Sheet 1 - Transformations

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1 Practical

Please find the solution in the accompanying .py file.

2 Theoretical

Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^{\mathbf{3}}$

2.1 a)

Show that $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

Let $\mathbf{d} = \mathbf{a} \times (\mathbf{b} + \mathbf{c}) - (\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c})$

$$
\begin{aligned}
\mathbf{d^2} =& \mathbf{d} \cdot (\mathbf{a} \times (\mathbf{b} + \mathbf{c}) - \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c}) \\
=& \mathbf{d} \cdot (\mathbf{a} \times (\mathbf{b} + \mathbf{c})) - \mathbf{d} \cdot (\mathbf{a} \times \mathbf{b}) - \mathbf{d} \cdot (\mathbf{a} \times \mathbf{c}) && \text{using(1)} \\
=& (\mathbf{a} \times \mathbf{d}) \cdot (\mathbf{b} + \mathbf{c}) - \mathbf{b} \cdot (\mathbf{a} \times \mathbf{d}) - \mathbf{c} \cdot (\mathbf{a} \times \mathbf{d}) \\
=& (\mathbf{a} \times \mathbf{d}) \cdot (\mathbf{b} + \mathbf{c}) - (\mathbf{a} \times \mathbf{d}) \cdot (\mathbf{b} + \mathbf{d}) \\
=& 0
\end{aligned}
$$

$$
\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^{\mathbf{3}} | \mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \mathbf{y} \cdot (\mathbf{x} \times \mathbf{z}) \tag{1}
$$

2.2 b)

Show that $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$

$$
\begin{aligned}
\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = & \begin{pmatrix} \mathbf{a_1} \\ \mathbf{a_2} \\ \mathbf{a_3} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a_2 b_3 - a_3 b_2} \\ \mathbf{a_3 b_1 - a_1 b_3} \\ \mathbf{a_1 b_2 - a_2 b_1} \end{pmatrix} \\
=& \mathbf{a_1 (a_2 b_3 - a_3 b_2) + a_2 (a_3 b_1 - a_1 b_3) + a_3 (a_1 b_2 - a_2 b_1)} \\
=& \mathbf{a_1 a_2 b_3 - a_1 a_2 b_3 + a_1 a_3 b_2 - a_1 a_3 b_2 + a_1 a_3 b_2 - a_1 a_3 b_2} \\
=& 0
\end{aligned}
$$

2.3 c)

$$\mathbf{a} \times \mathbf{b} \overset{def}{=} \begin{vmatrix} \mathbf{e_1} & \mathbf{a_x} & \mathbf{b_x} \\ \mathbf{e_2} & \mathbf{a_y} & \mathbf{b_y} \\ \mathbf{e_3} & \mathbf{a_z} & \mathbf{b_z} \end{vmatrix}$$

$$= \mathbf{e_1} \begin{vmatrix} \mathbf{a_y} & \mathbf{b_y} \\ \mathbf{a_z} & \mathbf{b_z} \end{vmatrix} - \mathbf{e_2} \begin{vmatrix} \mathbf{a_x} & \mathbf{b_x} \\ \mathbf{a_z} & \mathbf{b_z} \end{vmatrix} + \mathbf{e_3} \begin{vmatrix} \mathbf{a_x} & \mathbf{b_x} \\ \mathbf{a_y} & \mathbf{b_y} \end{vmatrix} \qquad \text{usingLaplace}$$

$$\text{Thus } |\mathbf{a} \times \mathbf{b}| = \left| e_1 \begin{vmatrix} \mathbf{a_y} & \mathbf{b_y} \\ \mathbf{a_z} & \mathbf{b_z} \end{vmatrix} - \mathbf{e_2} \begin{vmatrix} \mathbf{a_x} & \mathbf{b_x} \\ \mathbf{a_z} & \mathbf{b_z} \end{vmatrix} + \mathbf{e_3} \begin{vmatrix} \mathbf{a_x} & \mathbf{b_x} \\ \mathbf{a_y} & \mathbf{b_y} \end{vmatrix} \right|$$

For second part we know that $\det(\mathbf{A}) = \det(\mathbf{A}^{\mathrm{T}})$**and as such it is correct per definition given on the slides.**

2.4 d)

Show that the tripleproduct $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ **holds.**

$$\begin{aligned} \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = & \begin{pmatrix} \mathbf{c_1} \\ \mathbf{c_2} \\ \mathbf{c_3} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{a_2 b_3 - a_3 b_2} \\ \mathbf{a_3 b_1 - a_1 b_3} \\ \mathbf{a_1 b_2 - a_2 b_1} \end{pmatrix} \\ = & \mathbf{a_2 b_3 c_1 - a_3 b_2 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 + a_1 b_2 c_3 - a_2 b_1 c_3} \\ = & \mathbf{a_2 (b_3 c_1 - b_1 c_3) + a_3 (a_1 c_2 - b_2 c_1) + a_1 (b_2 c_3 - b_3 c_2)} \\ = & \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \end{aligned}$$