Sheet 8 - PCA and SVD

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Assignment 1) Eigendecomposition, SVD and PCA

a)

(i) wrong. The notion of eigenvalues λ_i and eigenvectors $v_i \in \mathbb{R}^n$ is only defined for quadratic matrices. This is easy to verify if we take a look at the formula $Av_i = \lambda_i v_i$. The LHS of this equation has to be a vector of the same size as v_i for the equation to be well-defined. Therefore A has to be an endomorphism of $\mathbb{R}^n$, i.e. $A \in \mathbb{R}^{n \times n}$.

(ii) true. This follows directly from the spectral theorem for symmetric matrices:

> If $A \in \mathbb{R}^{n \times n}$ is symmetric, there exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors v_i of A and the corresponding eigenvalues λ_i are real.

This means we can choose $\Phi = (v_1|...|v_n)$ and $\Lambda = \text{diag}(\lambda_1, ..., \lambda_n)$ with $\lambda_1 \geq ... \geq \lambda_n$. Φ is by definition an orthogonal matrix. For $A = U\Sigma V^T$ with $U = V = \Phi$ to be a SVD, the singular values have to be non-negative. This is satisfied if we require Λ to be positive-semidefinite.

(iii) wrong. Given a basis $v_1, ..., v_n \in \mathbb{R}^n \setminus \{0\}$ of real eigenvectors of A, we know that an eigendecomposition of A exists. It suffices to show that the eigenvalues are real. Assume there exists a non-real eigenvalue $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$ corresponding to the eigenvector v_i. First, observe that

$$a \in \mathbb{C} \setminus \mathbb{R}, \quad b \in \mathbb{R} \setminus \{0\} \implies a \cdot b \in \mathbb{C} \setminus \mathbb{R}.$$

Using this, we get

$$Av_i \in \mathbb{R}^n \text{ and } \lambda_i v_i \in \mathbb{C}^n \setminus \mathbb{R}^n.$$

But then $Av_i = \lambda_i v_i$ can not be true, as $\mathbb{R}^n \cap (\mathbb{C}^n \setminus \mathbb{R}^n) = \emptyset$. Therefore λ_i has to be real.

b) Choosing $V = \Phi, \Lambda = \Sigma^2$, we get

$$\begin{aligned}
A^T A &= (U\Sigma V^T)^T U\Sigma V^T \\
&= V\Sigma^T U^T U\Sigma V^T \\
&\overset{(*)}{=} V\Sigma^2 V^T \\
&\overset{(*)}{=} V\Sigma^2 V^{-1} \\
&= \Phi\Lambda\Phi^{-1}.
\end{aligned}$$

$(*)$ use unitary property of U and V, respectively.

To get U, we rearrange the definition of the SVD:

$$A = U\Sigma V^T \iff AV\Sigma^{-1} = U.$$

Assignment 2) Singular Value Decomposition by hand

a) First we calculate B:

$$B = \frac{2}{3\sqrt{10}} \begin{pmatrix} 5 & 3 \\ 0 & 4 \end{pmatrix} \frac{2}{3\sqrt{10}} \begin{pmatrix} 5 & 0 \\ 3 & 4 \end{pmatrix} = \frac{4}{45} \begin{pmatrix} 17 & 6 \\ 6 & 8 \end{pmatrix}.$$

Determine the characteristic polynomial:

$$\begin{aligned}
p(\lambda) &= \det(B - \lambda \cdot I) \\
&= \det \begin{pmatrix} \frac{4\cdot 17}{45} - \lambda & \frac{4\cdot 6}{45} \\ \frac{4\cdot 6}{45} & \frac{4\cdot 8}{45} - \lambda \end{pmatrix} \\
&= \det \begin{pmatrix} \frac{68}{45} - \lambda & \frac{24}{45} \\ \frac{24}{45} & \frac{32}{45} - \lambda \end{pmatrix} \\
&= \left(\frac{68}{45} - \lambda\right)\left(\frac{32}{45} - \lambda\right) - \left(\frac{24}{45}\right)^2 \\
&= \lambda^2 - \frac{20}{9}\lambda + \frac{64}{81}
\end{aligned}$$

Find the roots of p:

$$\begin{aligned}
&\lambda^2 - \frac{20}{9}\lambda + \frac{64}{81} = 0 \\
\implies \lambda &= \frac{20}{9}\cdot\frac{1}{2} \pm \sqrt{\left(\frac{20}{9}\cdot\frac{1}{2}\right)^2 - \frac{64}{81}} \\
\implies \lambda &= \frac{10}{9} \pm \sqrt{\frac{10^2-64}{81}} \\
\implies \lambda &= \frac{10}{9} \pm \frac{6}{9} \\
\implies \lambda_1 &= \frac{16}{9}, \quad \lambda_2 = \frac{4}{9}.
\end{aligned}$$

b) For (2×2)-matrices we can use a closed-form solution to determine the eigenvectors. We choose $\alpha_1, \alpha_2 \in \mathbb{R}$ after the fact s.t. the vectors are normalized.

$$\begin{aligned}
v_1 =& \alpha_1 \begin{pmatrix} \lambda_1 - \frac{32}{45} \\ \frac{24}{45} \end{pmatrix} = \alpha_1 \begin{pmatrix} \frac{80}{45} - \frac{32}{45} \\ \frac{24}{45} \end{pmatrix} = \alpha_1 \begin{pmatrix} \frac{48}{45} \\ \frac{24}{45} \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \\
v_2 =& \alpha_2 \begin{pmatrix} \lambda_2 - \frac{32}{45} \\ \frac{24}{45} \end{pmatrix} = \alpha_2 \begin{pmatrix} \frac{20}{45} - \frac{32}{45} \\ \frac{24}{45} \end{pmatrix} = \alpha_2 \begin{pmatrix} -\frac{12}{45} \\ \frac{24}{45} \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.
\end{aligned}$$

c) From assignment 1) we know that

$$V = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad \Sigma = \operatorname{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}).$$

Calculate U:

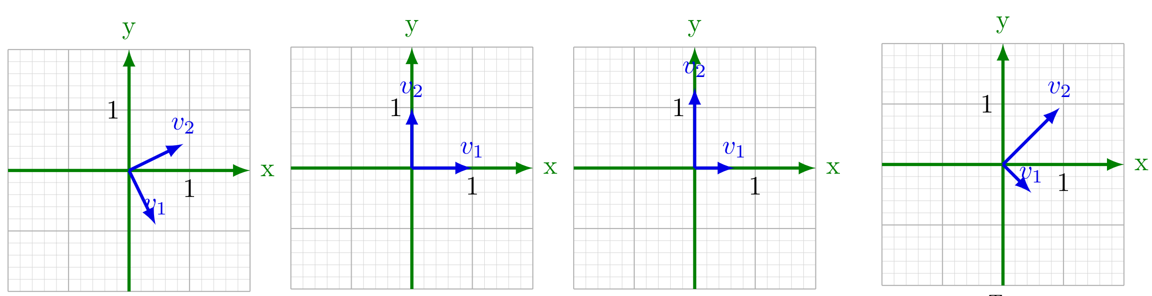
$$\begin{aligned}
U = AV\Sigma^{-1} &= \frac{2}{3\sqrt{10}}\begin{pmatrix} 5 & 0 \\ 3 & 4 \end{pmatrix} \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} \end{pmatrix} \\
&= \frac{2}{3\sqrt{50}}\begin{pmatrix} 5 & 0 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} \frac{6}{4} & -\frac{3}{2} \\ \frac{3}{4} & \frac{6}{2} \end{pmatrix} = \frac{1}{\sqrt{50}}\begin{pmatrix} 5 & -5 \\ 5 & 5 \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
\end{aligned}$$

d) Checking the correctness:

$$
\begin{aligned}
UU^T &= \frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\
VV^T &= \frac{1}{5}\begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix} = \frac{1}{5}\begin{pmatrix}4+1 & 2-2\\ 2-2 & 1+4\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\\
U\Sigma V^T &= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\frac{4}{3} & 0\\ 0 & \frac{2}{3}\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}\\
&= \frac{1}{\sqrt{10}}\begin{pmatrix}\frac{4}{3} & -\frac{2}{3}\\ \frac{4}{3} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}\\
&= \frac{2}{3\sqrt{10}}\begin{pmatrix}2 & -1\\ 2 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}\\
&= \frac{2}{3\sqrt{10}}\begin{pmatrix}5 & 0\\ 3 & 4\end{pmatrix}.
\end{aligned}
$$

Assignment 3) SVD interpretation

a)



$I[v_1, v_2] = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}$ $\quad$ $V^T[v_1, v_2] = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ $\quad$ $\Sigma V^T[v_1, v_2] = \begin{pmatrix}2/3 & 0\\ 0 & 4/3\end{pmatrix}$ $\quad$ $U\Sigma V^T[v_1, v_2] = \begin{pmatrix}\sqrt{2}/3 & 2\sqrt{2}/3\\ -\sqrt{2}/3 & 2\sqrt{2}/3\end{pmatrix}$

b) We know that Σ is a diagonal matrix with nonnegative real numbers and as such it will only applies scaling. Both U, V are unitary and as such will not affect the norm (length) but only rotate or mirror.

Assignment 4) Principal Component Analysis and Compression

a)^

Using PCA we generate a new basis for our data.
Let $X \in \mathbb{R}^{m\times n}$ then we get $X = U\Sigma V^T$ where $U \in \mathbb{R}^{m\times m}$, $\Sigma \in \mathbb{R}^{m\times n}$, $U \in \mathbb{R}^{n\times n}$.

The principal components in Σ determining the variation of the data on each

of our newly acquired axis. This allows us to choose the components with the highest variation in the data which will already approximate original data as it contains most of the information. Assuming that the matrices are sorted by singular values we choose the first r component to represent the data.
Let $X \in \mathbb{R}^{m \times n}$ then we get $X \approx U \Sigma V^T$ where $U \in \mathbb{R}^{m \times r}$, $\Sigma \in \mathbb{R}^{r \times r}$, $U \in \mathbb{R}^{r \times n}$. Choosing $r << m$ we can save significant amounts storage.