\ \frac{24}{45} & \frac{32}{45} - \lambda \end{pmatrix} \\
&= \left(\frac{68}{45} - \lambda\right)\left(\frac{32}{45} - \lambda\right) - \left(\frac{24}{45}\right)^2 \\
&= \lambda^2 - \frac{20}{9}\lambda + \frac{64}{81}
\end{aligned}$$

Find the roots of p:

$$
\begin{aligned}
&\lambda^2 - \frac{20}{9}\lambda + \frac{64}{81} = 0 \\
\Longrightarrow\ &\lambda = \frac{20}{9}\cdot\frac{1}{2} \pm \sqrt{\left(\frac{20}{9}\cdot\frac{1}{2}\right)^2 - \frac{64}{81}} \\
\Longrightarrow\ &\lambda = \frac{10}{9} \pm \sqrt{\frac{10^2-64}{81}} \\
\Longrightarrow\ &\lambda = \frac{10}{9} \pm \frac{6}{9} \\
\Longrightarrow\ &\lambda_1 = \frac{16}{9}, \quad \lambda_2 = \frac{4}{9}.
\end{aligned}
$$

b) For (2×2)-matrices we can use a closed-form solution to determine the eigenvectors. We choose $\alpha_1, \alpha_2 \in \mathbb{R}$ after the fact s.t. the vectors are normalized.

$$
\begin{aligned}
v_1 =& \alpha_1 \begin{pmatrix} \lambda_1 - \frac{32}{45} \\ \frac{24}{45} \end{pmatrix} = \alpha_1 \begin{pmatrix} \frac{80}{45} - \frac{32}{45} \\ \frac{24}{45} \end{pmatrix} = \alpha_1 \begin{pmatrix} \frac{48}{45} \\ \frac{24}{45} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
v_2 =& \alpha_2 \begin{pmatrix} \lambda_2 - \frac{32}{45} \\ \frac{24}{45} \end{pmatrix} = \alpha_2 \begin{pmatrix} \frac{20}{45} - \frac{32}{45} \\ \frac{24}{45} \end{pmatrix} = \alpha_2 \begin{pmatrix} -\frac{12}{45} \\ \frac{24}{45} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix}.
\end{aligned}
$$

c) From assignment 1) we know that

$$
V = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad \Sigma = \operatorname{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}).
$$

Calculate U:

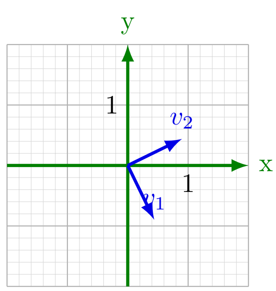
$$
\begin{aligned}
U = AV\Sigma^{-1} &= \frac{2}{3\sqrt{10}} \begin{pmatrix} 5 & 0 \\ 3 & 4 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} \end{pmatrix} \\
&= \frac{2}{3\sqrt{50}} \begin{pmatrix} 5 & 0 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} \frac{6}{4} & -\frac{3}{2} \\ \frac{3}{4} & \frac{6}{2} \end{pmatrix} = \frac{1}{\sqrt{50}} \begin{pmatrix} 5 & -5 \\ 5 & 5 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
\end{aligned}
$$

d) Checking the correctness:

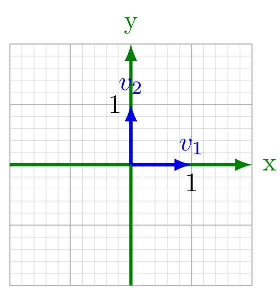
$$
\begin{aligned}
UU^T &= \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\
VV^T &= \frac{1}{5}\begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix} = \frac{1}{5}\begin{pmatrix}4+1 & 2-2\\ 2-2 & 1+4\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\
U\Sigma V^T &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\frac{4}{3} & 0\\ 0 & \frac{2}{3}\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}\\
&= \frac{1}{\sqrt{10}}\begin{pmatrix}\frac{4}{3} & -\frac{2}{3}\\ \frac{4}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}\\
&= \frac{2}{3\sqrt{10}}\begin{pmatrix}2 & -1\\ 2 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}\\
&= \frac{2}{3\sqrt{10}}\begin{pmatrix}5 & 0\\ 3 & 4\end{pmatrix}.
\end{aligned}
$$

Assignment 3) SVD interpretation

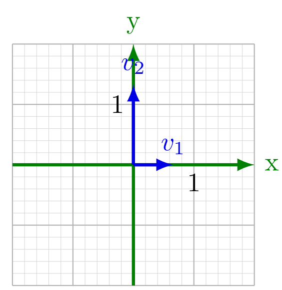
a)



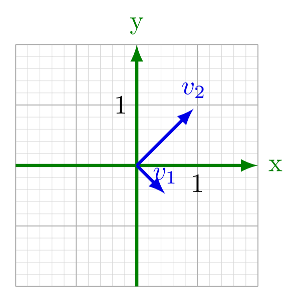
$I[v_1, v_2] = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}$



$V^T[v_1, v_2] = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$



$\Sigma V^T[v_1, v_2] = \begin{pmatrix}2/3 & 0\\ 0 & 4/3\end{pmatrix}$



$U\Sigma V^T[v_1, v_2] = \begin{pmatrix}\sqrt{2}/3 & 2\sqrt{2}/3\\ -\sqrt{2}/3 & 2\sqrt{2}/3\end{pmatrix}$

b) We know that Σ is a diagonal matrix with nonnegative real numbers and as such it will only applies scaling. Both U, V are unitary and as such will not affect the norm (length) but only rotate or mirror.

Assignment 4) Principal Component Analysis and Compression