上海交通大学硕士学位论文

两层网络上的社会舆论竞争

硕士研究生：

学　　　　号：

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申　请　学　位：学术硕士

学　　　　科：控制科学与工程

所　在　单　位：电子信息与电气工程学院

答　辩　日　期：2020 年 2 月

授予学位单位：上海交通大学

Dissertation Submitted to Shanghai Jiao Tong University
for the Degree of Master

COMPETITION OF SOCIAL OPINIONS ON TWO-LAYER NETWORKS

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Date of Defence:	Feb, 2020
Degree-Conferring-Institution:	Shanghai Jiao Tong University

两层网络上的社会舆论竞争

摘 要

不同的群体通常会有不同的意见，比如在社会问题、总统竞选等投票过程中意见相左，竞争是不可避免的。互联网络的竞争一直是复杂网络和社会行为领域的热点问题。本文研究了两层网络中的竞争，研究了网络结构、更新规则以及关键节点对竞争结果的影响。

首先，采用两层网络对两个群体的竞争进行建模，其中 A 层为意见形成组，B 层为决策组。从 A 层所有节点都有正面观点，B 层所有节点都有负面观点的极化竞争状态出发，分析了网络结构、内部度和外部度对竞争状态的影响。仿真结果表明，内外部环节在竞争中都起着至关重要的作用。值得注意的是，增加一个层面上的外部和内部联系，可以很容易战胜另一个群体，达成共识。

其次，在前面的两层意见模型的基础上，研究了更新规则的影响。根据层、节点、边等不同层次，考虑了更新规则，包括顺序更新规则和同时更新规则。观察到同步更新规则更可能具有共存状态并且容易被改变为相反状态，而顺序更新规则可以更快地实现共识。

此外，研究了关键节点对竞争的影响。利用 Pagerank、degree、eigenvector、betweenness、closeness 等中心度指标及其组合选择关键节点。通过仿真，发现关键节点对网络结构和观点动态的影响是不同的。此外，用于选择关键节点的单中心性指标和多中心性指标都能很好地说服其他群体的代理改变观点。

关键词：complex network, interconnected network, opinion dynamics, competition, consensus

COMPETITION OF SOCIAL OPINIONS ON TWO-LAYER NETWORKS

ABSTRACT

Different groups usually have different opinions, such as opposite opinions during votes on social issues, presidential campaigns, and so on, where competition is unavoidable. Competition on the interconnected networks has always been a hot topic in the field of complex networks and social behavior. In this paper, we investigate competition in a two-layer network and study the influence of network structures, updating rules as well as key nodes on the competition results.

First, a two-layer network is used to model the competition of two groups, where layer A is an opinion formation group, and layer B is a decision-making group. Starting with a polarized competition state, where all nodes in layer A have positive opinions while all nodes in layer B have negative opinions, the influences of network structures, internal degrees, and external degrees are analyzed. Simulation results show that both internal and external links play vital roles in the competition. Notably, increasing the number of external and internal links on one layer can make it easy to prevail over the other group and reach consensus.

Second, the influence of updating rules is investigated based on the previous two-layer opinion model. The updating rules, including sequential order and simultaneous order, are considered according to different levels, such as layers, nodes, and edges. It is observed that a simultaneous updating rule is more likely to have a coexistence state and easily be changed to the opposite state, while a sequential updating rule can enable consensus more quickly.

Moreover, the influence of critical nodes on the competition is studied by fixing their states during the evolution of opinion. Some centrality indexes, including Pagerank, degree, eigenvector, betweenness, closeness, and their combinations, are used to select the key nodes. Through simulations, it is found that the influence of the key nodes is different according to network structures and opinion dynamics. Besides, both indexes with single centrality and multiple centralities for selecting key nodes have an excellent performance on persuading the other group of agents to change their opinion.

KEY WORDS: complex network, interconnected network, opinion dynamics, competition, consensus

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Chapter 1 Introduction

1.1 Introduction

People have their own opinions, and sometimes they change their opinions in response to others who have views on those issues. Their opinions are reflected in the leaders to make laws and make necessary decisions. These phenomena can be found in some cases, such as voting, legislation, and the adoption of new policies. It is widely recognized that both opinion formation and decision-making formation have mutual interaction as interconnected networks.[1-7]. Sometimes, opinion formation could be opposed to decision-making formation. These situations often give rise to social conflicts and confusion. In order to figure out these social conflicts, it is necessary to understand and analyze the competition of interconnected networks. So far, physics and computer science have researched these social conflicts by modeling and analyzing complex systems[8-11]. The researches include opinion dynamics, voter model, game theory, and many more.[12-18] Competition of interconnected networks has been researched in various ways. These networks can be applied to the dissemination of computer viruses, messages, opinions, memes, diseases, and rumors[19-26]. Opinion dynamics on interconnected networks has been investigated with various network models such as *Abrams-Strogatz(AS)* model[27, 28] and *M* model[25]. Based on the previous research, this work studies the main features of competition in two-layer networks by changing network structures, changing the updating rules, and selecting the key nodes. It is proven and analyzed that these different conditions cause different results.

1.2 Competition on interconnected networks

In this research, we focus on the competition on a two-layer network or an interconnected network. If compared with a single-layer network, the interconnected network has two dynamics, two parameters, and includes internal edges and external edges, as shown in Fig. 1–2. Therefore, the interconnected network interaction is more complex than single-layer network interaction.

In order to make a two-layer network under competition, each layer is made up of different dynamics and parameters. Network dynamics are based on previous research,

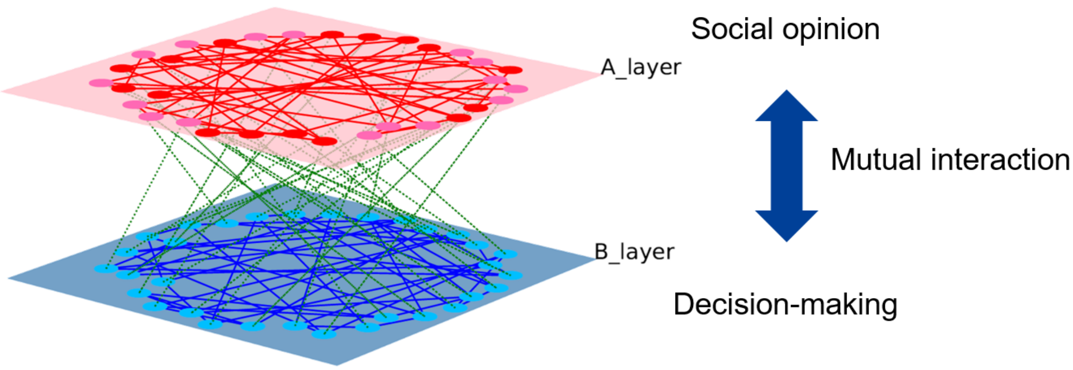
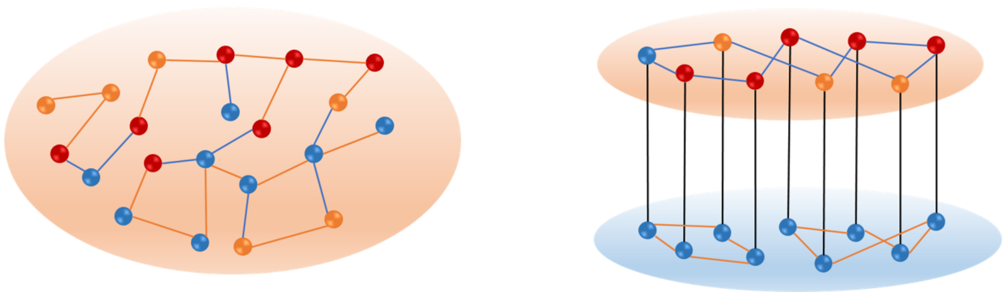


Figure 1–1 The example of competition on the two-layer network



Single layer network	Two-layer network
1 dynamics	2 dynamics (each layer has it own dynamics)
1 parameter	2 parameter
Internal edge	Internal edge + external edge

Figure 1–2 Comparison between a single-layer network and a two-layer network

such as [22]. The top layer has the function of social opinion and its dynamics. Some opinion models provide a social mechanism through the compromise process.[29] Other opinion models represent the persuasive process.[30] In this research, the social opinion layer is affected by the opinion dynamics, which are also known as M-model[25], which includes compromise function and persuasion function. The bottom layer has the function of decision-making and its dynamics. The dynamics of the decision-making layer is the language competition dynamics that are also called as the *Abrams-Strogatz* model[27, 28, 31]. This model is useful to decide only one opinion from two opinions. In order to make the competition condition of these two layers, the initial states of the two layers are assumed to be in opposite states, that the social opinion layer has all positive states and the decision-making layer has all negative states.

So far, main researchers have focused on what factors make a consensus or dissent(coexistence), which have shown that the system can make positive consensus, negative consensus, or coexistence under a certain range of parameters, such as volatility, reinforcement, and prestige.[22] Moreover, the interconnected competition of the social network has been researched by finding the threshold or critical point for consensus.[22-24] Also, it has been found out that the thresholds make the transition of states, and they can explain and analyze the social phenomena in the real world, such as the legislation, election, and social conflicts.[16, 22, 24]

In [23], it is shown that the transition from localized to mixed status occurs through a cascade from poorly connected nodes in the layers to the highly connected ones and the external degree is critical to change the state of the network. Besides, the main features, such as transition and cascade, found in Monte Carlo simulation, are precisely characterized by the mean-field theory and magnetization[16, 22-24].

Based on all this previous research, the competitions of interconnected networks are analyzed by three main topics, such as network structures, updating rules, and selection of key nodes. Before simulations, backgrounds for three topics are explained as follows. First, network structures are investigated. Networks can be largely divided into a regular network, random network[32], small-world network[33], scale-free network[34], and others. Fig. 1–3 shows the structures of various networks. A regular network has a lattice structure, and each node has the same number of links. A random network is made up of edges such that two nodes are connected with probability p in the systems with K nodes. A small-world network is a network graph in which most nodes are not neighbors of one another, but most nodes can be reached from every other node by a small number of links. A small-world network can be made by eliminating the edges with probability p and connecting two random nodes that are not connected in a regular network. A small-world network has all characteristics of a regular network and random network. A scale-free network has a model such that the distribution of edges follows power function. Examples of a scale-free network are the World Wide Web (WWW), the Internet, movie star networks, protein interactions, metabolism, and so on. There are several ways to create a scale-free network. Among them, the most typical way is the *Barabasi-Albert* model. The *Barabasi-Albert* model is growing networks in which nodes continue to be added, and connections between nodes have a preferential attachment. The process of creating this model repeats the following two processes: First, add one

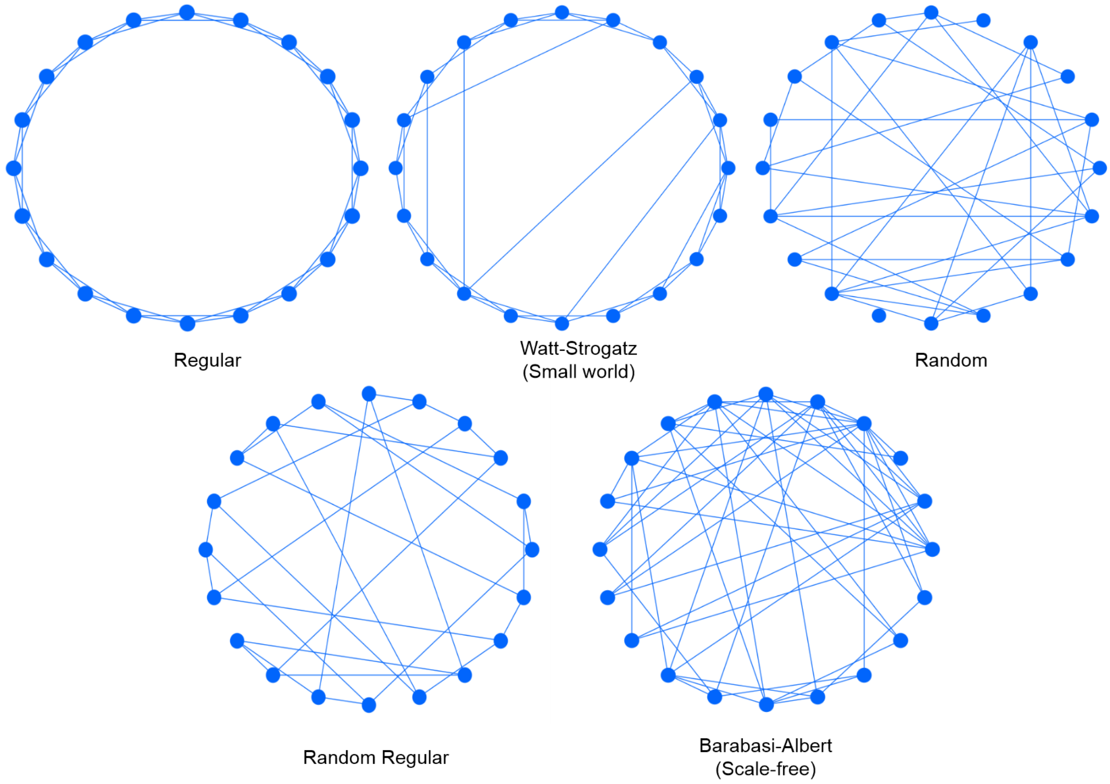


Figure 1–3 Various structures of the network

node with a constant number of edges to the system. Second, edges of the added nodes are connected in proportion to the edge number of the pre-existing nodes. In this work, two types of general networks are applied, such as a random regular(RR) network and *Barabasi-Albert*(BA) network.

Second, dynamics orders and updating rules are also studied. For further understanding of the competition on a two-layer network, it is crucial to investigate the interaction between nodes or layers. Methods of interaction between nodes are various.[35] However, related to time, the types of interactions would be divided into two categories, such as simultaneous interaction, and sequential interaction. In economics and social networks, it has been proven that there exist different results between simultaneous and sequential interaction.[36, 37] In [36], it was researched how experimental subjects update induced prior information when receiving two information signals simultaneously or receiving the same signals sequentially. As the experimental results, the simultaneous treatment is very different from sequential treatment, and under sequential information, subject's mean estimates of the two treatments(good news preceding bad news or vice versa) are also significantly different from each other. In conclusion, both the sequencing of process and the order of information matters. Moreover, in [37], the usual random sequential updating rule is replaced by the simultaneous updating rule on the *Sznajd* model. It is found out that this change makes a complete consensus much more difficult. The reason is analyzed as some agents with the simultaneous updating rule receive conflicting messages from different neighbor pairs and thus refuse to change their opinion. In this work, both simultaneous and sequential updating rules are applied to layers, nodes, and links.

Third, network centralities would be researched to select key nodes on a two-layer network. Network centrality means the index to measure how close each node is to the center of a network. That answers the question, "What characterizes an important node?". The concept of network centrality was first introduced in the field of social network analysis.[38] After that, it has expanded to various areas where the concept of the network is related and has been used to identify which nodes are important in the network. So far, various criteria for assessing network centrality have been presented. Generally, well-known network centralities include degree centrality, betweenness centrality, closeness centrality, eigenvector centrality, and Pagerank.[13, 39, 40] Degree centrality is the simplest but the most reliable concept. It is defined as the number of interacting neighbor nodes (or edges). Betweenness centrality is the concept of using the shortest path between two nodes on a network. It is explained as the concept to define two different node sets on the network (set 1, 2) and quantify how often each node appears on the shortest path for all combinations of nodes in set 1 and set 2. Closeness centrality is derived from that the shorter the path that one node reaches all the other nodes is, the more influential the node is. Eigenvector centrality is the concept that the more a node is connected with critical nodes, the more critical it is. Pagerank measures the convergent value by repeating the process of propagating each node's influence on the other nodes. So far, many researchers have tried to select critical nodes in a social network.[41-45] Based on a node centrality, some algorithms for identifying key nodes has been found out. In [43, 45], it has been found out that optimally combining multiple measures of nodal importance may provide a robust tool for identifying key nodes of interest, particularly in large graphs. Here, based on previous research, we select the key nodes by using a single node centrality and combined node centrality.

In this work, for single indicator methods to select key nodes, network centralities would be applied, such as Pagerank, degree, eigenvector, betweenness, and closeness. As multiple indicator methods recognize key nodes, several combined node centralities are applied, such as *PR+DE, PR+BE, DE+BE, PR+DE+BE* that are based on single indicators. By using these centralities(Pagerank, degree, eigenvector, closeness, betweenness, and combined node centralities), it is discussed and shown that which method is the most influential for changing the state of network on various models.

1.3 Motivation and organization

In this work, opinion dynamics of a competing two-layer social network would be investigated based on the pre-existed research[22-25]. We develop modeling and research to find out the characteristics of interconnected networks.

This research has four main directions to investigate the features of the competition model. First, it is shown how to make up competition models and how to measure the consensus for analysis. Second, we find out what factors make consensus by changing network structures. Third, it is analyzed how dynamics orders and updating rules influence the state of the two-layer network. Fourth, based on network centralities, it would be investigated which method is the most effective to identify key nodes.

This study can help to explain social network phenomena, such as a social conflict between two opinions. Therefore, this research can be used as a tool for making an efficient decision-making system, solving the social conflict, and analyzing social network problems such as legislation and vote system.

This paper is organized as follows. In the chapter 2, it is introduced to how competition model of two-layers is made up and how the dynamics of each layer works. Moreover, some indexes are provided to measure and evaluate the simulation results. In the chapter 3, with changing network structure, it is shown how the network structures influence the consensus of the two-layer network. In the chapter 4, considering the dynamics orders and updating rules, simulation results are compared and analyzed. In the chapter 5, it is researched which nodes are critical for affecting the state of the network by using single indicators and multiple indicators. Finally, in the chapter 6, all simulation results are summarized, and our findings are concluded.

Chapter 2 A two-layer network model

In this chapter, a basic model is introduced for competition on a two-layer network. It is also described that how each layer is made up and what kind of function and dynamics it has. Also, several indexes are provided to analyze and measure the interaction between two-layers.

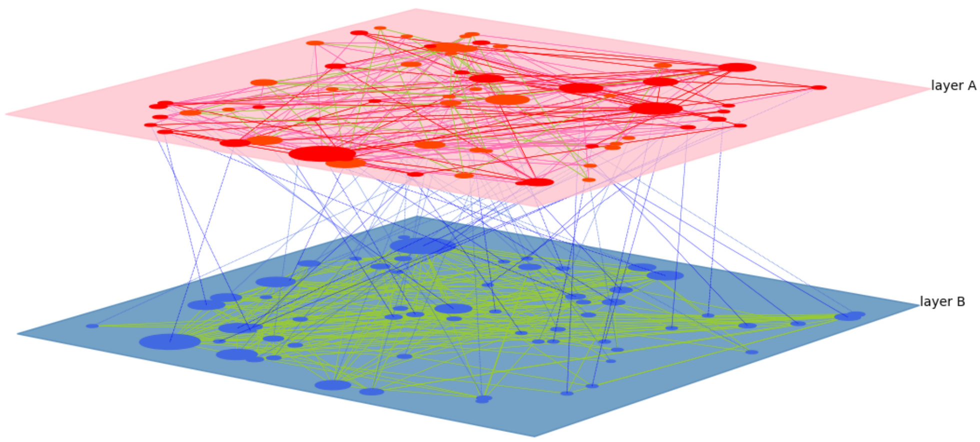


Figure 2–1 Competition of Interconnected Network

2.1 Modeling of a two-layer network

The model consists of two layers, and each layer has different dynamics. For layer A, the node changes its state according to the M model, as introduced in [25]. Here, we choose $M = 2$, that each node can have one of four states $(-2, -1, +1, +2)$. For each link (k, j) belong to layer A, the dynamics are designed as follows:

- Compromise: if two nodes connected with link(k, j) have opposite orientations, their states become more moderate with probability q :

$$\text{if } S_k < 0 \text{ and } S_j > 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^r, S_j^l) \text{ with } prob.q, \tag{2–1}$$

$$\text{if } S_k > 0 \text{ and } S_j < 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^l, S_j^r) \text{ with } prob.q. \tag{2–2}$$

If $S_k = \pm 1$ and $S_j = \mp 1$, one switches orientation at random:

$$(\pm 1, \mp 1) \rightarrow \begin{cases} (+1, +1) \text{ with } prob.q/2, \\ (-1, -1) \text{ with } prob.q/2. \end{cases} \tag{2–3}$$

- Persuasion: if two nodes connected with link(k, j) have the same orientation, their states become more extreme with probability p :

$$\text{if } S_k < 0 \text{ and } S_j < 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^l, S_j^l) \text{ with } prob.p, \tag{2–4}$$

$$\text{if } S_k > 0 \text{ and } S_j > 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^r, S_j^r) \text{ with } prob.p. \tag{2–5}$$

For each external link (k, j) with k belong to layer A, the state of node k is updated according to :

- $S_k \cdot S_j < 0$:

$$\text{if } S_k < 0 \text{ and } S_j > 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^r, S_j) \text{ with } prob.q, \tag{2–6}$$

$$\text{if } S_k > 0 \text{ and } S_j < 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^l, S_j) \text{ with } prob.q. \tag{2–7}$$

- $S_k \cdot S_j > 0$:

$$\text{if } S_k < 0 \text{ and } S_j < 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^l, S_j) \text{ with } prob.p, \tag{2–8}$$

$$\text{if } S_k > 0 \text{ and } S_j > 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^r, S_j) \text{ with } prob.p. \tag{2–9}$$

Here, S_k^r and S_k^l denote the right and left neighboring states of node k, defined as

$$S_k^r = \begin{cases} +1, \text{ for } S_k = -1 \\ +2, \text{ for } S_k = +2 \\ S_k + 1, \text{ otherwise }, \end{cases} \qquad S_k^l = \begin{cases} -1, \text{ for } S_k = +1 \\ -2, \text{ for } S_k = -2 \\ S_k - 1, \text{ otherwise }. \end{cases} \tag{2–10}$$

The sign of S^A represents the opinion orientation of a node, and its absolute value $|S^A|$ measures the intensity of its opinion. So, $|S^A| = 2$ represents a positive or negative extremist, while $|S^A| = 1$ corresponds to a moderate opinion of each side. In case of internal link (k, j) belong to layer A, when the nodes have the same orientation$(S_k S_j > 0)$, if the states of nodes are moderate, then they become extreme$(S_k = \pm 1 \rightarrow \pm 2, S_j = \pm 1 \rightarrow \pm 2)$ with probability p. If they are already extreme, they remain extreme$(S_k = \pm 2 \rightarrow \pm 2, S_j = \pm 2 \rightarrow \pm 2)$. On the other hand, when the nodes have opposite orientations$(S_k S_j < 0)$, if they are extreme, the states of nodes become moderate$(S_k = \pm 2 \rightarrow \pm 1, S_j = \pm 2 \rightarrow \pm 1)$ with probability q. If they are already moderate, they switch orientations individually($S_k = \pm 1 \rightarrow \mp 1, S_j = \pm 1 \rightarrow \mp 1$). In case of interaction between a node in layer A and a node in layer B, a node in layer A follows opinion dynamics formula, but the state of a node in layer B does not change. In other words, the state of layer B affects layer A, but layer A dynamics do not affect the state of a node in layer B. For example, one node in the layer A, $S_k = +2$ is connected with $S_j = -1$ node of layer B. Here, S_k will change into $S_k = +1$ with $prob.q$, but S_j would not change, which indicates that the states of layer B influence the states of layer A though the state of a node in layer B is not changed.

The dynamics of layer B follows the decision-making dynamics as introduced in [27, 28]. The state of node i in layer B can be $+1$ or -1, and it is updated according to

$$P_B(S_i \rightarrow -S_i) = \begin{cases} \left(\left(\frac{i_i + e_i}{n^{-S_i}} \right) \cdot \left(\frac{n^{-S_i}}{i_i + e_i} \right) \right)^{1/v}, \text{ if } v \neq 0 \\ 0, \text{ if } v = 0 \\ 0, \text{ if } n^{-S_i} = 0 \end{cases}, \tag{2–11}$$

where i_i is an internal degree of node i and e_i is an external degree of node i. n^{-S_i} is the number of neighbors of node i with opposite state $-S_i$. v represents the volatility that measures how prone the state of a node is changed. The scale of v is from 0 to 1. If $v \simeq 0$, a node is unlikely to change its state. On the other hand, if $v \simeq 1$, a node is very likely to change its state. Also, this formula shows that the more the edges connected with the opposite state is, the easier the nodal state is to be changed into the opposite state.

2.2 Simulations and Analysis

This model is nonlinear, and the applied dynamics are switched according to the states of nodes. In this model, helpful mathematical tools are no longer applicable, and that rigorous analytical results are difficult to obtain.[46, 47]. For that reason, we try to carry out the analysis of the above nonlinear model to a large extent by simulations on the computer.

To start with a polarized competition, as the initial conditions, nodes in layer A are all positive states, and nodes in layer B are all negative states, as shown in Fig. 2–1. For nodes in layer A, it begins with the states where half of the nodes are $+1$, and the others are $+2$. The initial states of nodes in layer B have only -1.

There are two parameters, p and q, in the dynamics of layer A. To represent the

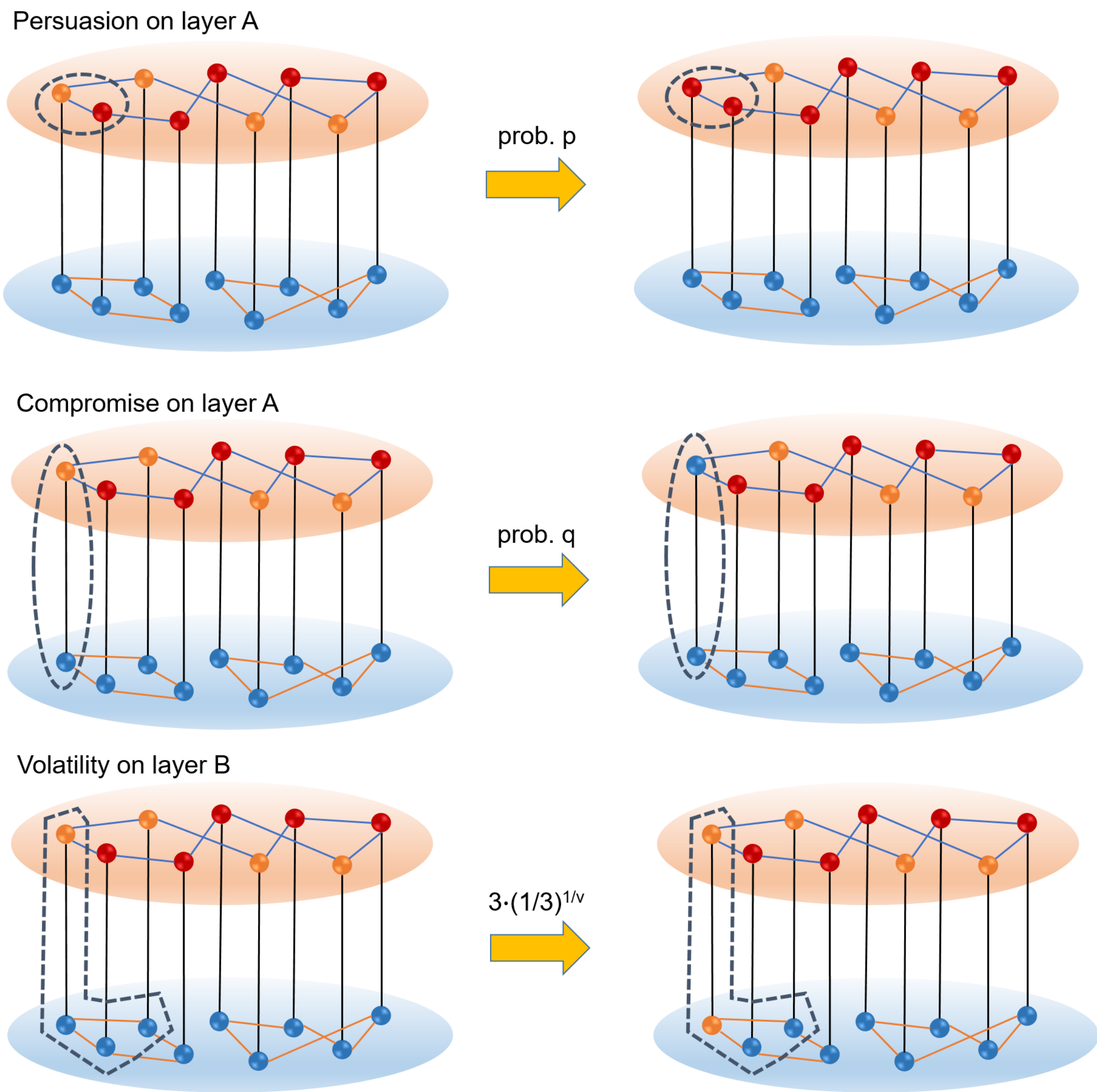


Figure 2–2 Dynamics on two layers

probability p and probability q together simply, we set $p + q = 1$. So, p represents the reinforcement or strength of opinion, such as extreme and moderate, which is scaled to be 0 to 1. And, there is only one parameter, v in the dynamics of layer B. The scale of v is also 0 to 1 as p. v represents the volatility, which means how prone the state of a node can be changed into the opposite state.

In order to implement the interconnected dynamics, one step consists of two layers dynamics, where every node in layer A is checked with opinion dynamics, and every node in layer B updates its state according to the decision-making dynamics. The dynamics order follows updating the state of layer B after updating the state of layer A. The dynamics orders and updating rules of the two-layer network are explicitly discussed in chapter 4.

Each simulation takes 100 steps for the opinion evolution, and 100 simulations are considered for average results. To analyze the simulation results, we use *'Average State'(AS)* to measure the average states of network and *'Consensus Index'(CI)* to measure how close the state of the network is to consensus. The formulas follow as

$$AS = avg\left(\sum_{i}^{K^A} S_i^A/4\right) + avg\left(\sum_{i}^{K^B} S_i^B/2\right), \tag{2–12}$$

$$CI = \frac{({K_+}^A \cdot {K_-}^B) + ({K_-}^A \cdot {K_+}^B)}{K^A \cdot K^B}. \tag{2–13}$$

In these formulas, S_i^A means the state of node i in layer A, and K^A is the number of nodes in layer A. ${K_+}^A$ represents the number of nodes with the positive state in layer A.

With AS, it could be verified whether a consensus happens under the change of p and v. If a positive consensus happens, the AS would be close to the value of $+1$. If a negative consensus happens, the AS would be close to the value of -1. And, the medium values between $+1$ and -1 mean that the state of the network belongs to the coexistence or dissent part.

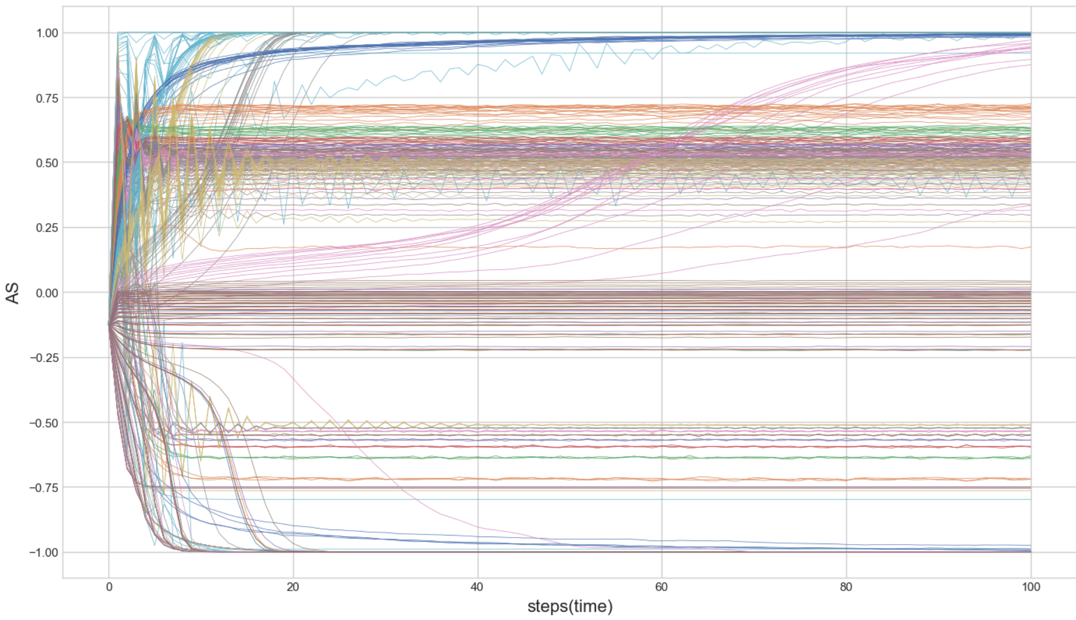


Figure 2–3 AS values per each step according to all parameters

Fig. 2–3 shows that *AS* values are convergent to +1, −1 or other values as step(time) goes by. +1 means making a positive consensus. −1 means making a negative consensus. The other values mean a coexistence or dissent state.

With *CI*, it could be measured how close the state of the network is to consensus. If the *CI* is close to 0, the state of the network is close to a positive or negative consensus. If the *CI* is close to 1, a state of the network is a separated coexistence, where states of all nodes in layer A is opposed to states of all nodes in layer B. If the *CI* is close to 0.5, a state of the network is a mixed coexistence, where each layer has both positive and negative states of nodes.

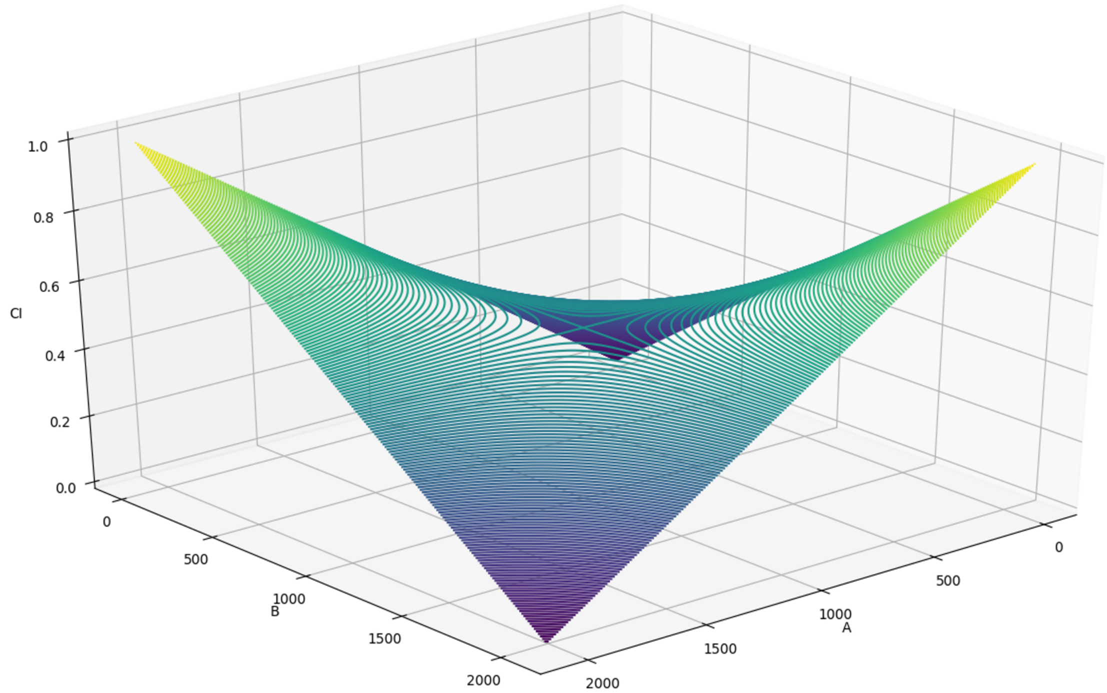


Figure 2–4 CI values according to all ${K_+}^A$ and ${K_+}^B$

Fig. 2–4 shows the characteristics of *CI*. Same orientation states in two layers make *CI* 0. Opposite orientation states between two layers make *CI* 1. Moreover, mixed states in two layers make *CI* close to 0.5.

As Fig. 2–5 shown, *CI* values are convergent to +1, 0, or other values as step(time) goes by. 0 means positive or negative consensus of two-layer. +1 means separated opposite states of two-layers. The other values mean mixed states of two-layers. By using *CI*, coexistence states can be divided into two categories, separated state and mixed state.

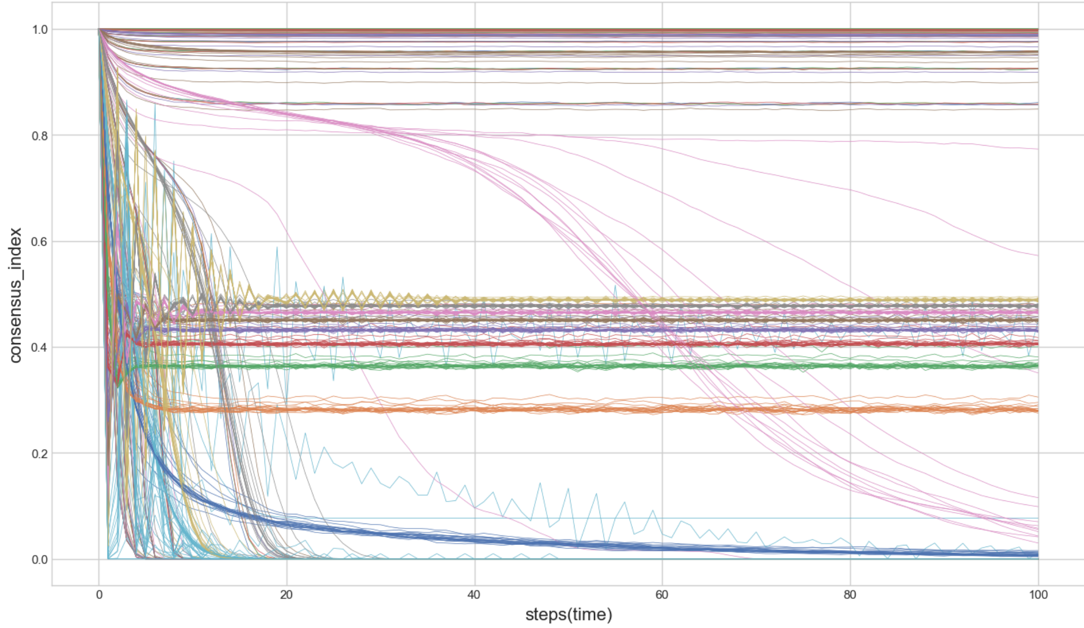


Figure 2–5 CI values per each step according to all parameters

To measure and evaluate the consensus results regarding to two parameters p and v, we use four kinds of indexes, including *'AS total'*, *'Positive Consensus Ratio'(PCR)*, *'Negative Consensus ratio'(NCR)*, and *'Consensus Ratio'(CR)*. *AS total* means the summation of AS for all ps and all vs. *PCR* is the ratio of positive consensus over all simulations. Similarly, *NCR* is the ratio of experiments with negative consensus. *CR* is the ratio of experiments reaching consensus, i.e., the summation of *PCR* and *NCR*.

$$AS\ total = \frac{\sum_j^m \sum_i^n AS_{p_i,v_j}}{n \times m}, \quad \begin{array}{l} p = \{p_1, p_2, \cdots, p_n\} \\ v = \{v_1, v_2, \cdots, v_m\} \end{array}. \tag{2–14}$$

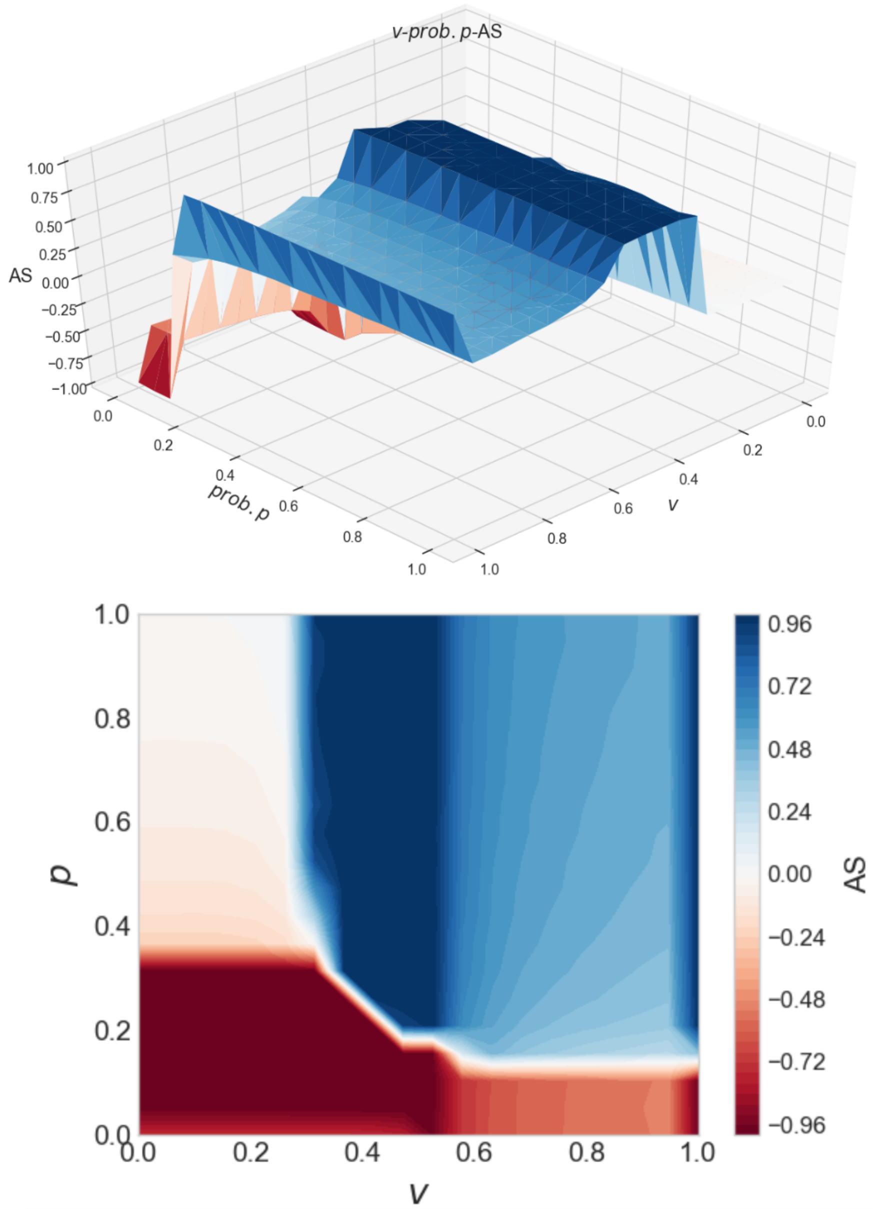
In Eq(2–14), AS_{p_i,v_j} means AS value according to parameters p_i and v_j, which shows the total orientation and intensity of interconnected network.

$$PCR = \frac{\sum_j^m \sum_i^n (AS_{p_i,v_j} \simeq 1)}{n \times m}. \tag{2–15}$$

In Eq(2–15), $AS_{p_i,v_j} \simeq 1$ means positive consensus.

$$NCR = \frac{\sum_j^m \sum_i^n (AS_{p_i,v_j} \simeq -1)}{n \times m}. \tag{2–16}$$

In Eq(2–16), $AS_{p_i,v_j} \simeq -1$ means negative consensus.



Div	AS total (Average values)	PCR(Blue part)	NCR(Red part)	CR(Blue+Red part)
BA-BA network	0.1799	0.1750	0.1450	0.3200

Figure 2–6 The example of simulation：BA-BA network

Fig. 2–6 shows the states of the interconnected network according to all ps and all vs. The X-axis is the p, the Y-axis is the v, and the Z-axis represents AS. The closer the color is to blue color, the more it has a positive consensus. Moreover, the closer the color is to red color, the more it has a negative consensus. Light-colored and white areas have coexistence with both positive states and negative states. Here, we can measure the consensus by using indexes, *'AS total'*, *'PCR'*, *'NCR'*, and *'CR'*. The average value of this figure means *'AS total'*. The blue part area means *'PCR'*, the red part area means *'NCR'*, and the summation of both *'PCR'* and *'NCR'* means *'CR'*.

Chapter 3　Competition on a two-layer network with various structures

In this chapter, based on the competition model described in the chapter 2, simulations are implemented with changing the network structures. As the basic model, the interconnected network with a random regular network on each layer is also provided. And then, the structure of interconnected networks is altered by changing the number of internal edges, the number of external edges, and network types. Finally, all simulations are compared and analyzed with the indexes, *AS total*, *PCR*, *NCR*, and *CR*.

3.1　Competition on Random Regular Networks

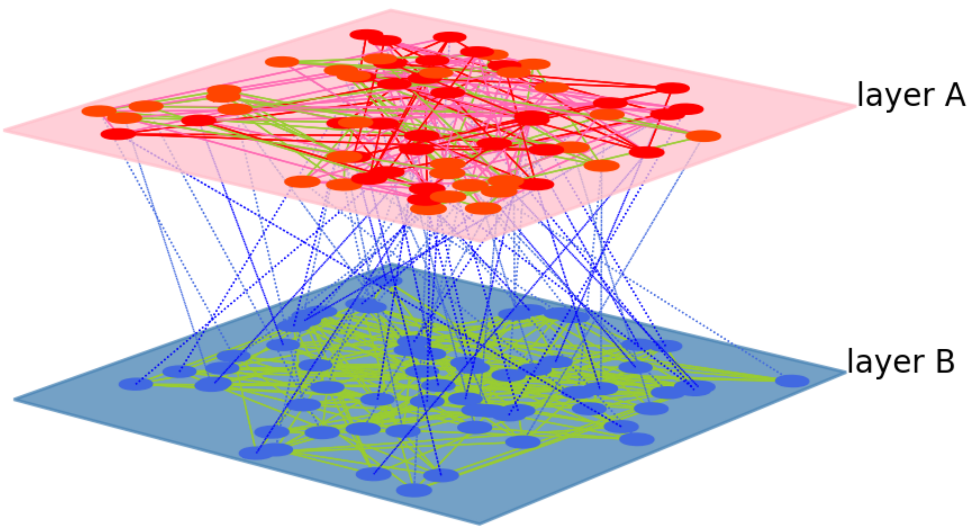


Figure 3–1 Competition on random regular network

In this section, simulation results on a two-layer network with random regular networks are provided to analyze the competition of two layers. Each layer consists of a random regular network that has N nodes with k internal edges as introduced in

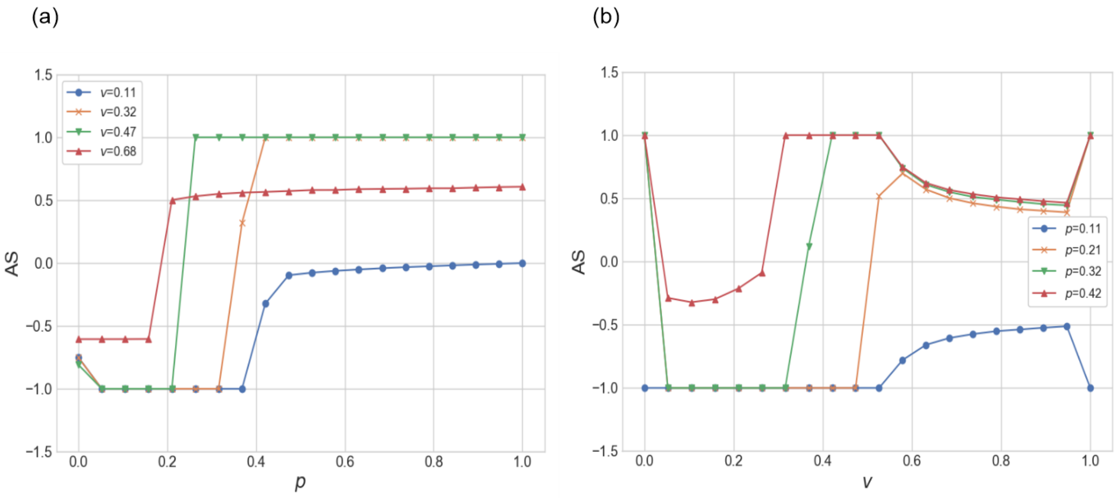


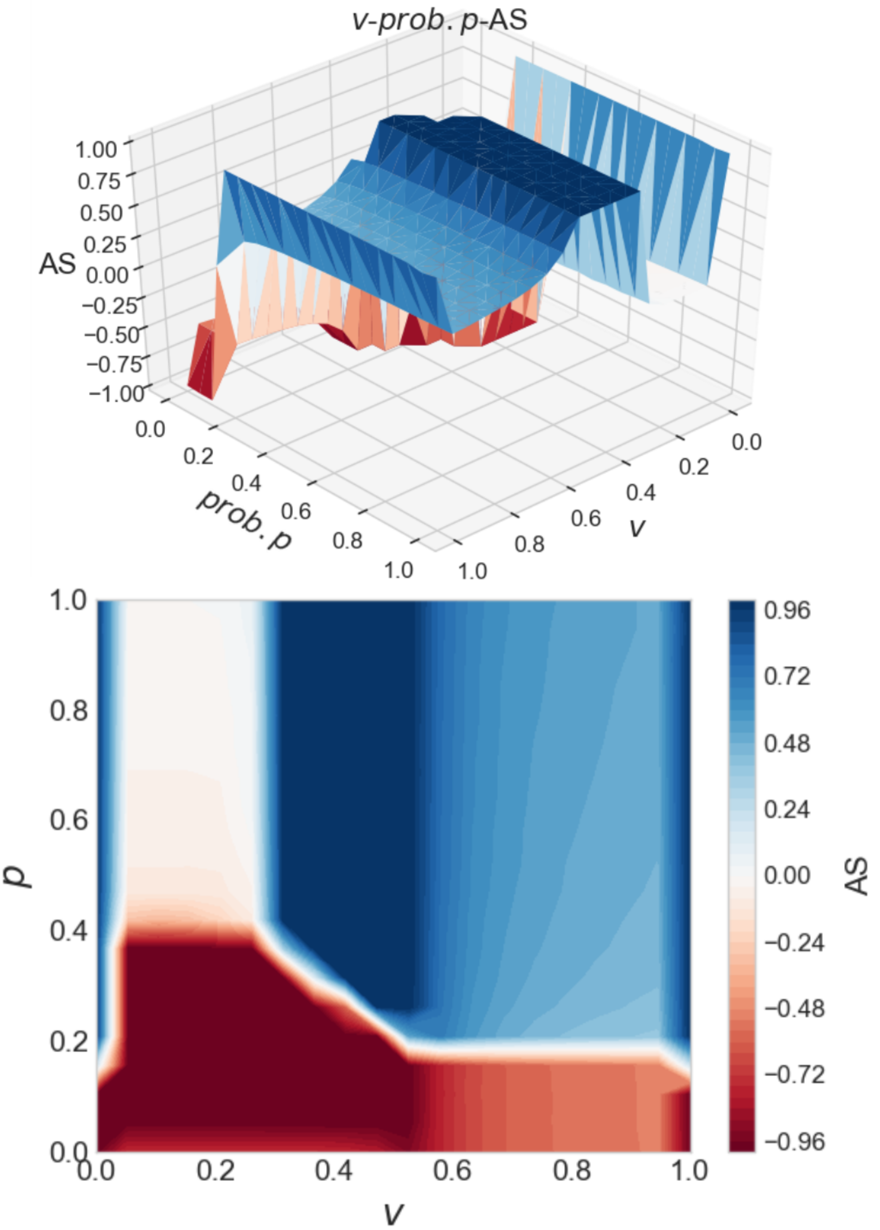
Figure 3–2 (a) *p-AS* chart according to specific v values. (b) *v-AS* chart according to specific p values.

[48-50]. Each node of one layer is connected with a random node on the other layer. That means each node has only one external edge. Simulations are performed on the two-layer network with $N = 2048$ and $k = 5$ on each layer.

The simulation results are shown in Fig. 3–2 and Fig. 3–3. Fig. 3–2 presents how the ‘Average State’(*AS*) is changed according to the other parameter(v or p) when one parameter(p or v) is constant. So we can know how each parameter works on the network. Fig. 3–3 provides total results with all parameters. Through these figures, the characteristics of the network are analyzed.

Fig. 3–2(a) shows that when $p > 0.2$, $0.32 < v < 0.47$, it normally tends to have positive consensus. We find that if v is lower or larger than a certain value, it does not make consensus. In Fig. 3–2(b), as v increases, it normally changes from negative to positive consensus. However, it is found out that when p is small enough($p \leq 0.11$), it does not make a positive consensus. To sum up, when p is large enough, it tends to make a positive consensus. However, when v is small enough, it tends to be changed into a negative consensus.

Fig. 3–3 shows the states of two layers according to all ps and all vs. As previously described in the chapter 2, blue areas are for positive consensus, red areas are for negative consensus, and light-colored and white areas are for coexistence. Moreover, indexes for consensus are also measured. *PCR* value is 0.2475, and *NCR* value is 0.1575. The ratio of coexistence is $1 - CR = 0.5950$. By using these values and figures, this model is



Div	AS total (Average values)	PCR (Blue part)	NCR (Red part)	CR (Blue+Red part)
RR-RR network	0.2034	0.2475	0.1575	0.4050

Figure 3–3 AS according to all ps and vs

compared with networks of various structures in the next section. Through these figures, the characteristics of parameters can be arranged as follows: First, large p tends to make positive consensus, and small p tends to make negative consensus. Second, small v tends to make negative consensus, and large v tends to make coexistence state.

3.2 Competition on Networks with different network structures

In this section, simulation results on a two-layer network with various structures are provided and compared with the basic model described in the section 3.1. The network structures are switched by changing internal degrees, external degrees, and network types.

3.2.1 Competition on Networks with different number of external links

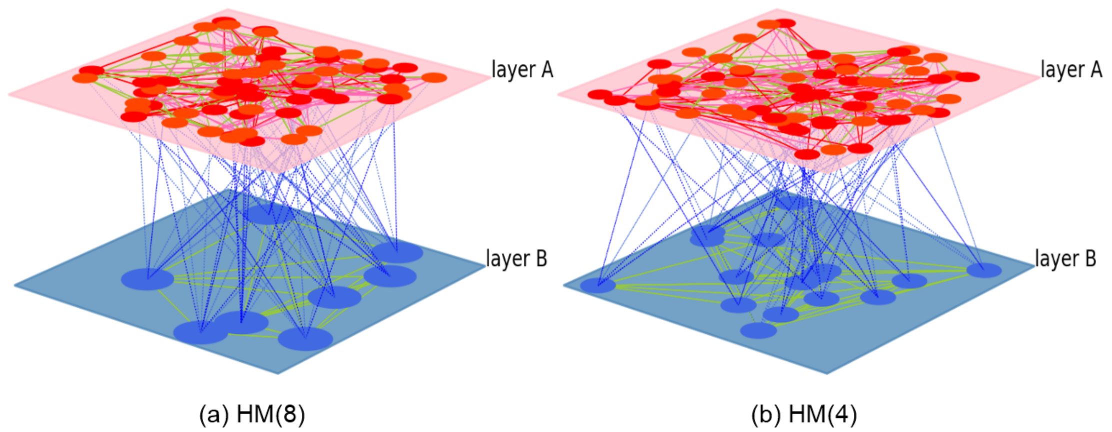


Figure 3–4 Competition on *Hierarchical Model*

First, we consider the influence of external links. Based on the *RR-RR* model in section 3.1, we reduce the number of nodes in layer B at a specific rate and increase the external links from nodes in layer B accordingly, as shown in Fig. 3–4. We denote *HM(n)* as a *Hierarchical Model* with a level n, which means that the number of nodes in layer B is $1/n$ of the number of nodes in layer A, and the number of external links from a node in layer B is n in view that the number of external links from a node in layer A is 1. In other words, each node in layer A has one external edge, but each node in layer B has n external edges for *HM(n)*, which means one node in layer B can interact with n nodes in layer A.

Various *HM(n)s* are simulated. The simulation results of 8 *HM(n)*s, *HM(2), HM(4), HM(8), HM(16), HM(32), HM(64), HM(128), HM(256)* are arranged, as shown in Fig. 3–

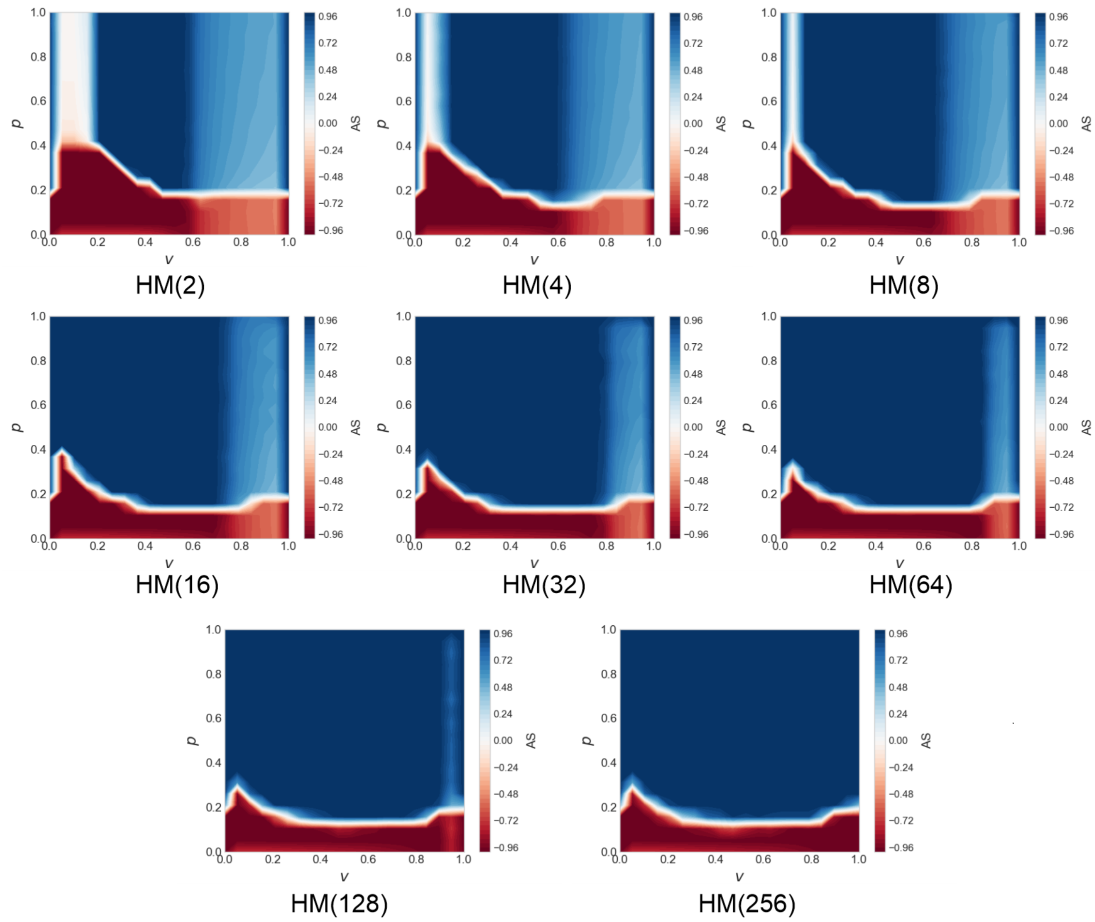


Figure 3–5 *AS total* on various *Hierarchical Models*

5. Fig. 3–5 shows that *HM(2)* has the most significant area for the coexistence part(light-colored and white area), and *HM(256)* has the most significant area for the consensus part(blue and red area). As *n* in *HM(n)* is increased, the coexistence area is decreased, and the consensus area is increased. Notably, the positive consensus area is significantly increased, the negative consensus area is slightly decreased.

To find out the difference between models, we use the indexes, *PCR, NCR, AS total*. Fig. 3–6 shows the results to analyze *HM(n)* with indexes. The blue color bar is for PCR, the red color bar is for NCR, and the green color bar is for AS total. Comparing *HMs* with the *Basic model(RR(5)-RR(5))*, *CR PCR*, and *AS total* are all increased remarkably. *HMs* have a larger area for positive consensus than *RR(5)-RR(5)*. Moreover, *HMs* have a smaller area for negative consensus than *RR(5)-RR(5)*.

In summary, all the *Hierarchical Model* has more massive *CR* than *Random Regular*

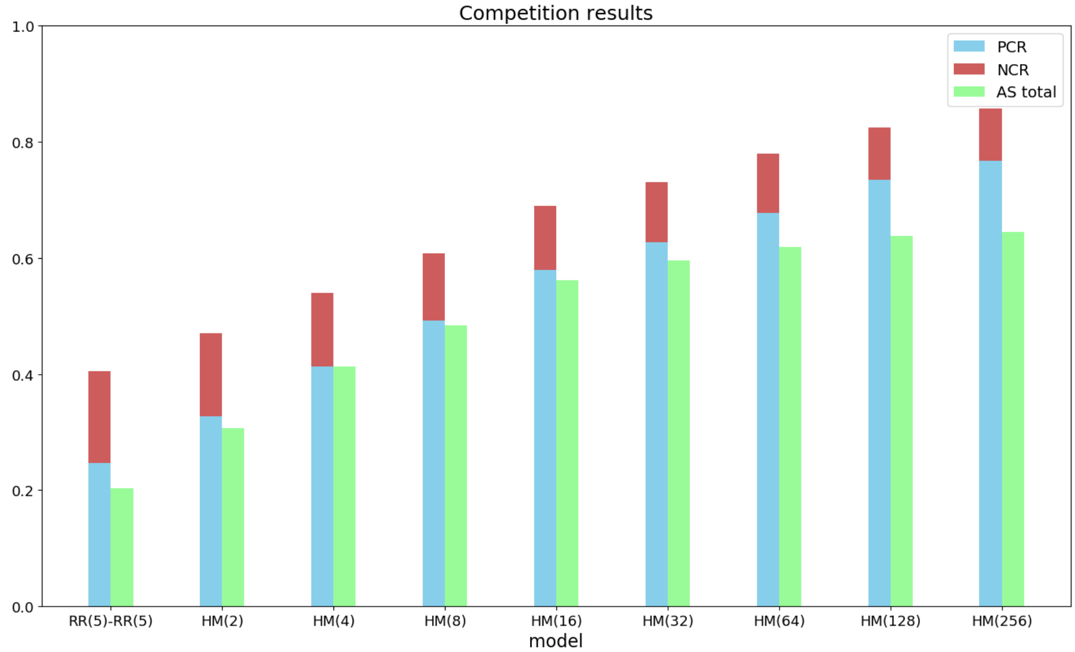


Figure 3–6 Histogram for *PCR, NCR, AS total* of *Hierarchical Models(HM(n))*

Network. However, *PCR* is increased, but *NCR* is decreased. It is found out that as the number of nodes in layer B is decreased as a larger ratio, the network makes it easier to have a positive consensus and harder to have a negative consensus. In the real world, it can be analyzed that as the number of leaders is much smaller, social conflict is decreased, and the opinion is convergent to social opinion(layer A). However, sometimes there are some dangers to ignore the leader opinions(layer B), or to cause more social conflicts when there are stubborn leaders; that case is simulated in the chapter 5.

3.2.2 Competition on Networks with different number of internal links

Next, the interconnected networks are simulated with various internal degrees in order to define and evaluate the influence of internal degrees. The random regular network is applied, and the internal degrees on each node is switched to various numbers, as shown in Fig. 3–7. However, there is no change in external degree, which is fixed to only 1. Here, *RR(n)-RR(m)* represents layer A has a random regular network with n internal edges per node, and layer B has a random regular network with m internal edges per node.

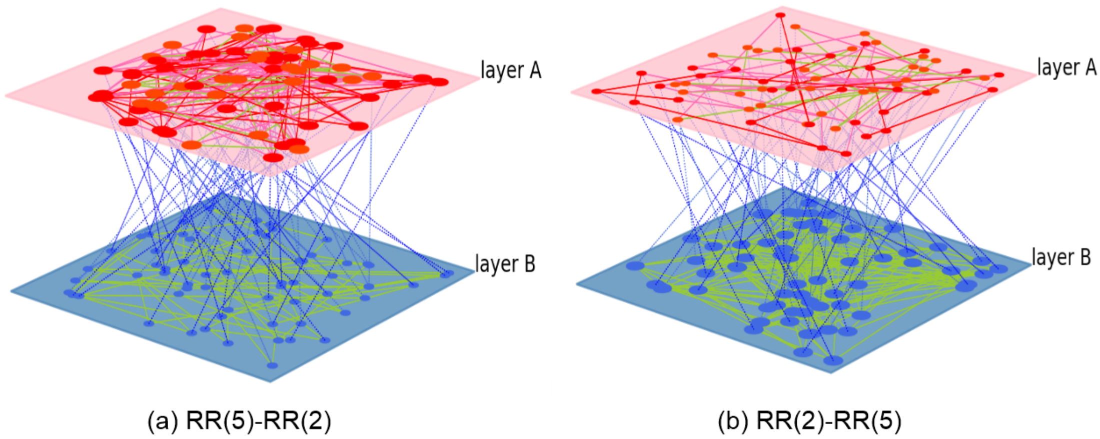


Figure 3–7 Competition on interconnected networks with different internal edges

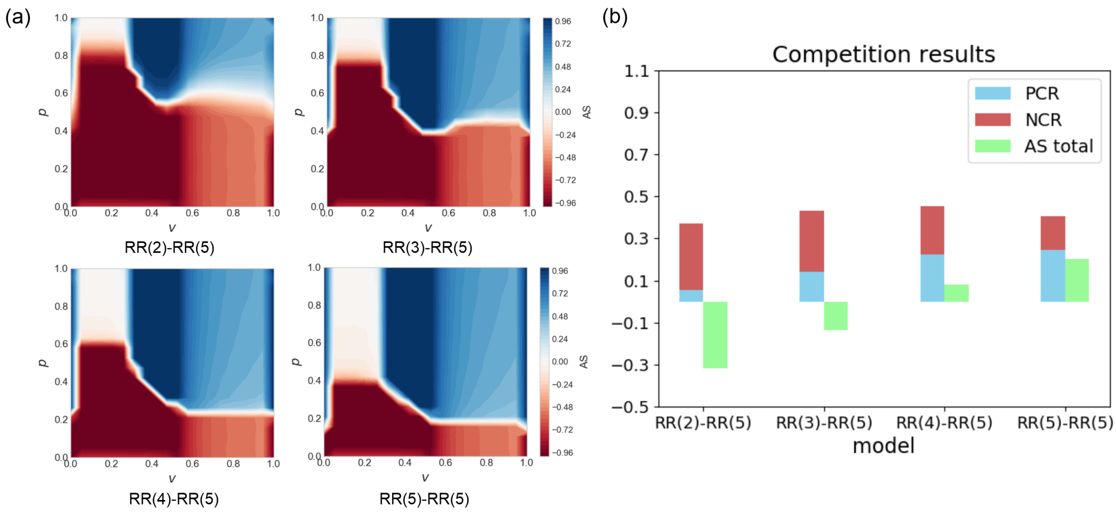


Figure 3–8 Simulation results with different internal degrees on layer A

First, the internal degrees on layer A is changed. The internal degree on layer B is fixed to 5,120, which means each node has 5 internal edges on layer B, and the internal degree on layer A is switched into 2,048, 3,072, 4,096, or 5,120, which means each node has 2, 3, 4, or 5 internal edges on layer A. Fig. 3–8 shows the simulation results according to changing the internal degree on layer A. As shown in Fig. 3–8 (a), as the internal degree on layer A is increased, the red area is decreased, and the blue area is increased. Moreover, the results are presented with the indexes, *PCR, NCR, AS total* in Fig. 3–8 (b), which shows that as the internal degree on layer A is increased, the negative consensus is decreased, and the positive consensus is increased. As shown in Fig. 3–8, RR(5)-RR(5) has the largest *PCR*, and RR(2)-RR(5) has the largest NCR. However, all models in Fig. 3–8 have almost the same *CR*. It can be analyzed that the internal degree on layer A tends to keep a positive state and to change a negative state into a positive state.

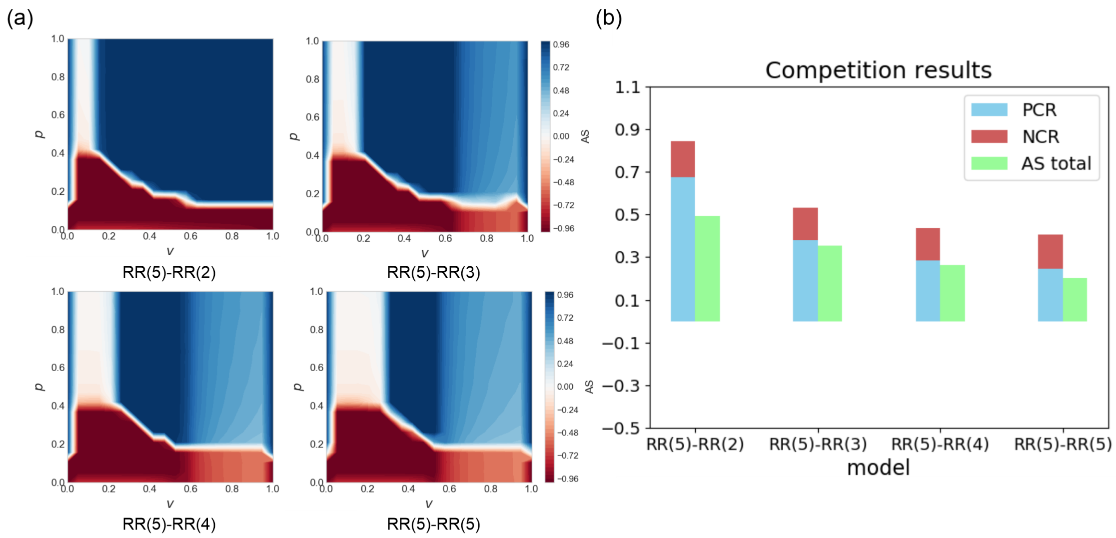


Figure 3–9 Simulation results with different internal degrees on layer B

Next, the internal degree on layer B is switched. The internal degree on layer A is fixed to 5, 120, which means each node has 5 internal edges on layer A, and the internal degree on layer B is switched into 2, 048, 3, 072, 4, 096, or 5, 120, which means each node has 2, 3, 4, or 5 internal edges on layer B. Fig. 3–9 shows the results simulated with changing the internal degree on layer B. As shown in Fig. 3–9 (a), as the internal degree on layer B is increased, the blue area is decreased, the white and light-colored area is increased, and the red area is almost the same, though the shape of the red area is a little changed. As shown in Fig. 3–9 (b), *RR(5)-RR(2)* has the largest *PCR* and *CR*, and *RR(5)-RR(5)* has the smallest PCR and *CR*. However, all models in Fig. 3–9 have almost the same *NCR*. It can be analyzed that the internal degree on layer B has the tendency to hinder the positive consensus state and has an inverse relation with *CR*. As the internal degrees on layer B is increased, *PCR* and *CR* are inversely decreased.

Considering two cases where an internal degree of layer A is changed and where an internal degree of layer B is changed, it is recognized that the role of internal degree on layer A is different with an internal degree on layer B. The internal degree on layer A has the function to keep the state of layer A and the internal degree on layer B has the function to restrain the consensus state of layer A and make a coexistence state.

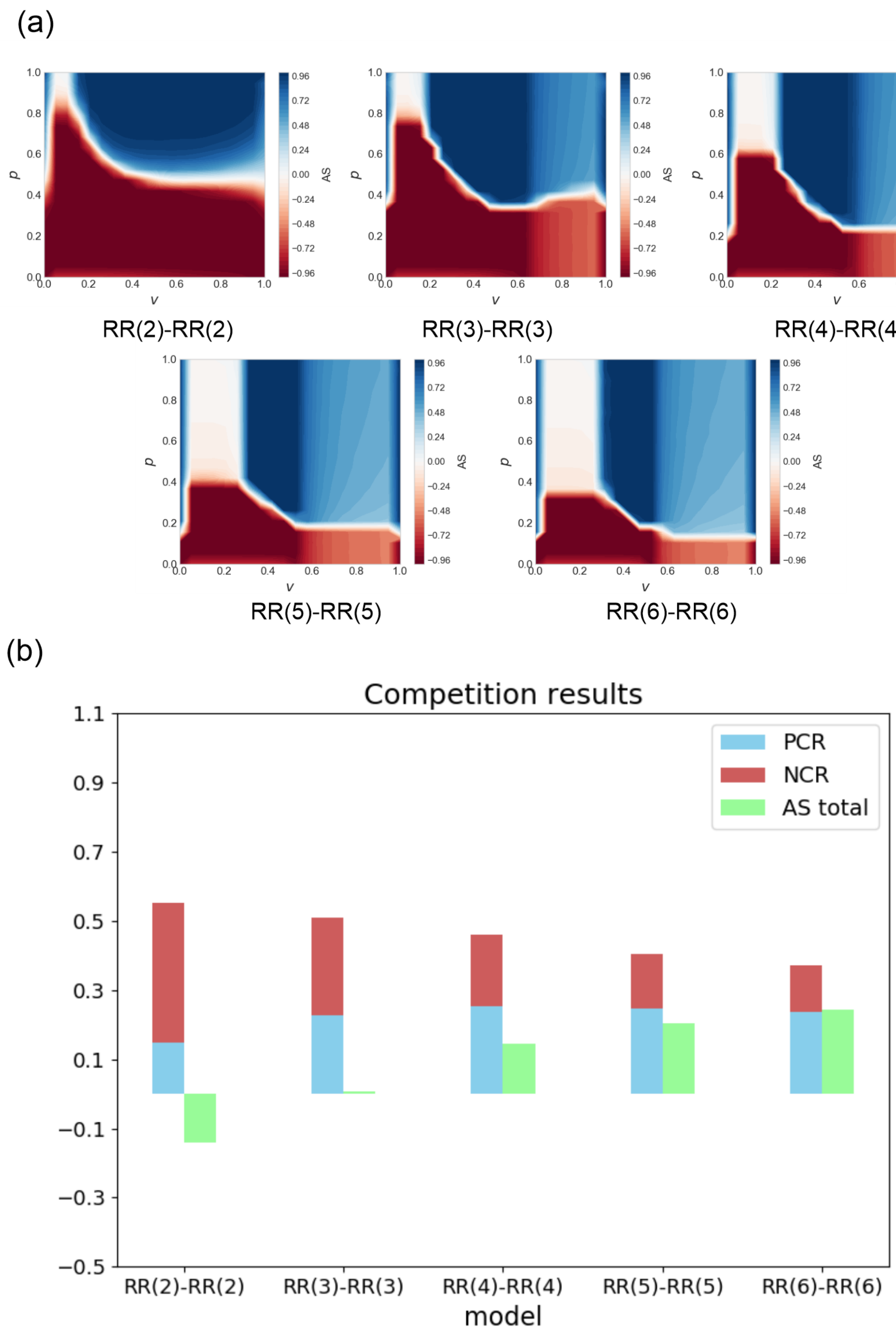


Figure 3–10 Simulation results with changing internal degrees on both layers

Next, it is simulated that internal degrees are changed on both layer A and layer

B, such as *RR(2)-RR(2), RR(3)-RR(3), RR(4)-RR(4), RR(5)-RR(5)* and *RR(6)-RR(6)*. Through these simulations, it is shown how a total internal degree on both layer A and layer B also affects the state of the interconnected network.

Fig. 3–10 shows the influence of a total internal degree on both layers. As the total internal degree is increased, *CR* is inversely decreased, and the ratio of *PCR*(the ratio of the blue bar in a histogram) is increased, but the ratio of *NCR*(the ratio of the red bar in a histogram) is decreased. It can be analyzed that a decrease in *CR* is caused by an increase in internal degree on layer B, and an increase in the ratio of *PCR* is brought out by an increase in internal degree on layer A. But, when the total internal degrees is increased, *PCR, NCR, CR* indexes are decreased. It can be analyzed that a massive internal degree on both layers makes it hard for the state of the network to reach consensus.

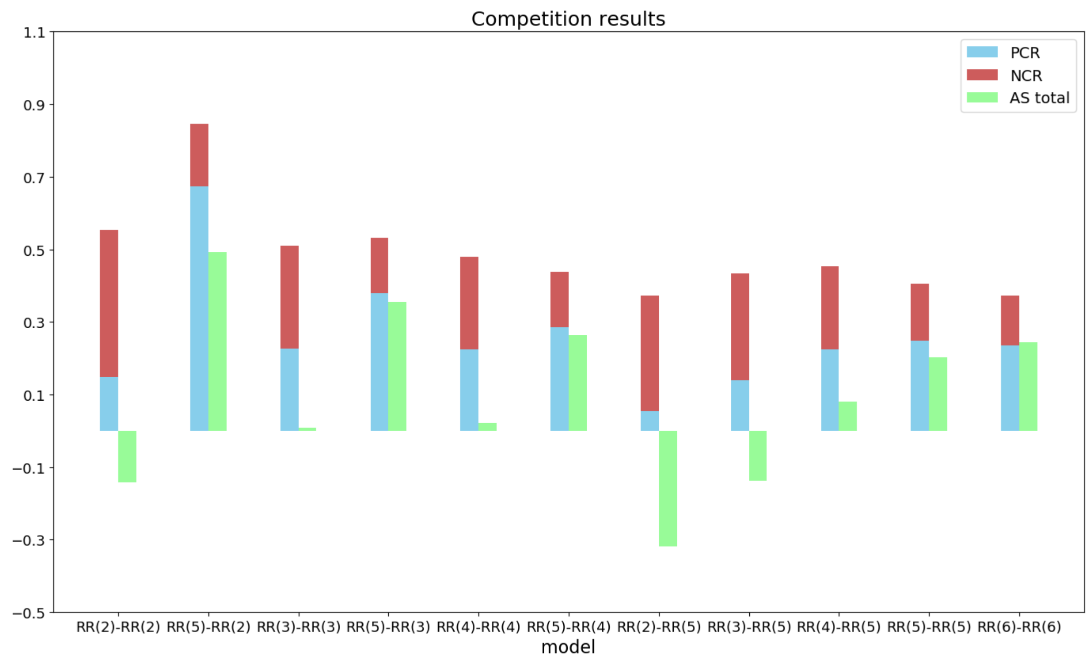


Figure 3–11 Total results with different internal degrees on two layers

In summary, three main simulations have been implemented to find out the influence of internal degree on an interconnected network by changing the internal degree on layer A, changing the internal degree on layer B and changing the internal degrees on both layers. The results are arranged as follows. First, it is found out that internal degrees on layer A tend to keep a positive state and to change a negative state into a positive state. Second, it is shown that the number of internal degrees on layer B has the tendency to hinder the positive consensus state and has an inverse relation with *CR*. Third, a massive internal degree makes it hard for the state of the network to reach consensus.

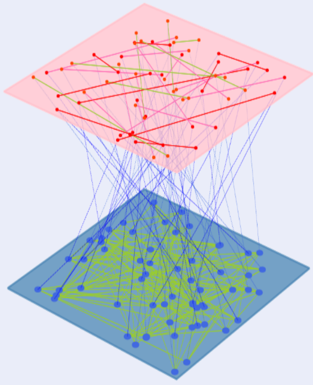
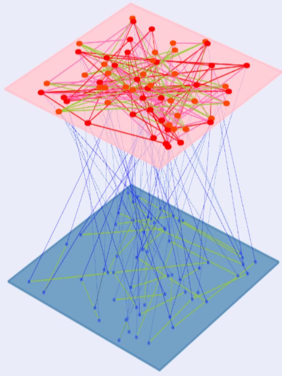
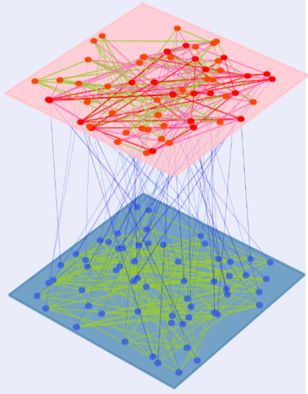
state	negative dominant	positive dominant	coexistence dominant
figure			
internal degrees on layer A	small	large	large
internal degrees on layer B	large or small	small	large

Figure 3–12 Categorizing the state of the network according to internal degrees on two layers

Fig. 3–11 shows the result for all simulations. Through these simulation results, we can analyze how the state of the network is changed according to the internal degrees. Several conclusions can be arranged, as shown in Fig. 3–12. First, it is easy to reach negative consensus(negative dominant) when the internal degrees on layer A is relatively small(the internal degrees on layer B does not matter). Second, it is easy to make positive consensus(positive dominant) when the internal degrees on layer A is relatively large, and the internal degrees on layer B is relatively small. Third, it is easy to make coexistence state(coexistence dominant) when the internal degrees on both layers are too large.

3.2.3 Competition on Networks with different network types

So far, the interconnected networks have been simulated with only *RR(random regular networks)* that have the same number of edges for each node. Now, the simulations are implemented on different network types. Here, we use the *Barabasi-Albert network(BA)* structure, as introduced in [34]. *Barabasi-Albert(BA)* network has N nodes with attaching

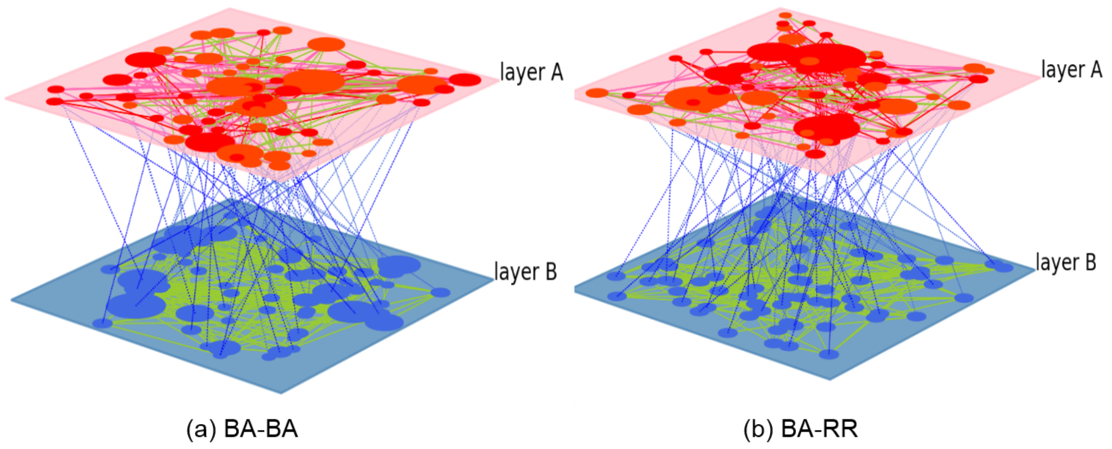


Figure 3–13 Competition on networks with different structures

new nodes, each with K edges that are preferentially attached to existing nodes with high degrees. However, there is no change in the external degree, which is fixed to only 1.

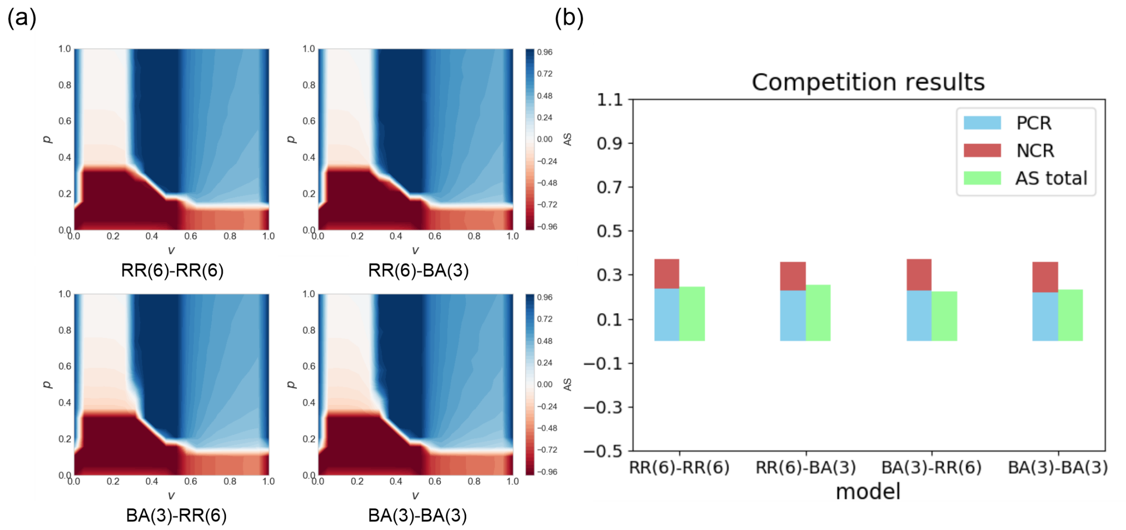


Figure 3–14 Simulation results with different network types

Four simulations are implemented with switching network structures. The *BA* or *RR* network is applied for both layers or switched on each layer. In order to restrain the influence of an internal degree, the number of internal edges in *BA* is set up to be similar to the number of internal edges in *RR*. So, simulations are implemented with $K = 6$ on the *RR* network and $K = 3$ on the *BA* network. The number of internal edges in the *BA* is $6,135$, and the number of internal edges in the *RR* is $6,144$.

The simulation results are shown in Fig. 3–14. The results of all simulations have

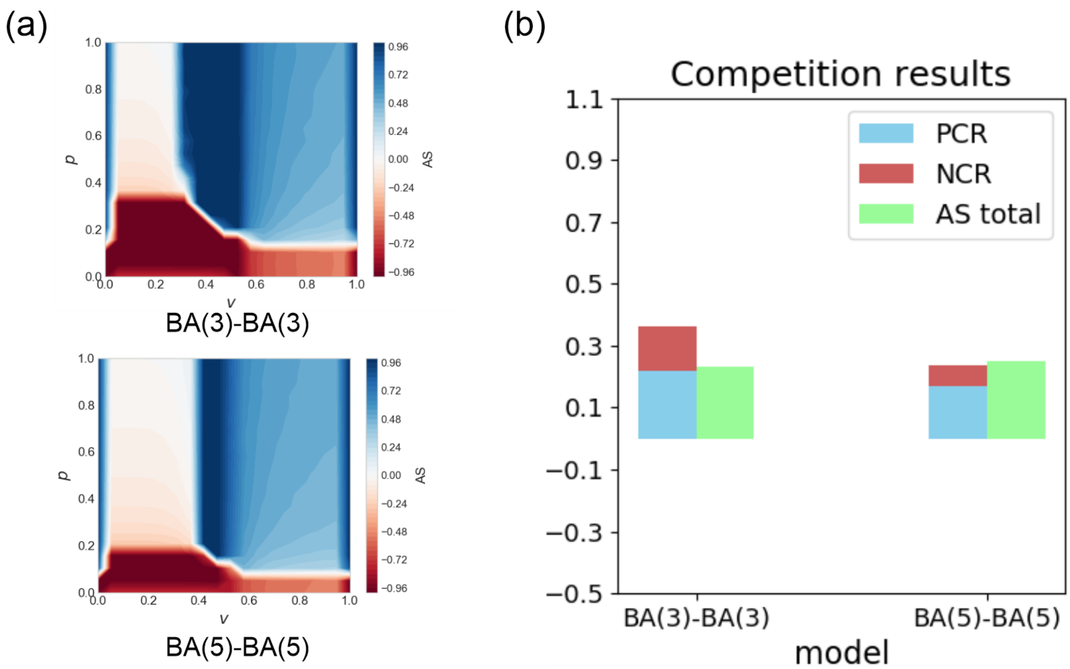


Figure 3–15 Simulation results of BA-BA networks with different internal degrees

almost the same features. The *PCR, NCR,* and *CR* gaps between simulation results are less than 0.02 individually. The structure of the network makes no noticeable difference in consensus results.

Next, the number of internal edges is increased on the network, where it consists of two *BA* networks. It is found out how the number of internal edges works on the different network types with the *RR* network. Two models, *BA(3)-BA(3)* and *BA(5)-BA(5)*, are simulated. *BA(3)-BA(3)* model has 6, 135 internal edges on each layer, and *BA(5)-BA(5)* model has 10, 215 internal edges on each layer.

As shown in Fig. 3–15, *BA(5)-BA(5)* has a larger coexistence area than *BA(3)-BA(3)* because of too many internal edges. It is shown that the influence of internal degree is more important for changing the state of the network and making consensus than the influence of network type. However, if there are stubborn nodes, which are nodes whose states are fixed during the evolution of opinion, on the networks, the simulation results are different because the centralities of stubborn nodes are changed according to network types. Key nodes selection by using stubborn nodes is simulated and analyzed in the chapter 5.

3.3 Conclusion

Various simulations are simulated to find out the role of internal and external degrees and the influence of network types. All results of the simulations are shown in Table 3–1.

Table 3–1 Consensus properties of simulation models

Div	A nodes	B nodes	A edges	B edges	AS total	PCR	NCR	CR
RR(2)-RR(5)	2,048	2,048	2,048	5,120	-0.3186	0.0550	0.3175	0.3725
RR(3)-RR(5)	2,048	2,048	3,072	5,120	-0.1368	0.1400	0.2925	0.4325
RR(4)-RR(5)	2,048	2,048	4,096	5,120	0.0804	0.2250	0.2275	0.4525
RR(5)-RR(2)	2,048	2,048	5,120	2,048	0.4927	0.6725	0.1725	0.8450
RR(5)-RR(3)	2,048	2,048	5,120	3,072	0.3555	0.3800	0.1525	0.5325
RR(5)-RR(4)	2,048	2,048	5,120	4,096	0.2633	0.2850	0.1525	0.4375
RR(2)-RR(2)	2,048	2,048	2,048	2,048	-0.1412	0.1475	0.4050	0.5525
RR(3)-RR(3)	2,048	2,048	3,072	3,072	0.0084	0.2275	0.2825	0.5100
RR(4)-RR(4)	2,048	2,048	4,096	4,096	0.1448	0.2525	0.2075	0.4600
RR(5)-RR(5)	2,048	2,048	5,120	5,120	0.2034	0.2475	0.1575	0.4050
RR(6)-RR(6)	2,048	2,048	6,144	6,144	0.2444	0.2350	0.1375	0.3725
RR(6)-BA(3)	2,048	2,048	6,144	6,135	0.2541	0.2275	0.1300	0.3575
BA(3)-RR(6)	2,048	2,048	6,135	6,144	0.2242	0.2300	0.1425	0.3725
BA(3)-BA(3)	2,048	2,048	6,135	6,135	0.2329	0.2200	0.1400	0.3600
BA(5)-BA(5)	2,048	2,048	10,215	10,215	0.2496	0.1675	0.0675	0.2350
HM(2)	2,048	1,024	5,120	2,560	0.3073	0.3275	0.1425	0.4700
HM(4)	2,048	512	5,120	1,280	0.4128	0.4125	0.1275	0.5400
HM(8)	2,048	256	5,120	640	0.4846	0.4925	0.1150	0.6075
HM(16)	2,048	128	5,120	320	0.5610	0.5800	0.1100	0.6900
HM(32)	2,048	64	5,120	160	0.5959	0.6275	0.1025	0.7300
HM(64)	2,048	32	5,120	80	0.6185	0.6775	0.1025	0.7800
HM(128)	2,048	16	5,120	40	0.6379	0.7350	0.0900	0.8250
HM(256)	2,048	8	5,120	20	0.6454	0.7675	0.0900	0.8575

Through the simulation results, several facts could be arranged, like the following.

If there are no stubborn nodes, network types do not make different results for the state of network and consensus. However, we can provide four conclusions about the roles of internal and external degrees. First, *Hierarchical Models* show that it is easy to make consensus on two-layer when the number of external edges in the decision-making layer is more than the opinion layer, and the number of nodes in the decision-making layer is less than the opinion layer. Second, the number of internal edges on layer A tends to keep a positive state and to change a negative state into a positive state. Third, the number of internal edges on layer B tends to hinder the positive consensus state. Fourth, too many internal edges on each layer can cause inner conflict, and that makes it hard to have consensus state. Those facts could be used to make network structures and organizations in the real world.

Chapter 4 Competition with different updating rules

In this chapter, we control dynamics orders and updating rules for nodes and edges. With changing these updating rules, it is investigated how the states of network are changed. Moreover, it is shown how the updating rules are analyzed in the real world, and which updating rule is more influential for changing a state of the network.

4.1 Updating rules

When considering updating rules on two-layer networks, there are many ways to update the state of nodes. Dynamics orders of two layers can be considered whether layer A works first, or layer B works first or both layers work together. Moreover, orders of nodes can be thought as to whether the states of nodes are changed simultaneously or sequentially or randomly. Orders of edges connected with a node also can be deliberated as to whether edges are activated on a node sequentially or simultaneously or randomly. However, in layer B dynamics, the order of edges in one node always follows the simultaneous updating rule, because dynamics formula already considers the states of all connected neighbor nodes simultaneously. To sum up, as shown in Table 4–1, 25 updating rules are considered according to layers, nodes, and edges.

Updating rules are indicated as follows. 'O' and 'D' represent 'Opinion layer' and 'Decision Making layer' individually. 'o' and 's' indicate sequential updating rule and simultaneous updating rule individually. Moreover, the arrow direction indicates the order of layers. In the table remarks, 'O(o, o) → D(s)' represents 'Opinion layer(nodes: sequential order updating, edges: sequential order updating) → Decision-Making layer(node: simultaneous updating)', which means according to the arrow direction, all nodes in opinion layer are updated with the order of nodes and edges, and then all nodes in the decision-making layer are updated with the order of nodes. 'O(o, o) ⇔ D(o)' means that one node in opinion layer is updated, and then one node in the decision-making layer is updated until all nodes are updated. Dynamics with 25 updating rules are simulated with the parameters such as $p = 0.4$ and $v = 0.4$.

<table>
<tr><th rowspan="2">Order of layers</th><th colspan="2">Layer A</th><th>Layer B</th><th rowspan="2">remarks</th></tr>
<tr><th>Order of nodes</th><th>Order of edges</th><th>Order of nodes</th></tr>
<tr><td rowspan="10">Layer A → Layer B</td><td rowspan="4">Sequential</td><td rowspan="2">Sequential</td><td>Sequential</td><td>O(o, o) → D(o)</td></tr>
<tr><td>Simultaneous</td><td>O(o, o) → D(s)</td></tr>
<tr><td rowspan="2">Simultaneous</td><td>Sequential</td><td>O(o, s) → D(o)</td></tr>
<tr><td>Simultaneous</td><td>O(o, s) → D(s)</td></tr>
<tr><td rowspan="4">Simultaneous</td><td rowspan="2">Sequential</td><td>Sequential</td><td>O(s, o) → D(o)</td></tr>
<tr><td>Simultaneous</td><td>O(s, o) → D(s)</td></tr>
<tr><td rowspan="2">Simultaneous</td><td>Sequential</td><td>O(s, s) → D(o)</td></tr>
<tr><td>Simultaneous</td><td>O(s, s) → D(s)</td></tr>
<tr><td rowspan="2">Random</td><td rowspan="2">Random</td><td>Sequential</td><td>O(r, r) → D(o)</td></tr>
<tr><td>Simultaneous</td><td>O(r, r) → D(s)</td></tr>
<tr><td rowspan="10">Layer A ← Layer B</td><td rowspan="4">Sequential</td><td rowspan="2">Sequential</td><td>Sequential</td><td>O(o, o) ← D(o)</td></tr>
<tr><td>Simultaneous</td><td>O(o, o) ← D(s)</td></tr>
<tr><td rowspan="2">Simultaneous</td><td>Sequential</td><td>O(o, s) ← D(o)</td></tr>
<tr><td>Simultaneous</td><td>O(o, s) ← D(s)</td></tr>
<tr><td rowspan="4">Simultaneous</td><td rowspan="2">Sequential</td><td>Sequential</td><td>O(s, o) ← D(o)</td></tr>
<tr><td>Simultaneous</td><td>O(s, o) ← D(s)</td></tr>
<tr><td rowspan="2">Simultaneous</td><td>Sequential</td><td>O(s, s) ← D(o)</td></tr>
<tr><td>Simultaneous</td><td>O(s, s) ← D(s)</td></tr>
<tr><td rowspan="2">Random</td><td rowspan="2">Random</td><td>Sequential</td><td>O(r, r) ← D(o)</td></tr>
<tr><td>Simultaneous</td><td>O(r, r) ← D(s)</td></tr>
<tr><td rowspan="2">Layer A ↔ Layer B</td><td rowspan="2">Simultaneous</td><td>Sequential</td><td>Simultaneous</td><td>O(s, o) ↔ D(s)</td></tr>
<tr><td>Simultaneous</td><td>Simultaneous</td><td>O(s, s) ↔ D(s)</td></tr>
<tr><td rowspan="3">Layer A ⇔ Layer B</td><td rowspan="2">Sequential</td><td>Sequential</td><td>Sequential</td><td>O(o, o) ⇔ D(o)</td></tr>
<tr><td>Simultaneous</td><td>Sequential</td><td>O(o, s) ⇔ D(o)</td></tr>
<tr><td>Random</td><td>Random</td><td>Random</td><td>O(r, r) ⇔ D(r)</td></tr>
</table>

Table 4–1 25 updating rules according to the order of layers, nodes, and edges

4.2 Competition results

As the conditions for simulations, each layer consists of *Barabasi-Albert(BA)* network that has N nodes with attaching new nodes, each with K edges that are preferentially attached to existing nodes with a high degree as introduced in [34]. Each node of one layer is connected with a random node on the other layer. That means each node has only one external edge. Simulations are performed on the network with $N = 2048$ and $K = 3$. Simulation results are divided by the order of layers, nodes, and edges.

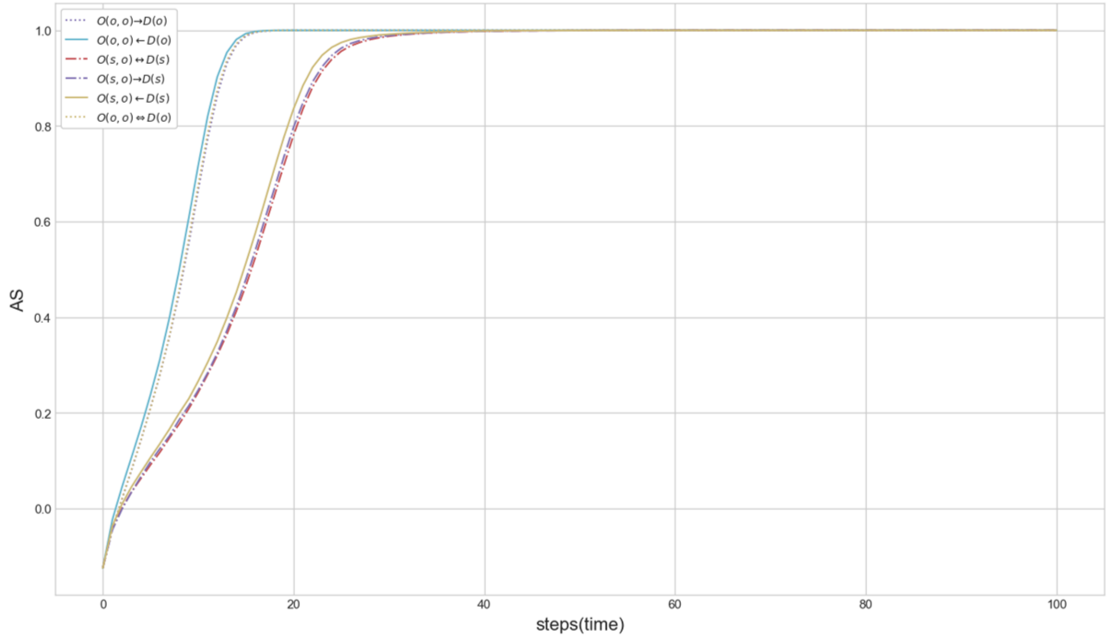
4.2.1 Order of layers

There exist two layers on the interconnected network. Moreover, each layer has its dynamics, such as *M-Model* and *AS-Model*. Two dynamics can be operated simultaneously or sequentially. If two layers act sequentially, the dynamics of layer A can act first, or the dynamics of layer B can work previously. If two layers are operated simultaneously, the order of nodes becomes the simultaneous updating rule automatically because the states of nodes are also changed according to the dynamics of layers. Otherwise, regardless of layers' order, nodes of two layers can interact mutually, i.e., one node in layer A is updated, and then one node in layer B is updated until all nodes are updated. In this case, the order of nodes becomes the sequential updating rule automatically.

There are four ways in order of two layers, *Layer A* $\longrightarrow$ *Layer B(sequential), Layer A* $\longleftarrow$ *Layer B(sequential), Layer A* $\leftrightarrow$ *Layer B(simultaneous), Layer A* $\Leftrightarrow$ *Layer B(interaction regardless of layers' order)*.

Fig. 4–1 shows simulation results related to orders of layers. The graph line indicates *AS* value per each step. If the line reaches to 1 or -1, that means the state of the network has a positive or negative consensus state.

As seen in Fig. 4–1, it is shown that there is little difference according to orders of layers, but there is a significant difference according to nodes' order. The order of nodes is described in the next subsection 4.2.2. Though consensus time is a little faster when the decision-making layer works first, two orders of layers have almost the same consensus time and result. Regardless of dynamics orders, when other conditions such as updating rules of nodes and edges are the same, the results of the dynamics are also very similar. It is observed that the dynamics order of layers does not have a significant influence on the state of the network.

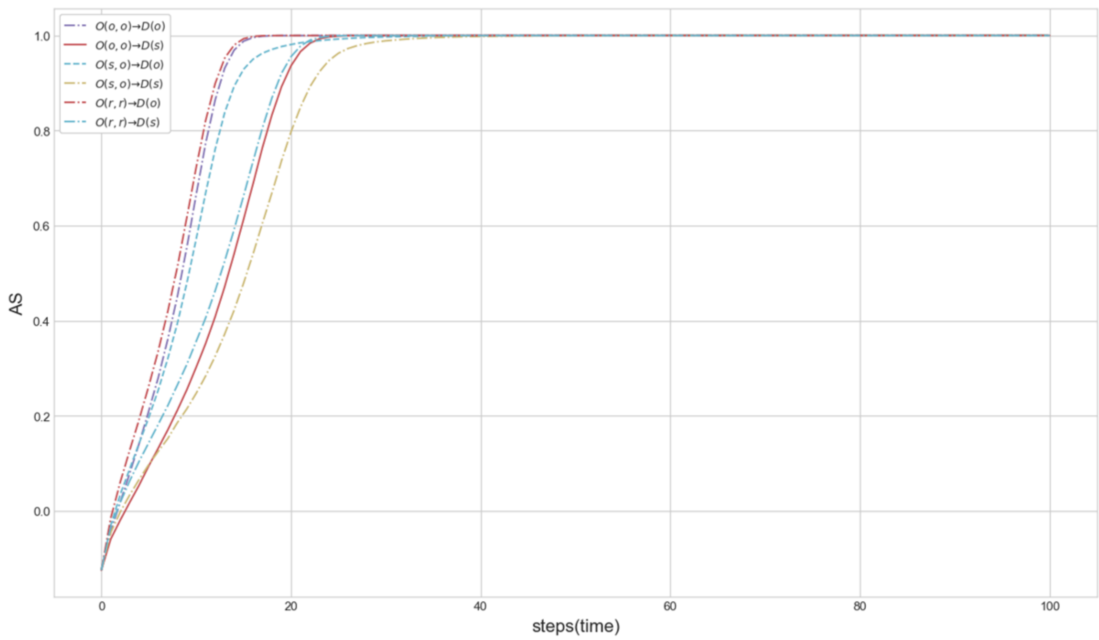


Div	Fast Consensus	Slow Consensus
Orders	① O(o, o) ← D(o) ② O(o, o) ⇔ D(o) ③ O(o, o) → D(o)	④ O(s, o) ← D(s) ⑤ O(s, o) → D(s) ⑥ O(s, o) ↔ D(s)

Figure 4–1 Simulation results according to orders of layers: Comparison between orders of layers under the same conditions, such as orders of nodes and edges.

4.2.2 Order of nodes

In the simulation models, each layer has 2048 nodes, and each node has interactions with other nodes. Now, the interaction order of nodes is considered. One node can be updated sequentially after neighbor nodes are updated. Otherwise, every node can be updated simultaneously. Moreover, nodes also can be updated randomly. As the method of random order, one edge is selected randomly and updated until all edges are considered regardless of nodes' orders or edges' orders.(For layer B, random order can not be applied because it has the formula that all edges of a node are considered together) Therefore, simulations are implemented according to three orders of nodes, such as sequential order, simultaneous order, and random order. The interaction orders of nodes can be analyzed as communication methods in the real world. If networks follow a sequential updating rule of nodes, communication methods of networks might be translated as discussion or conversation with enough time. However, if networks follow simultaneous updating rules of nodes, communication methods of networks might be analyzed as vote or election.



Div	Fast Consensus	Slow Consensus
Orders	① O(r, r) → D(o) ② O(o, o) → D(o) ③ O(s, o) → D(o)	④ O(r, r) → D(s) ⑤ O(o, o) → D(s) ⑥ O(s, o) → D(s)

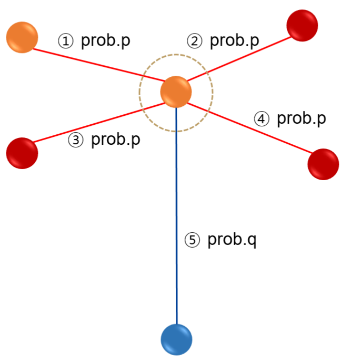
Figure 4–2 Simulation results according to orders of nodes: Comparison between orders of nodes under the same conditions, such as orders of layers and edges.

Fig. 4–2 shows simulation results according to the interaction orders of nodes. The results are classified into two categories, fast consensus and slow consensus. It is shown that simultaneous interaction between nodes makes a slow consensus. Simultaneous updating rule of nodes in layer A does not make a significant difference with other updating rules of nodes in layer A, but it makes consensus slightly slow. However, simultaneous interaction between nodes in layer B makes consensus much slower than layer A. Random order has similar results with sequential order and does not make different states.

In conclusion, it is found out that the simultaneous order of nodes makes a slow consensus, and the sequential order of nodes makes a fast consensus. Also, interaction order of nodes in layer B has more influence on consensus time than in layer A. To make quick social consensus, both opinion layer and decision-making layer need sequential updating rule of nodes, such as conversation and discussion.

4.2.3 Order of edges

Each node has several edges connected with other nodes. Updating rules also can be divided according to whether edges are activated sequentially or simultaneously. If the edges work sequentially, a state of the node is changed whenever each edge is activated. Otherwise, if edges of a node are activated simultaneously, a state of the node is changed considering all connected nodes. In the real world, the order of edges in one node can be analyzed as characteristics of nodes. If the order of edges is sequential, the node is analyzed as 'rash' because a state of the node is changed whenever edges are activated. If the order of edges is simultaneous, the node is analyzed as 'considerate' because it considers all connected nodes together.



Order of edges	Next state
Sequential	Hard to calculate (According to order of activated edges, final state is changed)
Simultaneous	Unchanged: prob.p(activated) = prob.q(activated) Persuasion : prob.p(activated) > prob.q(activated) Compromise: prob.p(activated) < prob.q(activated)

Figure 4–3 Order of edges: One node connected with other nodes is updated according to the sequential or simultaneous order of edges

For example, considering the case that one node is connected with other nodes, as shown in Fig. 4–3, we can think about how a state of the node is changed according to edges' orders. If the edges follow the sequential updating rule, it is hard to calculate the probabilities because a state of the node is continuously changed according to sequential edges' order. Therefore, the next states of nodes are found out by using computer simulation.

If the edges follow the simultaneous updating rule, it needs some assumptions as follows:

1. If the number of activated *prob.p* is more than the number of activated *prob.q*, persuasion dynamics works.
2. If the number of activated *prob.p* is the same as the number of activated *prob.q*, the state is unchanged.
3. If the number of activated *prob.p* is less than the number of activated *prob.q*, compromise dynamics works.

Through these assumptions, we can calculate probabilities for changing a state of the node by considering all cases as these formulas.

$$K = \{k \quad | \quad 0, \cdots, n^{-S_i}\}, \quad L = \{l \quad | \quad 0, \cdots, n^{S_i}\}, \quad M = \{m \quad | \quad k - l\},$$
$$P_A(S_i \mapsto S'_i) = \begin{cases} \text{unchanged}(k = l) : \sum p^{n^{-S_i}+m} \cdot (1-p)^{n^{S_i}-m} \cdot {}_{n^{S_i}}C_k \cdot {}_{n^{-S_i}}C_l \\ \text{compromise}(k > l) : \sum p^{n^{-S_i}+m} \cdot (1-p)^{n^{S_i}-m} \cdot {}_{n^{S_i}}C_k \cdot {}_{n^{-S_i}}C_l \\ \text{persuasion}(k < l) : \sum p^{n^{-S_i}+m} \cdot (1-p)^{n^{S_i}-m} \cdot {}_{n^{S_i}}C_k \cdot {}_{n^{-S_i}}C_l \end{cases} \tag{4–1}$$

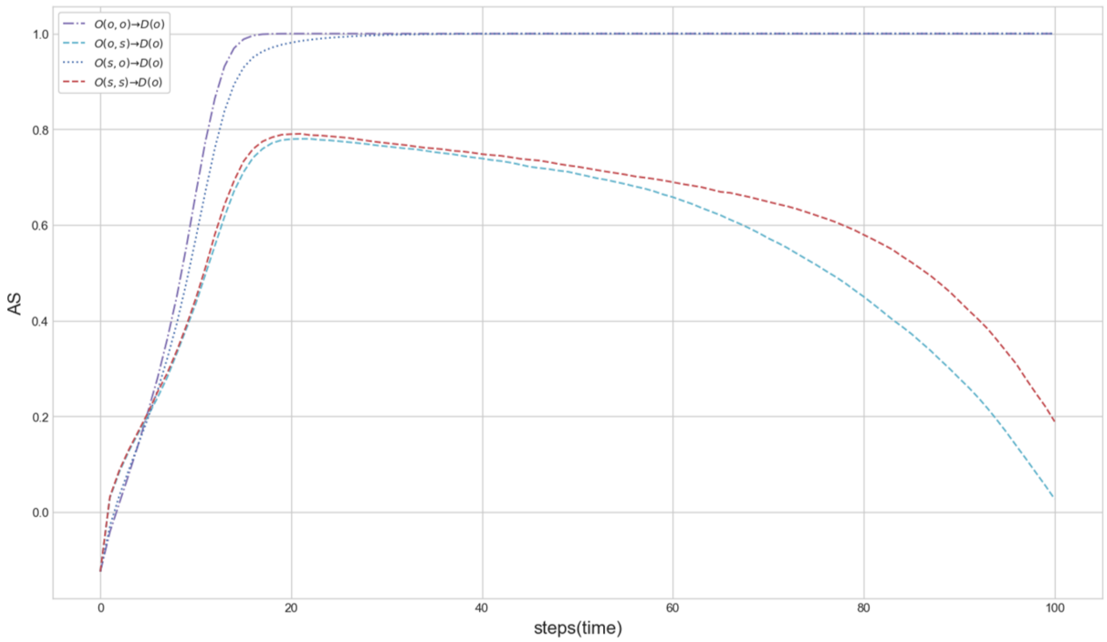
In Eq.(4–1), K means the set of an integer from 0 to the number of nodes with the opposite state(n^{-S_i}}). L means the set of an integer from 0 to the number of nodes with the same state(n^{S_i}}). By using permutations and combinations, these formulas are derived.

Fig. 4–4 shows the simulation result according to edges' orders. The results can be categorized into consensus and coexistence(not reaching consensus). The sequential updating rule of edges makes consensus under the same conditions, such as orders of nodes and layers, i.e., rash nodes make consensus. However, the simultaneous updating rule of edges makes it hard to reach consensus under the same conditions such as orders of nodes and layers, i.e., considerate nodes do not make consensus. It can be analyzed that the rash node is extreme and makes it easy to reach consensus, but the considerate node is moderate and makes it hard to reach consensus.

4.2.4 Comparison and Analysis

It is found out that there are different simulation results according to orders of layers, nodes, and edges. To sum up all updating rules, they can be categorized into three parts, positive consensus, coexistence, and negative consensus, as shown in Fig. 4–5.

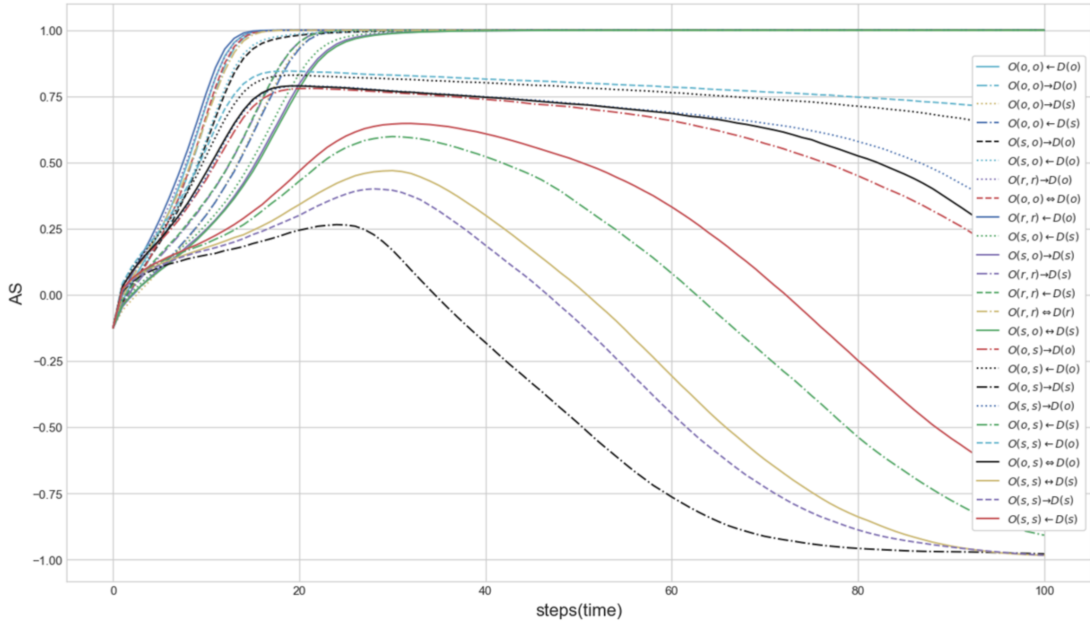
The results can be analyzed by using *CI*, which measures how close the state of



Div	Consensus	Not reaching consensus
Orders	① O(o, o) → D(o) ② O(s, o) → D(o)	③ O(o, s) → D(o) ④ O(s, s) → D(o)

Figure 4–4 Simulation results according to orders of edges: Comparison between orders of edges under the same conditions, such as orders of layers and nodes

the network is to consensus, as shown in Fig. 4–6. In this figure, there are three branch points. In the first branch point, the results are divided according to whether the order of nodes in layer B is sequential or simultaneous. The first branch point makes the results divided into fast opinion convergence and slow opinion convergence. In the second and third branch points, the results are divided according to whether the edges' order in layer A is sequential or simultaneous. The second branch point makes the results divided into consensus and coexistence. The third branch point makes the results divided into positive consensus and negative consensus. To sum up, simulations results are classified into four categories, such as fast positive consensus, slow positive consensus, coexistence, and slow negative consensus. The factors that make branch points have a vital influence on the final state of networks. That means the order of nodes(communication method) in layer B and the order of edges(node characteristics) in layer A have a critical role in determining consensus time and the final state of network individually. It can be analyzed that the communication method on the decision-making layer makes fast or slow



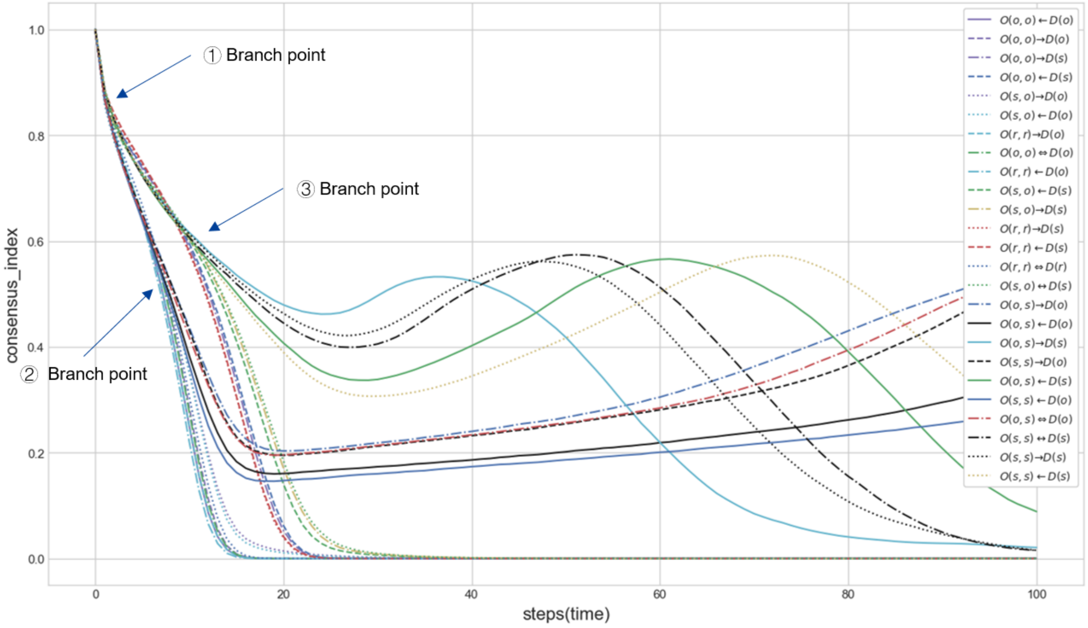
Div	Positive Consensus (close to positive)	Coexistence	Negative Consensus (close to negative)
Orders	O(r, r) ← D(o) O(r, r) → D(o) O(o, o) ← D(o) O(o, o) ⇔ D(o) O(o, o) → D(o) O(r, r) ⇔ D(r) O(r, r) → D(s) O(r, r) ← D(s) O(o, o) → D(s) O(o, o) ← D(s) O(s, o) ← D(o) O(s, o) → D(o) O(s, o) ← D(s) O(s, o) → D(s) O(s, o) ↔ D(s)	O(s, s) ← D(o) O(o, s) ← D(o) O(s, s) → D(o) O(o, s) ⇔ D(o) O(o, s) → D(o)	O(s, s) ← D(s) O(o, s) ← D(s) O(o, s) → D(s) O(s, s) → D(s) O(s, s) ↔ D(s)

Figure 4–5 Total results of 25 updating rules with measuring *AS*

opinion convergence and node characteristics on the opinion layer makes the final state of networks such as positive consensus, negative consensus, and coexistence.

4.3 Conclusion

Through these results, several important facts can be arranged. First, networks with simultaneous updating rules tend to make slow consensus or coexistence, sometimes



<table>
<tr><th colspan="2">Div.</th><th>States</th><th>Orders</th></tr>
<tr><td rowspan="4">① Branch point</td><td rowspan="2">② Branch point :
Sequential order of node in layer B</td><td>Fast positive consensus :
Sequential order of edge</td><td>O(r, r) ← D(o)
O(r, r) → D(o)
O(o, o) ← D(o)
O(o, o) ⇔ D(o)
O(o, o) → D(o)
O(r, r) ⇔ D(r)
O(s, o) ← D(o)
O(s, o) → D(o)</td></tr>
<tr><td>Coexistence :
Simultaneous order of edge</td><td>O(s, s) ← D(o)
O(o, s) ← D(o)
O(s, s) → D(o)
O(o, s) ⇔ D(o)
O(o, s) → D(o)</td></tr>
<tr><td rowspan="2">③ Branch point :
Simultaneous order of node in layer B</td><td>Slow positive consensus :
Sequential order of edge</td><td>O(r, r) → D(s)
O(r, r) ← D(s)
O(o, o) → D(s)
O(o, o) ← D(s)
O(s, o) ← D(s)
O(s, o) → D(s)
O(s, o) ↔ D(s)</td></tr>
<tr><td>Slow negative consensus :
Simultaneous order of edge</td><td>O(s, s) ← D(s)
O(o, s) ← D(s)
O(o, s) → D(s)
O(s, s) → D(s)
O(s, s) ↔ D(s)</td></tr>
</table>

Figure 4–6 Total results of 25 updating rules with measuring *CI*

make the transition to opposite orientation. On the other hand, networks with sequential updating rules tend to make fast consensus. Second, dynamics order between layers does not influence network state, though there exists a tiny consensus time gap. Third, the order of nodes in layer B has more influence on network states than the order of nodes in layer A because the order of nodes in layer B makes the first branch point that has a vital role in making fast or slow opinion convergence. That means the communication method in the decision-making layer is very important for determining consensus time. Fourth, the order of edges in layer A is very influential so that it makes the second and third branch points that determine the final state of the network. It can be analyzed as those characteristics of nodes in layer A, such as 'rash' and 'considerate' has the same orientation consensus or make the transition to coexistence or opposite orientation consensus.

Chapter 5 Influences of key nodes on competition

In this chapter, it is investigated which nodes have the most important influence for keeping or changing their orientation on a two-layer network. There exist many methods to select key nodes, such as Pagerank, degree centrality, eigenvector centrality, betweenness centrality, and closeness centrality. Moreover, in [43, 45], it has been proved that multiple indicators that use the rank difference of several node centralities are useful to identify key nodes and prevent the slow way to identify critical nodes. Based on these methods, such as single-node centrality(single indicators) and combined node centrality(multiple indicators), it is researched which method is the most effective and the most influential for selecting key nodes.

5.1 Method for selecting key nodes

As the initial conditions for selecting key nodes, each layer is made of a *BA* network with 512 nodes, $K = 3$, and 1 external edge. Each simulation takes 100 steps for opinion evolution, and 100 simulations are considered for average results. In order to demonstrate the influence of key nodes, the parameters such as p and v are set to be the opposite consensus state to the initial state of a layer for identifying key nodes. And then the stubborn nodes that do not change their states during the opinion evolution are selected by using methods for selecting key nodes, and the ratio of stubborn nodes is increased until a state of the network is changed into the same consensus state with the initial state of the layer for identifying key nodes. Under these conditions, the most powerful method is the fastest method to reach the same consensus state with the initial state of a layer for selecting key nodes. For example, for selecting key nodes on layer A(positive opinion), the parameters are set to be a negative consensus state. Then, as the stubborn nodes on layer A are selected by node centrality or other methods, and the ratio of the stubborn nodes is increased, a state of the network is gradually changed into a positive state. Inversely for selecting key nodes on layer B(negative opinion), the parameters are set to be a positive consensus state. Then, as the stubborn nodes on layer B are selected by the method for recognizing key nodes, and the ratio of stubborn nodes is increased, a state of the network is gradually changed into a negative state. Here, we find the fastest and most powerful method.

As the method to select stubborn nodes, we use two kinds of indicators, single indicators, and multiple indicators. As single indicators, node centralities are applied, such as Pagerank, degree, eigenvector, closeness, and betweenness. As multiple indicators, combined node centralities that consist of several node centralities are applied.

First, here is the way to select key nodes by using a single node centrality.

1. All nodes are ranked by five node centralities(Pagerank, degree, eigenvector, closeness, betweenness).
2. The nodes are deactivated from high ranked order until the state of network has a significant difference, i.e., the stubborn nodes are selected according to high ranked order, and the ratio of stubborn nodes is increased.
3. The results are compared according to node centralities. When a node centrality makes the state of network reach the fastest to the opposite consensus state with the initial condition or have the most significant change, it is the most powerful method for selecting key nodes.
4. As the ratio of stubborn nodes is increased, the summation of *AS*, which represents the 'Average States' of a network, is calculated on every single indicator. It is recognized that the larger the *AS* value is on layer A, the more influential that indicator is, inversely the smaller the *AS* value is on layer B, the more influential that indicator is.

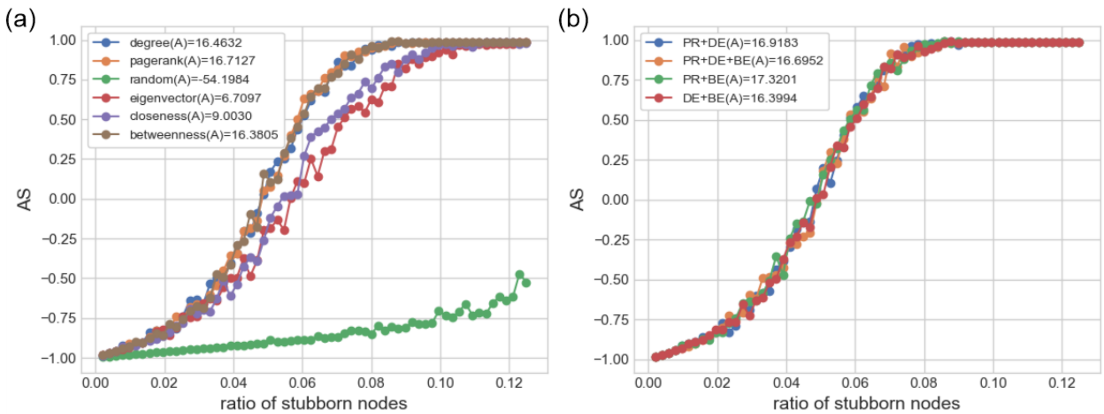
Furthermore, we research the way to recognize critical nodes by using multiple indicators such as combined node centralities(*PR+DE, PR+BE, DE+BE, PR+DE+BE*). Combined node centralities are made up of several selected node centralities. When it is proven that a node centrality is useful for selecting key nodes through the simulations, it is selected as a factor of combined node centrality. Here, 2 or 3 node centralities are selected, such as Pagerank, degree, and betweenness. The way to recognize key nodes by using combined node centrality follows like this steps.

1. Each selected node centrality ranks all nodes. All nodes have the ranks as the number of selected node centralities.
2. Combined node centrality is calculated by the summation of all ranks which a node has.
3. All nodes are ranked again by combined node centrality. The smaller the combined node centrality is, the higher a node is ranked.

4. The nodes are deactivated from high ranked order until the state of network has a significant difference, i.e., the stubborn nodes are selected according to high ranked order, and the ratio of stubborn nodes is increased.

It has been already proven that a single node centrality has good performance to identify key nodes.[13, 39, 40]. However, identifying key nodes by multiple indicators is still an open problem because there are lots of ways to set up and optimize the weight of each node centrality.[45] Here, we simplify the method by setting the weights as equal and calculate the summation of ranks. Although our multiple indicators need to be researched further, the multiple indicators are evaluated and compared with single indicators. The method for measuring and evaluating key nodes on *BA*-*BA* network follows as 5.1.1 and 5.1.2.

5.1.1 Key nodes on layer A



Div(single)	DE	PR	RA	EI	CL	BE
AS	16.4632	16.7127	-54.1984	6.7097	9.0030	16.3805

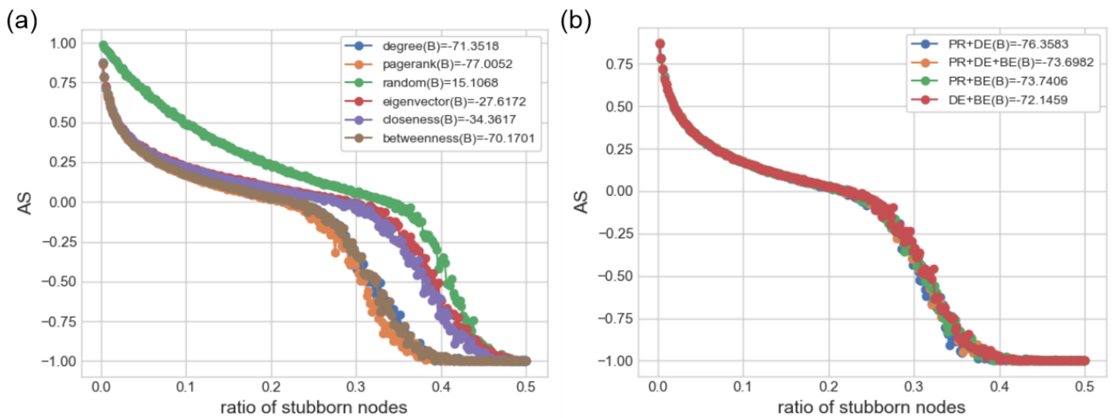
Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	16.9183	16.6952	17.3201	16.3994

Figure 5–1 Key nodes on layer A in *BA(3)-BA(3)* network($p = 0.2, v = 0.4$): (a) Single indicator methods, (b) Multiple indicator methods

To select key nodes on layer A, parameters are set to be negative consensus state like $p = 0.2, v = 0.4$. As single indicators, five node centralities(Pagerank, degree, eigenvector, closeness, betweenness) are used, and the influence of randomly selected nodes is also compared with five node centralities. As multiple indicators, 2 or 3 node centralities are combined, such as Pagerank, degree, and betweenness, which have good performance as single indicators. In combined node centralities, we denote Pagerank, degree, and betweenness as *PR, DE, BE*. Moreover, when they are combined, the methods are denoted as *PR+DE, PR+BE, DE+BE, PR+DE+BE* by using +.

Fig. 5–1 shows the simulation result for recognizing key nodes on layer A. As a single indicator, Pagerank has the best performance. The next ranks are degree and betweenness. As multiple indicators, *PR+BE* has the most effective result. The next is *PR+DE*. These two methods of multiple indicators work better than Pagerank. Compared with all methods, the best method is *PR+BE*. It can be found out that some multiple indicators are more useful for selecting key nodes than single indicators.

5.1.2 Key nodes on layer B



Div(single)	DE	PR	RA	EI	CL	BE
AS	-71.3518	-77.0052	15.1068	-27.6172	-34.3617	-70.1701

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	-76.3583	-73.6982	-73.7406	-72.1459

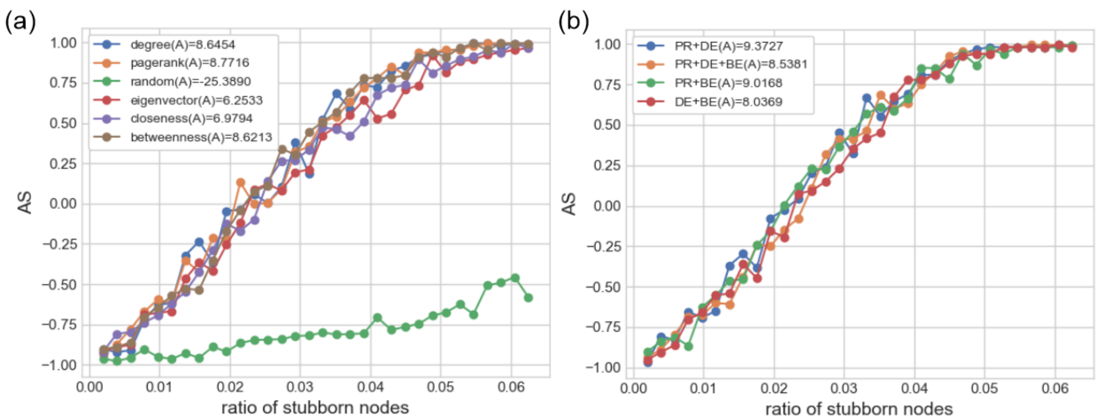
Figure 5–2 Key nodes on layer B in *BA(3)-BA(3)* network($p = 0.3, v = 0.5$): (a) Single indicator methods, (b) Multiple indicator methods

To select key nodes on layer B, parameters are set to be positive consensus state such as $p = 0.3, v = 0.5$. Fig. 5–2 shows the simulation result for identifying key nodes on layer B. As a single indicator, the most effective way to recognize important nodes is Pagerank centrality. The next ranks are degree and betweenness. As multiple indicators, *PR+DE* has the best performance. Pagerank is the most effective method for selecting key nodes on layer B. But, all multiple indicators work better than degree centrality, the second rank in single indicators. It can be found out that combined node centralities also have a good performance for selecting key nodes, though they are not the best.

5.2 Key nodes on two-layer networks with different structures

In this section, we select the key nodes in the networks with various structures that are described in the chapter 3. Node centralities and combined node centralities are also used as the methods for selecting key nodes. First, *Hierarchical Model* is applied to identify critical nodes. Second, we consider the case that each layer has a different network type, such as *BA-RR* or *RR-BA* networks. Third, the case is considered that each layer has a different number of internal edges. Layer A can have more internal links, or layer B can have more internal links. Both cases are simulated.

5.2.1 Key nodes in Hierarchical Model

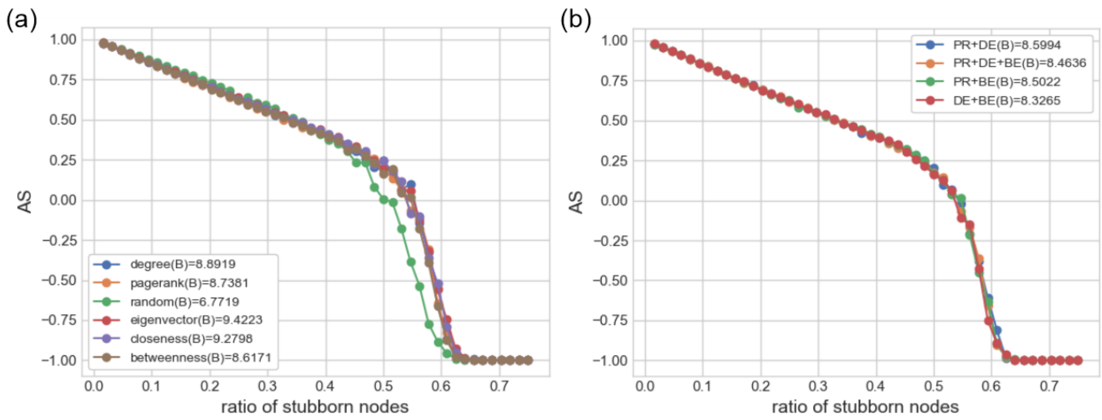


Div(single)	DE	PR	RA	EI	CL	BE
AS	8.6454	8.7716	-25.3890	6.2533	6.9794	8.6213

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	9.3727	8.5381	9.0168	8.0369

Figure 5–3 Key nodes on layer A in *Hierarchical Model(8)*($p = 0.2, v = 0.2$): (a) Single indicator methods, (b) Multiple indicator methods

As described in the chapter 3, *Hierarchical Model* is the two-layer network that the number of nodes in layer B is reduced at a specific rate, and the external links from nodes



Div(single)	DE	PR	RA	EI	CL	BE
AS	8.8919	8.7381	6.7719	9.4223	9.2798	8.6171

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	8.5994	8.4636	8.5022	8.3265

Figure 5–4 Key nodes on layer B in *Hierarchical Model(8)*$(p = 0.25, v = 0.3)$: (a) Single indicator methods, (b) Multiple indicator methods

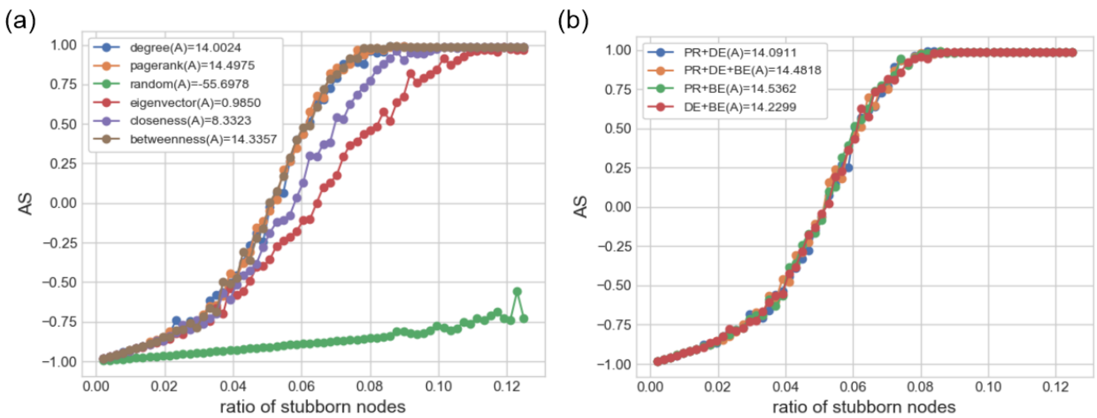
in layer B are increased accordingly. Here, each layer consists of a *BA* network with $k = 3$. Layer A has 512 nodes, and layer B has 64 nodes. We denote this model as *HM(8) with BA(3)*.

Fig. 5–3 shows the simulation result of key nodes on layer A. Simulation result represents that *PR+DE* is the best method for recognizing key nodes on *HM(8) with BA(3)*. The next ranks are *PR+BE* and Pagerank. The curve of changing the network states shown in Fig. 5–3 is more straight than Fig. 5–1. That means the speed of changing network states(consensus time) is much faster.

Fig. 5–4 shows the simulation result of key nodes on layer B. However, the result is different from other simulation results. The best performance method is a random method. That means node centralities do not work on this model. Furthermore, the curve of changing the network states shown in Fig. 5–4 is also more straight than Fig. 5–2, which means the consensus is much easier, and the consensus time is much shorter. It is found out that the *Hierarchical Model* makes it hard to recognize key nodes on layer B and make it easy to reach a consensus of two-layer by key nodes.

5.2.2 Key nodes on the two-layer network with different network types

Here, we consider two types of networks, *BA-RR* and *RR-BA*. The number of internal links on each layer is set up as the same or almost the same number to exclude the influence of internal degrees. These models are compared with the *BA-BA* to find out the influence of network types under the same conditions, such as p, v, and $the ratio of stubborn nodes$.



Div(single)	DE	PR	RA	EI	CL	BE
AS	14.0024	14.4975	-55.6978	0.9850	8.3323	14.3357

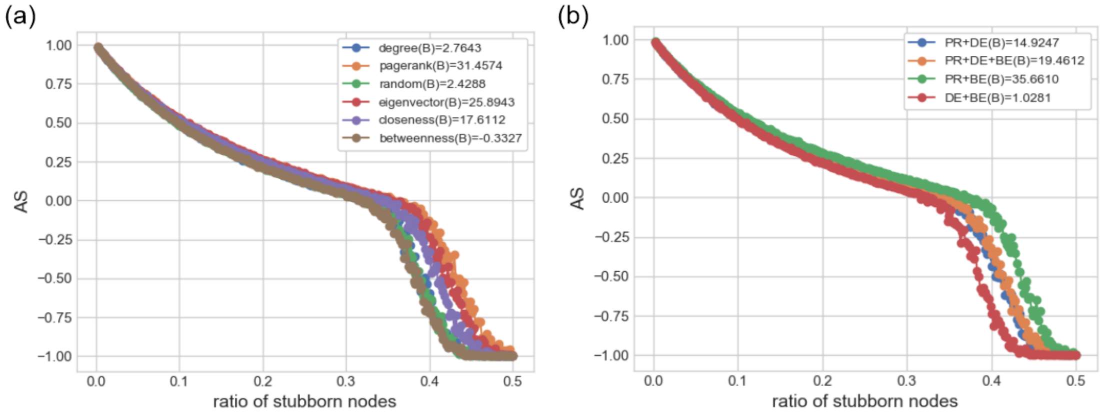
Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	14.0911	14.4818	14.5362	14.2299

Figure 5–5 Key nodes on layer A in *BA(3)-RR(6)* network($p = 0.2, v = 0.4$): (a) Single indicator methods, (b) Multiple indicator methods

First, the *BA-RR* network is investigated. Fig. 5–5 shows the simulation result of key nodes on layer A.*PR+BE* is the most powerful method. The next rank is Pagerank as a single indicator. Compared with the *BA(3)-BA(3)* shown in Fig. 5–1, *BA(3)-RR(6)* has smaller *AS* values and a more gentle curve to change the state of the network.

Fig. 5–6 shows the simulation result of key nodes on layer B. Betweenness is the best method for identifying key nodes on layer B in the *BA-RR* network. In this model, the degree centrality is not an exact method for the selection of key nodes because the degree of each node is the same in the *RR* network. However, random and degree method is the third and fourth method for recognizing key nodes. That means other methods except for betweenness do not work for identifying key nodes. Compared with the *BA(3)-BA(3)* shown in Fig. 5–2, the *BA(3)-RR(6)* has more massive *AS* values and a more gentle curve to change the state of the network.

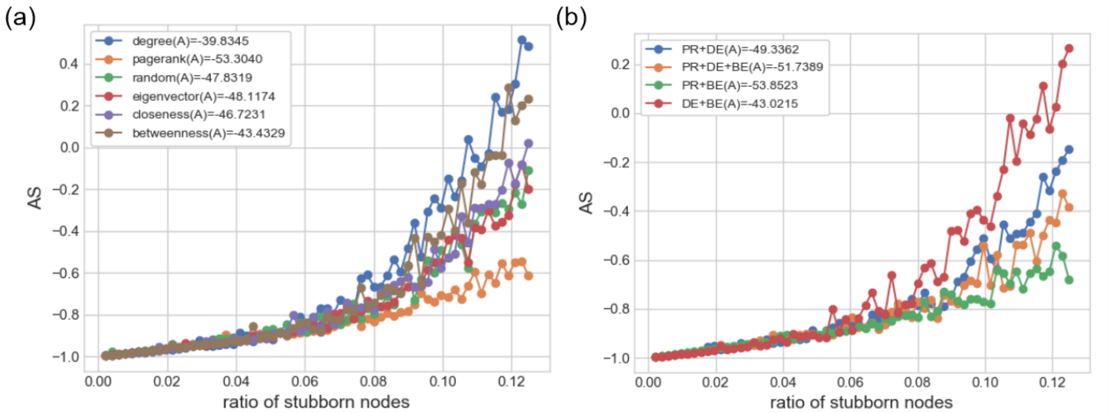
Next, the *RR-BA* network is considered. Fig. 5–7 shows the simulation result of



Div(single)	DE	PR	RA	EI	CL	BE
AS	2.7643	31.4574	2.4288	25.8943	17.6112	-0.3327

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	14.9247	19.4612	35.6610	1.0281

Figure 5–6 Key nodes on layer B in *BA(3)-RR(6)* network($p = 0.3, v = 0.5$): (a) Single indicator methods, (b) Multiple indicator methods

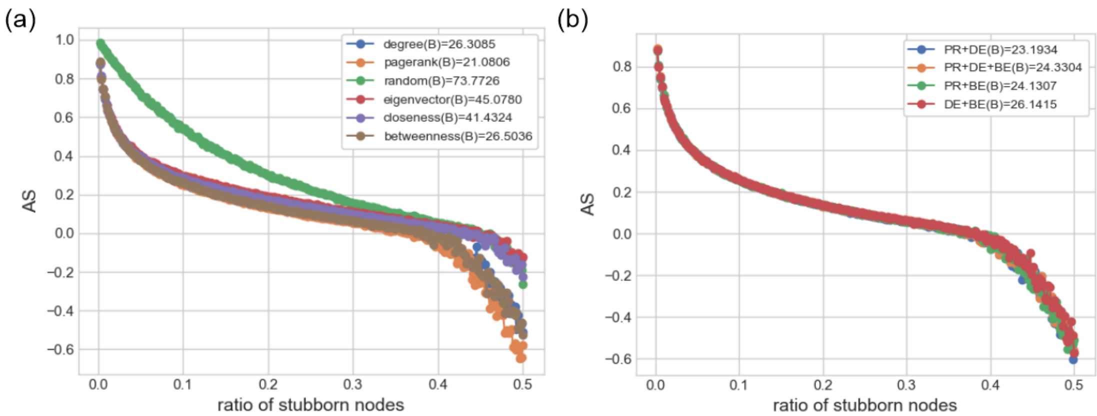


Div(single)	DE	PR	RA	EI	CL	BE
AS	-39.8345	-53.3040	-47.8319	-48.1174	-46.7231	-43.4329

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	-49.3362	-51.7389	-53.8523	-43.0215

Figure 5–7 Key nodes on layer A in *RR(6)-BA(3)* network($p = 0.2, v = 0.4$): (a) Single indicator methods, (b) Multiple indicator methods

key nodes on layer A. The best method is degree centrality. However, in this model, degree centrality is not meant for recognizing key nodes because all nodes in layer A have the same degree. Here, the reason why degree centrality has excellent performance is analyzed as those dynamics are very efficient because nodes are sequentially changed



Div(single)	DE	PR	RA	EI	CL	BE
AS	26.3085	21.0806	73.7726	45.0780	41.4324	26.5036

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	23.1934	24.3304	24.1307	26.1415

Figure 5–8 Key nodes on layer B in *RR(6)-BA(3)* network($p = 0.3, v = 0.5$): (a) Single indicator methods, (b) Multiple indicator methods

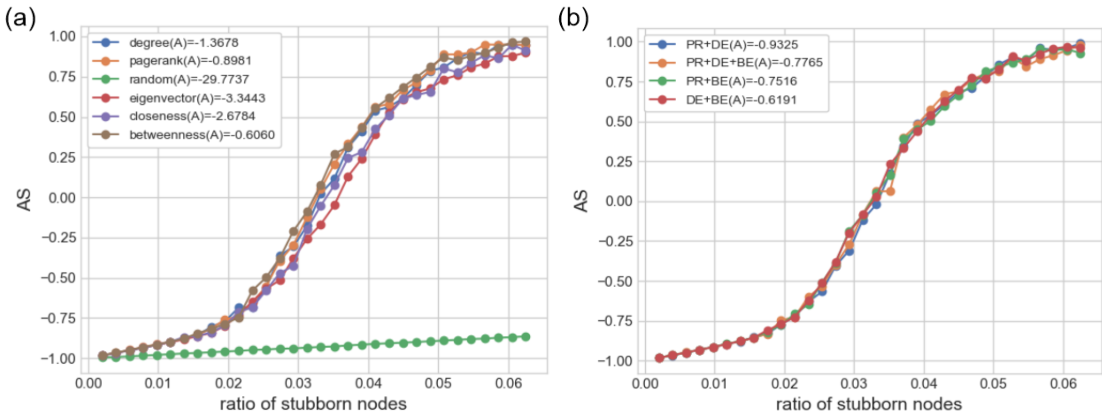
into the stubborn node and interacted(when nodes have the same node centrality, nodes are changed into stubborn nodes sequentially according to interaction order under given algorithm). Moreover, other single indicators have similar *AS* values with the random method. That means node centralities do not work for identifying key nodes though betweenness has better performance than other methods. Compared with the *BA(3)-BA(3)* shown in Fig. 5–1, *RR(6)-BA(3)* has smaller *AS* values and does not reach the opposite consensus yet.

Fig. 5–8 shows the simulation result of key nodes on layer B. Pagerank has the best performance. The next rank is *PR+DE*. Compared with the *BA(3)-BA(3)* shown in Fig. 5–2 , the *RR(6)-BA(3)* has more massive *AS* values and a more gentle curve to change the state of the network.

Compared with the *BA-BA* network, both *BA-RR* and *RR-BA* have a more gentle curve line to change the state of the network. It can be analyzed that the *RR* network makes it slow for crucial nodes to change the state of the network and makes it hard to select critical nodes though betweenness has excellent performance on the *RR* network.

5.2.3 Key nodes on the two-layer network with different number of internal links

Next, the case is considered that each layer has a different number of internal edges. In case that layer A has a more massive number of internal links, layer A consists of a *BA* network with $k = 4$, but layer B consists of a *BA* network with $k = 2$. Inversely, in case that layer B has a more massive number of internal links, layer B consists of a *BA* network with $k = 4$, but layer A consists of a *BA* network with $k = 2$.



Div(single)	DE	PR	RA	EI	CL	BE
AS	-1.3678	-0.8981	-29.7737	-3.3443	-2.6784	-0.6060

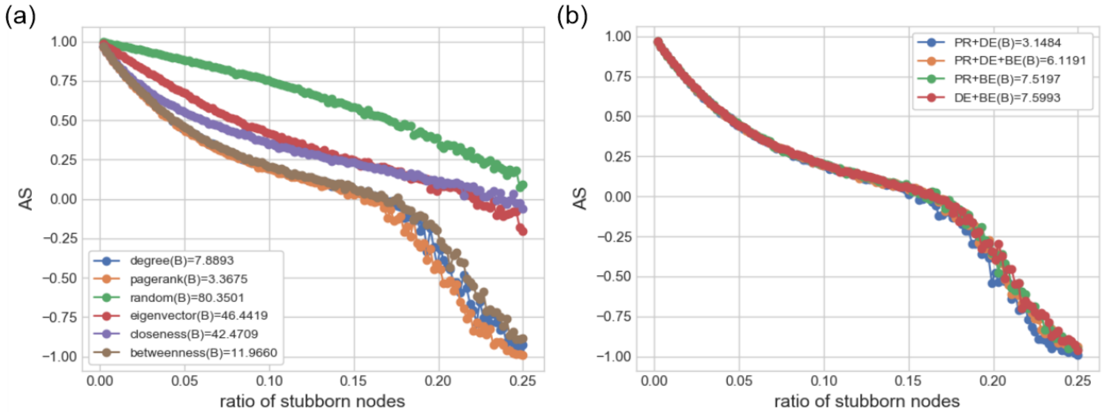
Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	-0.9325	-0.7765	-0.7516	-0.6191

Figure 5–9 Key nodes on layer A in *BA(4)-BA(2)* network($p = 0.15, v = 0.3$): (a) Single indicator methods, (b) Multiple indicator methods

First, the case of more internal links on layer A than layer B is investigated. Fig. 5–9 shows the simulation result of key nodes on layer A in the *BA(4)-BA(2)* network. Betweenness has the best performance for selecting key nodes. The next ranks are *DE+BE*, *PR+BE*, and *PR+DE+BE*. Compared with the *BA(2)-BA(4)* network shown in Fig. 5–11, the curve of changing the state that is shown in Fig. 5–9 is much more straight-line. That means consensus time is short, and it is easy to have consensus.

Fig. 5–10 shows the simulation result of key nodes on layer B in the *BA(4)-BA(2)* network. *PR+DE* is the most powerful method. The next ranks are Pagerank, *PR+DE+BE*, and *PR+BE*. Compared with the *BA(2)-BA(4)* network shown in Fig. 5–12, the curve of changing the state that is shown in Fig. 5–10 is also more straight-line.

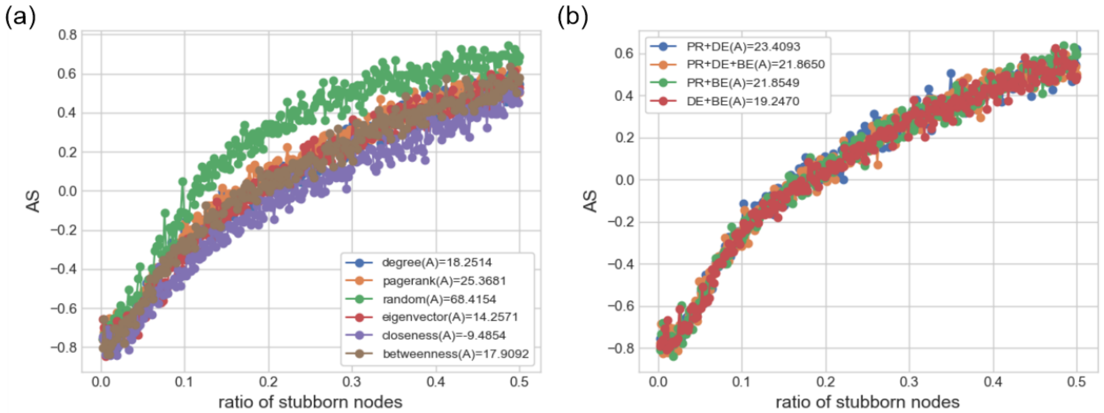
Compared with the *BA(2)-BA(4)* network, it can be analyzed that more internal edges on layer A make it easy to have a consensus by key nodes.



Div(single)	DE	PR	RA	EI	CL	BE
AS	7.8893	3.3675	80.3501	46.4419	42.4709	11.9660

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	3.1484	6.1191	7.5197	7.5993

Figure 5–10 Key nodes on layer B in *BA(4)-BA(2)* network($p = 0.2, v = 0.4$): (a) Single indicator methods, (b) Multiple indicator methods

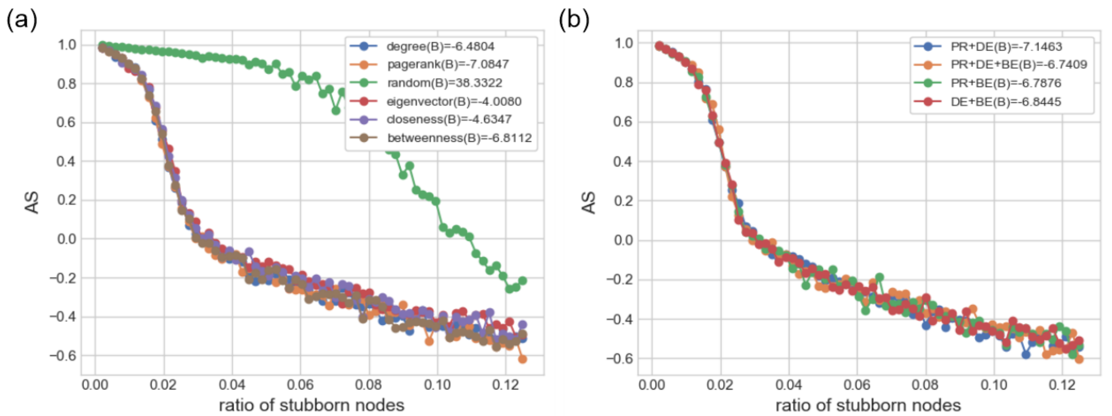


Div(single)	DE	PR	RA	EI	CL	BE
AS	18.2514	25.3681	68.4154	14.2571	-9.4854	17.9092

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	23.4093	21.8650	21.8549	19.2470

Figure 5–11 Key nodes on layer A in *BA(2)-BA(4)* network($p = 0.57, v = 0.37$): (a) Single indicator methods, (b) Multiple indicator methods

Next, the case of more internal links on layer B than layer A is researched. Fig. 5–11 shows the simulation result of key nodes on layer A in the *BA(2)-BA(4)* network. However, the simulation results are different from other results because the random method has the best performance. That means node centralities do not work on this model. Compared



Div(single)	DE	PR	RA	EI	CL	BE
AS	-6.4804	-7.0847	38.3322	-4.0080	-4.6347	-6.8112

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	-7.1463	-6.7409	-6.7876	-6.8445

Figure 5–12 Key nodes on layer B in *BA(2)-BA(4)* network($p = 0.6, v = 0.4$): (a) Single indicator methods, (b) Multiple indicator methods

with the *BA(4)-BA(2)* network shown in Fig. 5–9, the curve of changing the state that is shown in Fig. 5–11 is much slower and more gentle.

Fig. 5–12 shows the simulation result of key nodes on layer B in the *BA(2)-BA(4)* network. *PR+DE* has the most effective performance. The next ranks are Pagerank, *DE+BE*, and betweenness. Compared with the *BA(4)-BA(2)* network shown in Fig. 5–10, the curve of changing the state that is shown in Fig. 5–12 is much faster at the beginning but much slower at the end. Besides, consensus does not happen in this model.

Compared with the *BA(4)-BA(2)* network, it can be analyzed that the larger number of internal edges on layer B makes consensus by key nodes hard. Moreover, decreasing internal edges on layer A makes it hard to select key nodes on layer A.

5.3 Conclusion

By using node centrality and combined node centrality, key nodes on each layer have been recognized on networks with various structures. Table 5–1 shows total simulation results for selecting key nodes on various interconnected networks.

Here, we find several facts from these simulation results. First, it can be found out that the best and most powerful method for select key nodes is different accord-

Table 5–1 Effective method for selecting key nodes on various networks

Div	A nodes	B nodes	A edges	B edges	layer	1st method	2nd method	3rd method	remarks
BA(3)-BA(3)	512	512	1,527	1,527	A	PR+BE	PR+DE	Pagerank	
					B	Pagerank	PR+DE	PR+BE	
BA(3)-RR(6)	512	512	1,527	1,536	A	PR+BE	Pagerank	PR+DE+BE	
					B	betweenness	DE+BE	random	not working
RR(6)-BA(3)	512	512	1,536	1,527	A	degree	DE+BE	betweenness	not working
					B	Pagerank	PR+DE	PR+BE	
BA(4)-BA(2)	512	512	2,032	1,020	A	betweenness	DE+BE	PR+BE	
					B	PR+DE	Pagerank	PR+DE+BE	
BA(2)-BA(4)	512	512	1,020	2,032	A	random	Pagerank	PR+DE	not working
					B	PR+DE	Pagerank	DE+BE	
HM(8) with BA(3)	512	64	1,527	183	A	PR+DE	PR+BE	Pagerank	
					B	random	DE+BE	PR+DE+BE	not working

ing to network structures and layers. Second, as single indicators, Pagerank, degree and betweenness are an excellent method to select key nodes on a two-layer network. Third, as multiple indicators, combined node centralities have an excellent performance to recognize the critical nodes on various networks. Combined node centralities are the first or second effective methods on all simulation models.(except not working methods) Fourth, as the results are shown in interconnected networks with a different number of internal edges on each layer, the larger number of links on layer A makes it easy to have a consensus by key nodes, and the larger number of links on layer B makes it hard to make a consensus by key nodes. Besides, decreasing internal edges on layer A makes it hard to recognize key nodes on layer A. Fifth, as the results are shown in the *HM(8) with BA(3)* network, decreasing the number of nodes on layer B and increasing the number of external edges on layer B make it hard to identify key nodes on layer B and makes it easy to reach consensus by key nodes. Sixth, as the results are shown in interconnected networks with different network types, network types influence whether a network can make consensus by key nodes or not. Notably, it is found out that the *RR* network makes it slow to have a consensus by key nodes and makes it hard to recognize critical nodes.

Chapter 6 Conclusion

In the competition of a two-layer network, by changing network structures, switching updating rules, and selecting key nodes, the features of competition on a two-layer network have been researched.

6.1 Summary

Many simulations have been carried out. In summary, it could be arranged as follows. In the chapter 2, the interconnected network with different dynamics on each layer is introduced to understand the competition on a two-layer network. Also, many indexes are provided to measure how a state of the network is changed and to evaluate the consensus on a two-layer network. Based on this modeling, various simulations have been implemented according to 3 main topics as follows.

- Competition on a two-layer network with various structures
- Competition with different updating rules
- Influences of key nodes on competition

In the chapter 3, we investigated competition dynamics on a two-layer network with various structures. With changing network structures, it is measured and evaluated how the state of the network is changed and whether the networks reach consensus or not. As the method to revise the network structure, three ways were provided, such as changing internal degrees, changing external degrees, and switching network types. First, as the result of changing the internal degrees, it is found out that an internal degree on each layer has a different function. The number of internal degrees on layer A tends to keep a positive state and to change a negative state into a positive state. Moreover, the number of internal degrees on layer B tends to hinder a positive consensus state. Second, as the result of changing the external degrees, *Hierarchical Models* are provided. *Hierarchical Models* show that it is easy to make consensus on both layers when the number of external edges in the decision-making layer is larger than the opinion layer, and the number of nodes in the decision-making layer is smaller than the opinion layer. Third, as the result of switching the network type, there is no noticeable difference in the final state of the network. That means if there are no stubborn nodes, network type does not matter. However, it is found out that the number of internal edges has a more influential role in changing the state of the network than network types.

In the chapter 4, it is researched how the updating rules influence the competition of a two-layer network. Though updating rules are very various, we consider time-related updating rules, such as simultaneous updating rule and sequential updating rule. According to where the updating rules are applied, the simulations of three categories are implemented, such as orders of layers, orders of nodes, and orders of links. Through simulation results, several conclusions are formed. First, dynamics order between layers does not have a significant influence on changing the state of the network. Second, an order of edges in the layer A that can be analyzed as characteristics of nodes such as 'rash' and 'considerate', has a vital influence on determining the final state of the network, such as the same orientation consensus, coexistence, and opposite orientation consensus. Third, an order of nodes in layer B that can be analyzed as a communication method is more influential for changing the state of network than an order of nodes in layer A because it makes opinion convergence slow or fast. That means the communication method in the decision-making layer has a vital role in determining consensus time. Fourth, networks with simultaneous updating rules are easy to make slow consensus and coexistence or to change into the opposite state. Otherwise, networks with sequential updating rules are easy to make fast consensus.

In the chapter 5, it is studied that how the key nodes can be selected on the various two-layer networks. To select key nodes on the various networks, we use single indicators and multiple indicators on various networks described in the chapter 3. Through the simulation results, several conclusions could be arranged as follows. First, the most effective method to identify key nodes is different according to network structures and layers, as shown in Table 5–1. Second, as single indicators, Pagerank, degree, and betweenness work well for selecting key nodes. Third, as multiple indicators, combined node centralities have good results to recognize the key nodes on various interconnected networks. Fourth, the larger number of links on layer A makes it easy to have a consensus by key nodes, and the larger number of links on layer B makes it hard to make a consensus by key nodes. Fifth, as shown in *Hierarchical Models*, decreasing the number of nodes on layer B and increasing the number of external edges on layer B make it hard to identify key nodes on layer B and make it easy to reach consensus by key nodes. Sixth, network types influence whether the network can make consensus by key nodes or not. Notably, it is observed that the *RR* network is harder to reach consensus by key nodes and to recognize key nodes than the *BA* network.

6.2 Discussion

The competition of a two-layer network has been researched and analyzed under various conditions. It has been observed that how network structures influence the consensus of a two-layer network, how the updating rules affect the state of the network, what nodes have more influential for affecting the state of the network, and which method is a more effective way to identify critical nodes. Through these results, the state of a two-layer network might be controlled by managing the number of edges and the method of updating rules. Furthermore, for the best and fastest way to change the state of networks, the critical nodes might be recognized and controlled by using the method to select key nodes. In the real world, we can find out the phenomenon of these competitions, such as election, legislation, adoption of new policies, and making-decision on social conflict issues. These competitions of the real world have similar characteristics with our simulation results. Therefore, based on simulation results, these competition models can be applied to solve social conflicts. As future work, it would be very interesting to make a generalized competition model with various structures and updating rules and to recognize key nodes on the generalized competition model.

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