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COMPETITION OF SOCIAL OPINIONS ON TWO-LAYERS NETWORK

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COMPETITION OF SOCIAL OPINIONS ON TWO-LAYERS NETWORK

ABSTRACT

Social conflict could be modeled based on competition on two-layers network. To understand and analyze social conflict, we would research the features of a competition model by changing conditions such as network structures and updating rules and selecting key nodes on the networks with various structures. The competition model consists of two-layer opinions, where the first layer is opinion formation layer and the second layer is decision making layer. Starting with a polarized competition case where layer A has all the positive opinion and layer B has all the negative opinion, the states of two-layers network are changed by dynamics that each layer has. Competition results are analyzed and compared by indexes that measure the state of network and consensus. Based on this modeling and analysis method, models under various conditions are simulated.

By changing network structures such as internal links, external links and network types, it is researched how the network structures have influence on the final state of network. These simulation results shows that both internal and external links play a vital role for consensus. Especially, increasing the number of external and internal links on one side layer make it easy to reach consensus.

By changing updating rules such as dynamics order, sequential updating rule and simultaneous updating rule, it is investigated how the updating rules affect the final state of network and what features each updating rule has. These simulations provides that the networks with simultaneous updating rules are easy to have slow consensus and coexistence or to change into the opposite state, on the other hands, the networks with sequential updating rules are easy to make fast consensus.

By selecting key nodes on two-layers network, it is analyzed that which method is the most effective and best for recognizing key nodes. To identify key nodes on the various networks, single indicators and multiple indicators are applied. As single indicators, node centralities are used such as pagerank, degree, eigenvector, betweenness and closeness. As multiple indicators, 2 or 3 node centralities are combined and calculated. As the simulation results, it is found out that the most effective method to select key nodes is different according to network structures and layers. In addition, not only single indicators but multiple indicators also have good performance for recognizing key nodes.

This study could help to analyze social networks, such as legalization of social issues and vote results. Further more, it could contribute to understanding social conflicts and social network structure.

KEY WORDS: opinion dynamics, competition, consensus

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Chapter 1 Introduction

1.1 Introduction

People have their own opinions, and sometimes they change their opinions in response to others that hold views on given issues. Their opinions are reflected to the leader to make laws and vital decision. These phenomena can be found out in some cases, such as voting, legislation and adoption of new policies. It is widely recognized that opinion formation and decision making formation have mutual interaction as interconnected networks.[1-7]. And sometimes, opinion formation could be opposed to decision making formation. These situations often make social conflict and cause social confusion. To figure out these social conflicts, it is needed to understand and analyze the competition of interconnected networks. So far, physics and computer science have researched these social conflict by modeling and analyzing the complex systems[8-11]. The researches include opinion dynamics, voter model, game theory and many more.[12-18] Competition of interconnected networks has been researched in many ways. These networks can be applied to the dissemination of computer viruses, messages, opinions, memes, diseases and rumors[19-26]. Opinion dynamics on interconnected networks has been investigated with various network models such as *Abrams-Strogatz(AS)* model[27, 28] and *M* model[25]. Based on the previous researches, we would study the main features of competing two-layer networks by changing network structures, changing the updating rules on two-layers, and selecting the key nodes on two-layers. It would be proven and analyzed that these different conditions cause different results.

1.2 Related Work

In this research, we focus on the competition on two layer network or interconnected network. Comparing with single layer, interconnected network has 2 dynamics, 2 parameters and include internal edge and external edge as shown in Fig. 1–2. Therefore, multi-layers network interaction would be more complex than single layer network interaction.

To make two layer networks under competition, each layer is made up with different dynamics and parameter. Network dynamics are based on previous research such as [22].

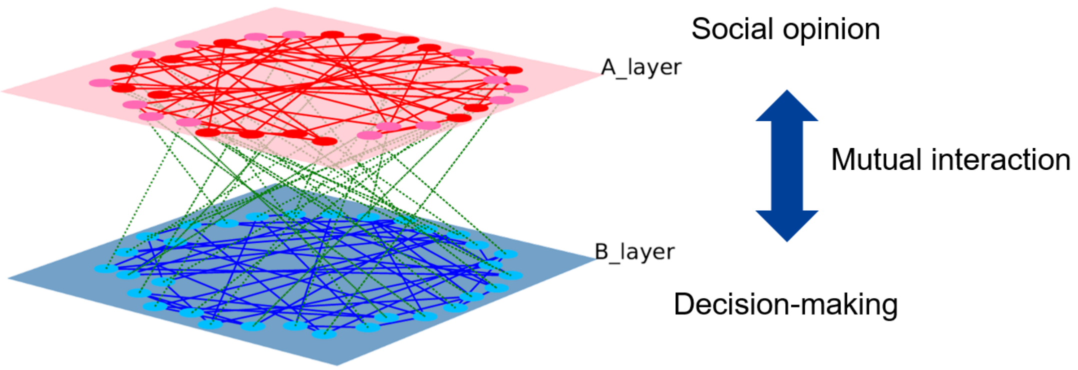
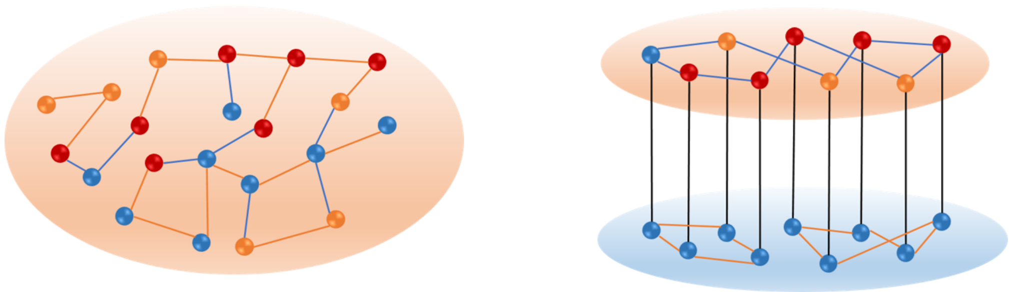


Figure 1–1 The example of competition on two-layer network



Single layer	Multi layer(interconnected network)
1 dynamics	2 dynamics (each layer has it own dynamics)
1 parameter	2 parameter
Internal edge	Internal edge + external edge

Figure 1–2 Comparison between single layer and multi-layer

One layer has the function of social opinion and its own dynamics. Some opinion models provide social mechanism by means of a compromise process.[29] Other opinion models represent persuasive process.[30] In this paper, the social opinion layer is affected by the opinion dynamics which are also known as M-model[25], that includes compromise function and persuasion function. The other layer has the function of decision-making and its own dynamics. The dynamics of the decision making layer is the language competition dynamics that are also called as Abrams-Strogatz model[27, 28, 31]. This model is useful to decide only one opinion from two opinions. To make competition condition of these two layers, the initial status of the two layers is assumed to be in opposite states, that social opinion layer has all positive states and decision making layer has all negative states.

So far, main researches have focused on what factors make a consensus or dissent(coexistence), which have shown that the system can make positive consensus, negative consensus or coexistence under certain range of parameters, such as volatility, reinforcement and prestige.[22] And interconnected competition of the social network have been researched by finding the threshold or critical point for consensus.[22-24] Also, it has been found out that the thresholds make the transition of states and they can explain and analyze the social phenomena in real world such as the legislation, election and social conflicts.[16, 22, 24]

In [23], it is shown that the transition from localized to mixed status occurs through a cascade from poorly connected nodes in the layers to the highly connected ones and the number of external degree is very important to change the state of layers. In addition, the main features, such as transition and cascade, found in Monte Carlo simulation are exactly characterized by the mean-field theory and magnetization[16, 22-24].

Based on these pre-existed researches, the competition of interconnected network would be analyzed by 3 main topics, such as network structures, updating rules and node centralities. Prior to simulations, backgrounds for 3 topics would be explained as follows. First, network structures would be investigated. Networks could be largely divided into regular network, random network[32], small world network[33], scale free network[34] and others. Fig. 1–3 shows the structures of various networks.

Regular network has lattice structure, and each node has exactly the same number of links. Random network is made up with edges that two node are connected with probability p in the systems with K nodes. Small world network is a network graph in which most nodes are not neighbors of one another, but most nodes can be reached from every other node by a small number of links. Small world network can be made by eliminating the edges with probability p and connecting two random nodes that are not connected in a regular network. Small world network has all characteristics of regular network and random network. Scale free network has the model that distribution of edges follows power function. Examples of scale free network are the World Wide Web (WWW), the Internet, movie star networks, protein interactions, metabolism, and so on. There are several ways to create a scale free network. Among them, the most typical way is *Barabasi-Albert* models. The *Barabasi-Albert* model is growing networks in which nodes continue to be added, and connections between nodes has preferential attachment. The process of creating this model repeats the following two processes: First, add one

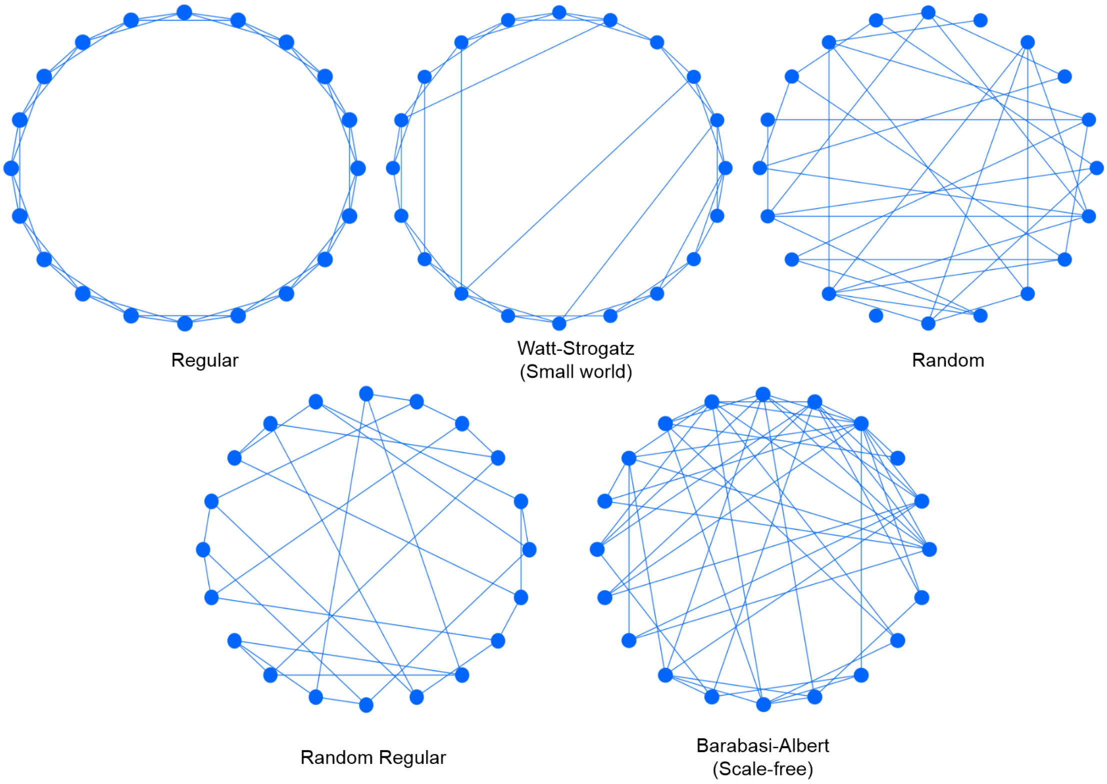


Figure 1–3 Various structures of network

node with a constant number of edges to the system. Second, edges of the added nodes are connected in proportion to edges number of the pre-existing nodes.
In this paper, two type of general network would be applied such as random regular(RR) network and *Barabasi-Albert*(BA) network.

Second, dynamics orders and updating rules would be studied. For further understanding the competition on two layer network, it is very important to investigate the interaction between nodes or layers. Methods of interaction between nodes are very various.[35] But, related to time, the types of interactions would be divided into two categories, simultaneous interaction and sequential interaction. In economics and social networks, it has been proven that there exists different results between simultaneous and sequential interactions.[36, 37] In [36], it was researched that how experimental subjects update induced prior information when receiving two information signals simultaneously or receiving the same signals sequentially. As the experimental results, the simultaneous treatment is very different from sequential treatment, and under sequential information,

subjects' mean estimates of the two treatments(good news preceding bad news or vice versa) are also significantly different from each other. In conclusion, both sequencing of process and the order of information suggest which one arrives matters. And, in [37], the usual random sequential updating rule is replaced by simultaneous updating on the Sznajd model. As the results, it is found out that this change makes a complete consensus much more difficult. The reason is analyzed as that for simultaneous updating some agents simultaneously receive conflicting messages from different neighbor pairs and thus refuse to change their opinion.
In this paper, both simultaneous and sequential updating rules would be applied to layers, nodes and links.

Third, network centralities would be researched to select key nodes on two layers network. Network centrality means the index to measure how close each node is to the center of the network. That means answers to the question "What characterizes an important node?". The concept of network centrality was first introduced in the field of social network analysis.[38] After that, it has expanded to various areas where the concept of the network is related and has been used to identify which nodes are important in the network. So far, various criteria for assessing network centrality have been presented. Generally well-known network centralities include degree centrality, betweeness centrality, closeness centrality, eigenvector centrality and pagerank centrality.[13, 39, 45]
Degree centrality is the simplest but the most reliable concept. it is defined as the number of interacting neighbor nodes (or edges). Betweenness centrality is the concept of using the shortest path between two nodes on a network. It is explained as the concept to define two different node sets on the network (set 1, 2) and quantify how often each node appears on the shortest path for all combinations of nodes in set 1 and set 2. Closeness centrality is derived from that the shorter the path that one node reaches all the other nodes is, the more important the node is. Eigenvector centrality is the concept that the more a node is connected with critical nodes, the more important it is. Pagerank centrality measures the convergent value by repeating the process of propagating each node's influence to the other nodes.
So far, many researchers have been trying to find important nodes in social network.[40-44] Based on node centrality, some algorithms for identifying key nodes has been found out. In [42, 44], it has been found out that optimally combining multiple measures of

nodal importance may provide a robust tool for identifying key nodes of interest, particularly in large graphs. Here, based on previous research, we would try to find the key nodes by using single node centrality and combined node centrality.

In this paper, as the single indicator methods to select key nodes, network centralities would be applied such as pagerank, degree centrality, eigenvector centrality, betweenness and closeness. As the multiple indicator methods to recognize key nodes, several combined node centralities would be applied such as *PR+DE, PR+BE, DE+BE, PR+DE+BE* that are based on single indicators. By using these centralities(pagerank, degree, eigenvector, closeness, betweenness and combined node centralities), it would be found out that which method is the most influential to change the network states on various models.

1.3 Motivation and organization

In this paper, opinion dynamics of a competing two-layer social network would be investigated on the basis of the pre-existed research[22-25]. We would develop the previous modeling and research to find out the characteristics of interconnected networks. By switching the network structure and updating rule of each layer, we can see how the consensus or coexistence states change and what conditions make the social consensus or dissent. In addition, trying to identifying key nodes on various network structures, it would be found out which method works well.

This Research has 4 main directions to solve the problems. First, it would be provided how to make up competition models and how to measure the consensus for analysis. Second, it would be found out what factors make consensus by changing network structures. Third, it would be analyzed how dynamics orders and updating rules have an influence on the state of two-layers network. Fourth, it would be investigated which method is the most effective to identify key nodes based on node centralities.

These study can help to explain social networks phenomena, such as social conflict between social opinion and the congress. Therefore, this research could be used as a tool for analyzing social network problems such as legislation and vote system, making efficient decision-making system and solving the social conflict.

This paper is organized as follows. In chapter.2, it is introduced that how competing two-layers are made up and how the dynamics of each layer works. And some indexes are provided to measure and evaluate the simulation results. In chapter.3, with changing network structure, it would be found out that how the network structures have the influence on the consensus of two layers. In chapter.4, considering the dynamics orders and updating rules, simulation results would be compared and analyzed. In chapter.5, it would be researched that which nodes are important for affecting the state of network by using single indicators and multiple indicators. Finally, in chapter.6, all simulation results will be summarized and our findings are concluded. And it is considered that what would be researched as future works and how we can be applied to real world.

Chapter 2 Modeling and Analysis

In this chapter, a basic model would be introduced for competition on two-layers network. It would be also described that how each layer is made up and what kind of function and dynamics it has. In addition, several indexes would be provided to analyze and measure the interaction between two-layers.

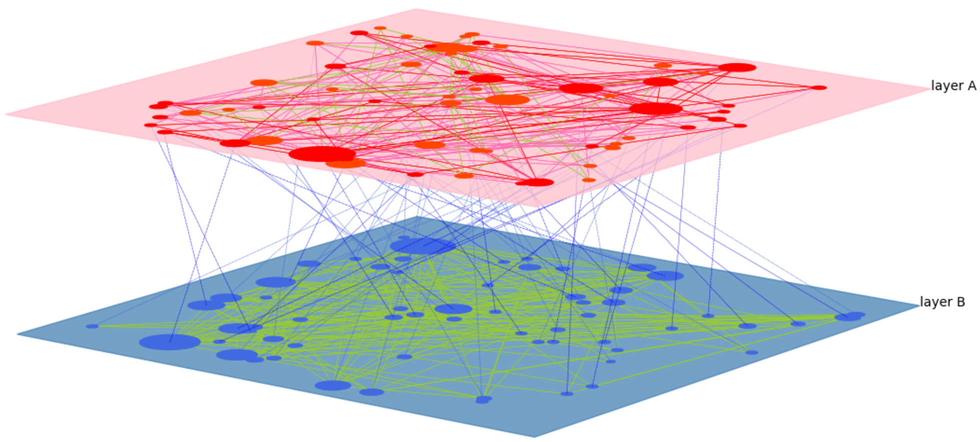


Figure 2–1 Competition of Interconnected Network

2.1 Modeling of two-layers network

The model consists of two-layers, and each layer has different dynamics. For layer A, the node changes its states according to M model as introduced in [25]. Here, we choose $M = 2$, that each node has four states $(-2, -1, +1, +2)$. For each link (k, j) belong to layer A, the dynamics are designed as follows:

- Compromise : if two nodes connected with link(k, j) have opposite orientations, their states become more moderate with probability q :

$$\text{if } S_k < 0 \text{ and } S_j > 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^r, S_j^l) \text{ with } prob.q, \tag{2–1}$$

$$\text{if } S_k > 0 \text{ and } S_j < 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^l, S_j^r) \text{ with } prob.q. \tag{2–2}$$

If $S_k = \pm 1$ and $S_j = \mp 1$, one switches orientation at random:

$$(\pm 1, \mp 1) \rightarrow \begin{cases} (+1, +1) \text{ with } prob.q/2, \\ (-1, -1) \text{ with } prob.q/2. \end{cases} \tag{2–3}$$

- Persuasion : if two nodes connected with link(k, j) have the same orientation, their states become more extreme with probability p :

$$\text{if } S_k < 0 \text{ and } S_j < 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^l, S_j^l) \text{ with } prob.p, \tag{2–4}$$

$$\text{if } S_k > 0 \text{ and } S_j > 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^r, S_j^r) \text{ with } prob.p. \tag{2–5}$$

For each external link (k, j) with k belong to layer A, the state of node k is updated according to :

- $S_k \cdot S_j < 0$:

$$\text{if } S_k < 0 \text{ and } S_j > 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^r, S_j) \text{ with } prob.q, \tag{2–6}$$

$$\text{if } S_k > 0 \text{ and } S_j < 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^l, S_j) \text{ with } prob.q. \tag{2–7}$$

- $S_k \cdot S_j > 0$:

$$\text{if } S_k < 0 \text{ and } S_j < 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^l, S_j) \text{ with } prob.p, \tag{2–8}$$

$$\text{if } S_k > 0 \text{ and } S_j > 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^r, S_j) \text{ with } prob.p. \tag{2–9}$$

Here, S_k^r and S_k^l denote the right and left neighboring states of node k, defined as

$$S_k^r = \begin{cases} +1, \text{ for } S_k = -1 \\ +2, \text{ for } S_k = +2 \\ S_k + 1, \text{ otherwise} , \end{cases} \qquad S_k^l = \begin{cases} -1, \text{ for } S_k = +1 \\ -2, \text{ for } S_k = -2 \\ S_k - 1, \text{ otherwise} . \end{cases} \tag{2–10}$$

The sign of S^A represents opinion orientation of a node and its absolute value $|S^A|$ measures the intensity of its opinion. So, $|S^A| = 2$ represents a positive or a negative extremist, while $|S^A| = 1$ corresponds to a moderate opinion of each side. In case of internal link (k, j) belong to layer A, when the nodes have the same orientation($S_k S_j > 0$), if the states of nodes are moderate, then they become extreme($S_k = \pm 1 \rightarrow \pm 2, S_j = \pm 1 \rightarrow \pm 2$) with probability p. If they are already extreme, they remain extreme($S_k = \pm 2 \rightarrow \pm 2, S_j = \pm 2 \rightarrow \pm 2$). On the other hand, when the nodes have opposite orientations($S_k S_j < 0$), if they are extreme, the states of nodes become moderate($S_k = \pm 2 \rightarrow \pm 1, S_j = \pm 2 \rightarrow \pm 1$) with probability q. If they are already moderate, they switch orientations individually($S_k = \pm 1 \rightarrow \mp 1, S_j = \pm 1 \rightarrow \mp 1$). In case of interaction between a node in layer A and a node in layer B, a node in layer A follows opinion dynamics formula, but the state of a node in layer B does not change. In other words, the state of layer B affects layer A, but layer A dynamics does not affect the state of node in layer B. For example, one of the layer A node, $S_k = +2$ is connected with $S_j = -1$ node of layer B. Here, S_k will change into $S_k = +1$ with $prob.q$, but S_j will not change, which indicates that the states of layer B have an influence on the states of layer A though state of a node in layer B is not changed.

The dynamics of layer B follows the decision-making dynamics as introduced in [27, 28]. The state of node i in layer B can be +1 and −1, and it is updated according to

$$P_B(S_i \rightarrow -S_i) = \begin{cases} \left(\left(\dfrac{i_i + e_i}{n^{-S_i}} \right) \cdot \left(\dfrac{n^{-S_i}}{i_i + e_i} \right) \right)^{1/v}, \text{ if } v \neq 0 \\ 0, \text{ if } v = 0 \\ 0, \text{ if } n^{-S_i} = 0 \end{cases}, \tag{2–11}$$

where i_i is internal degree of node i and e_i is external degree of node i. n^{-S_i} is the number of neighbors of node i with opposite state $-S_i$. v represents the volatility that measures how prone a node changes its state. The scale of v is from 0 to 1. If $v \simeq 0$, a node is unlikely to change its state. On the other hand, if $v \simeq 1$, a node is very likely to change its state. Also, this formula shows that the more the number of edges connected with the nodes of opposite state is, the easier the nodes are to change into the opposite state.

2.2 Simulations and Analysis

This model is nonlinear in that applied dynamics are switched according to the states of nodes. In this model, helpful mathematical tools are no longer applicable and it turns

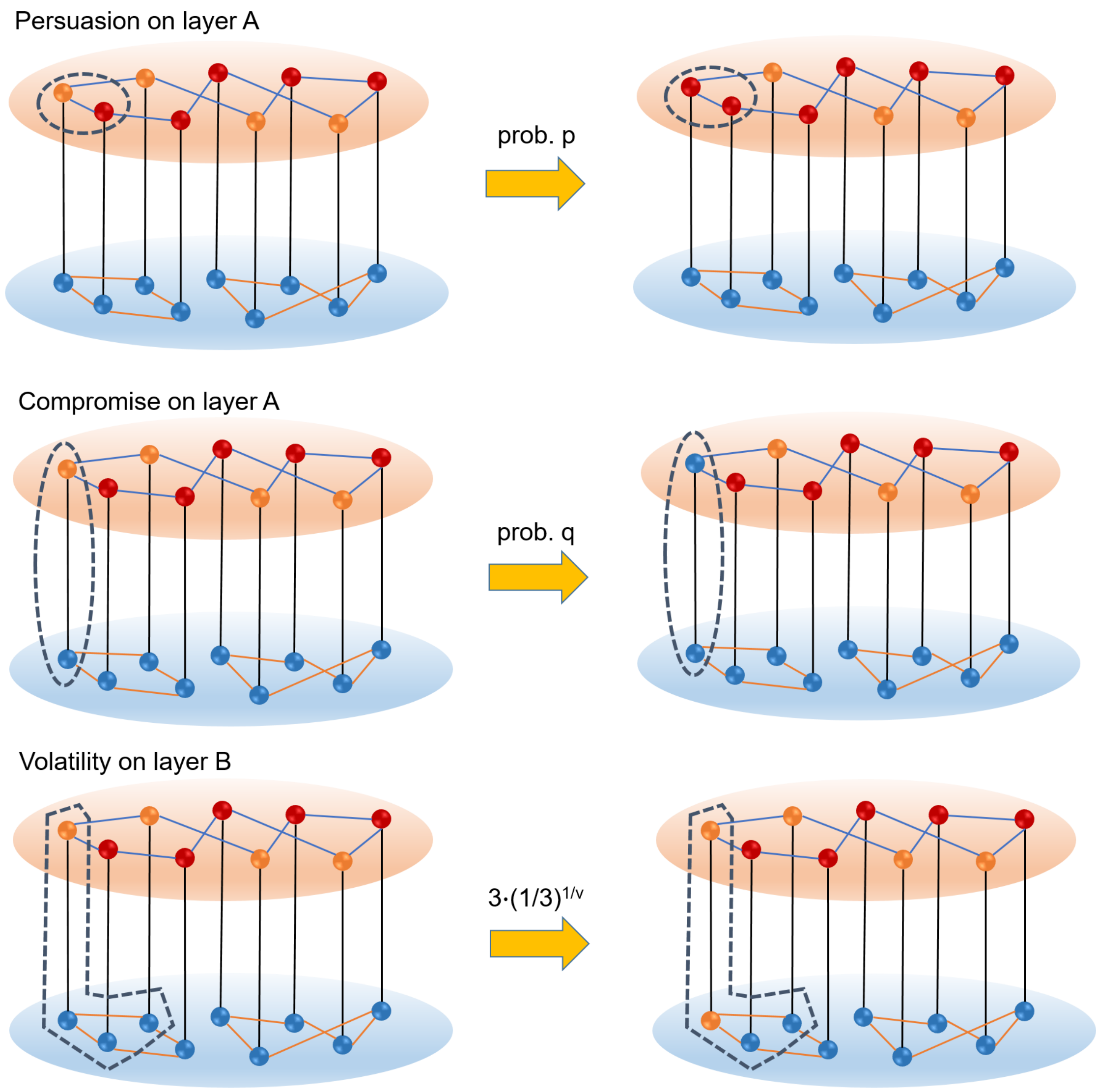


Figure 2–2 Dynamics on two layers

out, moreover, that rigorous analytical results are difficult to obtain.[46, 47]. For that reason, we would try to carry out the analysis of the above nonlinear model to a large extent by simulations on the computer.

To start with a polarized competition, as the initial conditions, nodes in layer A are all positive, and nodes in layer B are all negative as shown in Fig. 2–1. For nodes in layer A, it begins with the status where half of nodes are $+1$ and the others are $+2$. The initial states of nodes in layer B have only -1.

There are two parameters, p and q in the dynamics of layer A. To simply represent the probability p and probability q together, we set $p + q = 1$. So, p represents the reinforcement or strength of opinion such as extreme and moderate, which is scaled to be 0 to 1. And, there are only 1 parameter, v in the dynamics of layer B. The scale of v is also 0 to 1 as p. v represents the volatility, which means how prone the state can be changed into the opposite state.

To implement the interconnected dynamics, one step consists of two layers dynamics, where every node in layer A is checked with opinion dynamics, and every node in layer B updates its state according to the decision-making dynamics. Basically, the dynamics order follows updating state of layer B after updating state of layer A. The dynamics orders and updating rules would be discussed specifically in chapter.4.

Each simulation takes 100 steps, and 100 simulations are considered for average results. To analyze the simulation results, we use *'Average State'(AS)* to measure the average states of network and *'Consensus Index'(CI)* to measure how close the states of network is to consensus. The formula follow as

$$AS = avg\left(\sum_{i}^{K^A} S_i^A/4\right) + avg\left(\sum_{i}^{K^B} S_i^B/2\right), \tag{2–12}$$

$$CI = \frac{(K_+{}^A \cdot K_-{}^B) + (K_-{}^A \cdot K_+{}^B)}{K^A \cdot K^B}. \tag{2–13}$$

In these formula, S_i^A means the state of node i in layer A, and K^A is the number of nodes in layer A. $K_+{}^A$ represents the number of nodes with positive state in layer A.

With AS, it could be verified whether the consensus happens in accordance with the change of p and v. If the positive consensus happens, it would be close to the value of $+1$ and if the negative consensus happens, it would be close to the value of -1. And, the medium values between $+1$ and -1 mean that the states are belonging to the coexistence or dissent part. Fig. 2–3 shows that AS values are convergent to $+1$, -1 or other values as step(time) goes by. $+1$ means making positive consensus. -1 means making negative consensus. The other values mean coexistence or dissent state.

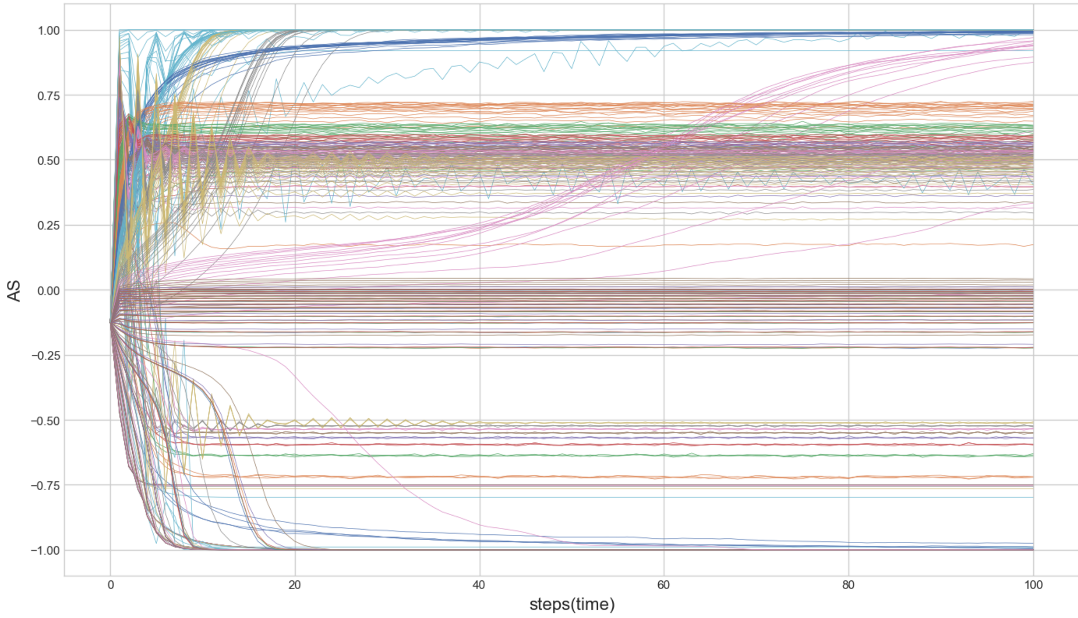


Figure 2–3 AS values per each step according to all parameters

With *CI*, it could be measured how close the network state is to consensus. If the *CI* is close to 0, the state of network is close to positive or negative consensus. If the *CI* is close to 1, the state of network is the separated coexistence where states of all nodes in layer A is opposed to states of all nodes in layer B. If the CI is close to 0.5, the state of network is the mixed coexistence where each layer has both positive and negative states of nodes.

Fig. 2–4 shows the characteristics of *CI*. Same orientation states in two layers make *CI* 0. Opposite orientation states between two layers make *CI* 1. And mixed states in two layers make *CI* close to 0.5.

As Fig. 2–5 shown, *CI* values are convergent to +1, 0, or other values as step(time) goes by. 0 means positive or negative consensus. +1 means opposite state between two layers. The other values means mixed state. By using *CI*, coexistence states can be divided into two categories, separated state and mixed state.

To measure and evaluate the consensus results regarding to two parameters p and v, we use four kinds of indexes including *'AS total'*, *'Positive Consensus Ratio'(PCR)*, *'Negative Consensus ratio'(NCR)*, and *'Consensus Ratio'(CR)*. *AS total* means the summation

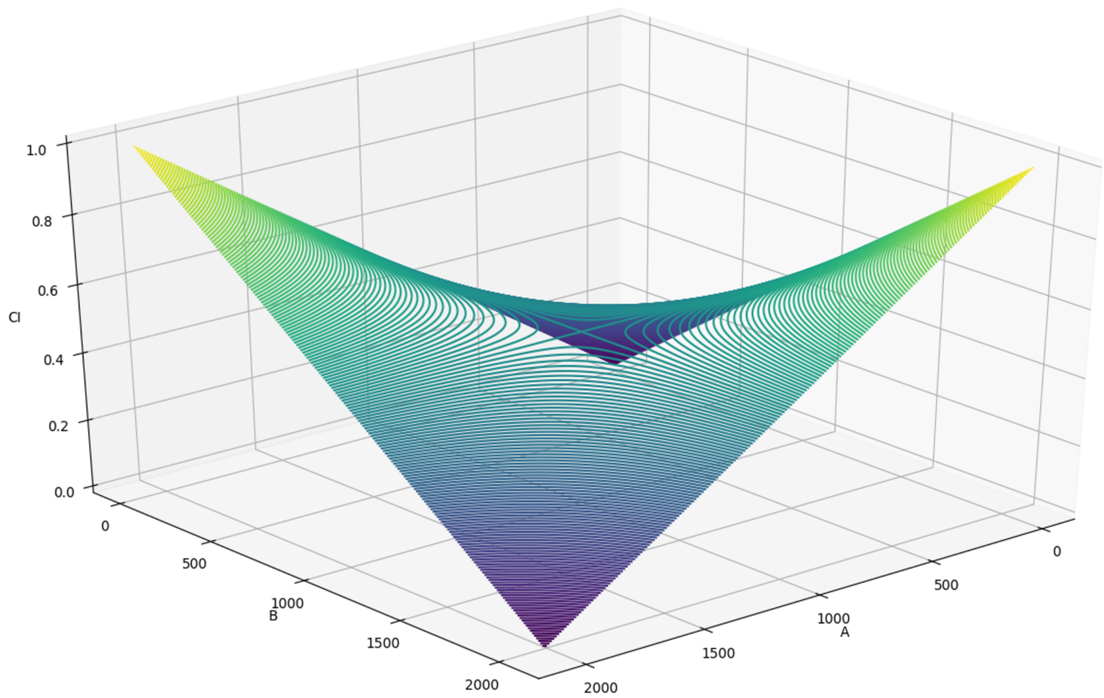


Figure 2–4 CI values according to all ${K_+}^A$ and ${K_+}^B$

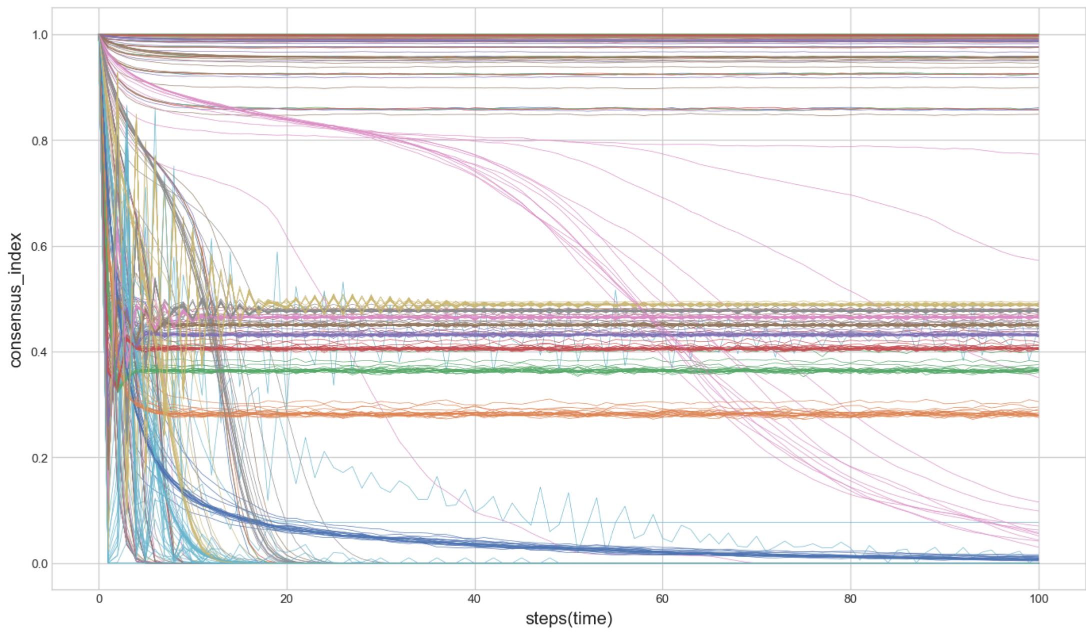


Figure 2–5 CI values per each step according to all parameters

of *AS* for all *p*s and all *v*s. *PCR* is the ratio of positive consensus over all simulations. Similarly, *NCR* is the ratio of experiments with negative consensus. *CR* is the ratio of experiments reaching consensus, i.e. summation of *PCR* and *NCR*.

$$AS\ total = \frac{\sum_{j}^{m}\sum_{i}^{n} AS_{p_i,v_j}}{n\times m}, \quad \begin{matrix} p=\{p_1,p_2,\cdots,p_n\} \\ v=\{v_1,v_2,\cdots,v_m\} \end{matrix}. \tag{2–14}$$

In Eq(2–14), AS_{p_i,v_j} means AS value according to parameters p_i and v_j, which shows the total orientation and intensity of interconnected network.

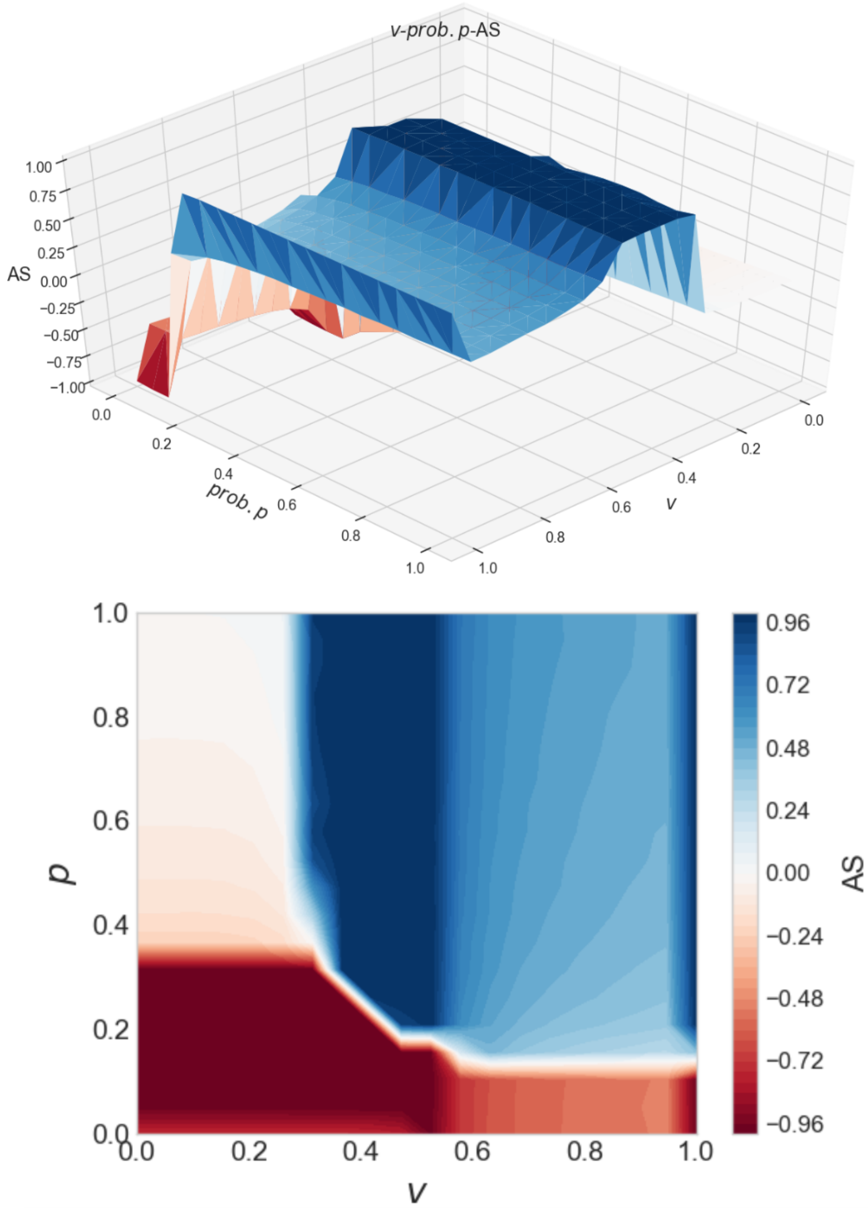
$$PCR = \frac{\sum_{j}^{m}\sum_{i}^{n}(AS_{p_i,v_j} \simeq 1)}{n \times m}. \tag{2–15}$$

In Eq(2–15), $AS_{p_i,v_j} \simeq 1$ means positive consensus.

$$NCR = \frac{\sum_{j}^{m}\sum_{i}^{n}(AS_{p_i,v_j} \simeq -1)}{n \times m}. \tag{2–16}$$

In Eq(2–16), $AS_{p_i,v_j} \simeq -1$ means negative consensus.

Fig. 2–6 shows the states of interconnected network according to all ps and all vs. The X-axis is the p and the Y-axis is the v, and the Z-axis represents AS. The closer the color is to blue, the more it has positive consensus. And the closer the color is to red, the more it has negative consensus. A light colored and white areas have coexistence with positive states and negative states. Here, we can measure the consensus by using indexes, *'AS total'*, *'PCR'*, *'NCR'*, and *'CR'*. The average value of this figure means *'AS total'*. The blue part area means *'PCR'*, the red part area means *'NCR'*, and the summation of those means *'CR'*.



Div	AS total (Average values)	PCR(Blue part)	NCR(Red part)	CR(Blue+Red part)
BA-BA network	0.1799	0.1750	0.1450	0.3200

Figure 2–6 The example of simulation：BA-BA network

Chapter 3 Competition on two-layers network with various structures

In this chapter, based on the competition model described in chapter.2, simulations would be implemented with changing the network structures. As the basic model, the interconnected network with random regular networks would be provided. And then, the structure of interconnected network would be altered by changing the internal edges, external edges and network types. Finally, all simulations would be compared and analyzed with the indexes, *AS total*, *PCR*, *NCR* and *CR*.

3.1 Competition on Random Regular Networks

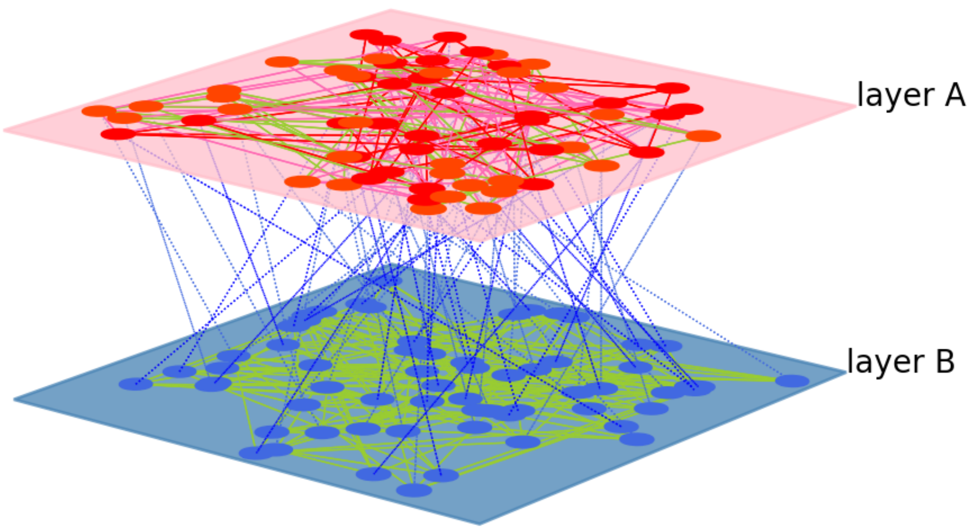


Figure 3–1 Competition on random regular network

In this section, simulation results on random regular network would be provided to comprehend the competition of two layers. Each layer consists of random regular network that has N nodes with k internal edges as introduced in [48-50]. Each node of

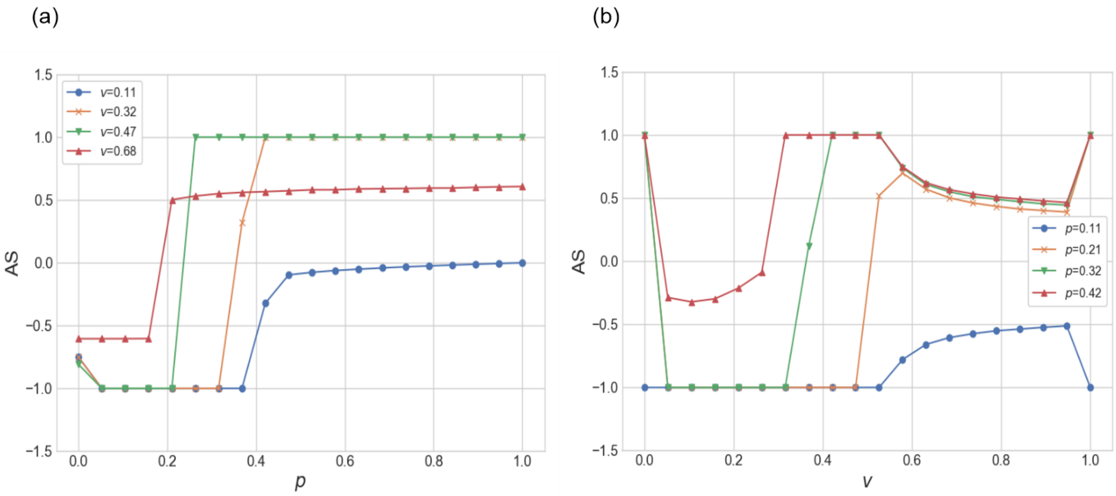


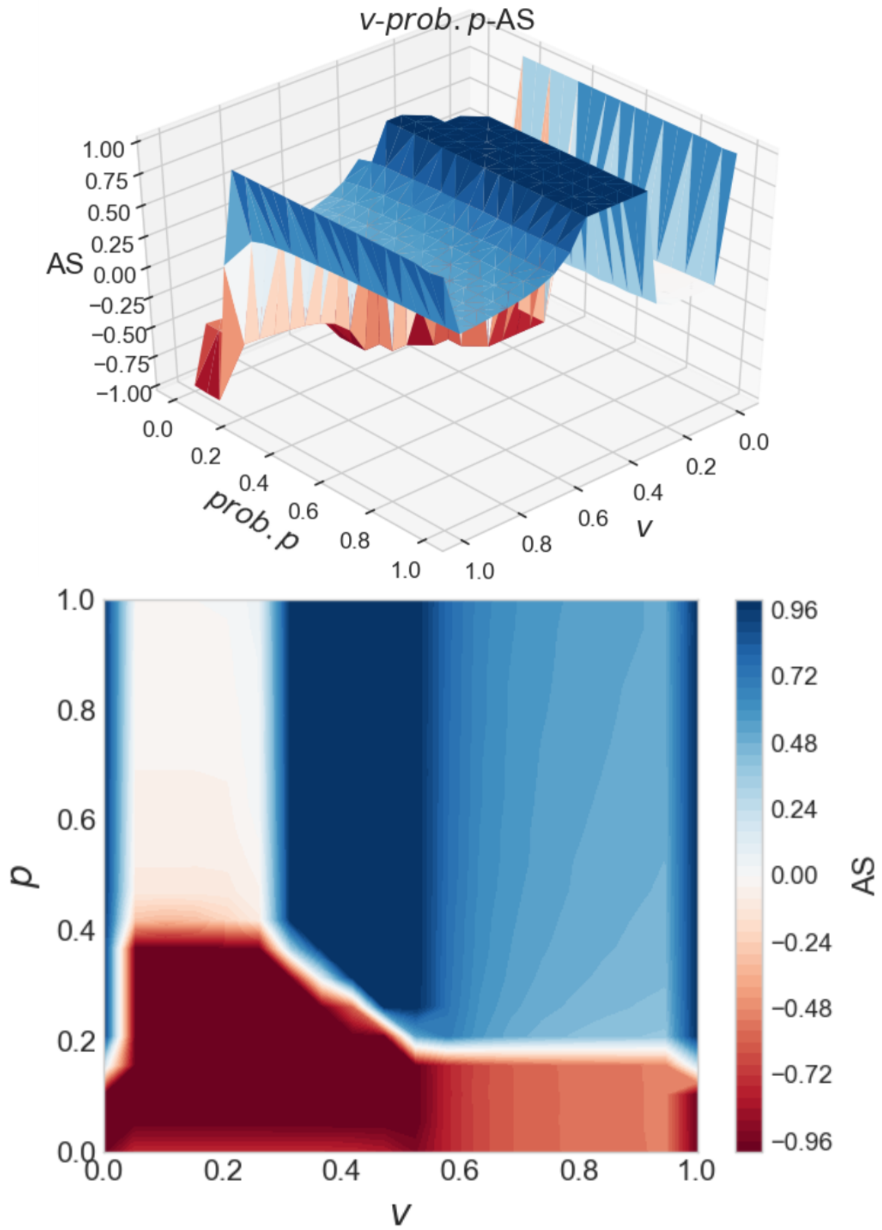
Figure 3–2 (a) p-AS chart according to certain v values. (b) v-AS chart according to certain p values.

one layer is connected with a random node on the other layer. This means each node has only 1 external un-directed edge. Simulations are performed on network with $N = 2048$, and $k = 5$.

The simulation results are shown in Fig. 3–2 and Fig. 3–3. Fig. 3–2 presents that how the 'Average State'(AS) are changed according to the other parameter(v or p), when one parameter(p or v) is constant. So we can know that how each parameter works on the network. Fig. 3–3 provides total results with all parameters. Through these figures, the characteristics of network would be analyzed.

Fig. 3–2(a) shows that when $p > 0.2$, $0.32 < v < 0.47$, it normally tends to positive consensus. Here, we could found out that if v is lower or larger than certain values, it doesn't make consensus. In Fig. 3–2(b), as v increases, it normally change from negative to positive consensus. But, it is found out that when p is very low($p \leq 0.11$), it doesn't make positive consensus. To sum up, when p is large enough, it tends to make positive consensus. But, when v is small enough, it tends to be changed into negative consensus.

Fig. 3–3 shows the states of two layers according to all ps and all vs. As previously described in chapter.2, blue area is for positive consensus, red area is for negative consensus, and light colored and white area is for coexistence. And indexes for consensus are also measured. Positive consensus area is 0.2475, and negative consensus area is



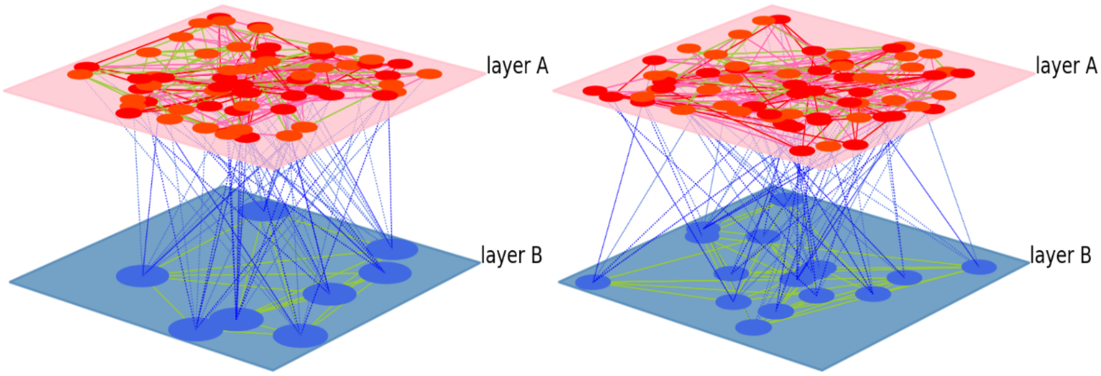
Div	AS total (Average values)	PCR (Blue part)	NCR (Red part)	CR (Blue+Red part)
RR-RR network	0.2034	0.2475	0.1575	0.4050

Figure 3–3 AS according to all ps and vs

0.1575. Coexistence area is $1 - CR = 0.5950$. By using these values and figures, this model would be compared with networks of various structures in next section. Through these figures, the characteristics of parameters can be arranged as follows. First, large p

tends to make positive consensus and small p tends to make negative consensus. Second, small v tends to make negative consensus, and large v tends to make coexistence state.

3.2 Competition on Networks with different number of external links



(a) HM(8) (b) HM(4)

Figure 3–4 Competition on *Hierarchical Model*

In this section, we consider the influence of external links. Based on *RR-RR* model in section 3.1, we reduce the number of nodes in layer B at a certain rate and increase the external links from nodes in layer B accordingly as shown in Fig. 3–4. We denote *HM(n)* as a *Hierarchical Model* with a level n, which means that the number of nodes in layer B is $1/n$ of the number of nodes in layer A, and the number of external links from node in layer B is n in view that the number of external links from node in layer A is 1. In other words, each node in layer A has one external edge, but each node in layer B has n external edges for *HM(n)*, which means one node in layer B can interact with n nodes in layer A.

To find out the significant influence of external edges, various *HM(n)s* were simulated. Totally, 8 *HM(n)*s, *HM(2), HM(4), HM(8), HM(16), HM(32), HM(64), HM(128), HM(256)* were arranged as shown in Fig. 3–5. Fig. 3–5 shows that *HM(2)* has the most area for coexistence part(light colored and white area) and *HM(256)* has the most are for consensus part(blue and red area). As n in *HM(n)* is increased, coexistence area is decreased and consensus area is increased. Particularly, positive consensus area is significantly increased, negative consensus area is slightly decreased.

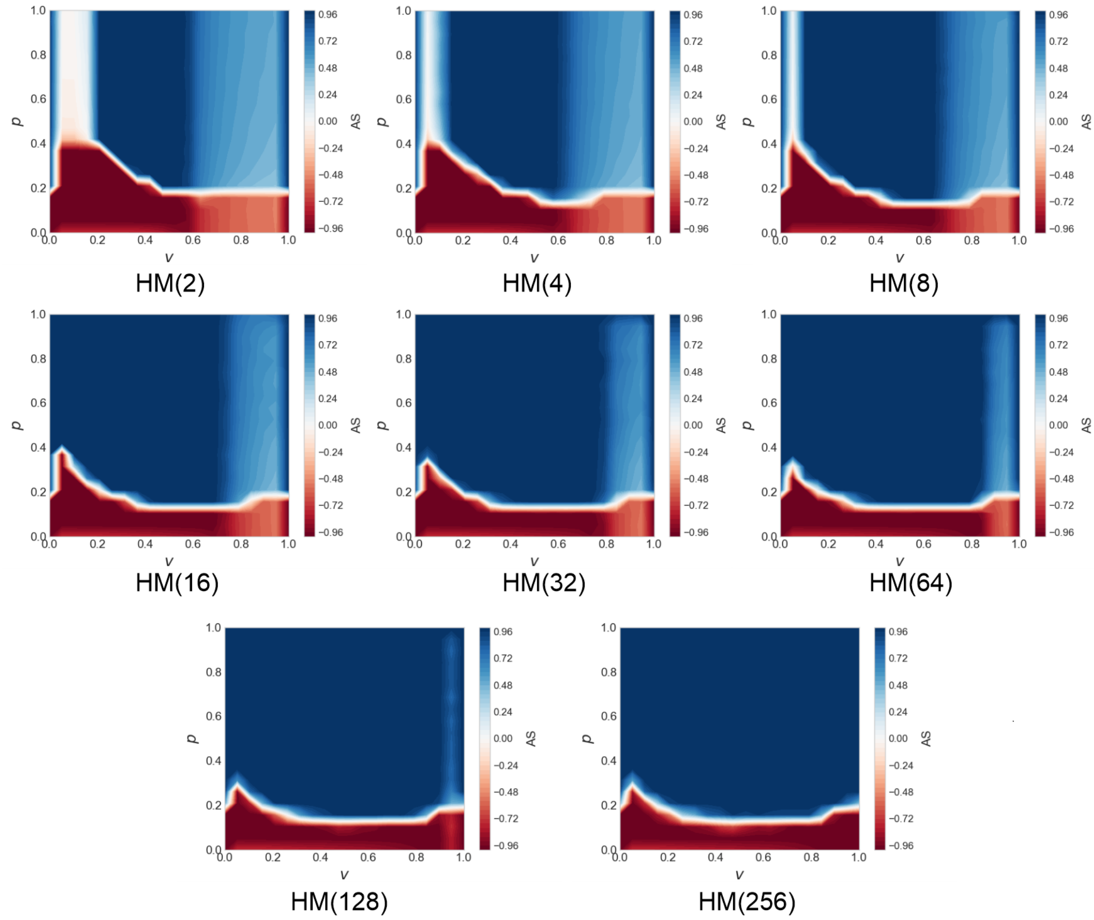


Figure 3–5 *AS total* on various *Hierarchical Models*

To clearly find out the difference between models, we use the indexes, *PCR, NCR, AS total*. Fig. 3–6 shows the results to analyze *HM(n)* with indexes. Blue color bar is for PCR, red color bar is for NCR, and green color bar is for AS total. Comparing *HMs* with *Basic model(RR(5)-RR(5))*, *CR PCR* and *AS total* are all increased remarkably. *HMs* have more positive consensus part than *RR(5)-RR(5)*. And, *HMs* have less negative consensus part than *RR(5)-RR(5)*.

In summary, all the *Hierarchical Models* have more consensus ratio than *Random Regular Networks* Model. However, positive consensus ratio is increased, but negative consensus ratio is decreased. It is found out that as the number of B nodes are more decreased, it makes easier to make positive consensus and harder to make negative consensus. In real world, it would be analyzed that as the number of leaders is less,

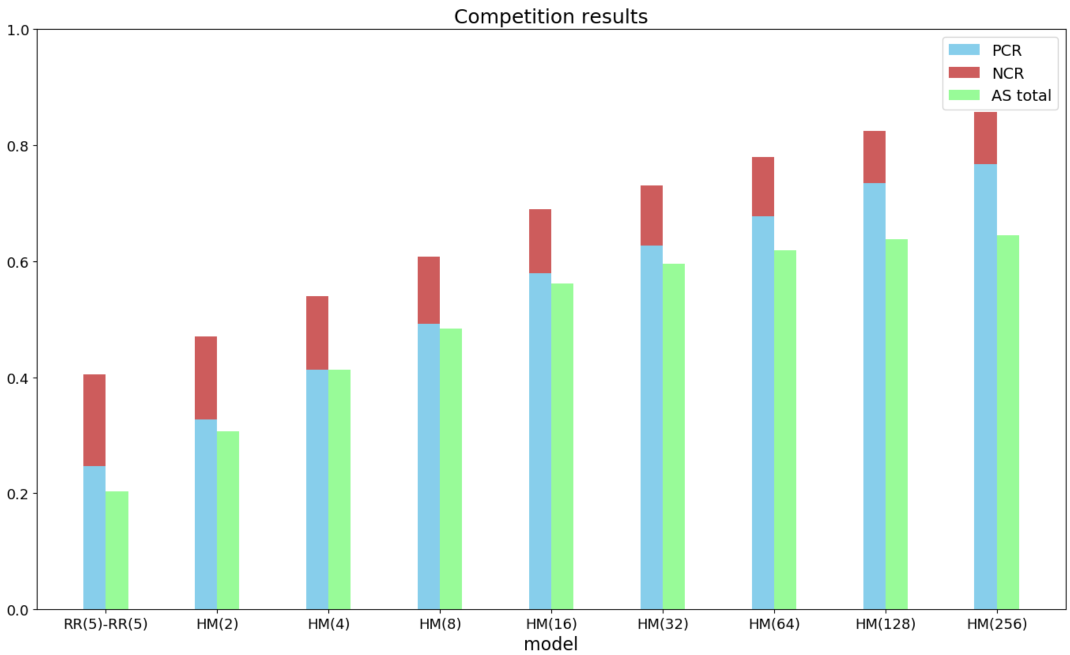


Figure 3–6 Histogram for *PCR, NCR, AS total* of *Hierarchical Models(HM(n))*

social conflict are decreased and the opinion is convergent to social opinion(layer A). But, sometimes there are some dangers to ignore the leader opinions(layer B), or to cause more social conflict when there are stubborn leaders, that would be simulated in chapter.5.

3.3 Competition on Networks with different number of internal links

Next, the interconnected networks are simulated with various internal degrees in order to define and evaluate the influence of internal degrees. Random regular network would be applied and the number of internal degrees on each node is switched to various numbers as shown in Fig. 3–7. But, there is no change on external degree, which would be fixed to only 1. Here, *RR(n)-RR(m)* represents layer A has a random regular network with n internal edges per a node and layer B has random regular network with m internal edges per a node. First, the internal degrees on layer A is changed. The internal degree on layer B is fixed to $5,120$, which means each node has 5 internal degrees on layer B, and the internal degree on layer A is switched into $2,048$, $3,072$, $4,096$, or $5,120$, which means each node has 2, 3, 4, or 5 internal degrees on layer A. Fig. 3–8 shows the simulation results for changing the internal degree on layer A. As shows in Fig. 3–8 (a),

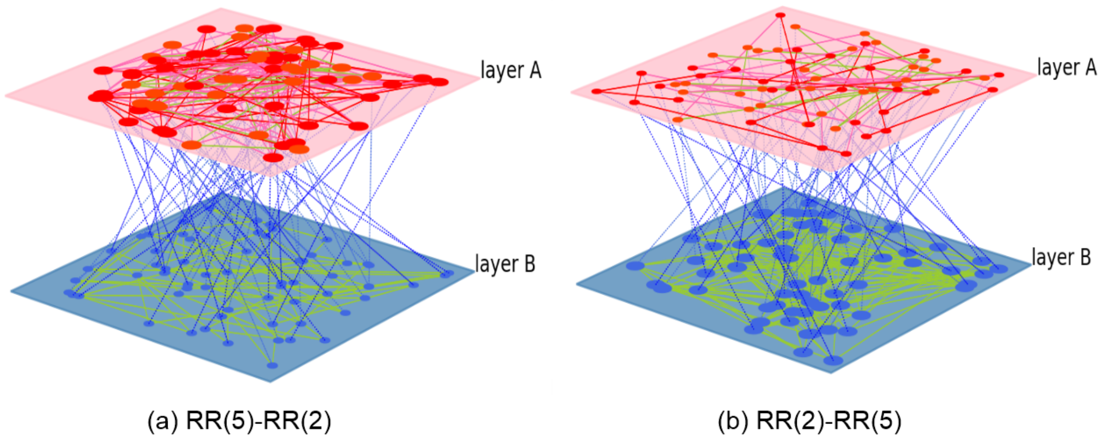


Figure 3–7 Competition on interconnected networks with different internal edges

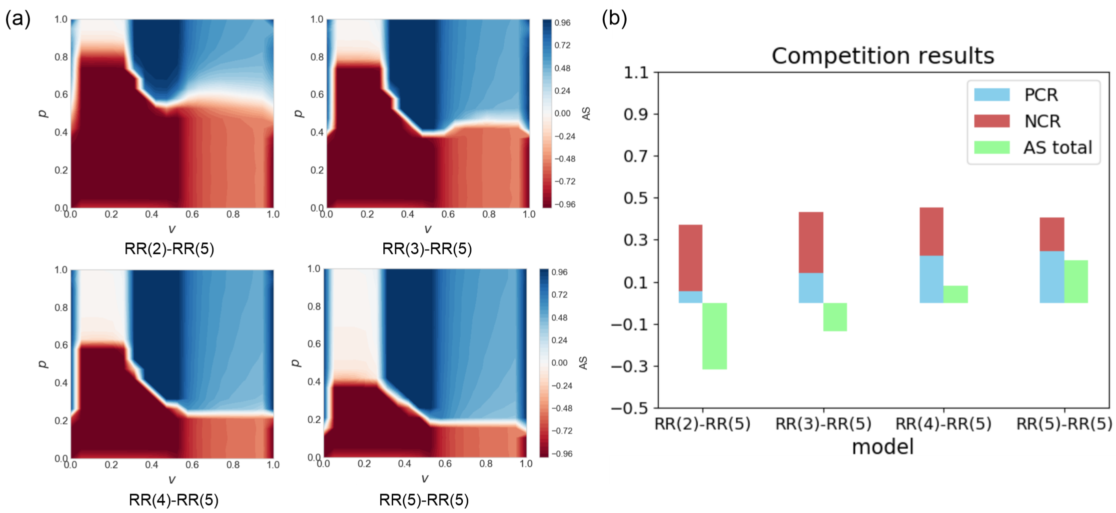


Figure 3–8 Simulation results with different internal degrees on layer A

as the internal degree on layer A is increased, the red part is decreased and the blue part is increased.

To clearly compare and analyze the results, the results are presented with the indexes, *PCR, NCR, AS total* in Fig. 3–8 (b), which shows that as the internal degree on layer A is increased, negative consensus is decreased and positive consensus is increased. As shown in Fig. 3–8, RR(5)-RR(5) has the largest *PCR*, and RR(2)-RR(5) has the largest NCR. However, all models in Fig. **??** have almost same *CR*. It can be analyzed that the internal degree on layer A has the tendency to keep positive state and to change negative state into positive state. Next, the internal degree on layer B is changed. The internal

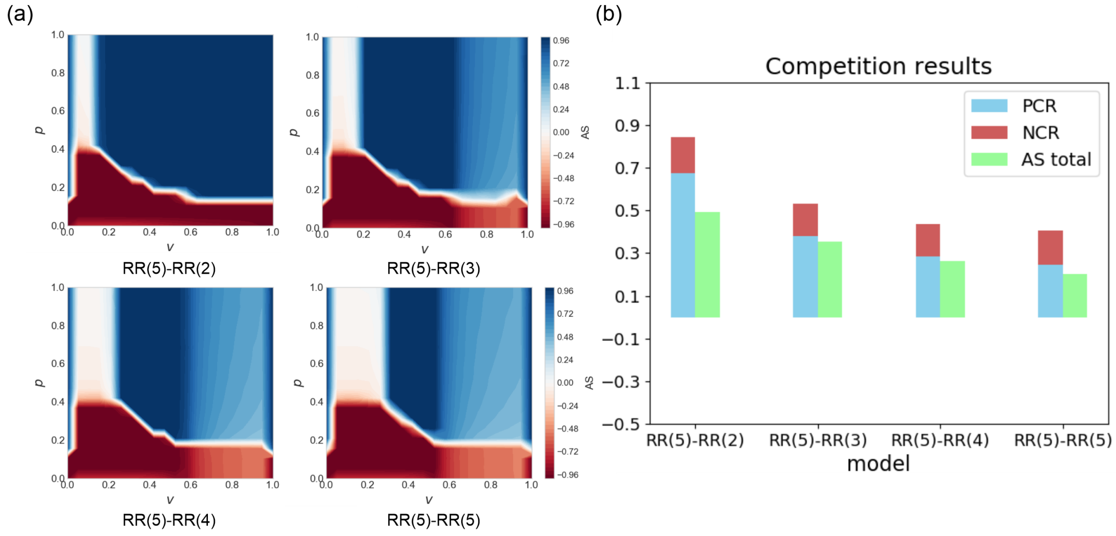


Figure 3–9 Simulation results with different internal degrees on layer B

degree on layer A is fixed to 5,120, which means each node has 5 internal degree on layer A, and the internal degree on layer B is switched into 2,048, 3,072, 4,096, or 5,120, which means each node has 2, 3, 4, or 5 internal degree on layer B. Fig. 3–9 shows the simulation results with changing the internal degree on layer B. As shows in Fig. **??** (a), as the internal degree on layer B is increased, the blue part is decreased, the white and light color part is increased, and the red part is almost same, though the shape of red area is changed. As shown in Fig. 3–9 (b), *RR(5)-RR(2)* has the largest *PCR* and *CR*, and *RR(5)-RR(5)* has the smallest PCR and *CR*. However, all models in Fig. **??** have almost same *NCR*. It can be analyzed that the internal degree on layer B has the tendency to hinder positive consensus state and has the inverse relation with *CR*. As the internal degrees on layer B is increased, *PCR* and *CR* are inversely decreased.

Considering two cases where an internal degree of layer A is changed and where an internal degree of layer B is changed, it is recognized that the role of internal degree on layer A is different with internal degree on layer B. The internal degree on layer A has the function to keep the state of layer A, and the internal degree on layer B has the function to restrain the consensus state of layer A and makes coexistence part. Next, it is simulated that internal degrees are changed on both layer A and layer B, such as *RR(2)-RR(2), RR(3)-RR(3), RR(4)-RR(4), RR(5)-RR(5)* and *RR(6)-RR(6)*. Through these simulations, it would be found out that how total internal degree on both layer A and layer B affect the interconnected network. Fig. 3–10 shows the influence of internal degree on

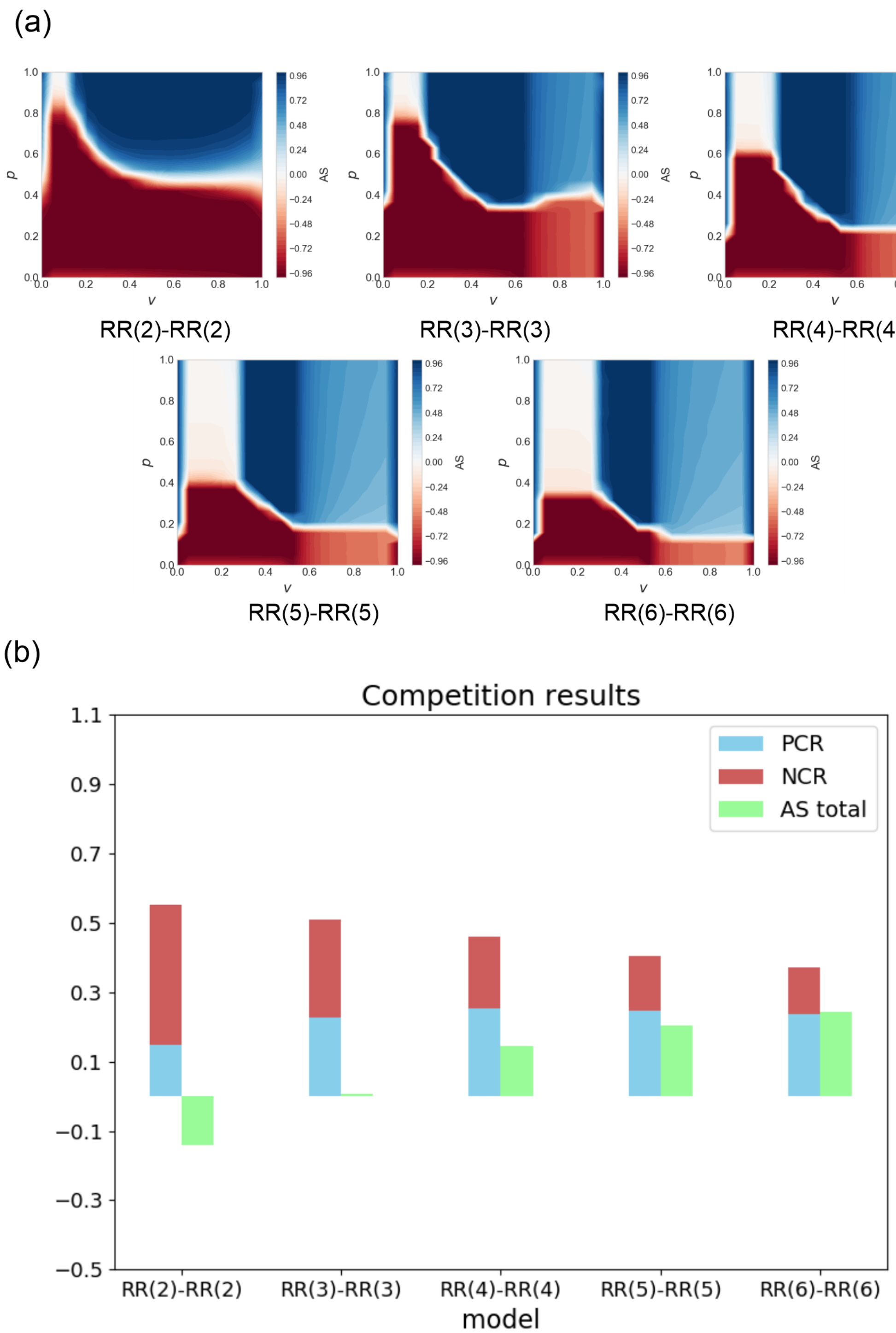


Figure 3–10 Simulation results with changing internal degrees on both layers

both layers. As the total internal degree is increased, *CR* is inversely decreased, and the ratio of positive consensus(blue area) is increased, but the ratio of negative consensus(red area) is decreased. It can be analyzed that a decrease in *CR* is caused by increase in internal degree on layer B, and an increase in ratio of *PCR* is brought out by an increase in internal degree on layer A. But, when the total internal degrees is increased, *PCR, NCR, CR* indexes are decreased. It can be analyzed that too large internal degree on both layers makes it hard to reach consensus. In summary, 3 main simulations have been

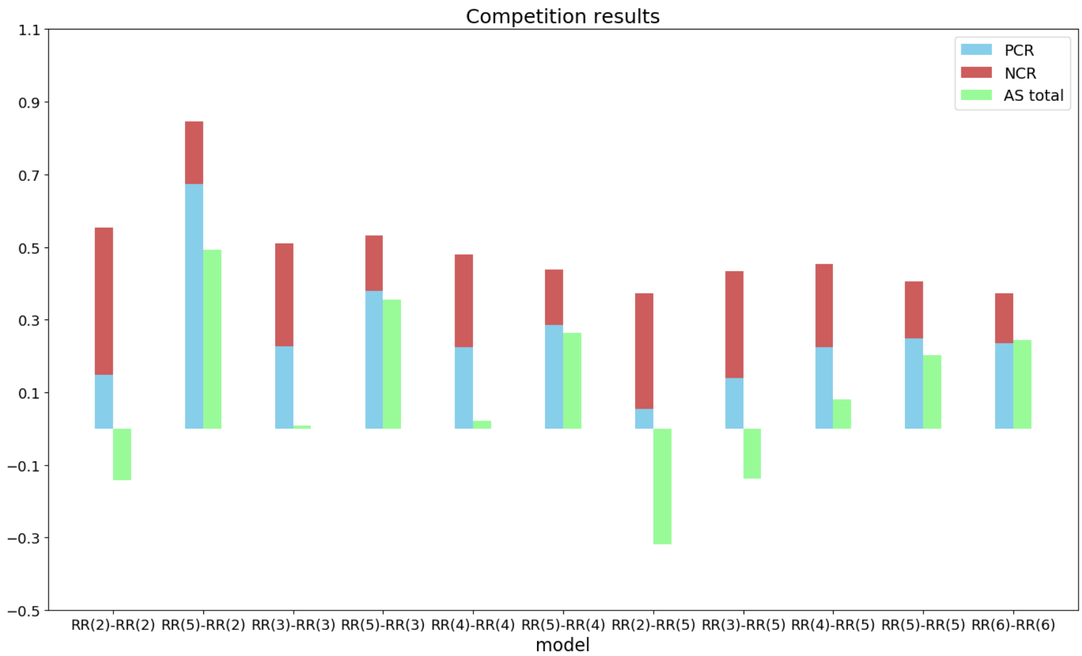


Figure 3–11 Total results with different internal degrees on two layers

implemented to find out the influence of internal degree on interconnected network by changing the internal degree on layer A, changing the internal degree on layer B and changing the internal degrees on both layers. The results are arranged as follows. First, it is found out that internal degrees on layer A has the tendency to keep positive state and to change negative state into positive state. Second, it is shown that the number of internal degrees on layer B has the tendency to hinder positive consensus state and has the inverse relation with *CR*. Third, Too large internal degree makes it hard to reach consensus.

Fig. 3–11 shows the result for all simulations. Through these simulation results, we can analyze that how the state of network is changed according to the internal degrees. Several conclusions can be arranged as shown in Fig. 3–12. First, it is easy to reach negative consensus when the internal degrees on layer A is relatively small(the internal degrees on layer B doesn't matter). Second, it is easy to make positive consensus when the internal

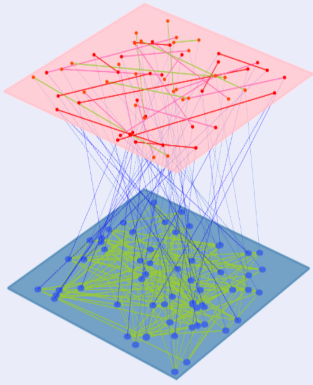
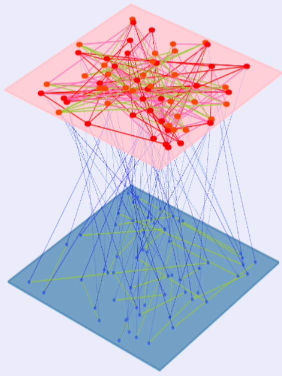
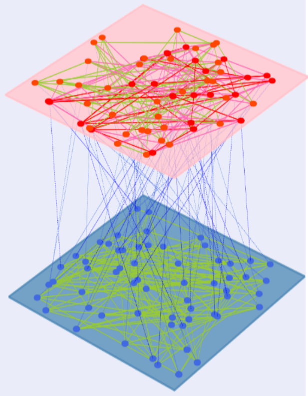
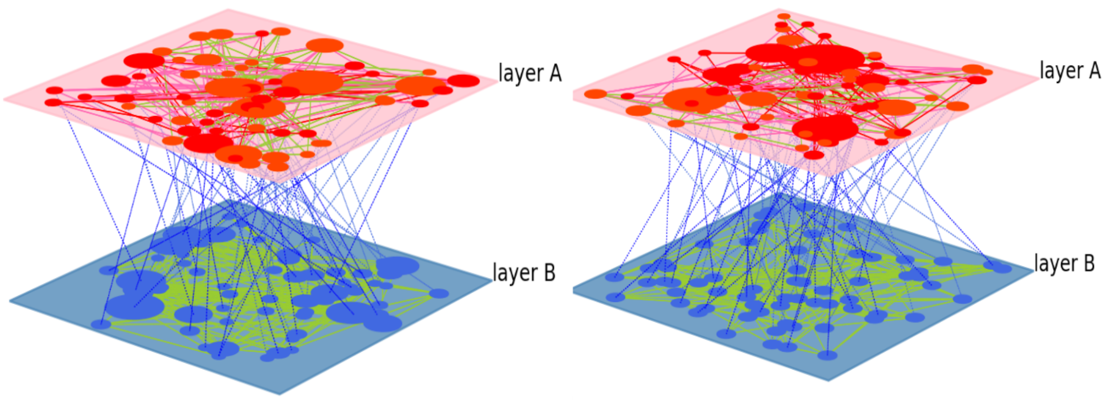
state	negative dominant	positive dominant	coexistence dominant
figure			
internal degrees on layer A	small	large	large
internal degrees on layer B	large or small	small	large

Figure 3–12 Categorizing the state of network according to internal degrees on two layers

degrees on layer A is relatively large and the internal degrees on layer B is relatively small. Third, social conflict can be caused when the internal degrees on both layers are too large.

3.4 Competition on networks with different structures



(a) BA-BA (b) BA-RR

Figure 3–13 Competition on networks with different structures

So far, each layer of the interconnected network consisted of *RR(random regular networks)* that has the same number of edges for each node. Now, the simulation would be implemented on different network type. Here, we use *Barabasi-Albert network(BA)* structure as introduced in [34]. *Barabasi-Albert(BA)* network has N nodes with attaching new nodes each with K edges that are preferentially attached to existing nodes with large degrees. But, there is no change on external degree, which would be fixed to only 1. To

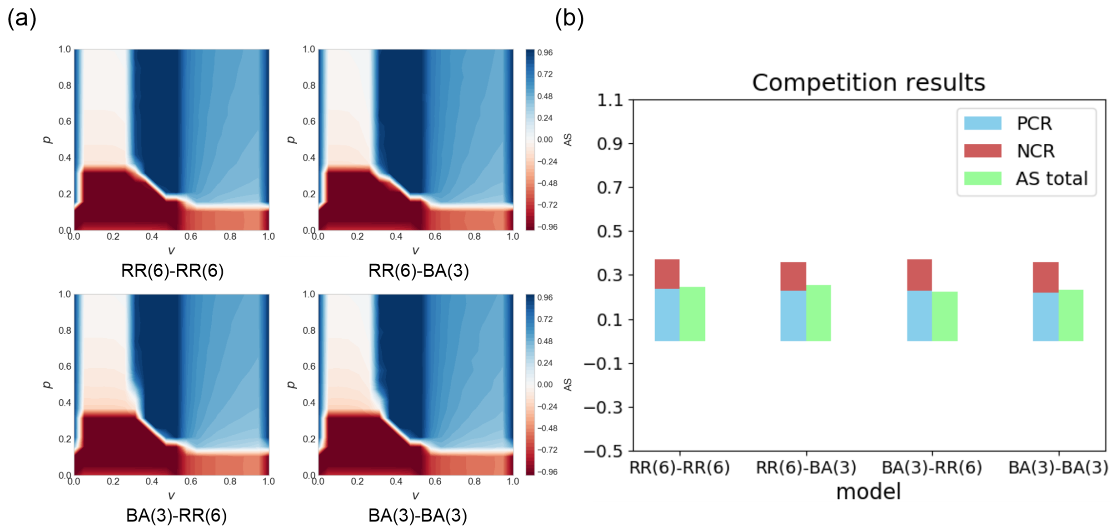


Figure 3–14 Simulation results with different network types

evaluate the influence of network structure, 4 simulations are implemented with switching network structures. The *BA* or *RR* network is applied for both layers or switched on each layer. To restrain the influence of internal degree, the number of internal edges in *BA* is set up to be similar with the number of internal edges in *RR*. The number of internal edges in *BA* is 6, 135, and the number of internal edges in *RR* is 6, 144.

The simulation results are shown in Fig. 3–14. The results of all simulations have almost the same features. The gap of *PCR, NCR* and *CR* is less than 0.02. The structure of network make no obvious difference of consensus results.

Next, the number of internal edges would be increased on the network, where consists of two *BA*. It would be found out that how the number of internal edges work on the different network type with *RR*. 2 models, *BA(3)-BA(3)* and *BA(5)-BA(5)* would be simulated. *BA(3)-BA(3)* model has 6, 135 internal edges on each layer, and *BA(5)-BA(5)* model has 10, 215 internal edges on each layer.

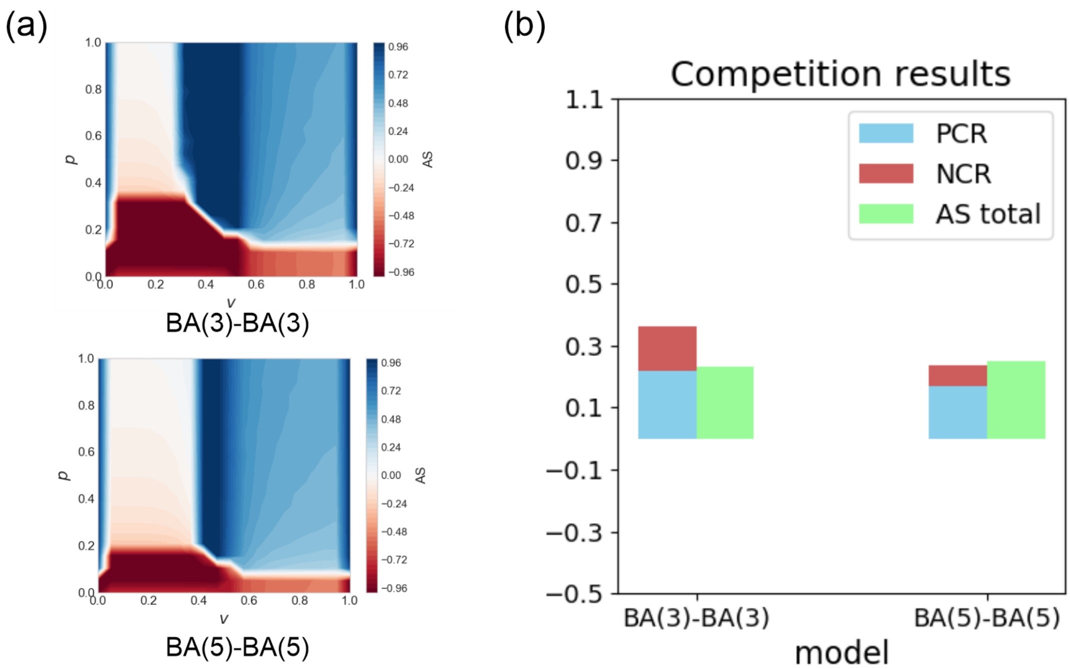


Figure 3–15 Simulation result of BA-BA networks with different internal degrees

As shown in Fig. 3–15, *BA(5)-BA(5)* has more coexistence area than *BA(3)-BA(3)*because of too many internal edges. It is shown that the influence of internal and external degrees is more important for changing the state of network and making consensus than the influence of network type. However, If there are stubborn nodes on networks, the simulation results would be different because the centrality of stubborn nodes would be changed according to network type. Selecting key nodes on interconnected networks would be simulated and analyzed in chapter.5.

3.5 Conclusion

Various simulations have been simulated to find out the role of internal and external degrees and the influence of network types. All results of simulations are shown in Table. 3–1. Through the simulation results, several facts could be arranged like the followings. If there are no stubborn nodes, network types do not make different result for the state of network and consensus. But, we can provide four conclusions about the roles of internal and external degrees. First, *Hierarchical Models* show that it is easy to make consensus on two-layers when the number of external edges in decision making

Table 3–1 Consensus properties of simulation models

Div	A nodes	B nodes	A edges	B edges	AS total	PCR	NCR	CR
RR(2)-RR(5)	2,048	2,048	2,048	5,120	-0.3186	0.0550	0.3175	0.3725
RR(3)-RR(5)	2,048	2,048	3,072	5,120	-0.1368	0.1400	0.2925	0.4325
RR(4)-RR(5)	2,048	2,048	4,096	5,120	0.0804	0.2250	0.2275	0.4525
RR(5)-RR(2)	2,048	2,048	5,120	2,048	0.4927	0.6725	0.1725	0.8450
RR(5)-RR(3)	2,048	2,048	5,120	3,072	0.3555	0.3800	0.1525	0.5325
RR(5)-RR(4)	2,048	2,048	5,120	4,096	0.2633	0.2850	0.1525	0.4375
RR(2)-RR(2)	2,048	2,048	2,048	2,048	-0.1412	0.1475	0.4050	0.5525
RR(3)-RR(3)	2,048	2,048	3,072	3,072	0.0084	0.2275	0.2825	0.5100
RR(4)-RR(4)	2,048	2,048	4,096	4,096	0.1448	0.2525	0.2075	0.4600
RR(5)-RR(5)	2,048	2,048	5,120	5,120	0.2034	0.2475	0.1575	0.4050
RR(6)-RR(6)	2,048	2,048	6,144	6,144	0.2444	0.2350	0.1375	0.3725
RR(6)-BA(3)	2,048	2,048	6,144	6,135	0.2541	0.2275	0.1300	0.3575
BA(3)-RR(6)	2,048	2,048	6,135	6,144	0.2242	0.2300	0.1425	0.3725
BA(3)-BA(3)	2,048	2,048	6,135	6,135	0.2329	0.2200	0.1400	0.3600
BA(5)-BA(5)	2,048	2,048	10,215	10,215	0.2496	0.1675	0.0675	0.2350
HM(2)	2,048	1,024	5,120	2,560	0.3073	0.3275	0.1425	0.4700
HM(4)	2,048	512	5,120	1,280	0.4128	0.4125	0.1275	0.5400
HM(8)	2,048	256	5,120	640	0.4846	0.4925	0.1150	0.6075
HM(16)	2,048	128	5,120	320	0.5610	0.5800	0.1100	0.6900
HM(32)	2,048	64	5,120	160	0.5959	0.6275	0.1025	0.7300
HM(64)	2,048	32	5,120	80	0.6185	0.6775	0.1025	0.7800
HM(128)	2,048	16	5,120	40	0.6379	0.7350	0.0900	0.8250
HM(256)	2,048	8	5,120	20	0.6454	0.7675	0.0900	0.8575

layer is more than opinion layer and the number of nodes in decision making layer is less than opinion layer. Second, the number of internal edges on layer A has the tendency to keep positive state and to change negative state into positive state. Third, the number of internal edges on layer B has the tendency to hinder positive consensus state. Fourth, too many internal edges on each layer can cause inner conflict, and that makes it hard to have consensus state. We could apply these facts to make network structures or organizations in real world.

Chapter 4 Competition with different updating rules

In this chapter, we would control dynamics orders between layers and updating rules of nodes and edges. With changing these updating rules, it would be investigated how the states of network are changed. Here, each layer consists of *Barabasi-Albert(BA)* network that has N nodes with attaching new nodes each with K edges that are preferentially attached to existing nodes with high degree as introduced in [34]. Each node of one layer is connected with a random node on the other layer. This means each node has only 1 external edge. Simulations are performed on network with $N = 2048$, and $K = 3$.

When considering updating rules on two-layer networks, there are many ways to update the state of nodes. Dynamics orders of two-layers can be considered whether layer A works first or layer B works first or both layers work together. And, orders of nodes can be thought as whether the states of nodes are changed simultaneously or sequentially or randomly. Orders of edges connected with a node also can be deliberated as whether edges are activated on a node sequentially or simultaneously or randomly. However, in layer B dynamics, order of edges in one node always follows the simultaneous updating rule, because dynamics formula already considers the states of all connected neighbor nodes simultaneously. To sum up, as shown in Table.4–1, 25 updating rules would be considered according to layers, nodes and edges.

Updating rules are indicated as follows. ‘O’ and ‘D’ represent ‘Opinion layer’ and ‘Decision Making layer’ individually. ‘o’ and ‘s’ indicate sequential updating rule and simultaneous updating rule individually. And the arrow direction indicates order of layers.

In table remarks, ‘O(o, o) → D(s)’ indicates ‘Opinion layer(nodes : sequential order updating, edges : sequential order updating) → Decision Making layer(node : simultaneous updating)’, that means according to the arrow direction, all nodes in Opinion layer are updated with order of nodes and edges and then all nodes in Decision Making layer are updated with order of nodes. ‘O(o, o) ⇔ D(o)’ means that one node in Opinion layer is updated and then one node in Decision Making layer is updated until all nodes are updated. Dynamics with 25 updating rules are simulated with the parameters such as $p = 0.4$ and $v = 0.4$. Simulation results are divided by order of layers, nodes and edges.

Order of layers	Layer A		Layer B	remarks
	Order of nodes	Order of edges	Order of nodes	
Layer A → Layer B	Sequential	Sequential	Sequential	O(o, o) → D(o)
			Simultaneous	O(o, o) → D(s)
		Simultaneous	Sequential	O(o, s) → D(o)
			Simultaneous	O(o, s) → D(s)
	Simultaneous	Sequential	Sequential	O(s, o) → D(o)
			Simultaneous	O(s, o) → D(s)
		Simultaneous	Sequential	O(s, s) → D(o)
			Simultaneous	O(s, s) → D(s)
	Random	Random	Sequential	O(r, r) → D(o)
			Simultaneous	O(r, r) → D(s)
Layer A ← Layer B	Sequential	Sequential	Sequential	O(o, o) ← D(o)
			Simultaneous	O(o, o) ← D(s)
		Simultaneous	Sequential	O(o, s) ← D(o)
			Simultaneous	O(o, s) ← D(s)
	Simultaneous	Sequential	Sequential	O(s, o) ← D(o)
			Simultaneous	O(s, o) ← D(s)
		Simultaneous	Sequential	O(s, s) ← D(o)
			Simultaneous	O(s, s) ← D(s)
	Random	Random	Sequential	O(r, r) ← D(o)
			Simultaneous	O(r, r) ← D(s)
Layer A ↔ Layer B	Simultaneous	Sequential	Simultaneous	O(s, o) ↔ D(s)
		Simultaneous	Simultaneous	O(s, s) ↔ D(s)
Layer A ⇔ Layer B	Sequential	Sequential	Sequential	O(o, o) ⇔ D(o)
		Simultaneous	Sequential	O(o, s) ⇔ D(o)
	Random	Random	Random	O(r, r) ⇔ D(r)

Table 4–1 25 updating rules according to order of layers, nodes, and edges

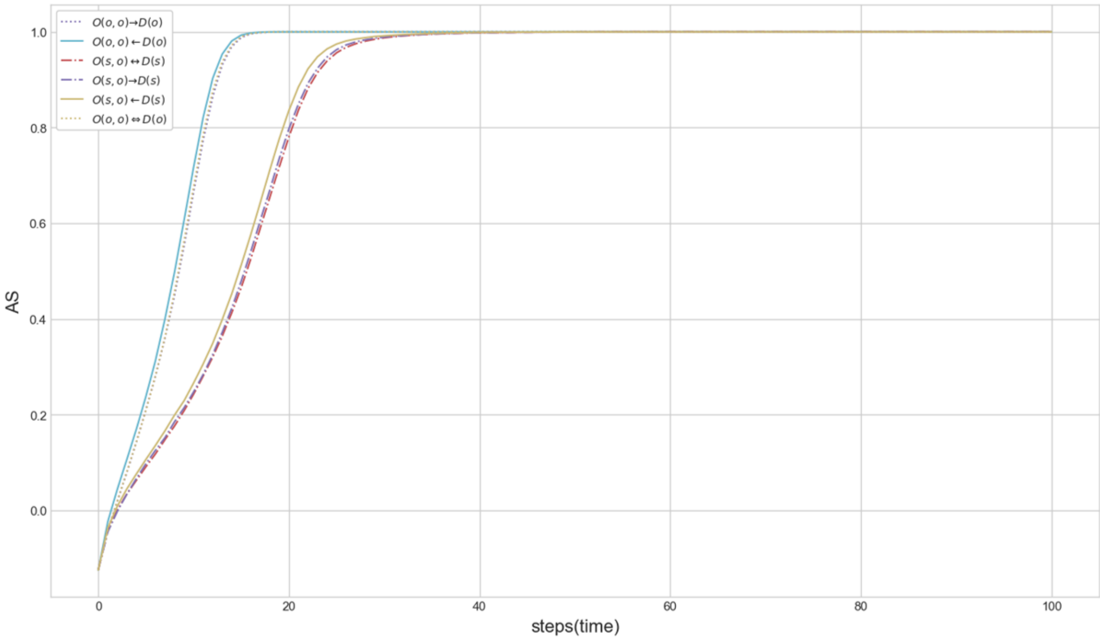
4.1 Order of layers

There exist two layers on interconnected network. And each layer have its own dynamics, such as *M-Model* and *AS-Model*. Two dynamics can be operated simultaneously or sequentially. If two-layers act sequentially, dynamics of layer A can act first or dynamics of layer B can work previously. If two-layers are operated simultaneously, order of nodes becomes the simultaneous updating rule automatically because the states of nodes are also changed according to dynamics of layers. Otherwise, regardless of layers' order, nodes of two layers can interact mutually, i.e. one node in layer A are updated and then one node in layer B are updated until all nodes are updated. In this case, order of nodes becomes the sequential updating rule automatically.

Considering all situations, there are 4 ways in order of two layers, *Layer A* → *Layer*

B(sequential), Layer A ← Layer B(sequential), Layer A ↔ Layer B(simultaneous), Layer A ⇔ Layer B(interaction regardless of layers' order).

Fig. 4–1 shows simulation results related to orders of layers. The graph line indicates



Div	Fast Consensus	Slow Consensus
Orders	① O(o, o) ← D(o) ② O(o, o) ⇔ D(o) ③ O(o, o) → D(o)	④ O(s, o) ← D(s) ⑤ O(s, o) → D(s) ⑥ O(s, o) ↔ D(s)

Figure 4–1 Simulation results according to orders of layers: Comparison between orders of layers under the same conditions such as orders of nodes and edges.

AS value per each step. If the line reaches to 1 or −1, that means the state of network has a positive or negative consensus state.

As seen in Fig. 4–1, it is shown that there is little difference according to orders of layers, but there is difference according to order of nodes. Order of nodes would be described in next section. Consensus time and result are almost same, though dynamics order is different. Regardless of dynamics orders, when other conditions such as updating rules of nodes and edges, are same, the dynamics results are also very similar. It could be found out that dynamics order of layers does not have an significant influence on the network state.

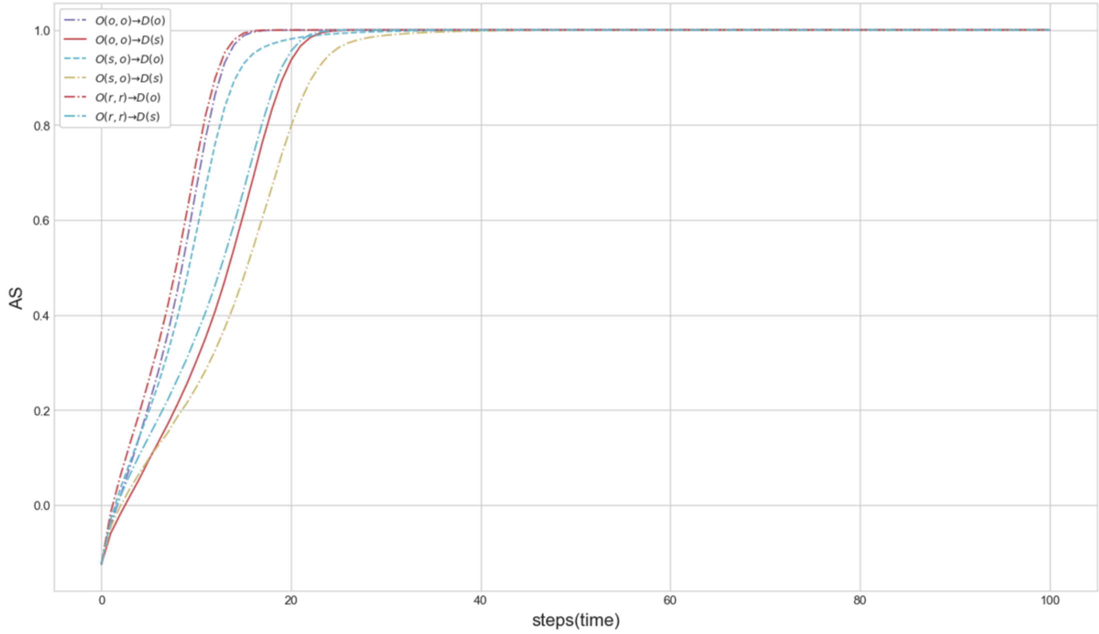
4.2 Order of nodes

In the simulation model, each layer has 2048 nodes, and each node has interactions with other nodes. Now, interaction order of nodes would be considered. One node can be updated sequentially after neighbor nodes are updated. Otherwise, every node can be updated simultaneously. And, nodes would be also updated randomly. As the method of random order, one edge is selected randomly and updated until all edges are considered regardless of nodes' orders or edges' orders.(For layer B, random order can not be applied because it has the formula that all edges of a node are considered together) Therefore, simulations would be implemented according to 3 orders of nodes, such as sequential order, simultaneous order and random order. The interaction orders of nodes could be analyzed as communication methods in real world. If networks follow sequential updating rule of nodes, communication methods of networks might be translated as discussion or conversation with enough time. However, if networks follow simultaneous updating rules of nodes, communication methods of networks might be analyzed as vote or election.

Fig. 4–2 shows simulation results according to interaction orders of nodes. The results are classified to two categories, fast consensus and slow consensus. It is shown that simultaneous interaction between nodes makes slow consensus. Simultaneous order in layer A does not make large difference with other orders in layer A, but it makes consensus slightly slow. Simultaneous interaction between nodes in layer B makes consensus slower than layer A. Random order has similar results with sequential order and does not make different states.
To sum up, it is found out that simultaneous order of nodes makes slow consensus and sequential order of nodes makes fast consensus. In addition, interaction order of nodes in layer B has more influence on consensus time than in layer A. To make quick social consensus, both opinion layer and decision making layer would need sequential updating rule, such as conversation and discussion.

4.3 Order of edges

Each node has several edges connected with other nodes. Updating rules can be divided according to that edges are activated sequentially or simultaneously. If edges of each node work sequentially, a state of node is changed whenever each edge is activated.



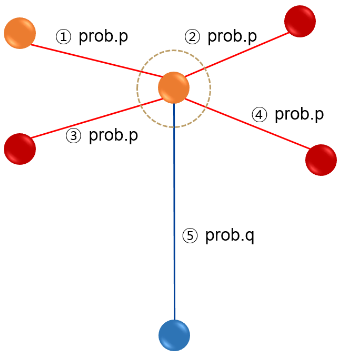
Div	Fast Consensus	Slow Consensus
Orders	① O(r, r) → D(o) ② O(o, o) → D(o) ③ O(s, o) → D(o)	④ O(r, r) → D(s) ⑤ O(o, o) → D(s) ⑥ O(s, o) → D(s)

Figure 4–2 Simulation results according to orders of nodes: Comparison between orders of nodes under the same conditions such as orders of layers and edges.

However, If edges of a node are activated simultaneously, a state of node would be changed considering all connected nodes. In real world, order of edges in one node can be analyzed as characteristics of nodes. If order of edges is sequential, the node would be analyzed as rash because the state of node is changed whenever edges are activated. If order of edges is simultaneous, the node would be analyzed as considerate because it considers all connected nodes together.

For example, considering the case that one node is connected with other nodes as shown in Fig. 4–3, we can think how the state of node changes according to edges' orders. If the edges follow the sequential updating rule, it is hard to calculate the probabilities because the state of node is continuously changed according to sequential edges' order. Therefore, the next states of nodes would be found by using computer simulation.
If the edges follow the simultaneous updating rule, it needs some assumptions as follows:

1. If the number of activated *prob.p* is more than the number of activated *prob.q*,



Order of edges	Next state
Sequential	Hard to calculate (According to order of activated edges, final state is changed)
Simultaneous	Unchanged: prob.p(activated) = prob.q(activated) Persuasion : prob.p(activated) > prob.q(activated) Compromise: prob.p(activated) < prob.q(activated)

Figure 4–3 Order of edges: One node connected with other nodes is updated according to the sequential or simultaneous order of edges

persuasion dynamics would work.

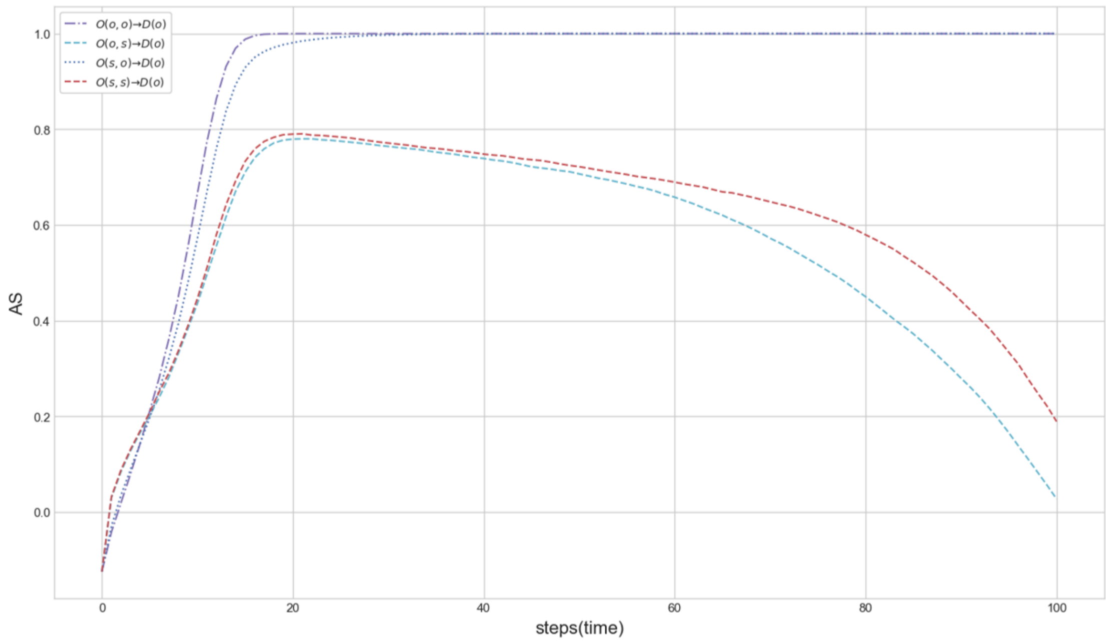
2. If the number of activated *prob.p* is same with the number of activated *prob.q*, the state would be unchanged.
3. If the number of activated *prob.p* is less than the number of activated *prob.q*, compromise dynamics would work.

Through these assumptions, we can calculate probabilities of changing the state of node by considering all cases as these formula.

$$
\begin{gathered}
K = \{k \mid 0, \cdots, n^{-S_i}\}, \quad L = \{l \mid 0, \cdots, n^{S_i}\}, \quad M = \{m \mid k - l\}, \\
P_A(S_i \mapsto S'_i) = \begin{cases} \text{unchanged}(k = l) : \sum p^{n^{-S_i}+m} \cdot (1-p)^{n^{S_i}-m} \cdot {}_{n^{S_i}}C_k \cdot {}_{n^{-S_i}}C_l \\ \text{compromise}(k > l) : \sum p^{n^{-S_i}+m} \cdot (1-p)^{n^{S_i}-m} \cdot {}_{n^{S_i}}C_k \cdot {}_{n^{-S_i}}C_l \\ \text{persuasion}(k < l) : \sum p^{n^{-S_i}+m} \cdot (1-p)^{n^{S_i}-m} \cdot {}_{n^{S_i}}C_k \cdot {}_{n^{-S_i}}C_l \end{cases}
\end{gathered}
\tag{4–1}
$$

In Eq.(4–1), K means the set of integer from 0 to the number of nodes with the opposite state(n^{-S_i}}). L means the set of integer from 0 to the number of nodes with the same state(n^{S_i}}). By using permutations and combinations, these formula are derived.

Fig. 4–4 shows the simulation result according to edges' orders. The results could be categorized to consensus and coexistence(not reaching consensus). Sequential updating rule of edges makes consensus under the same conditions such as orders of nodes and layers, i.e. rash nodes make consensus. But simultaneous updating rule of edges makes it hard to reach consensus under the same conditions such as orders of nodes and layers, i.e. considerate nodes do not make consensus. It can be analyzed that rash node is easy



Div	Consensus	Not reaching consensus
Orders	① O(o, o) → D(o) ② O(s, o) → D(o)	③ O(o, s) → D(o) ④ O(s, s) → D(o)

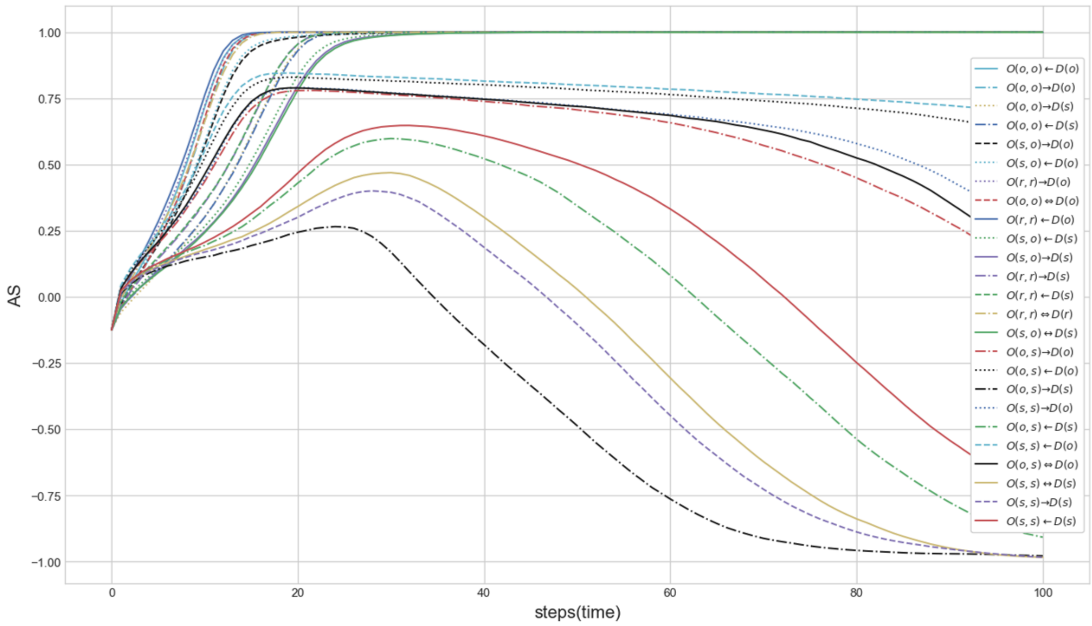
Figure 4–4 Simulation results according to orders of edges: Comparison between orders of edges under the same conditions such as orders of layers and nodes

to be extreme and make consensus, but considerate node is very moderate and makes it hard to reach consensus.

4.4 Comparison and Analysis

It is found out that there are different simulation results according to orders of layers, nodes, and edges. To sum up all updating rules, they can be categorized into 3 parts, positive consensus, coexistence and negative consensus as shown in Fig. 4–5.

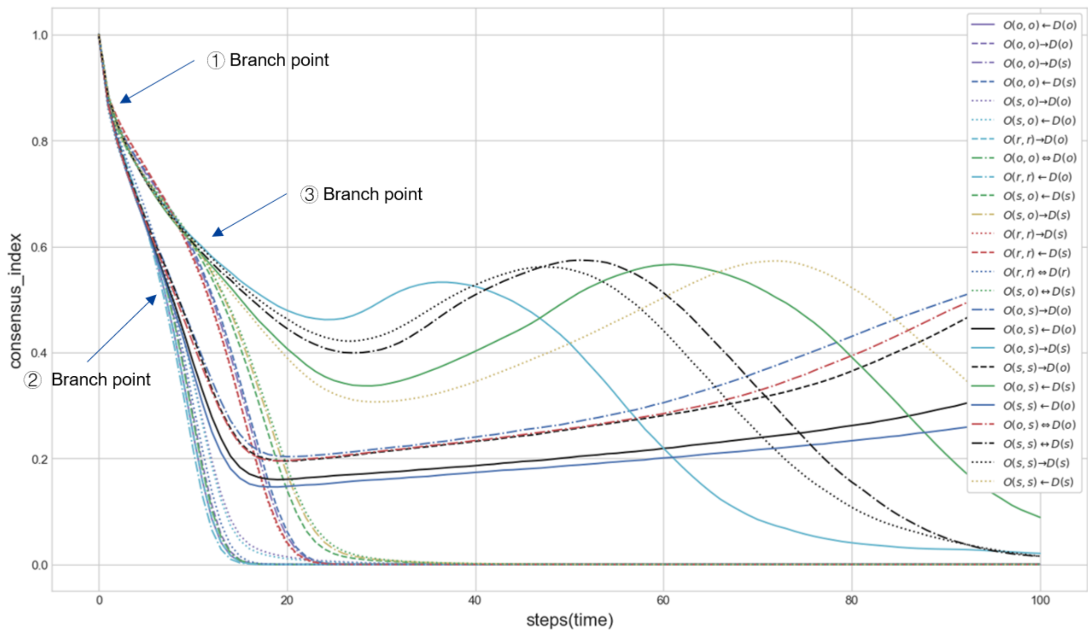
To clearly classify the state of two-layers, the results can be analyzed by using *CI*, that measures how close the state of network is to consensus, as shown in Fig. 4–6. In this figure, there are three branch points. In the first branch point, the results are divided according to whether order of nodes in layer B is sequential or simultaneous. First branch point makes the results divided into fast convergent or slow convergent. In the second



Div	Positive Consensus (close to positive)	Coexistence	Negative Consensus (close to negative)
Orders	O(r, r) ← D(o) O(r, r) → D(o) O(o, o) ← D(o) O(o, o) ⇔ D(o) O(o, o) → D(o) O(r, r) ⇔ D(r) O(r, r) → D(s) O(r, r) ← D(s) O(o, o) → D(s) O(o, o) ← D(s) O(s, o) ← D(o) O(s, o) → D(o) O(s, o) ← D(s) O(s, o) → D(s) O(s, o) ↔ D(s)	O(s, s) ← D(o) O(o, s) ← D(o) O(s, s) → D(o) O(o, s) ⇔ D(o) O(o, s) → D(o)	O(s, s) ← D(s) O(o, s) ← D(s) O(o, s) → D(s) O(s, s) → D(s) O(s, s) ↔ D(s)

Figure 4–5 Total results of 25 updating rules with measuring *AS*

and third branch point, the results are divided according to whether order of edges in layer A is sequential or simultaneous. Second branch point makes the results divided into consensus or coexistence. Third branch point makes the results divided into positive consensus or negative consensus. To sum up, simulations results are classified into 4 categories such as fast positive consensus, slow positive consensus, coexistence and slow negative consensus. The factors that make branch points have the vital influence for the state of networks. That means order of nodes(communication method) in layer B and



<table>
<tr><th colspan="2">Div.</th><th>States</th><th>Orders</th></tr>
<tr><td rowspan="4">① Branch point</td><td rowspan="2">② Branch point :
Sequential order of node in layer B</td><td>Fast positive consensus :
Sequential order of edge</td><td>O(r, r) ← D(o)
O(r, r) → D(o)
O(o, o) ← D(o)
O(o, o) ⇔ D(o)
O(o, o) → D(o)
O(r, r) ⇔ D(r)
O(s, o) ← D(o)
O(s, o) → D(o)</td></tr>
<tr><td>Coexistence :
Simultaneous order of edge</td><td>O(s, s) ← D(o)
O(o, s) ← D(o)
O(s, s) → D(o)
O(o, s) ⇔ D(o)
O(o, s) → D(o)</td></tr>
<tr><td rowspan="2">③ Branch point :
Simultaneous order of node in layer B</td><td>Slow positive consensus :
Sequential order of edge</td><td>O(r, r) → D(s)
O(r, r) ← D(s)
O(o, o) → D(s)
O(o, o) ← D(s)
O(s, o) ← D(s)
O(s, o) → D(s)
O(s, o) ↔ D(s)</td></tr>
<tr><td>Slow negative consensus :
Simultaneous order of edge</td><td>O(s, s) ← D(s)
O(o, s) ← D(s)
O(o, s) → D(s)
O(s, s) → D(s)
O(s, s) ↔ D(s)</td></tr>
</table>

Figure 4–6 Total results of 25 updating rules with measuring *CI*

order of edges(node characteristics) in layer A have very important role to make the state of network. It can be analyzed that communication method on decision-making layer makes fast or slow convergent for opinions and node characteristics on opinion layer makes the final state of networks such as positive consensus, negative consensus and coexistence.

4.5 Conclusion

Through these results, several important facts could be arranged. First, networks with simultaneous updating rules tend to make slow consensus or coexistence, sometimes make transition to opposite orientation. On the other hands, networks with sequential updating rules have a tendency to make fast consensus. Second, dynamics order between layers does not have an influence for network state, though there exists tiny consensus time gap. Third, order of nodes in layer B has more influence for network states than order of nodes in layer A because order of nodes in layer B makes the first branch point that has a vital role to make fast or slow convergent. That means the communication method in decision making layer is very important for determining consensus time. Fourth, order of edges in layer A is very influential so that it makes the second and third branch points that determine the final state of network. It can be analyzed as that characteristics of nodes in layer A, such as rash and considerate, has same orientation consensus or make transition to coexistence or opposite orientation consensus.

Chapter 5 Key nodes selection on two-layers network

In this chapter, it would be investigated that which nodes have the most important influence for keeping or changing their orientation on two-layer networks. There exist many methods to select key nodes, such as pagerank, degree centrality, eigenvector centrality, betweenness centrality and closeness centrality. And, in [42, 44], it has been proved that multiple indicators,that use the rank difference of several node centralities, are useful to identify key nodes and prevent the slow way to identify important nodes. Based on these methods such as single node centrality(single indicator) and combined node centrality(multiple indicator), it would be researched that which method is the most effective and the most influential for selecting key nodes.

5.1 Method for selecting key nodes

As initial condition for selecting key nodes, each layer is made of *BA* network with 512 nodes, $K = 3$, and 1 external edge. Each simulation takes 100 steps, and 100 simulations are considered for average results. To clearly demonstrate the influence of key nodes, the parameters such as p and v would be set to be opposite consensus state to a layer for identifying key nodes. And then the stubborn nodes that do not change their states are selected by using methods for selecting key nodes and the ratio of stubborn nodes is increased until the state of network is changed into same consensus state with the layer. Under these conditions, the most influential method would be the fastest method to reach the same consensus state with a layer for selecting key nodes. For example, for selecting key nodes on layer A, the parameters would be set to be negative consensus state. Then, as the stubborn nodes on layer A are selected by node centrality or other method and the ratio of the stubborn nodes is increased, the network state would be gradually changed into positive state. Inversely for selecting key nodes on layer B, the parameters would be set to be positive consensus state. Then, as the stubborn nodes on layer B are selected by the method for recognizing key nodes and the ratio of stubborn nodes is increased, the network state would be gradually changed into negative state. Here, we would try to find the fastest and the most influential method.

As the method to select stubborn nodes, we would use two kinds of indicators, single indicator and multiple indicator. As single indicators, node centralities would be applied such as pagerank, degree, eigenvector, closeness and betweenness. As multiple indicators, combined node centralities that consists of several node centralities would be applied.

First, here is the way to select key nodes by using a single node centrality.

1. All nodes are ranked by 5 node centralities(pagerank, degree, eigenvector, closeness, betweenness).
2. The nodes are deactivated from high ranked order until the state of network has significant difference, i.e. the stubborn nodes are selected according to high ranked order and the ratio of stubborn nodes is increased.
3. The results are compared according to node centralities. When a node centrality makes the state of network reach the fastest to the same consensus with the layer or have the largest change, it would be the most influential method for selecting key nodes.
4. To clarify which method is the most effective, each single indicator is calculated with summation of all *AS*, that represents 'Average States' of network, according to the ratio of stubborn nodes. It is recognized that the larger the *AS* value is on layer A, the more influential that indicator is, inversely the smaller the *AS* value is on layer B, the more influential that indicator is.

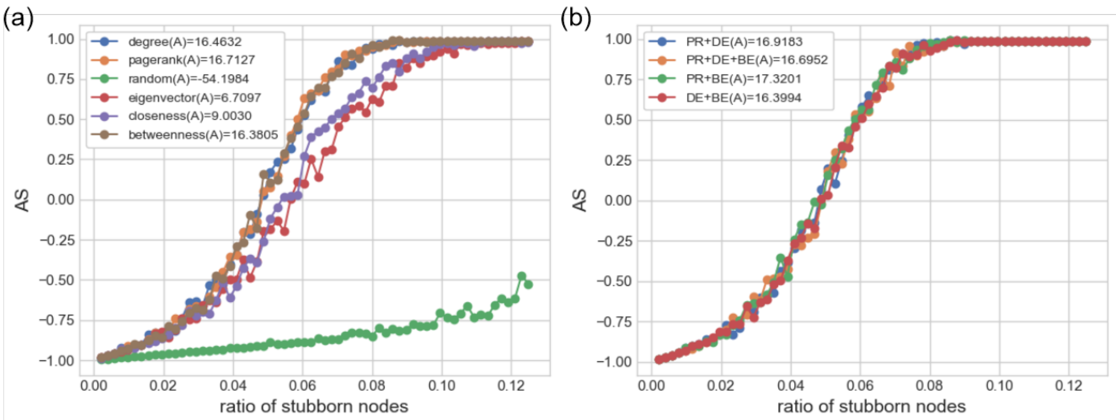
And, we would research the way to recognize important node by using multiple indicators such as combined node centralities(*PR+DE, PR+BE, DE+BE, PR+DE+BE*). Combined node centralities are made up with several selected node centralities. When it is proven that a node centrality is effective for selecting key nodes through the simulations, it would be selected as a factor of combined node centrality. Here, 2 or 3 node centralities would be selected such as pagerank, degree and betweenness. The way to recognize key nodes by using combined node centrality follows like this steps.

1. All nodes are ranked by each selected node centrality. All nodes has the ranks as the number of selected node centralities.
2. Combined node centrality is calculated by the summation of all ranks which a node has.
3. All nodes are ranked again by combined node centrality. The smaller the combined node centrality is, the higher a node are ranked.
4. The nodes are deactivated from high ranked order until the state of network has

significant difference, i.e. the stubborn nodes are selected according to high ranked order and the ratio of stubborn node is increased.

It has been already proven that single node centrality has good performance to identify key nodes.[13, 39, 45]. However, identifying key nodes by multiple indicators is still open problem because there are lots of ways to set up and optimize the weight of each node centrality.[44] Here, we simplify the method by setting the weights as equal and simply calculate the summation of ranks. Though our multiple indicators need to be researched further, the multiple indicators would be evaluated and compared with single indicators.

5.2 Key nodes on layer A



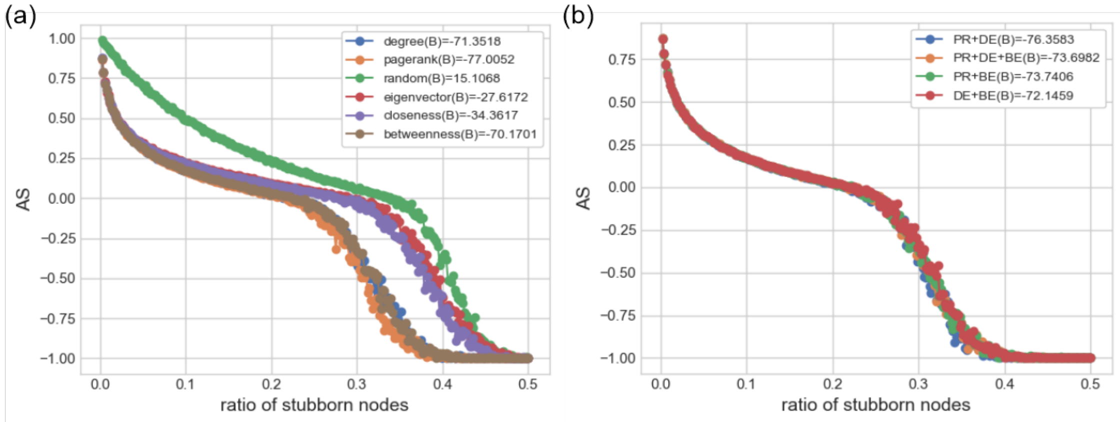
Div(single)	DE	PR	RA	EI	CL	BE
AS	16.4632	16.7127	-54.1984	6.7097	9.0030	16.3805

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	16.9183	16.6952	17.3201	16.3994

Figure 5–1 Key nodes on layer A in *BA(3)-BA(3)* network($p = 0.2, v = 0.4$): (a) Single indicator methods, (b) Multiple indicator methods

To select key nodes on layer A, parameters are set to be negative consensus state like $p = 0.2, v = 0.4$. As single indicators, 5 node centralities(pagerank, degree, eigenvector, closeness, betweenness) are used, and randomly selected nodes are compared with 5 node centralities. As multiple indicators, 2 or 3 node centralities are combined such as pagerank, degree and betweenness which have the good performance as single indicators. Fig. 5–1 shows the simulation result for recognizing key nodes on layer A. As a single indicator, pagerank has the best performance. The next ranks are degree and betweenness. As a multiple indicator, *PR+BE* has the most effective result. The next is *PR+DE*. These two methods of multiple indicators work better than pagerank. Totally, compared with all methods, the best method is *PR+BE*. It could be found out that some multiple indicators are more effective for selecting key nodes than single indicators.

5.3 Key nodes on layer B



Div(single)	DE	PR	RA	EI	CL	BE
AS	-71.3518	-77.0052	15.1068	-27.6172	-34.3617	-70.1701

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	-76.3583	-73.6982	-73.7406	-72.1459

Figure 5–2 Key nodes on layer B in *BA(3)-BA(3)* network($p = 0.3, v = 0.5$): (a) Single indicator methods, (b) Multiple indicator methods

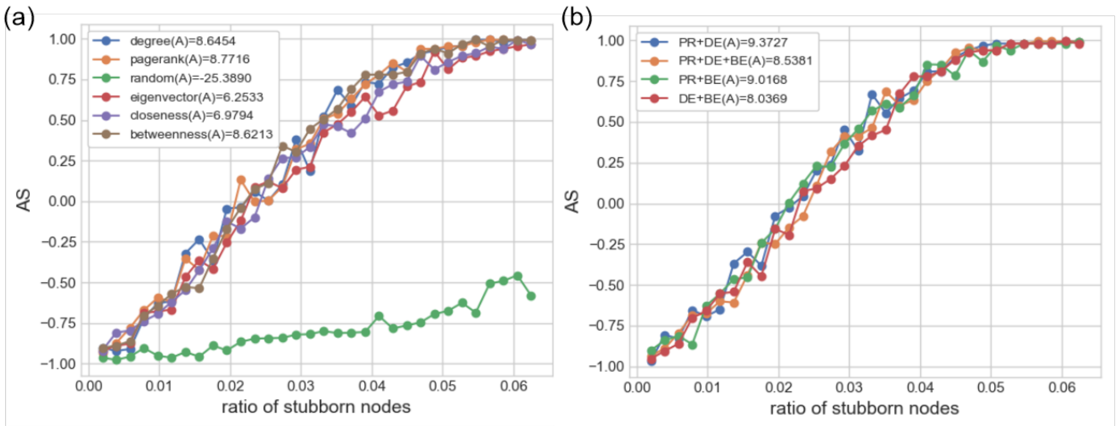
To select key nodes on layer B, parameters are set to be positive consensus state like $p = 0.3, v = 0.5$.

Fig. 5–2 shows the simulation result for identifying key nodes on layer B. As a single indicator, the most effective way to recognize important nodes is pagerank centrality. The next ranks are degree and betweenness. As a multiple indicator, *PR+DE* has the best performance. Totally, pagerank is the most effective method for selecting key nodes on layer B. However, all multiple indicators work better than degree centrality, the second rank in single indicators. It could be found out that combined node centralities also have the good performance for selecting key nodes, though they are not the best.

5.4 Key nodes on two layers with different structures

In this section, we would try to select the key nodes in the networks with various structures, that are described in chapter.2. Node centralities and combined node centralities are also used as the methods for selecting key nodes. First, *Hierarchical Model* would be applied to identify important nodes. Second, we would consider the case that each layer has different network type, such as *BA-RR* or *RR-BA* networks. Third, the case would be considered that each layer has different number of internal edges. Layer A could have more internal links or layer B could have more internal links. Both cases would be checked.

5.4.1 Key nodes in Hierarchical Model

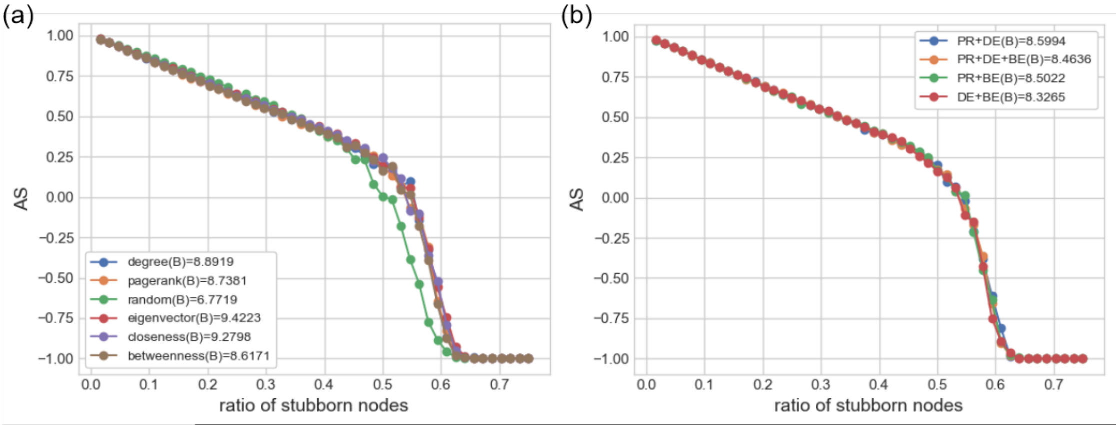


Div(single)	DE	PR	RA	EI	CL	BE
AS	8.6454	8.7716	-25.3890	6.2533	6.9794	8.6213

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	9.3727	8.5381	9.0168	8.0369

Figure 5–3 Key nodes on layer A in *Hierarchical Model(8)*$(p = 0.2, v = 0.2)$: (a) Single indicator methods, (b) Multiple indicator methods

Each layer consists of *BA* network with $k = 3$. Layer A has 512 nodes, and layer B has 64 nodes. We denote these models as *HM(8) with BA(3)*.
Fig. 5–3 shows the simulation result of key nodes on layer A. Simulations result shows that *PR+DE* is the best method for recognizing key nodes on *HM(8)*. Next ranks are *PR+BE* and pagerank. The curve of changing the network states shown in Fig. 5–3 is more straight than Fig. 5–1. That means the speed of changing network states is faster.



Div(single)	DE	PR	RA	EI	CL	BE
AS	8.8919	8.7381	6.7719	9.4223	9.2798	8.6171

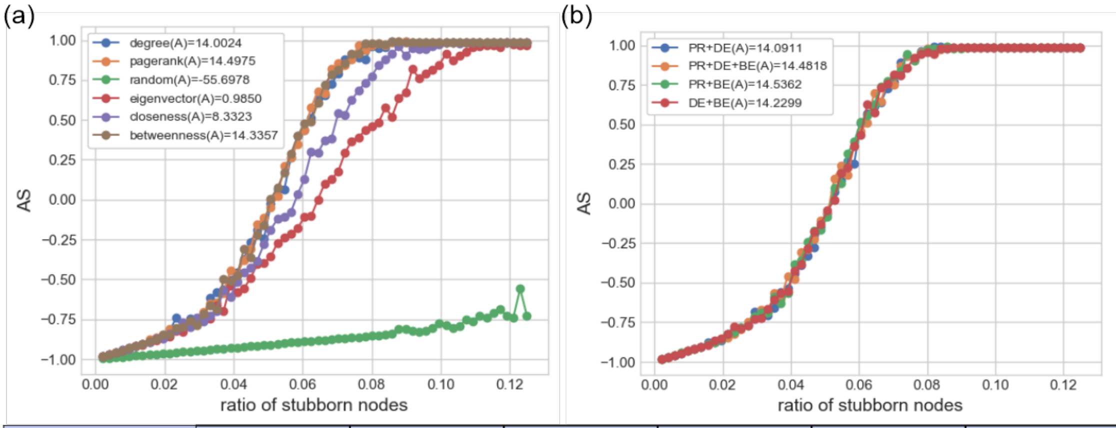
Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	8.5994	8.4636	8.5022	8.3265

Figure 5–4 Key nodes on layer B in *Hierarchical Model(8)*($p = 0.25, v = 0.3$): (a) Single indicator methods, (b) Multiple indicator methods

Fig. 5–4 shows the simulation result of key nodes on layer B. However, the result is different from other simulation results. The best performance method is random method. That means node centralities do not work on this model. And the curve of changing the network states shown in Fig. 5–4 is also more straight than Fig. 5–2, that means the consensus is easier and the consensus time is shorter. It is found out that *Hierarchical Models* make it hard to recognizing key nodes on layer B and make it easy to have consensus of two layers by stubborn nodes.

5.4.2 Key nodes on two layers with different network types

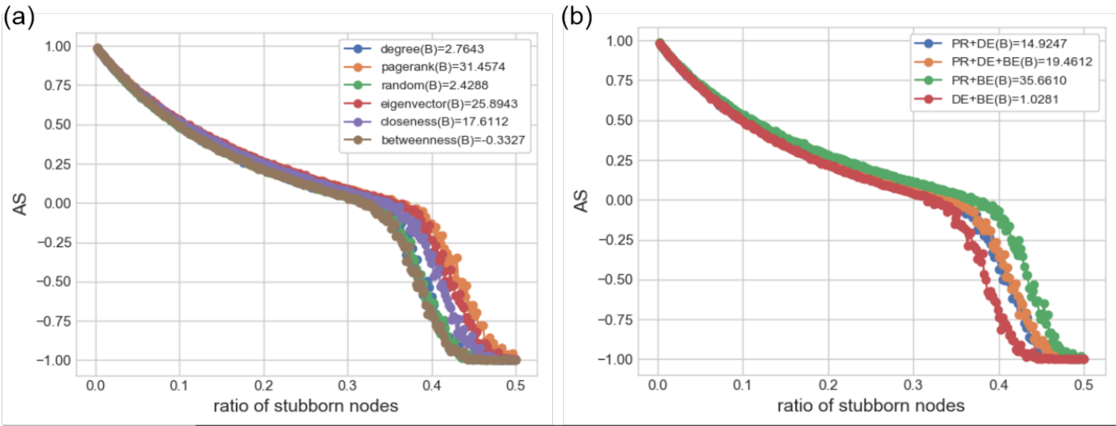
Here, we would consider two types of network, *BA-RR* and *RR-BA*. The number of internal links on each layer would be set up as same or almost same number to exclude the influence of internal links. These models would be compared with *BA-BA* to find out the influence of network types under same conditions, such as p, v, and $ratio of stubborn nodes$. First, *BA-RR* network would be investigated. Fig. 5–5 shows the simulation result of key nodes on layer A.*PR+BE* is the most influential method. Next rank is pagerank as a single indicator. Compared with *BA-BA* shown in Fig. 5–1, *BA-RR* has smaller *AS* values and more gentle curve to change the state of network.



Div(single)	DE	PR	RA	EI	CL	BE
AS	14.0024	14.4975	-55.6978	0.9850	8.3323	14.3357

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	14.0911	14.4818	14.5362	14.2299

Figure 5–5 Key nodes on layer A in *BA(3)-RR(6)* network($p = 0.2, v = 0.4$): (a) Single indicator methods, (b) Multiple indicator methods



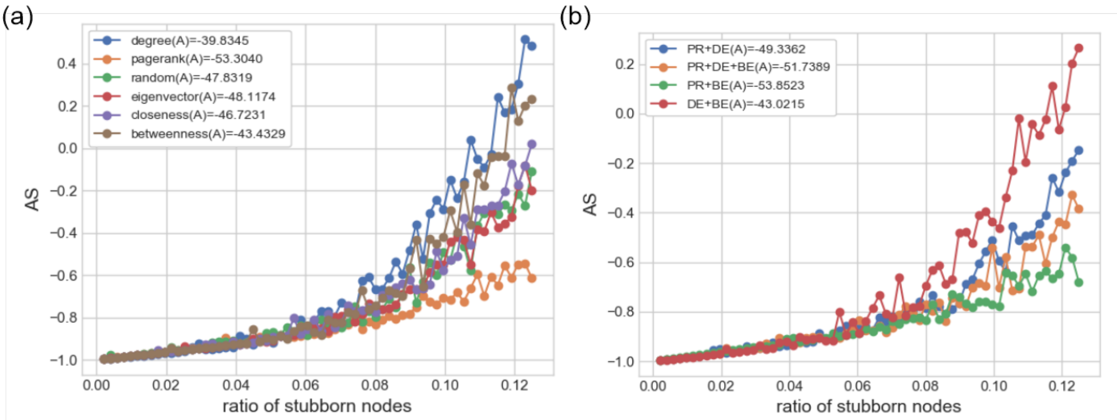
Div(single)	DE	PR	RA	EI	CL	BE
AS	2.7643	31.4574	2.4288	25.8943	17.6112	-0.3327

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	14.9247	19.4612	35.6610	1.0281

Figure 5–6 Key nodes on layer B in *BA(3)-RR(6)* network($p = 0.3, v = 0.5$): (a) Single indicator methods, (b) Multiple indicator methods

Fig. 5–6 shows the simulation result of key nodes on layer B. Betweenness is the best method for finding key nodes on layer B in *BA-RR* network. In this model, degree centrality is not meaningful method because degree of each node is same in *RR* network. However, random and degree method is the third and fourth method for recognizing key nodes. That means other methods except for betweenness do not work for identifying key nodes. Compared with *BA-BA* shown in Fig. 5–2, *BA-RR* has larger *AS* values and more gentle curve to change the state of network.

Next, *RR-BA* network would be considered. Fig. 5–7 shows the simulation result of



Div(single)	DE	PR	RA	EI	CL	BE
AS	-39.8345	-53.3040	-47.8319	-48.1174	-46.7231	-43.4329

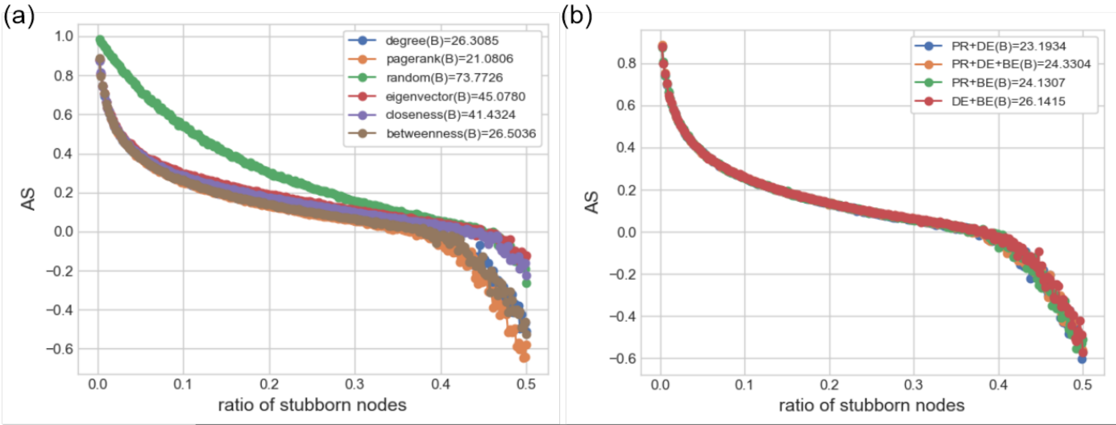
Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	-49.3362	-51.7389	-53.8523	-43.0215

Figure 5–7 Key nodes on layer A in *RR(6)-BA(3)* network($p = 0.2, v = 0.4$): (a) Single indicator methods, (b) Multiple indicator methods

key nodes on layer A. The best method is degree centrality. However, in this model, degree centrality is not meaningful for recognizing key nodes because all nodes in layer A have the same degree. Here, the reason why degree centrality has good performance is analyzed as that dynamics are very efficient because nodes are sequentially changed into stubborn node and interacted(when nodes have the same node centrality, nodes are changed into stubborn nodes sequentially according to interaction order under given algorithm). And other single indicators have similar *AS* values with random method. That means node centralities do not work for identifying key nodes though betweenness has better performance than other methods. Compared with *BA-BA* shown in Fig. 5–2 , *RR-BA* also has smaller *AS* values and does not reach the opposite consensus yet.

Fig. 5–8 shows the simulation result of key nodes on layer B. Pagerank has the best performance. Next rank is *PR+DE*. Compared with *BA-BA* shown in Fig. 5–2 , *RR-BA* also has larger *AS* values and more gentle curve to change the state of network.

Totally, compared with *BA-BA* network, both *BA-RR* and *RR-BA* have more gentle curve.



Div(single)	DE	PR	RA	EI	CL	BE
AS	26.3085	21.0806	73.7726	45.0780	41.4324	26.5036

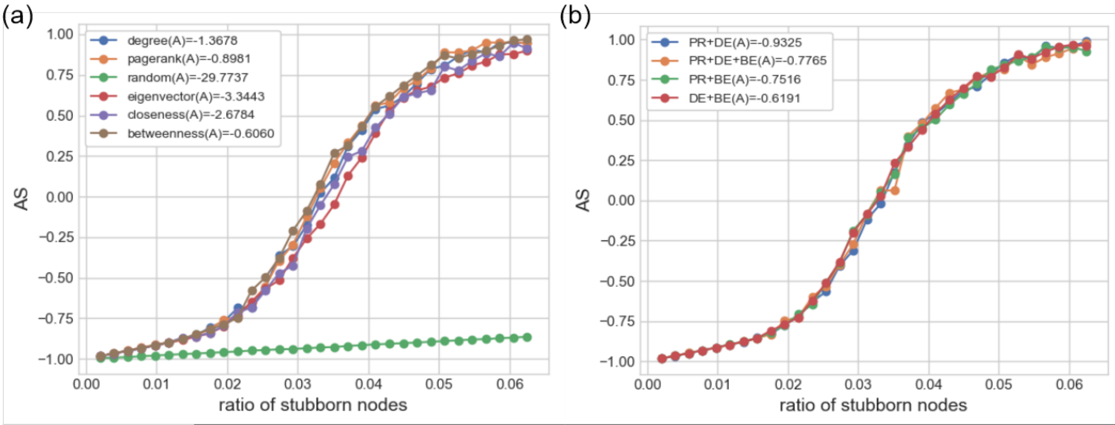
Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	23.1934	24.3304	24.1307	26.1415

Figure 5–8 Key nodes on layer B in *RR(6)-BA(3)* network($p = 0.3, v = 0.5$): (a) Single indicator methods, (b) Multiple indicator methods

It could be analyzed that *RR* network makes it slow for stubborn nodes to change the state of network and makes it hard to select key nodes though betweenness has good performance on *RR* network.

5.4.3 Key nodes on two layers with different number of internal links

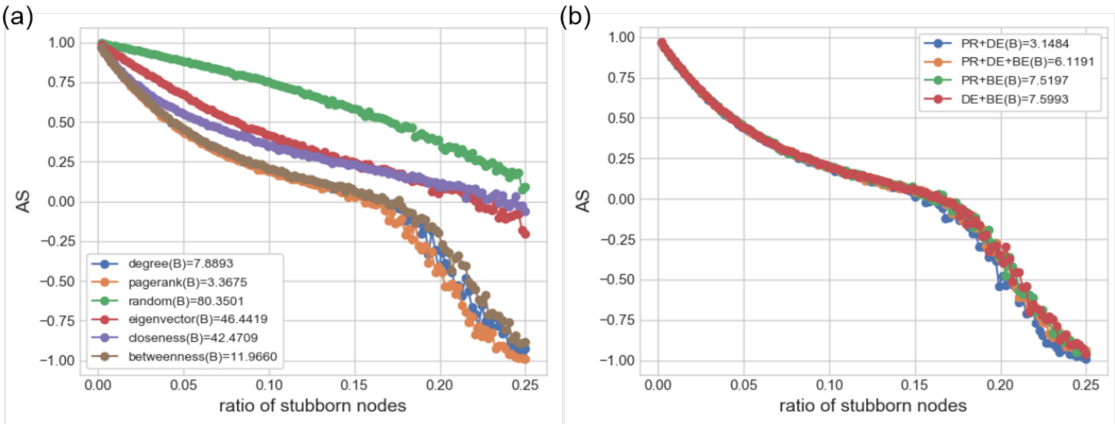
Next, the case would be considered that each layer has different number of internal edges. In case that layer A has more internal links, layer A consists of *BA* network with $k = 4$, but layer B consists of *BA* network with $k = 2$. Inversely, in case that layer B has more internal links, layer B consists of *BA* network with $k = 4$, but layer A consists of *BA* network with $k = 2$. First, the case of more internal links on layer A than layer B would be investigated. Fig. 5–9 shows the simulation result of key nodes on layer A in *BA(4)-BA(2)* network. Betweenness has the best performance for selecting key nodes. Next ranks are *DE+BE*, *PR+BE* and *PR+DE+BE*. Compared with the *BA(2)-BA(4)* network shown in Fig. 5–11, the curve of changing the state that is shown in Fig. 5–9 is much more straight. That means consensus time is very short and it is easy to have consensus. Fig. 5–10 shows the simulation result of key nodes on layer B in *BA(4)-BA(2)* network. *PR+DE* is the most influential method. Next ranks are pagerank, *PR+DE+BE* and



Div(single)	DE	PR	RA	EI	CL	BE
AS	-1.3678	-0.8981	-29.7737	-3.3443	-2.6784	-0.6060

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	-0.9325	-0.7765	-0.7516	-0.6191

Figure 5–9 Key nodes on layer A in *BA(4)-BA(2)* network($p = 0.15, v = 0.3$): (a) Single indicator methods, (b) Multiple indicator methods



Div(single)	DE	PR	RA	EI	CL	BE
AS	7.8893	3.3675	80.3501	46.4419	42.4709	11.9660

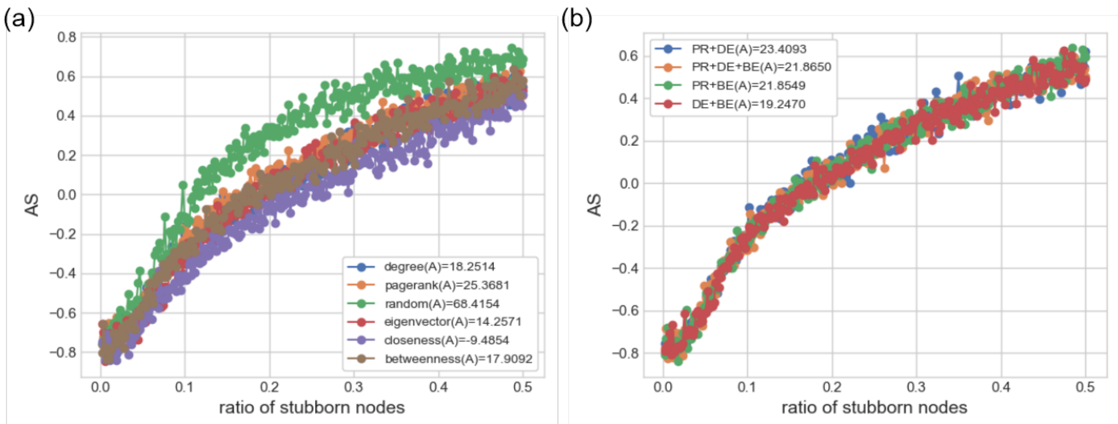
Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	3.1484	6.1191	7.5197	7.5993

Figure 5–10 Key nodes on layer B in *BA(4)-BA(2)* network($p = 0.2, v = 0.4$): (a) Single indicator methods, (b) Multiple indicator methods

PR+BE. Compared with *BA(2)-BA(4)* network shown in Fig. 5–12, the curve of changing the state that is shown in Fig. 5–10 is also more straight.

Totally, compared with *BA(2)-BA(4)* network, it could be analyzed that more internal edges on layer A make it easy to have consensus by stubborn nodes.

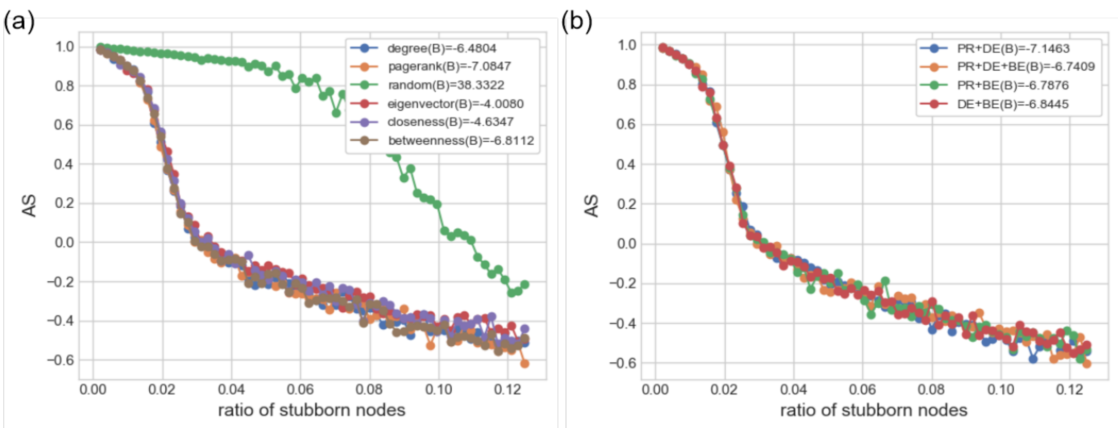
Next, the case of more internal links on layer B than layer A would be researched.



Div(single)	DE	PR	RA	EI	CL	BE
AS	18.2514	25.3681	68.4154	14.2571	-9.4854	17.9092

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	23.4093	21.8650	21.8549	19.2470

Figure 5–11 Key nodes on layer A in *BA(2)-BA(4)* network($p = 0.57, v = 0.37$): (a) Single indicator methods, (b) Multiple indicator methods



Div(single)	DE	PR	RA	EI	CL	BE
AS	-6.4804	-7.0847	38.3322	-4.0080	-4.6347	-6.8112

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	-7.1463	-6.7409	-6.7876	-6.8445

Figure 5–12 Key nodes on layer B in *BA(2)-BA(4)* network($p = 0.6, v = 0.4$): (a) Single indicator methods, (b) Multiple indicator methods

Fig. 5–11 shows the simulation result of key nodes on layer A in *BA(2)-BA(4)* network. However, the simulation results are different from other results, because random method has the best performance. That means node centralities do not work on this model. Compared with *BA(4)-BA(2)* network shown in Fig. 5–9, the curve of changing the state that is shown in Fig. 5–11 is much slower and more gentle.

Fig. 5–12 shows the simulation result of key nodes on layer B in *BA(2)-BA(4)* network. *PR+DE* has the most effective performance. Next ranks are pagerank, *DE+BE* and betweenness. Compared with *BA(4)-BA(2)* network shown in Fig. 5–10, the curve of changing the state that is shown in Fig. 5–12 is much faster. But consensus does not happen in this model.

Totally, compared with *BA(4)-BA(2)* network, it could be analyzed that the more number of internal edges on layer B makes consensus by stubborn nodes hard. And decreasing internal edges on layer A makes it hard to select key node on layer A.

5.5 Conclusion

By using node centrality and combined node centrality, key nodes on each layer have been recognized on networks of various structures. Table. 5–1 shows total simulation results for selecting key nodes on various interconnected networks. Here, we could found

Table 5–1 Effective methods for selecting key nodes on various networks

Div	A nodes	B nodes	A edges	B edges	layer	1st method	2nd method	3rd method	remarks
BA(3)-BA(3)	512	512	1,527	1,527	A	PR+BE	PR+DE	pagerank	
					B	pagerank	PR+DE	PR+BE	
BA(3)-RR(6)	512	512	1,527	1,536	A	PR+BE	pagerank	PR+DE+BE	
					B	betweenness	DE+BE	random	not working
RR(6)-BA(3)	512	512	1,536	1,527	A	degree	DE+BE	betweenness	not working
					B	pagerank	PR+DE	PR+BE	
BA(4)-BA(2)	512	512	2,032	1,020	A	betweenness	DE+BE	PR+BE	
					B	PR+DE	pagerank	PR+DE+BE	
BA(2)-BA(4)	512	512	1,020	2,032	A	random	pagerank	PR+DE	not working
					B	PR+DE	pagerank	DE+BE	
HM(8) with BA(3)	512	64	1,527	183	A	PR+DE	PR+BE	pagerank	
					B	random	DE+BE	PR+DE+BE	not working

out several facts from these simulation results. First, it could be found out that the best and most influential method is different according to network structures and layers. Second, as single indicators, pagerank, degree and betweenness are good method to select key nodes. Third, as multiple indicators, combined node centrality has good performance to recognize the key nodes on various networks. Combined node centralities are first or second effective method on all simulation models.(except not working methods) Fourth, as the results shown in interconnected networks with different number of internal edges on each layer, the more number of links on layer A makes it easy to have consensus by stubborn nodes, and the more number of links on layer B makes it hard to make consensus by stubborn nodes. In addition, decreasing internal edges on layer A makes it hard to recognize key nodes on layer A. Fifth, as the results shown in *HM(8) with BA(3)* network, decreasing the number of nodes on layer B and increasing the number of external edges on layer B make it hard to identify key nodes on layer B and makes it easy to have consensus by stubborn nodes. Sixth, as the results shown in interconnected networks with different network types, network types have the influence to whether network can make consensus by stubborn nodes or not. Especially, it is found out that *RR* network makes it slow to have consensus by stubborn nodes and makes it hard to recognize key nodes.

Chapter 6 Conclusion

We have researched the competition of two layer networks. By changing network structures, switching updating rules and selecting key nodes, the features of competition on two layers have been found out. We hope that deficiency of this research would be researched forward and developed.

6.1 Summary

So far, many simulations have been carried out. In summary, it could be arranged as follows. In chapter.2, interconnected networks with different dynamics on each layer were introduced to understand the competition on interconnected network. And some indexes were provided to measure how the state of network is changed and to evaluate the consensus on two-layers. Based on this modeling, various simulations have been implemented according to 3 main topics as follows.

- Competition on two-layers network with various structures
- Competition on with different updating rules
- Key nodes selection on two-layers network

In chapter.3, we have investigated competition dynamics on two-layers network with various structures. With changing network structures, it has been measured and evaluated that how the state of network was changed and whether the networks make consensus or not. As the method to revise the network structure, 3 ways were provided such as changing internal degrees, changing external degrees and switching network types. First, as the result of changing the internal degrees, it could be found out that internal degrees on each layer has different features. The number of internal degrees on layer A has the tendency to keep a positive state and to change a negative state into a positive state. And the number of internal degrees on layer B has the tendency to hinder a positive consensus state. Second, as the result of changing the external degrees, *Hierarchical Models* were provided. *Hierarchical Models* show that it is easy to make consensus on both layers when the number of external edges in decision making layer is more than opinion layer and the number of nodes in decision making layer is less than opinion layer. Third, as the result of switching the network type, there is no obvious difference on the final state of network. That means if there are no stubborn nodes, network types do not matter. However, it is found out that the number of internal edges has more influential role for changing the state of network than network types.

In chapter.4, it has been researched that how the updating rules have influence on the competition of two-layers network. Though updating rules are very various, we just have considered time-related updating rules, such as simultaneous updating rule and sequential updating rule. According to where the updating rules are applied, we have implemented the simulations of 3 categories, order of layers, order of nodes and order of links. Through simulation results, several conclusions could be arranged. First, dynamics order between layers does not have an significant influence for changing the state of network. Second, order of edges in layer A, that can be analyzed as characteristics of nodes such as rash and considerate, has a vital influence for determining the final state of network such as same orientation consensus, coexistence and opposite orientation consensus. Third, order of nodes in layer B, that can be analyzed as communication method, is more influential for changing the state of network than order of nodes in layer A because it makes fast opinion convergent or slow opinion convergent. That means the communication method in decision making layer is very important for determining consensus time. Fourth, networks with simultaneous updating rules are easy to make slow consensus and coexistence or to change into the opposite state, otherwise networks with sequential updating rules are easy to make fast consensus.

In chapter.5, it has been studied that how the key nodes could be selected on the various two-layers networks. To select key nodes on the various networks, we used single indicators and multiple indicators on various networks described in chapter.3. Through the simulation results, several conclusions could be arranged as follows. First, the most effective method to identify key nodes is different according to network structures and layers as shown in Table. 5–1. Second, as single indicators, pagerank, degree and betweenness work well for selecting key nodes. Third, as multiple indicators, combined node centrality totally has good results to recognize the key nodes on various interconnected networks. Fourth, the more number of links on layer A makes it easy to have consensus by stubborn nodes, and the more number of links on layer B makes it hard to make consensus by stubborn nodes. Fifth, as shown in *Hierarchical Models*, decreasing the number of nodes on layer B and increasing the number of external edges on layer B make it hard to identify key nodes on layer B and make it easy to have consensus by stubborn nodes. Sixth, network types have the influence on whether the network can make consensus by stubborn nodes or not. Especially, it is found out that *RR* network is harder to make consensus by stubborn nodes and to select key nodes than *BA* network.

6.2 Discussion

So far, the competition of two-layers network has been researched and analyzed under various conditions. It has been found out that how network structures have the influence on the consensus of two-layers, how the updating rules affect the state of network, what nodes have more influential to affect the state of network, and which method is more effective way to identify important nodes. Through these results, the state of two-layers network might be controlled by managing the number of edges and the method of updating rules. And for the best and fastest way to change the state of networks, the important nodes might be recognized and controlled by using the method to select key nodes. In real world, we can find out the phenomenon of these competitions, such as election, legislation, adoption of new policies and making decision on social conflict issues. These competitions of real world may have similar characteristics with our simulation results. Therefore, based on simulation results, these competition models can be applied to solve the social conflicts. As future work, it could be very interesting to make generalized competition models with various structures and updating rules, and to recognize key nodes on generalized competition models.

Bibliography

[1] KIVELÄ M, ARENAS A, BARTHELEMY M, et al. Multilayer networks[J]. J. Complex Networks, 2014, 2: 203-271.

[2] DANZIGER M M, BONAMASSA I, BOCCALETTI S, et al. Dynamic interdependence and competition in multilayer networks[J]. Nature Physics, 2019, 15: 178-185.

[3] NEWMAN M J. Networks: An Introduction[M]. [S.l.]: Oxford Scholarship Online, 2010.

[4] BOCCALETTI S, BIANCONI G, CRIADO R, et al. The structure and dynamics of multilayer networks[J]. CoRR, 2014, abs/1407.0742.

[5] DE DOMENICO M, SOLÉ-RIBALTA A, COZZO E, et al. Mathematical Formulation of Multilayer Networks[J/OL]. Phys. Rev. X, 2013, 3: 041022. https://link.aps.org/doi/10.1103/PhysRevX.3.041022. DOI: 10.1103/PhysRevX.3.041022.

[6] TOMASINI M. An Introduction to Multilayer Networks[C]//. [S.l. : s.n.], 2015. DOI: 10.13140/RG.2.2.16830.18243.

[7] VU N K. Robustness of Interconnected Complex Networks with Directed Dependency[D]. Yonsei University Department of Computer Science, 2017.

[8] WU F, HUBERMAN B A. Social Structure and Opinion Formation[J]., 2004.

[9] S Z A, Fedyanin. Models of opinion control for agents in social networks[J]. Automation and Remote Control, 2012, 73: 1753-1764.

[10] Et AL M F L. The dynamics of opinion in hierarchical organizations[J]. Physica A: Statistical Mechanics and its Applications, 2005, 351: 580-592.

[11] MASUDA N. Opinion control in complex networks[J/OL]. CoRR, 2014, abs/1412.2170arXiv: 1412.2170. http://arxiv.org/abs/1412.2170.

[12] SMYRNAKIS M, BAUSO D, HAMIDOU T. An evolutionary game perspective on quantised consensus in opinion dynamics[J/OL]. PLOS ONE, 2019, 14(1): 1-17. https://doi.org/10.1371/journal.pone.0209212. DOI: 10.1371/journal.pone.0209212.

[13] BIANCONI G. Multilayer Networks: Structure and Function[M/OL]. [S.l.]: Oxford University Press, 2018. https://books.google.co.kr/books?id=6v5cDwAAQBAJ.

[14] REDNER S. Dynamics of Voter Models on Simple and Complex Networks[J]. ArXiv e-prints, 2017, arXiv:1705.02249: arXiv:1705.02249arXiv: 1705.02249 [physics.soc-ph].

[15] HU H. Competing opinion diffusion on social networks[J/OL]. Royal Society Open Science, 2017, 4(11): 171160eprint: https://royalsocietypublishing.org/doi/pdf/10.1098/rsos.171160. https://royalsocietypublishing.org/doi/abs/10.1098/rsos.171160. DOI: 10.1098/rsos.171160.

[16] AMATO R, KOUVARIS N, SAN MIGUEL M, et al. Opinion competition dynamics on multiplex networks[J]. New Journal of Physics, 2017, 19: 123019. DOI: 10.1088/1367-2630/aa936a.

[17] QUATTROCIOCCHI W, CALDARELLI G, SCALA A. Opinion dynamics on interacting networks: Media competition and social influence[J]. Scientific reports, 2014, 4. DOI: 10.1038/srep04938.

[18] CASEY M. SCHNEIDER-MIZELL L M S. A Generalized Voter Model on Complex Networks[J]. J Stat Phys 136:59-71, DOI : 10.1007/s10955-009-9757-6, 2009.

[19] HUA J, WANG L, WANG X. An information diffusion model based on individual characteristics[C]//. [S.l. : s.n.], 2014: 2866-2871. DOI: 10.1109/ChiCC.2014.6897094.

[20] ZHOU S, SHI S, WANG L. Immunizations of Interacting Diseases[C]//. [S.l. : s.n.], 2018: 9686-9691. DOI: 10.23919/ChiCC.2018.8482798.

[21] ZHOU S, XU S, WANG L, et al. Propagation of interacting diseases on multilayer networks[J/OL]. Phys. Rev. E, 2018, 98: 012303. https://link.aps.org/doi/10.1103/PhysRevE.98.012303. DOI: 10.1103/PhysRevE.98.012303.

[22] ALVAREZ-ZUZEK L G, ROCCA C E L, VAZQUEZ F, et al. Interacting Social Processes on Interconnected Networks[C]//PloS one. [S.l. : s.n.], 2016.

[23] GÓMEZ-GARDEÑES J, DOMENICO M D, GUTIÉRREZ G, et al. Layer-layer competition in multiplex complex networks[J]. Philosophical transactions. Series A, Mathematical, physical, and engineering sciences, 2015, 373 2056.

[24] DIEP(LPTM) H T, KAUFMAN M, KAUFMAN S. Dynamics of two-group conflicts: A statistical physics model[J]. Physica A: Statistical Mechanics and its Applications, Elsevier, 2017, 469: 183-199.

[25] ROCCA C E L, BRAUNSTEIN L A, VAZQUEZ F. The influence of persuasion in opinion formation and polarization[J]. CoRR, 2014, abs/1403.3011.

[26] VELÁSQUEZ-ROJAS F, VAZQUEZ F. Opinion dynamics in two dimensions: Domain coarsening leads to stable bi-polarization and anomalous scaling exponents[J]. Journal of Statistical Mechanics: Theory and Experiment, 2018, 2018. DOI: 10.1088/1742-5468/aab1b4.

[27] ABRAMS D M, STROGATZ S H. Linguistics: Modelling the dynamics of language death[J]. Nature, 2003, 424: 900-900.

[28] VÁZQUEZ F, CASTELLÓ X, MIGUEL M S. Agent Based Models of Language Competition : Macroscopic descriptions and Order-Disorder transitions[J]. Journal of Statistical Mechanics: Theory and Experiment, 2010.

[29] E. BEN-NAIM S R, P. L. Krapivsky. Bifurcations and patterns in compromise processes[J]., 2003, 183: 190-204. DOI: 10.1016/S0167-2789(03)00171-4.

[30] CHAU H F, WONG C Y, CHOW F K, et al. Social judgment theory based model on opinion formation, polarization and evolution[J]., 2014, 415: 133-140. DOI: 10.1016/j.physa.2014.07.082.

[31] PATRIARCA M, CASTELLÓ X, URIARTE J R, et al. Modeling Two-Language Competition Dynamics[J]. Advances in Complex Systems, 2012, 15.

[32] P. ERDÖS A R. On the Evolution of Random Graphs[J]., 5: 17-61.

[33] WATTS D J, STROGATZ S H. Collective dynamics of 'small-world' networks[J/OL]. Nature, 1998, 393(6684): 440-442. http://dx.doi.org/10.1038/30918. DOI: 10.1038/30918.

[34] BARABÁSI A L, ALBERT R. Emergence of Scaling in Random Networks[J/OL]. Science, 1999, 286(5439): 509-512eprint: https://science.sciencemag.org/content/286/5439/509.full.pdf. https://science.sciencemag.org/content/286/5439/509. DOI: 10.1126/science.286.5439.509.

[35] SÍRBU A, LORETO V, SERVEDIO V, et al. Opinion Dynamics: Models, Extensions and External Effects[M]//. [S.l. : s.n.], 2017: 363-401. DOI: 10.1007/978-3-319-25658-0_17.

[36] HOFFMAN R, KAGEL J, LEVIN D. Simultaneous versus Sequential Information Processing[J]. Economics Letters, 2011, 112: 16-18. DOI: 10.1016/j.econlet.2011.03.006.

[37] STAUFFER D. DIFFICULTY FOR CONSENSUS IN SIMULTANEOUS OPINION FORMATION OF SZNAJD MODEL[J/OL]. The Journal of Mathematical Sociology, 2004, 28(1): 25-33eprint: https://doi.org/10.1080/00222500490278531. https://doi.org/10.1080/00222500490278531. DOI: 10.1080/00222500490278531.

[38] FREEMAN L. Centrality in Social Networks Conceptual Clarification[J]., 1979, 1: 215-239. DOI: http://dx.doi.org/10.1016/0378-8733(78)90021-7.

[39] KOSCHÜTZKI D, SCHREIBER F. Centrality analysis methods for biological networks and their application to gene regulatory networks[J]., 2008, 2: 193-201.

[40] EOM Y H, SHEPELYANSKY D L. Opinion formation driven by PageRank node influence on directed networks[J/OL]. Physica A: Statistical Mechanics and its Applications, 2015, 436: 707-715. http://dx.doi.org/10.1016/j.physa.2015.05.095. DOI: 10.1016/j.physa.2015.05.095.

[41] WHITE S, SMYTH P. Algorithms for Estimating Relative Importance in Networks[C/OL]//. Washington, D.C.: ACM, 2003: 266-275. http://doi.acm.org/10.1145/956750.956782. DOI: 10.1145/956750.956782.

[42] MESGARI I, KERMANI A, HANNEMAN R, et al. Identifying Key Nodes in Social Networks Using Multi-Criteria Decision-Making Tools[M]. [S.l. : s.n.], 2015: 137-150. DOI: 10.1007/978-3-319-16619-3_10.

[43] HWANG C, YOON K. Multiple Attribute Decision Making: Methods and Applications[M]. New York, NY, USA: Springer-Verlag, 1981. DOI: http://dx.doi.org/10.1007/978-3-642-48318-9.

[44] HUANG S, LV T, ZHANG X, et al. Identifying Node Role in Social Network Based on Multiple Indicators[J/OL]. PLOS ONE, 2014, 9(8): 1-16. https://doi.org/10.1371/journal.pone.0103733. DOI: 10.1371/journal.pone.0103733.

[45] RODRIGUES F A. Network Centrality: An Introduction[C]//. [S.l. : s.n.], 2019.

[46] LANCHIER N. Stochastic Modeling[M]. [S.l.]: Springer Nature, DOI:1007/978-3-319-50038-6, 2017.

[47] HEGSELMANN R, KRAUSE U. Opinion Dynamics and Bounded Confidence Models, Analysis and Simulation[J]. Journal of Artificial Societies and Social Simulation, 2002, 5.

[48] SANGWOO K. Structure and dynamics of complex networks[D]. Yonsei Univ, Department of physics, 2012.

[49] CHOI W. A Study on the Characteristics and the Transition of the Complex System Network Structure and Dynamics[D]. 2011.

[50] BOLLOBÁS B. Random Graphs[M]. 2nd ed. [S.l.]: Cambridge University Press, 2001. DOI: 10.1017/CBO9780511814068.

Publications

[1] Hyunchel Cho, Arfan Mahmood, Lin Wang. Competition of Social Opinions on Two layer Networks. In Proceeding of the 38th Chinese Control Conference, July 2019, Guangzhou, China, Accepted.