Sheet 2 - Physics

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1 Using Lyx

2 Practical part

Please find the solution in the accompanying .py file.

3 Particle systems

3.1 a)

We can w.l.o.g. simplify the following formulas by assuming $t_0 = 0$. As the particle is at rest up until t_0, we get $v(0) = 0$, $x(0) = (x_0, y_0)^T$,

$$a(t) = r \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \forall t \geq 0$$

and the following integrals

$$\begin{aligned} v(t_1) &= \int_0^{t_1} a(t)\, dt + v(0) = \int_0^{t_1} r \cdot \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} dt + 0 = r \cdot \left[\begin{pmatrix} \sin t \\ -\cos t \end{pmatrix} \right]_0^{t_1} \\ &= r \cdot \begin{pmatrix} \sin t_1 \\ -\cos t_1 + 1 \end{pmatrix} \\ x(t_1) &= \int_0^{t_1} v(t)\, dt + x(0) = \int_0^{t_1} r \cdot \begin{pmatrix} \sin t_1 \\ -\cos t_1 + 1 \end{pmatrix} dt + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= r \cdot \left[\begin{pmatrix} -\cos t \\ t - \sin t \end{pmatrix} \right]_0^{t_1} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = r \cdot \begin{pmatrix} -\cos t_1 + 1 + x_0 \\ t_1 - \sin t_1 + y_0 \end{pmatrix} \end{aligned}$$

3.2 b)

Euler integration takes the first derivative of a quantity and adds it to the current value to approximate the next value.

$$x_{n+1} = x_n + \dot{x_n}\Delta t$$

By comparison with the Taylor expansion we know that the Euler method approximates the value with a single step error of $O\left(\Delta t^2\right)$. Are approximate $x(t)$ from $x\left(t_0\right)$ we apply the Euler method k times with the step size of $\frac{t-t_0}{k}$ for an accumulated error of $O\left(k\right)$ over the approximation. For particle kinematics we need to apply the Euler method twice. Once to integrate acceleration to velocity and then again to integrate velocity to the position.

$$v_{n+1} = v_n + a_n\Delta t$$

$$x_{n+1} = x_n + v_{n+1}\Delta t$$

3.3 c)

Again, we assume $t_0 = 0$ and define $x_0 = x(0) = (1,0)^T, v_0 = v(0) = (0,0)^T$. For $\Delta t = h = \pi$ we get

$$v_1 = 0 + 1 \cdot \begin{pmatrix} \cos 0 \\ \sin 0 \end{pmatrix} \cdot \pi = \begin{pmatrix} \pi \\ 0 \end{pmatrix}$$
$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \pi \\ 0 \end{pmatrix} \cdot \pi = \begin{pmatrix} 1+\pi^2 \\ 0 \end{pmatrix}$$

3.4 d)

Calculating 4 steps with $h = \frac{\pi}{4}$ results in

$$\begin{aligned}
v_1 =& 0 + \begin{pmatrix} \cos 0 \\ \sin 0 \end{pmatrix} \cdot \frac{\pi}{4} = \frac{\pi}{4}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
x_1 =& \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\frac{\pi}{4}\right)^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
v_2 =& \frac{\pi}{4}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\frac{\pi}{4} \\ \sin\frac{\pi}{4} \end{pmatrix}\right) = \frac{\pi}{4}\begin{pmatrix} 1+\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \\
x_2 =& \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\frac{\pi}{4}\right)^2 \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1+\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\frac{\pi}{4}\right)^2 \begin{pmatrix} 2+\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
v_3 =& \frac{\pi}{4}\left(\begin{pmatrix} 1+\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} \cos\frac{\pi}{2} \\ \sin\frac{\pi}{2} \end{pmatrix}\right) = \frac{\pi}{4}\begin{pmatrix} 1+\frac{1}{\sqrt{2}} \\ 1+\frac{1}{\sqrt{2}} \end{pmatrix} \\
x_3 =& \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\frac{\pi}{4}\right)^2 \left(\begin{pmatrix} 2+\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} 1+\frac{1}{\sqrt{2}} \\ 1+\frac{1}{\sqrt{2}} \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\frac{\pi}{4}\right)^2 \begin{pmatrix} 3+\frac{2}{\sqrt{2}} \\ 1+\frac{2}{\sqrt{2}} \end{pmatrix} \\
v_4 =& \frac{\pi}{4}\left(\begin{pmatrix} 1+\frac{1}{\sqrt{2}} \\ 1+\frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} \cos\frac{3\pi}{4} \\ \sin\frac{3\pi}{4} \end{pmatrix}\right) = \frac{\pi}{4}\begin{pmatrix} 1 \\ 1+\frac{2}{\sqrt{2}} \end{pmatrix} \\
x_4 =& \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\frac{\pi}{4}\right)^2 \left(\begin{pmatrix} 3+\frac{2}{\sqrt{2}} \\ 1+\frac{2}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} 1 \\ 1+\frac{2}{\sqrt{2}} \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left(\frac{\pi}{4}\right)^2 \begin{pmatrix} 4+\frac{2}{\sqrt{2}} \\ 2+\frac{4}{\sqrt{2}} \end{pmatrix}
\end{aligned}
$$

3.5 e)

Using the analytic formula from above, we get

$$
x(\pi) = \begin{pmatrix} -\cos(\pi)+1+1 \\ \pi - \sin(\pi)+0 \end{pmatrix} = \begin{pmatrix} 3 \\ \pi \end{pmatrix}
$$

In Fig. 1 we see that the Euler method for step size $h = \pi$ strongly differs from the correct value. Reducing the step size to $h = \frac{\pi}{4}$ improves the outcome, but there is still a noticeable deviation from the analytically derived trajectory. These effects stem from the fact that the acceleration $a(t)$ changes significantly on the interval $[0, \pi]$. The sampling at four points is too coarse to get an accurate approximation.

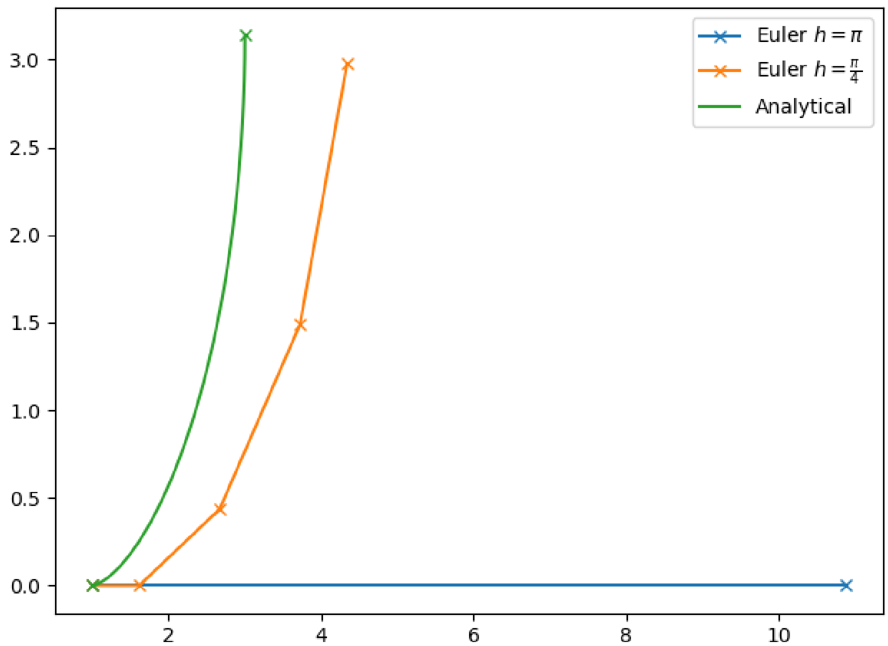

Figure 1: A plot of the integration results from c) (blue), d) (orange) and a) (green).