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COMPETITION OF SOCIAL OPINIONS ON TWO LAYER NETWORKS

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ABSTRACT

Social conflict could be explained with competition network of two layers. This research is investigated for a model with the competition between two-layer opinions, where the first layer is opinion formation and the second layer is decision making, on interconnected networks. Starting with a polarized competition case where layer A has all the positive opinion and layer B has all the negative opinion, competition simulations are considered with various network structures and various updating rules. And it would be also researched that which nodes is more influential to affect the state of network on various network structures. With Monte Carlos simulations, various structural models are compared with average state and consensus ratio, which shows that both internal and external links play a vital role for consensus. Especially, increasing the number of external and internal links on one side layer make it easy to reach consensus. And it is found out that the final state of networks could be different according to time-related updating rules, which is analyzed as communication methods and the characteristic of nodes. Last, it is provided that which way is the best and fastest to recognize important nodes on various network structures. This study could help to analyze social networks, such as legalization of social issues and prediction of vote results. Further more, it could contribute to understanding social conflicts and social network structure. **KEY WORDS:** opinion dynamics,

competition, consensus

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Chapter 1 Introduction

1.1 Introduction

People have their own opinions, and sometimes they change their opinions in response to others that hold views on given issues. Their opinions are reflected to the leader to make laws and vital decision. These phenomena can be found out in some cases, such as voting, legislation and adoption of new policies. It is widely recognized that opinion formation and decision making formation have mutual interaction as interconnected networks.[1-7]. And sometimes, opinion formation could be opposed to decision making formation. These situations often make social conflict and cause social confusion. To figure out these social conflicts, it is needed to understand and analyze the competition of interconnected networks. So far, physics and computer science have researched these social conflict by modeling and analyzing the complex systems[8-11]. The researches include opinion dynamics, voter model, game theory and many more.[12-18] Competition of interconnected networks has been researched in many ways. These networks can be applied to the dissemination of computer viruses, messages, opinions, memes, diseases and rumors[19-26]. Opinion dynamics on interconnected networks has been investigated with various network models such as *Abrams-Strogatz(AS)* model[27, 28] and *M* model[25]. Based on the previous researches, we would study the main features of competing two-layer networks by changing network structures, changing the time-related updating rules to interact on two-layers, and finding the key nodes on layers. The simulation results of changing network structures presents how the network structures, which include internal degrees, external degrees and network type, affect the state of network. Changing the way to interact on two-layers are related to updating rules, and it would be proven that different updating rules cause different results. In addition, it would be analyzed that what updating rules mean in real world. Last, finding key nodes show that which node centrality is the most influential to change network state on the interconnected network.

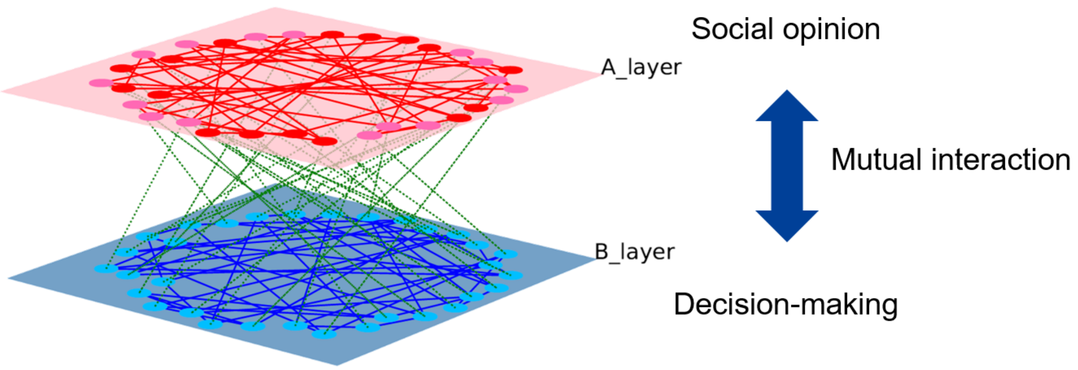
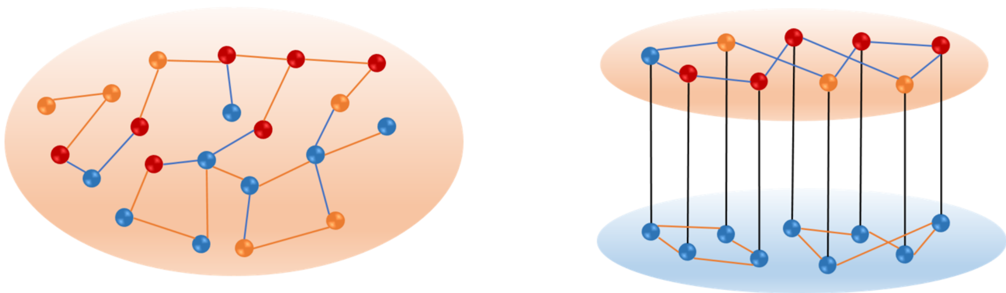


Figure 1–1 The example of competition on two-layer network

1.2 Related Work

In this research, we focus on the competition on two layer network or interconnected network. Comparing with single layer, interconnected network has 2 dynamics, 2 parameters and include internal edge and external edge as shown in Fig. 1–2. Therefore, multi-layer network interaction would be more complex than single layer network interaction. To make two layer networks under competition, each layer is made up with



Single layer	Multi layer(interconnected network)
1 dynamics	2 dynamics (each layer has it own dynamics)
1 parameter	2 parameter
Internal edge	Internal edge + external edge

Figure 1–2 Comparison between single layer and multi-layer

different dynamics and parameter. Network dynamics are based on previous research such as [22]. One layer has the function of social opinion and its own dynamics. Some opinion models provide social mechanism by means of a compromise process.[29] Other opinion models represent persuasive process.[30] In this paper, the social opinion layer is affected by the opinion dynamics which are also known as M-model[25], that includes compromise function and persuasion function. The other layer also has the function of decision-making and its own dynamics. The dynamics of the decision making layer is the language competition dynamics that are also called as Abrams-Strogatz model[27, 28, 31]. This model is useful to decide only one opinion from two opinions. For competition condition, the initial status of the two layers is assumed to be in opposite states, that social opinion layer has all positive states and decision making layer has all negative states. So far, main researches have focused on what factors make a consensus or dissent, which have shown that the system can make positive consensus, negative consensus or coexistence under certain range of parameters, such as volatility, reinforcement and prestige.[22] And interconnected competition of the social network have been researched by finding the threshold or critical point for consensus.[22-24] Also, it has been found out that the thresholds make the transition of states and they can explain and analyze the social phenomena in real world such as the legislation, election and social conflicts.[16, 22, 24] In [23], it is shown that the transition from localized to mixed status occurs through a cascade from poorly connected nodes in the layers to the highly connected ones and the number of external degree is very important to change the state of layers.[23] In addition, the main features, such as transition and cascade, found in Monte Carlo simulation are exactly characterized by the mean-field theory and magnetization[16, 22-24]. Based on these pre-existed researches, the competition of interconnected network would be analyzed by 3 main topics, such as network structures, updating rules and node centralities. Prior to simulations, backgrounds for 3 topics would be explained as follows. First, network structures would be investigated to change the structure of network. Networks could be largely divided into regular network, random network[32], small world network[33], scale free network[34] and others. Fig. 1–3 shows the graph of various networks. Regular network has lattice structure, and each node has exactly the same number of links. Random network is made up with edges that two node are connected with probability p in the systems with K nodes. Small world network is a network graph in which most nodes are not neighbors of one another, but most nodes can be reached from every other node by a small number of links. Small world network can be made by eliminating the edges with probability p and connecting two random nodes that are not connected in a regular network. Small world network has all characteristics of regular network and random network. Scale free network has the model that distribution of

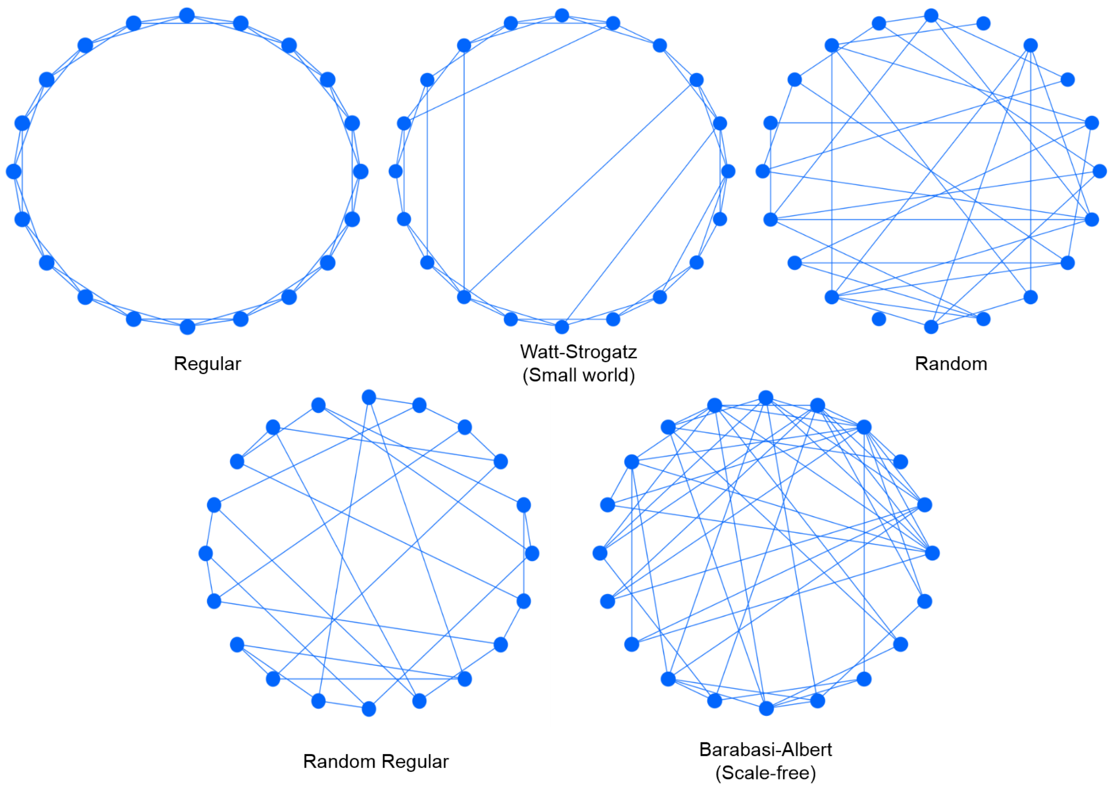


Figure 1–3 Various structures of network

edges follows power function. Examples of scale free network are the World Wide Web (WWW), the Internet, movie star networks, protein interactions, metabolism, and so on. There are several ways to create a scale free network. Among them, the most typical way is Barabasi-Albert models. The *Barabasi-Albert* model is growing networks in which nodes continue to be added, and connections between nodes has preferential attachment. The process of creating this model repeats the following two processes: First, add one node with a constant number of edges to the system. Second, edges of the added nodes are connected in proportion to edges number of the pre-existing nodes. In this paper, two type of general network would be applied such as random regular(RR) network and *Barabasi-Albert*(BA) network.

Second, dynamics orders and time-related updating rules would be studied. For further understanding the competition on two layer network, it is very important to investigate the interaction between nodes or layers. Methods of interaction between nodes are very various.[35] But, related to time, the types of interactions would be divided into two categories, simultaneous interaction and sequential interaction. In economics and social networks, it has been proven that there exists different results between simultaneous and sequential interactions.[36, 37] In [36], it was researched that how experimental subjects update induced prior information when receiving two information signals simultaneously or receiving the same signals sequentially. As the experimental results, the simultaneous treatment is very different from sequential treatment, and under sequential information, subjects’ mean estimates of the two treatments(good news preceding bad news or vice versa) are also significantly different from each other. In conclusion, both sequencing and the order of information processing suggest which one arrives matters. And, in [37], the usual random sequential updating rule is replaced by simultaneous updating on the Sznajd model. As the results, it is found out that this change makes a complete consensus much more difficult. The reason is analyzed as that for simultaneous updating some agents simultaneously receive conflicting messages from different neighbor pairs and thus refuse to change their opinion. In this paper, both simultaneous and sequential updating rules would be applied to layers, nodes and links.

Third, network centralities would be researched to find key nodes on two layers network. Network centrality means the index to measure how close each node is to the center of the network. Than means answers to the question "What characterizes an important node?". The concept of network centrality was first introduced in the field of social network analysis.[38] After that, it has expanded to various areas where the concept of the network is related and has been used to identify which nodes are important in the network. So far, various criteria for assessing network centrality have been presented. Generally well-known network centralities include degree centrality, betweeness centrality, closeness centrality, eigenvector centrality and pagerank centrality.[39] Degree centrality is the simplest but the most reliable concept. it is defined as the number of interacting neighbor nodes (or edges). Betweenness centrality is the concept of using the shortest path between two nodes on a network. It is explained as the concept to define two different node sets on the network (set 1, 2) and quantify how often each node appears on the shortest path for all combinations of nodes in set 1 and set 2. Closeness centrality is derived from that the shorter the path that one node reaches all the other nodes is, the more important the node is. Eigenvector centrality is the concept that the more a node is connected with critical nodes, the more important it is. Pagerank centrality measures the convergent value by repeating the process of propagating each node’s influence to the other nodes. So far, many researchers have been trying to find important nodes in social network.[40-44] Based on node centrality, some algorithms for identifying key nodes has been found out. In [42, 44], it has been found out that optimally combining multiple measures of nodal importance may provide a robust tool for identifying key nodes of interest, particularly in large graphs. Here, based on previous research, we would try to find the key nodes by using single node centrality and combined node centrality.

In this paper, as the single indicator methods to find key nodes, network centralities would be applied such as pagerank, degree centrality, eigenvector centrality, betweenness and closeness.[13, 45]. As the multiple indicator methods to recognize key nodes, several combined node centralities would be applied such as *PR+DE, PR+BE, DE+BE, PR+DE+BE* that are based on single indicators. By using these centralities(pagerank, degree, eigenvector, closeness, betweenness and combined node centralities), it would be found out that which method is the most influential to change the network states on various models.

1.3 Thesis Objective and Direction

In this paper, opinion dynamics of a competing two-layer social network are investigated on the basis of the pre-existed research[22-25]. We develop the previous modeling and research to find out the characteristics of interconnected networks. By switching the network structure and updating rule of each layer, we can see how the consensus or coexistence states change and what conditions make the social consensus or dissent. In addition, trying to identifying key nodes on various network structures, it would be found out which method works well.

This Research has 4 main directions. First, it would be provided how to make up competition models and how to measure the consensus for analysis. Second, it would be found out what factors make consensus by changing network structures. Third, it would be analyzed how dynamics orders and time-related updating rules have an influence on status of two-layer. Fourth, it would be investigated which method is the most effective to identify key nodes based on node centralities. These study can help to explain social networks phenomena, such as social conflict between social opinion and the congress. Therefore, this research could be used as a tool for analyzing legislation problems, making efficient decision-making system and solving the social conflict.

This paper is organized as follows. In chapter 2, it is introduced that how competing two layers are made up and how the dynamics of each layer works. And some indexes are provided to measure and evaluate the simulation results. In chapter 3, with changing network structure, it would be found out that how the network structures have the influence on the consensus of two layers. In chapter 4, considering the dynamics orders and time-related updating rules, simulation results would be compared and analyzed. In chapter 5, it would be researched that which nodes are important to affect the state of network by using single indicators and multiple indicators. Finally, in chapter 6, all simulation results will be summarized and our findings are concluded. And it is considered that what would be researched as future works and how we can be applied to real world.

Chapter 2 Modeling and Analysis

In this chapter, a basic model would be introduced for competition on two-layer network. It would be also explained that how each layer is made up and what kind of function and dynamics it has. After modeling, many simulations would be fulfilled under the various conditions. Several indexes would be provided to analyze the interaction between two-layers. Simulation results would be analyzed with these indexes.

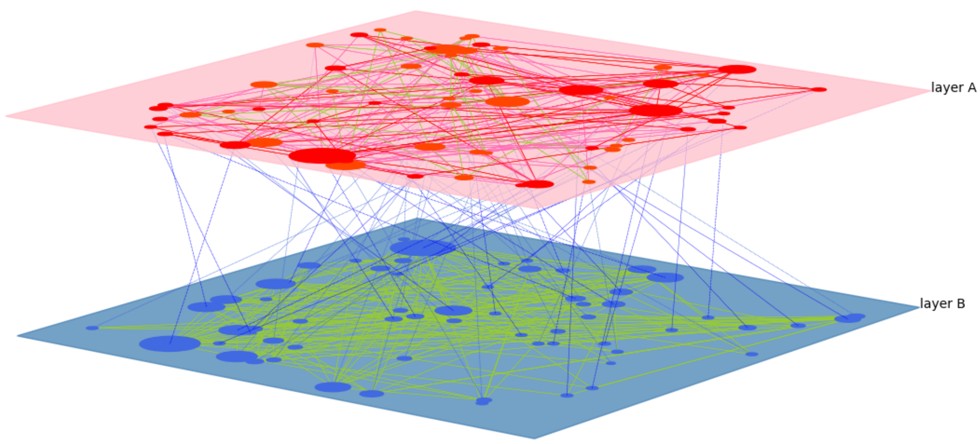


Figure 2–1 Competition of Interconnected Network

2.1 Modeling of two layer network

The model consists of two layers, and each layer has different dynamics. For layer A, the node change its states according to M model as introduced in [25]. Here, we choose $M = 2$, that each node has four states $(-2, -1, +1, +2)$. For each link (k, j) belong to layer A, the dynamics are designed as follows:

- Compromise : if two nodes connected with link(k, j) have opposite orientations, their states become more moderate with probability q :

$$\text{if } S_k < 0 \text{ and } S_j > 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^r, S_j^l) \text{ with } prob.q, \tag{2–1}$$

$$\text{if } S_k > 0 \text{ and } S_j < 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^l, S_j^r) \text{ with } prob.q. \tag{2–2}$$

If $S_k = \pm 1$ and $S_j = \mp 1$, one switches orientation at random:

$$(\pm 1, \mp 1) \rightarrow \begin{cases} (+1, +1) \text{ with } prob.q/2, \\ (-1, -1) \text{ with } prob.q/2. \end{cases} \tag{2–3}$$

- Persuasion : if two nodes connected with link(k, j) have the same orientation, their states become more extreme with probability p :

$$\text{if } S_k < 0 \text{ and } S_j < 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^l, S_j^l) \text{ with } prob.p, \tag{2–4}$$

$$\text{if } S_k > 0 \text{ and } S_j > 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^r, S_j^r) \text{ with } prob.p. \tag{2–5}$$

For each external link (k, j) with k belong to layer A, the state of node k is updated according to :

- $S_k \cdot S_j < 0$:

$$\text{if } S_k < 0 \text{ and } S_j > 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^r, S_j) \text{ with } prob.q, \tag{2–6}$$

$$\text{if } S_k > 0 \text{ and } S_j < 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^l, S_j) \text{ with } prob.q. \tag{2–7}$$

- $S_k \cdot S_j > 0$:

$$\text{if } S_k < 0 \text{ and } S_j < 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^l, S_j) \text{ with } prob.p, \tag{2–8}$$

$$\text{if } S_k > 0 \text{ and } S_j > 0 \Rightarrow (S_k, S_j) \rightarrow (S_k^r, S_j) \text{ with } prob.p. \tag{2–9}$$

Here, S_k^r and S_k^l denote the right and left neighboring states of k, defined as

$$S_k^r = \begin{cases} +1, \text{ for } S_k = -1 \\ +2, \text{ for } S_k = +2 \\ S_k + 1, \text{ otherwise }, \end{cases} \qquad S_k^l = \begin{cases} -1, \text{ for } S_k = +1 \\ -2, \text{ for } S_k = -2 \\ S_k - 1, \text{ otherwise }. \end{cases} \tag{2–10}$$

The sign of S^A represents its opinion orientation and its absolute value $|S^A|$ measures the intensity of its opinion. So, $|S^A| = 2$ represents a positive or a negative extremist, while $|S^A| = 1$ correspond to a moderate opinion of each side. In case of internal link (k, j) belong to layer A, when the nodes have the same orientation($S_k S_j > 0$), if the states of nodes are moderate, then they become extreme($S_k = \pm1 \rightarrow \pm2, S_j = \pm1 \rightarrow \pm2$) with probability p. If they are already extreme, they remain extreme($S_k = \pm2 \rightarrow \pm2, S_j = \pm2 \rightarrow \pm2$). On the other hand, when the nodes have opposite orientations($S_k S_j < 0$), if they are extreme, the states of nodes become moderate($S_k = \pm2 \rightarrow \pm1, S_j = \pm2 \rightarrow \pm1$) with probability q. If they are already moderate, they switch orientations individually($S_k = \pm1 \rightarrow \mp1, S_j = \pm1 \rightarrow \mp1$). In case of interaction between node in layer A and node in layer B, node in layer A follows opinion dynamics formula, but the state of node in layer B does not change. In other words, the state of layer B affects layer A, but layer A dynamics does not affect the state of node in layer B. For example, one of the layer A node, $S_k = +2$ is connected with $S_j = -1$ node of layer B. Here, S_k will change into $S_k = +1$ with $prob.q$. But S_j will not change, which indicates that the states of layer B will influence the states of layer A.

The dynamics of layer B follows the decision-making dynamics as introduced in [27, 28]. The state of node i in layer B can be $+1$ and -1, and it updates according to

$$P_B(S_i \rightarrow -S_i) = \begin{cases} \left(\left(\dfrac{i_i + e_i}{n^{-S_i}}\right) \cdot \left(\dfrac{n^{-S_i}}{i_i + e_i}\right)\right)^{1/v}, \text{ if } v \neq 0 \\ 0, \text{ if } v = 0 \\ 0, \text{ if } n^{-S_i} = 0 \end{cases}, \tag{2–11}$$

where i_i is the number of internal edges and e_i is the number of external edges. n^{-S_i} is the number of neighbors of i with opposite state $-S_i$. v represents the volatility that measures how prone a node change its state. The scale of v is from 0 to 1. If $v \simeq 0$, a node is unlikely to change its state. On the other hand, if $v \simeq 1$, a node is very likely to change its state. Also, this formula shows that the more the number of nodes connected with the opposite state is, the easier the nodes are to change into the opposite state.

2.2 Simulations and Analysis

This modeling is nonlinear in that the structure of the model changes with the states of nodes. In this model, helpful mathematical tools are no longer applicable and it turns out, moreover, that rigorous analytical results are difficult to obtain.[46, 47]. For that reason, we would try to carry out the analysis of the above nonlinear model to a large extent by simulations on the computer.

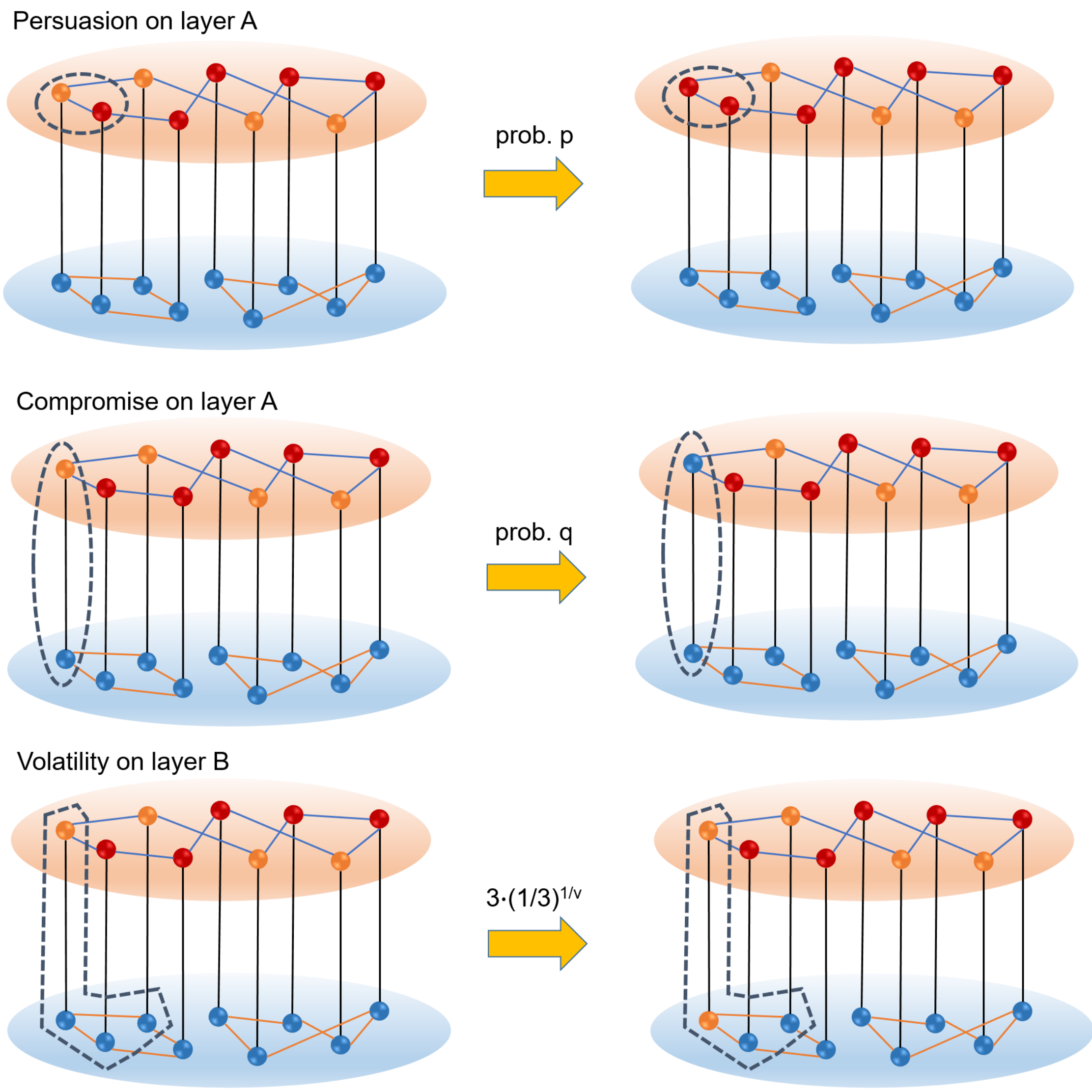


Figure 2–2 Dynamics on two layers

To start with a polarized competition, as the initial conditions, nodes in layer A are all positive, and nodes in layer B are all negative as shown in Fig. 2–1. For nodes in layer A, it begins with the status where half of nodes are +1 and the others are +2. The initial state of nodes in layer B have only −1.

There are two parameters in the dynamics of layer A. To simply represent the probability p and probability q together, we set $p + q = 1$. So, p represents the tendency of opinion such as extreme or moderate, which is scaled to be 0 to 1. And, the scale of v, in the dynamics of layer B, is also 0 to 1.

To implement the interconnected dynamics, one step consists of two layers dynamics, where every node in layer A is checked with opinion dynamics, and every node in layer B updates its state according to the decision-making dynamics. Basically, the dynamics order follows updating state of layer B after updating state of layer A. The dynamics orders and updating rules would be discussed specifically in chapter 4.

Each simulation takes 100 steps, and 100 simulations are considered for average results. In the following simulations, we use *'Average State'(AS)* and *'Consensus Index'(CI)* to measure the competition result.

$$AS = avg\left(\sum_{i}^{K^A} S_i^A/4\right) + avg\left(\sum_{i}^{K^B} S_i^B/2\right). \tag{2–12}$$

$$CI = \frac{({K_+}^A \cdot {K_-}^B) + ({K_-}^A \cdot {K_+}^B)}{K^A \cdot K^B}. \tag{2–13}$$

In these formula, S_i^A means the state of node i in layer A, and K^A is the number of nodes in layer A. ${K_+}^A$ represents the number of nodes with positive state in layer A.

With AS, it could be verified whether the consensus happens in accordance with the change of p and v. If the positive consensus happens, it would be close to the value of $+1$ and if the negative consensus happens, it would be close to the value of -1. The values between $+1$ and -1 mean the states are belonging to the coexistence part. Figure. 2–3

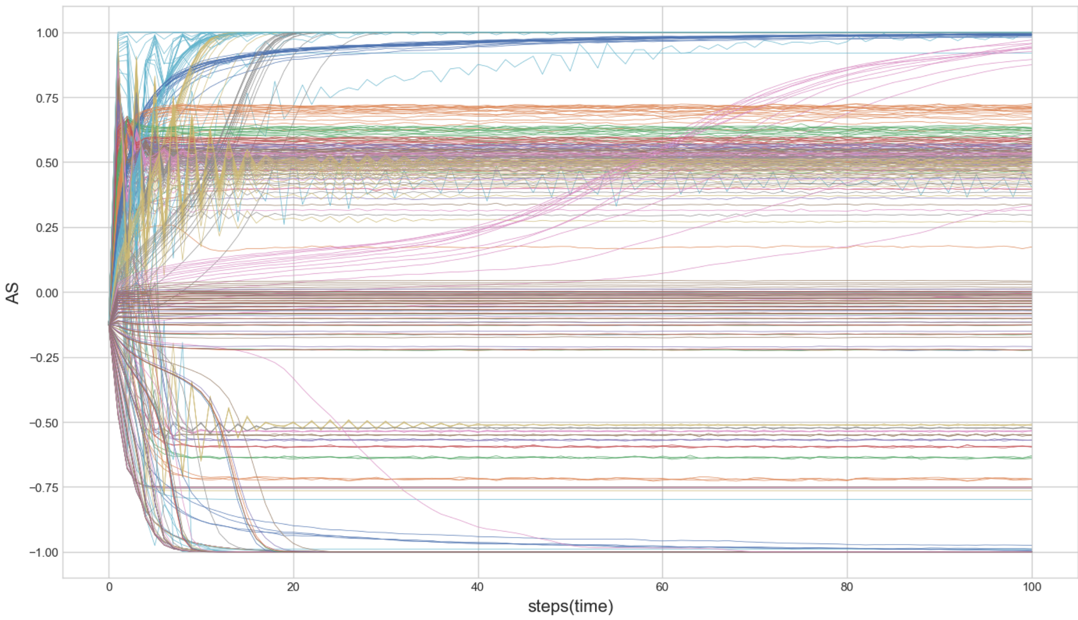


Figure 2–3 AS values per each step according to all parameters

shows that AS values are convergent to +1, −1 or other values as step(time) goes by. +1 means making positive consensus. −1 means making negative consensus. The other values mean coexistence state.

With *CI*, it could be measured how close the network state is to consensus. If the CI is close to 0, the state is close to positive or negative consensus. If the CI is close to 1, the state is separated coexistence where states of all nodes in layer A is opposed to states of all nodes in layer B. If the CI is close to 0.5, the state is mixed coexistence where each layer has both positive and negative states of nodes.

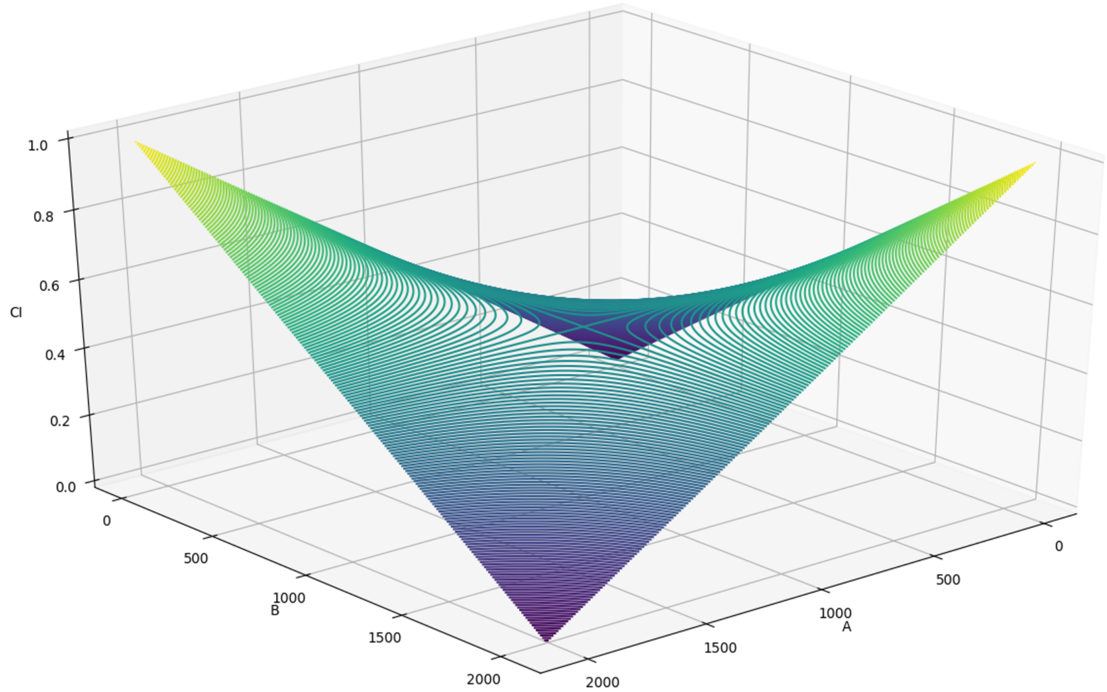


Figure 2–4 CI values according to all ${K_+}^A$ and ${K_+}^B$

Figure. 2–4 shows the characteristics of *CI*. Same orientation in two layers makes *CI* 0. Opposite orientation between two layers makes *CI* 1. And Mixed orientation in two layers makes *CI* close to 0.5.

As Figure. 2–5 shown, *CI* values are convergent to +1, 0, or other values as step(time) goes by. 0 means positive or negative consensus. +1 means opposite state between two layers. The other values means mixed state. By using *CI*, coexistence states can be divided into two categories, opposite state and mixed state.

To estimate and evaluate the consensus results regarding to different parameters p and v, we use four kinds of measures including *'AS total'*, *'Positive Consensus Ra-*

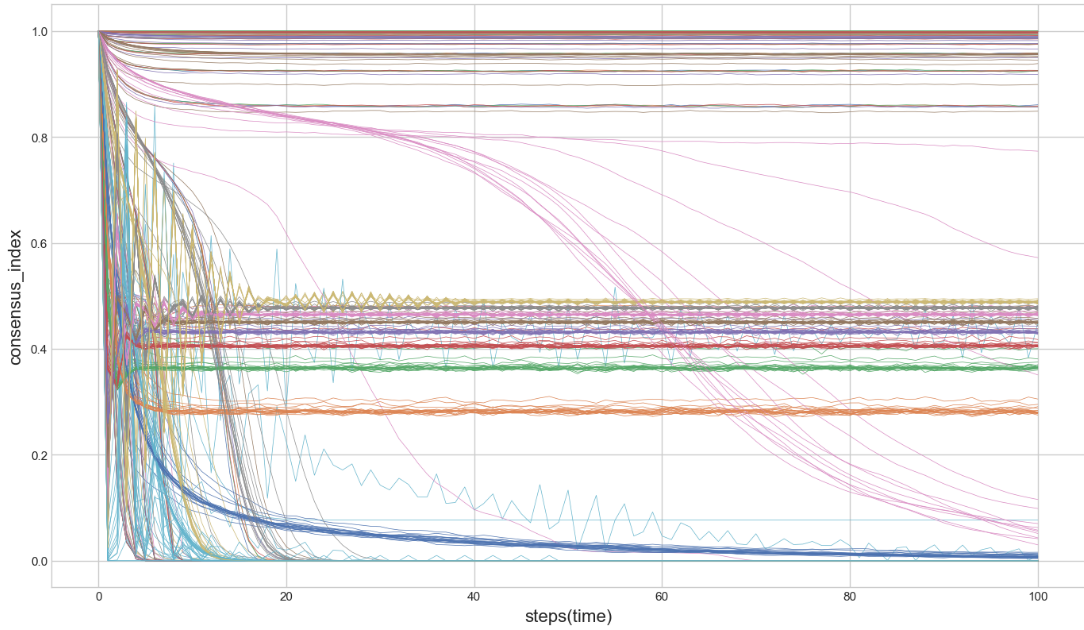


Figure 2–5 CI values according to all ${K_+}^A$ and ${K_+}^B$

tio'(PCR), *'Negative Consensus ratio'(NCR)*, and *'Consensus Ratio'(CR)*. *AS total* means the summation of *AS* for all *p*s and all *v*s. *PCR* is the ratio of positive consensus over all simulations. Similarly, *NCR* is the ratio of experiments with negative consensus. *CR* is the ratio of experiments reaching consensus, i.e. summation of *PCR* and *NCR*.

$$AS\ total = \frac{\sum_j^m \sum_i^n AS_{p_i,v_j}}{n \times m}, \quad \begin{matrix} p = \{p_1, p_2, \cdots, p_n\} \\ v = \{v_1, v_2, \cdots, v_m\} \end{matrix} . \tag{2–14}$$

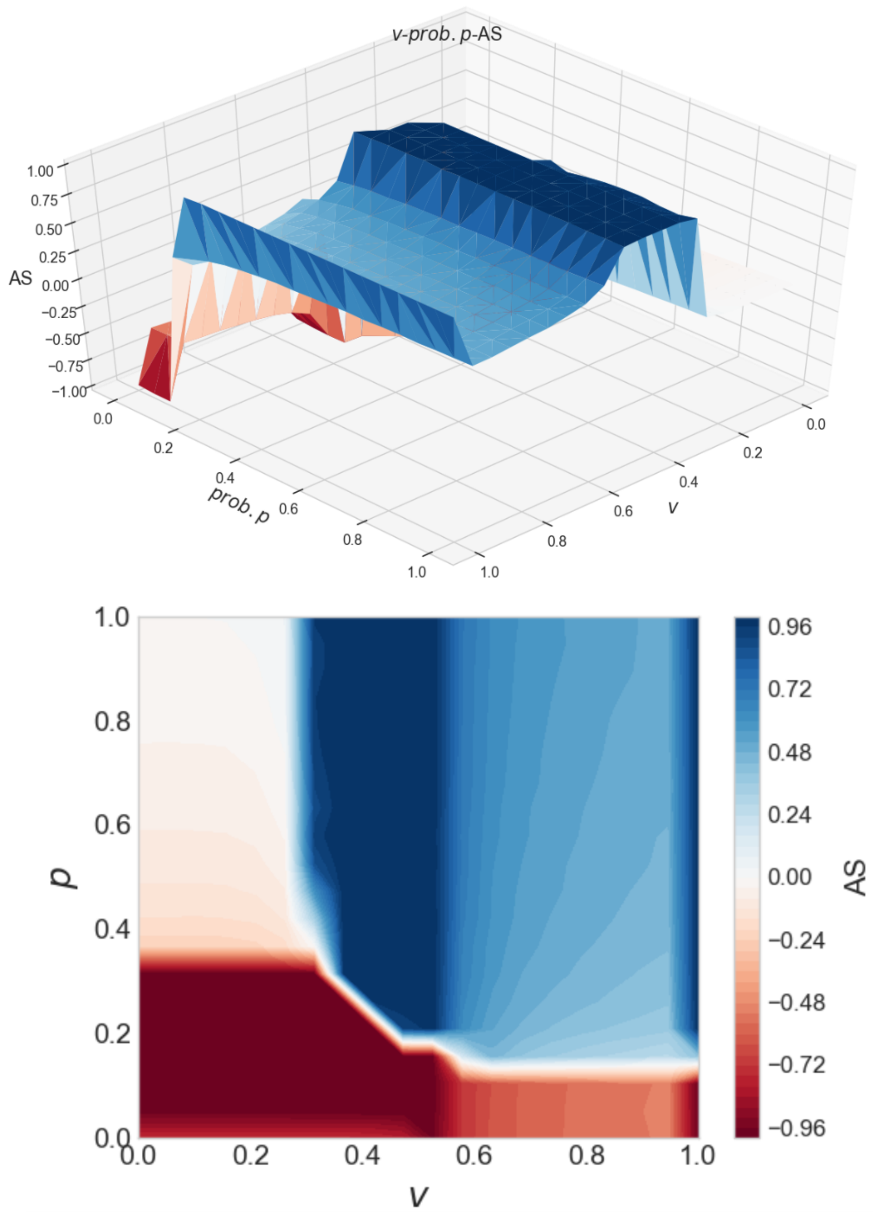
In Eq(2–14), AS_{p_i,v_j} means *AS* value with parameters p_i and v_j, which shows the total orientation and intensity of interconnected network.

$$PCR = \frac{\sum_j^m \sum_i^n (AS_{p_i,v_j} \simeq 1)}{n \times m}. \tag{2–15}$$

In Eq(2–15), $AS_{p_i,v_j} \simeq 1$ means positive consensus.

$$NCR = \frac{\sum_j^m \sum_i^n (AS_{p_i,v_j} \simeq -1)}{n \times m}. \tag{2–16}$$

In Eq(2–16), $AS_{p_i,v_j} \simeq -1$ means negative consensus.



Div	AS total (Average values)	PCR(Blue part)	NCR(Red part)	CR(Blue+Red part)
BA-BA network	0.1799	0.1750	0.1450	0.3200

Figure 2–6 The example of simulation : BA-BA network

Fig. 2–6 shows the states of two layers according to all ps and all vs. The X-axis is the p and the Y-axis is the v, and the Z-axis represents AS. The closer the color is to blue, the more it has positive consensus. And the closer the color is to red, the more it has negative consensus. A light and white areas have coexistence with positive states and negative states. Here, we can measure the consensus by using indexes, *‘AS total’*, *‘PCR’*, *‘NCR’*, and *‘CR’*. The average value of this chart means *‘AS total’*. The blue part area means *‘PCR’*, the red part area means *‘NCR’*, and the summation of those means *‘CR’*.

Chapter 3　Competition on two layer with various structural network

In this chapter, based on the competition model described in chapter.2, simulation would be implemented with changing the network structures. As the basic model, interconnected layer with random regular networks would be provided. And then, the structure of interconnected network would be altered by changing the internal edges, external edges and network types. Finally, all simulations would be compared and analyzed with the indexes, *AS total*, *PCR*, *NCR* and *CR*

3.1　Competition on Random Regular Networks

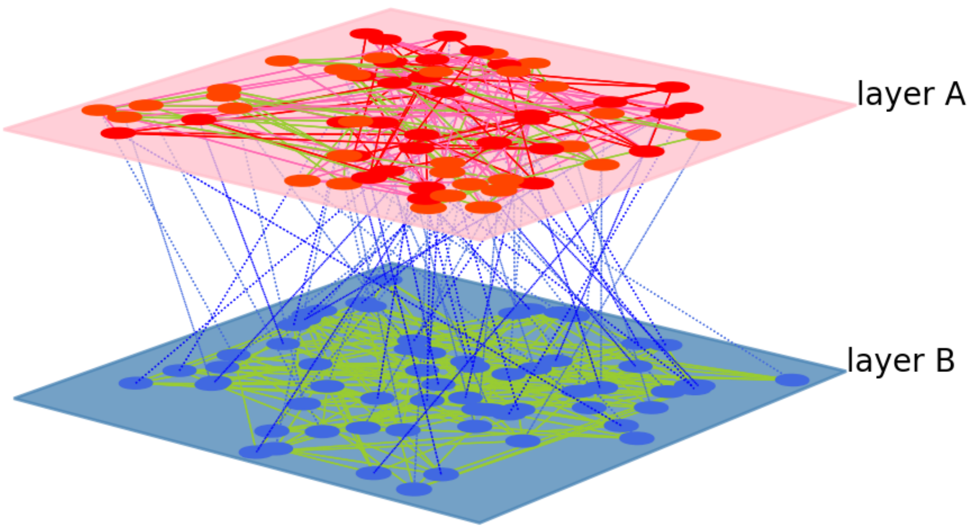


Figure 3–1 Competition on random regular network

In this section, simulation results on random regular network would be provided to comprehend the competition of two layers. Each layer consists of random regular network that has N nodes with k internal edges as introduced in [48-50]. Each node of one layer is connected with a random node on the other layer. This means each node has

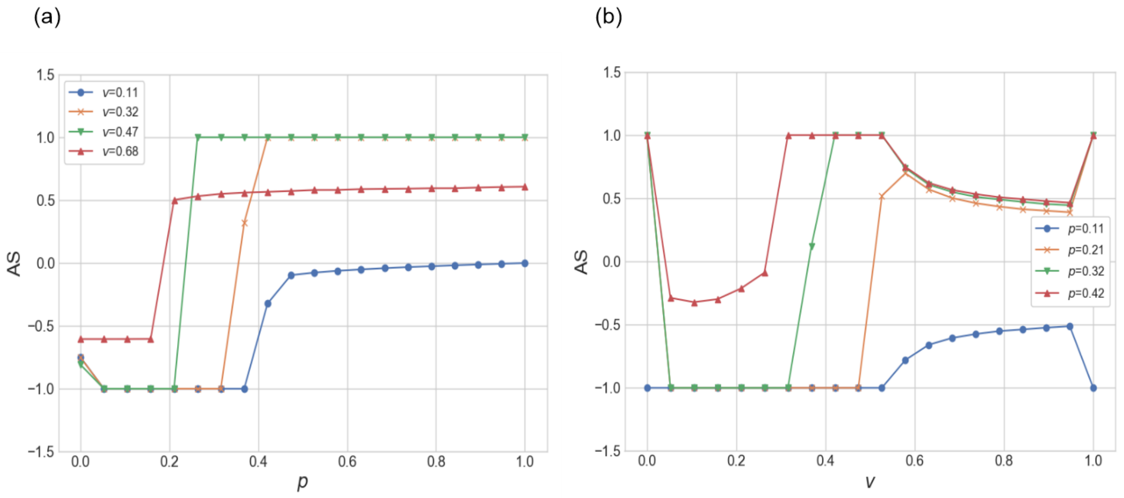


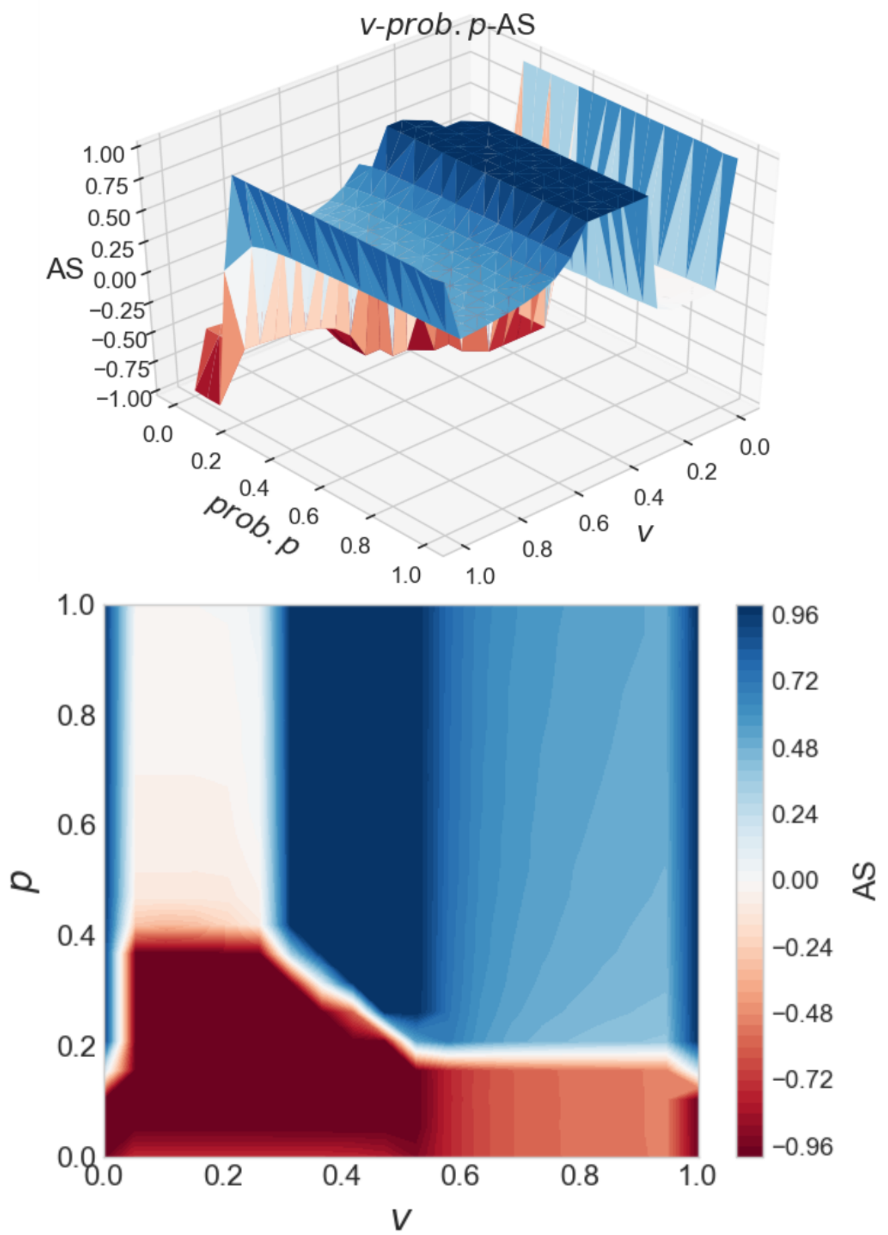
Figure 3–2 (a) p-AS chart according to certain v values. (b) v-AS chart according to certain p values.

only 1 external un-directed edge. Simulations are preformed on network with $N = 2048$, and $k = 5$.

The simulation results are shown in Fig. 3–2 and Fig. 3–3. Fig. 3–2 presents that how the Average State(AS) are changed according to each parameter, p and v. So we can know that how each parameter works on the network. Fig. 3–3 provides total results with all parameters. Through these figures, the characteristics of network would be analyzed.

Fig. 3–2(a) shows that when $p > 0.2$, $0.32 < v < 0.47$, it normally tends to positive consensus. But, if v is lower or larger than certain values, it doesn't make consensus. In Fig. 3–2(b), as v increases, it normally change from negative to positive consensus. But, when p is very low($p \leq 0.11$), it doesn't make positive consensus. On the other hand, when p is large enough, it makes positive consensus. But, when v is small enough, it is changed into negative consensus. When both of p and v are large enough, the state is in a coexistence part.

Fig. 3–3 shows the states of two layers according to all ps and all vs. As previously described in chapter. 2, blue area is for positive consensus, red area is for negative consensus, and light colored and white area is for coexistence. And indexes for consensus are also measured. Positive consensus area is 0.2475, and negative consensus area is 0.1575. Coexistence area is $1 - CR = 0.5950$. By using these values and figures, this model would be compared with various structural networks in next section. Through these figures, several facts can be arranged. First, large p tends to make positive consensus and small p tends to make negative consensus. Second, small v tends to make negative



Div	AS total (Average values)	PCR (Blue part)	NCR (Red part)	CR (Blue+Red part)
RR-RR network	0.2034	0.2475	0.1575	0.4050

Figure 3–3 *AS* with changing with all *p* and *v*

consensus, and large *v* tends to make coexistence state.

3.2 Competition on Networks with different number of external links

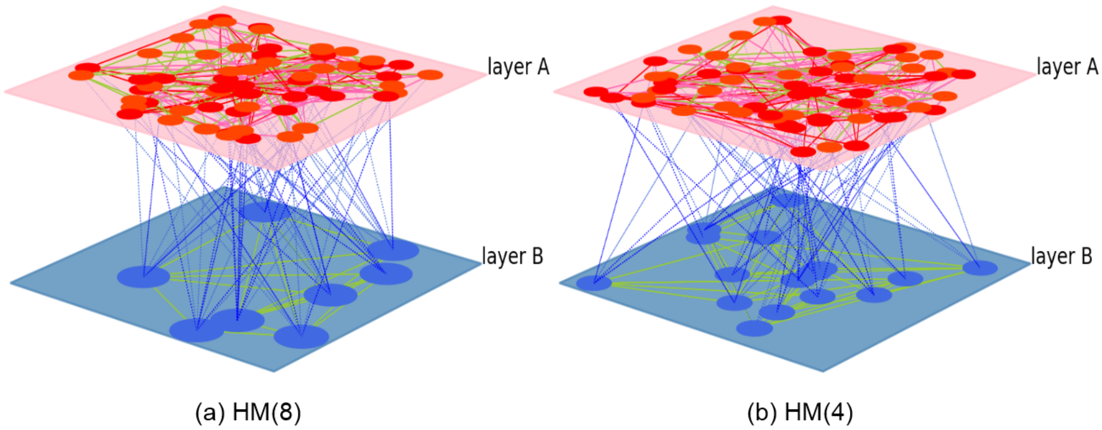


Figure 3–4 Competition on hierarchical model

In this section, we consider the influence of external links. Based on *RR-RR* model in section 3.1, we reduce the number of nodes in layer B at a certain rate and increase the external links from nodes in layer B accordingly as shown in Fig. 3–4. We denote *HM(n)* as a hierarchical model with a level n, which means that the number of nodes in layer B is $1/n$ of the number of nodes in layer A, and the number of external links from node in layer B is n in view that the number of external links from node in layer A is 1. In other words, each node in layer A has one external edge, but each node in layer B has n external edges for *HM(n)*, which means one node in layer B can be influenced by n nodes in layer A. To find out the significant influence of external edges, various *HM(n)s* were simulated. Totally, 8 *HM(n)*, *HM(2), HM(4), HM(8), HM(16), HM(32), HM(64), HM(128), HM(256)* were arranged as shown in Fig. 3–5. Fig. 3–5 shows that *HM(2)* has the most area for coexistence part(light and white area) and *HM(256)* has the most are for consensus part(blue and red area). As n in *HM(n)* is increased, coexistence area is decreased and consensus area is increased. Particularly, positive consensus area is significantly increased, negative consensus area is slightly decreased. To clearly find out the difference between models, we use the indexes, *PCR, NCR, AS total*. Fig. 3–6 shows the results to analyze *HM(n)* with indexes. Blue color bar is for PCR, red color bar is for NCR, and green color bar is for AS total. Comparing *HMs* with *Basic model(RR(5)-RR(5))*, *CR PCR* and *AS total* are all increased remarkably. *HMs* have more positive consensus part than *RR(5)-RR(5)*. But, *HMs* have less negative consensus part

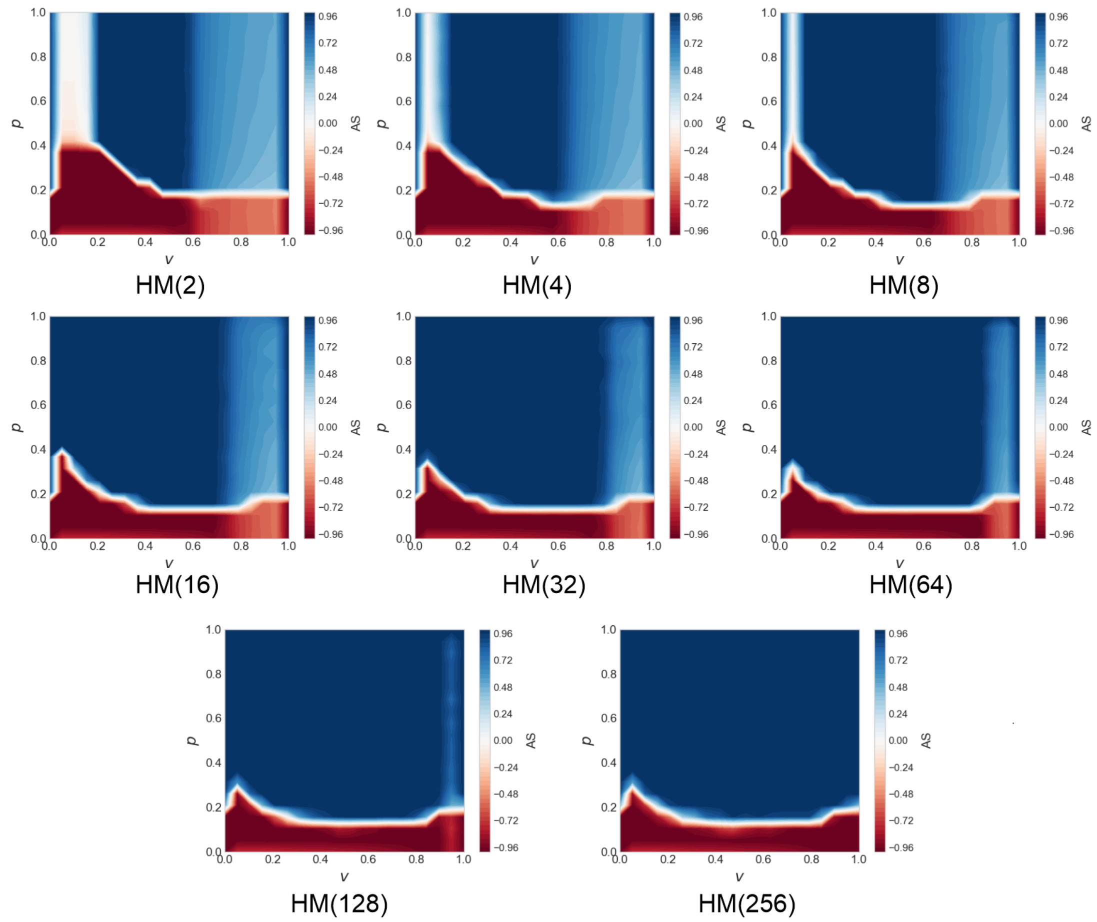


Figure 3–5 *AS total* on various hierarchical models

than *RR(5)-RR(5)*. It shows that as the number of B nodes are decreased, it is easy to make positive consensus(layer A opinion) and hard to make negative consensus(layer B opinion).

In summary, all the Hierarchical Models have more consensus ratio than *Random Regular Networks* Model. However, positive consensus ratio is increased, but negative consensus ratio is decreased. It is found out that as the number of B nodes are more decreased, it makes easier to make positive consensus and harder to make negative consensus. In real world, it would be analyzed that as the number of leaders is less, social conflict are decreased and the opinion is convergent to social opinion(layer A). But, sometimes there are some dangers to ignore the leader opinions(layer B), or to cause more social conflict when there are stubborn leaders, that would be simulated in chapter.5.

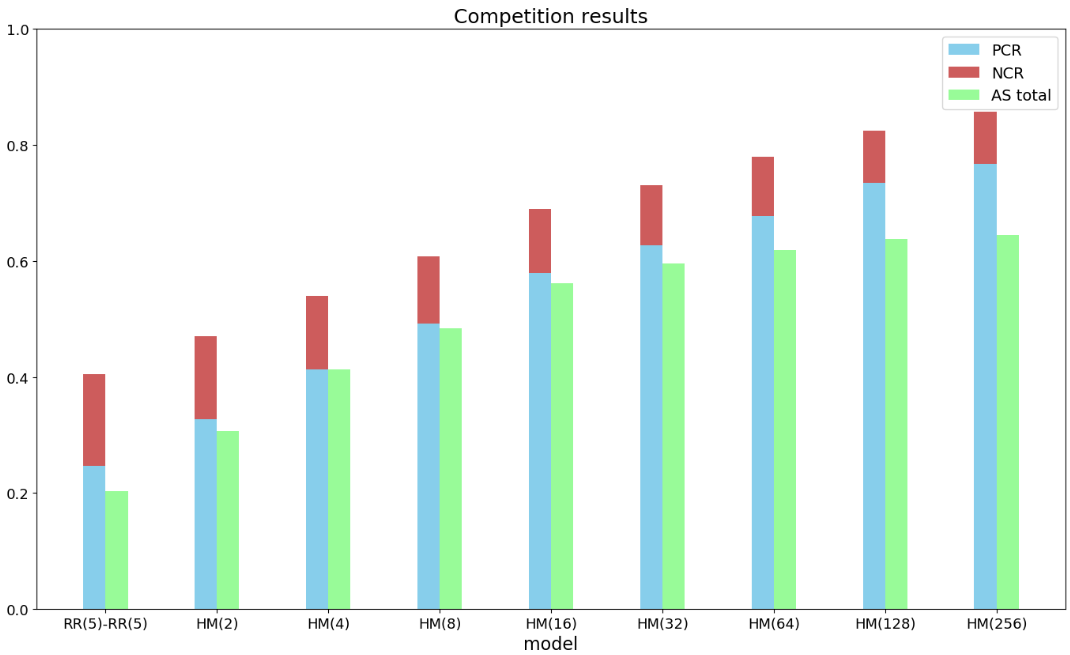


Figure 3–6 Histogram for *PCR, NCR, AS total* of Hierarchical Models(*HM(n)*)

3.3 Competition on Networks with different number of internal links

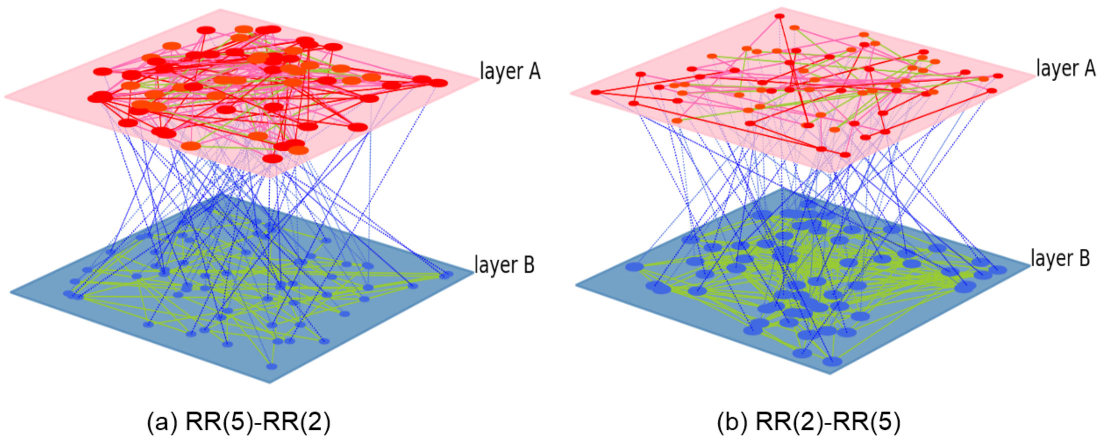


Figure 3–7 Competition on interconnected networks with different internal edges

Next, the interconnected networks are simulated with different internal degrees in order to define and evaluate the influence of internal degrees. Random regular network would be applied. And the number of internal degrees on each node is switched to various numbers as shown in Fig. 3–7. But, there is no change on external degree, which would be fixed to only 1. Here, *RR(n)-RR(m)* represents layer A has random regular network with *n* internal edges, layer B has random regular network with *m* internal edges. Firstly,

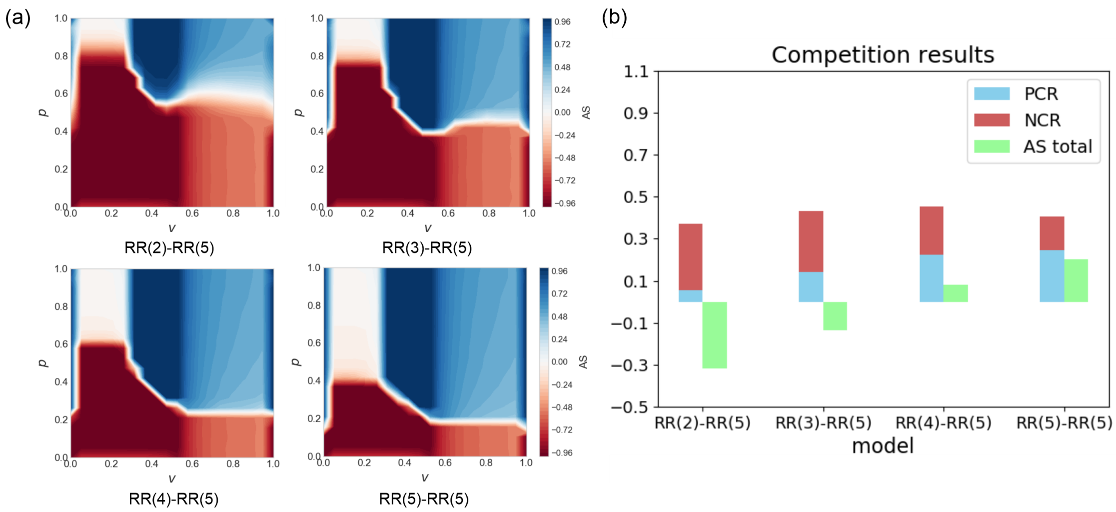


Figure 3–8 Simulation results with different internal degrees on layer A

the internal degrees on layer A are changed. The internal degrees on layer B are fixed to 5,120, which means each node has 5 internal degrees on layer B, and the internal degrees on layer A are switched into 2,048, 3,072, 4,096, or 5,120, which means each node has 2, 3, 4, or 5 internal degrees on layer A. Fig. 3–8 shows the simulation results for changing the internal degrees on layer A. As shows in Fig. 3–8 (a), as the number of internal degrees on layer A is increased, the red part is decreased and the blue part is increased.

To clearly compare and analyze the results, the results are presented with the indexes, *PCR, NCR, AS total* in Fig. 3–8 (b), which shows that as the number of internal degrees on layer A is increased, negative consensus is decreased and positive consensus is increased. As shown in Fig. 3–8, RR(5)-RR(5) has the most *PCR*, and RR(2)-RR(5) has the most NCR. However, *CR* is almost same with all models in Fig. **??**. It can be analyzed that the number of internal degrees on layer A has the tendency to keep positive state and to change negative state into positive state. Next, the internal degrees on layer B are changed. The internal degrees on layer A are fixed to 5,120, which means each node has 5 internal degrees on layer A, and the internal degrees on layer B are switched into 2,048, 3,072, 4,096, or 5,120, which means each node has 2, 3, 4, or 5 internal degrees on layer B. Fig. 3–9 shows the simulation results with changing the number of internal

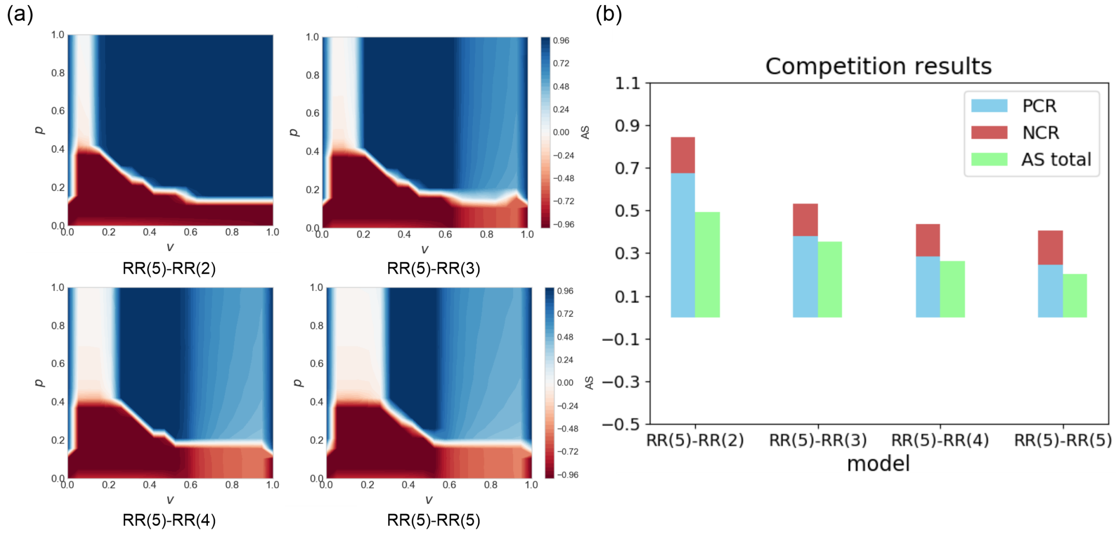


Figure 3–9 Simulation results with different internal degrees on layer B

degrees on layer B. As shows in Fig. **??** (a), as the number of internal degrees on layer B is increased, the blue part is decreased, the white and light color part is increased, and the red part is almost same, though the shape of red area is changed. As shown in Fig. 3–9 (b), RR(5)-RR(2) has the most *PCR* and *CR*, and RR(5)-RR(5) has the least PCR and *CR*. However, *NCR* is almost same with all models in Fig. **??**. It can be analyzed that the number of internal degrees on layer B has the tendency to hinder positive state and has the inverse relation with *CR*. As the number of internal degrees on layer B is increased, *PCR* and *CR* is inversely decreased. It is recognized that the role of internal degrees on layer A is different with internal degrees on layer B. The internal degrees on layer A has the function to keep the state of layer A, and the internal degrees on layer B has the function to restrain the state of layer A and make coexistence part. Next, it is simulated that internal degrees are changed on both layer A and layer B, such as RR(2)-RR(2), RR(3)-RR(3), RR(4)-RR(4), RR(5)-RR(5) and RR(6)-RR(6). Through these simulations, it would be found out that how total internal degrees affect the interconnected network. Fig. 3–10 shows the influence of internal degrees on both layers. As the total number of internal degrees is increased, *CR* is inversely decreased, and the ratio of positive consensus(blue area) is increased, but the ratio of negative consensus(red area) is decreased. It can be analyzed that a decrease in *CR* is caused by increase in internal degrees on layer B, and an increase in ratio of *PCR* is brought out by an increase in internal degrees on layer A. But, when the total number of internal degrees is increased, *PCR, NCR, CR* indexes

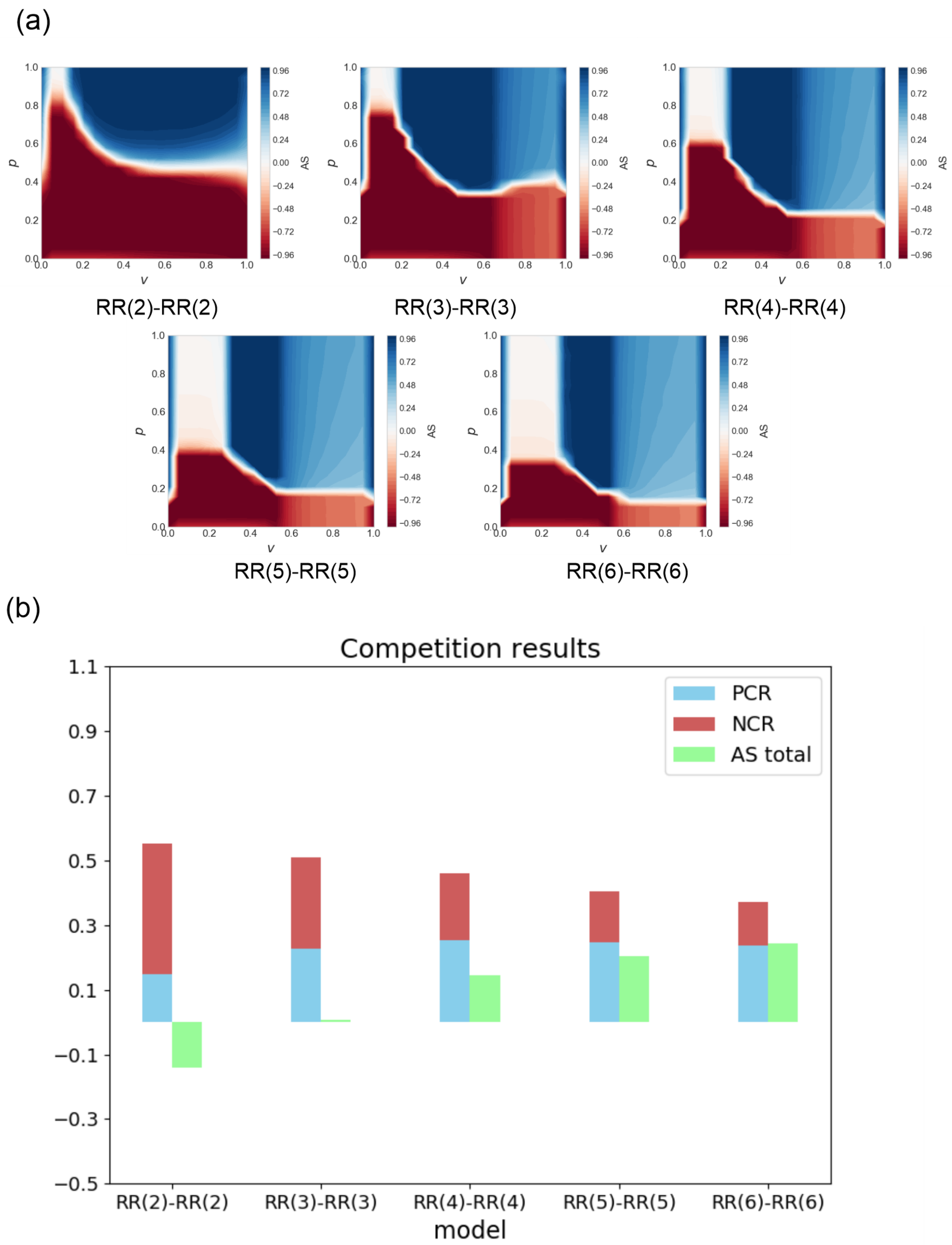


Figure 3–10 Simulation results with changing internal degrees on both layers

are decreased. It can be analyzed that too many internal degrees on both layers make it hard to reach consensus. In summary, 3 main simulations are implemented to find

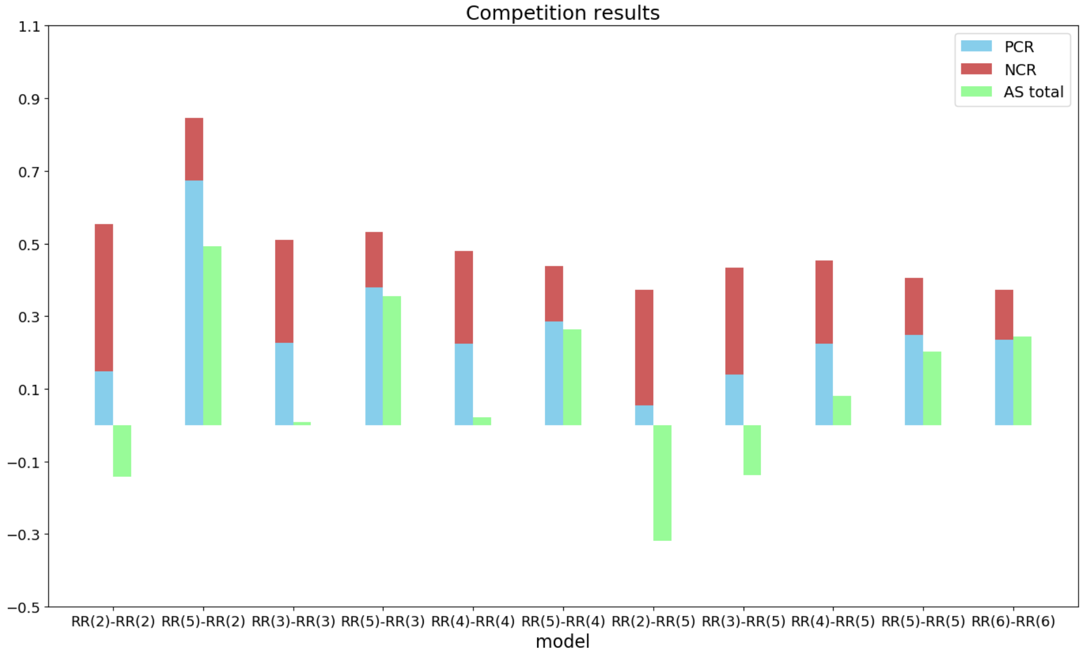


Figure 3–11 Total results with different internal degrees on two layers

out the influence of internal degrees on interconnected network. First, the number of internal degrees on layer A are changed, and it is found out that the number of internal degrees on layer A has the tendency to keep positive state and to change negative state into positive state. Second, the number of internal degrees on layer B are switched, and it is found out that the number of internal degrees on layer B has the tendency to hinder positive state and has the inverse relation with *CR*. Third, the number of internal degrees on both layers are changed, and it is found out that too many internal degrees make it hard to reach consensus. Fig. 3–11 shows the result for all simulations. Through these simulation results, we can analyze that how network state is changed according to the number of internal degrees. So, several conclusions can be arranged as shown in Fig. 3–12. First, it is easy to reach negative consensus when the internal degrees on layer A is relatively small, but the internal degrees on layer B doesn't matter. Second, it is easy to make positive consensus when the internal degrees on layer A is relatively large and the internal degrees on layer B is relatively small. Third, social conflict can be caused when the internal degrees on both layers are too large.

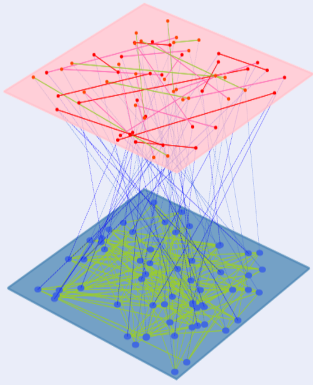
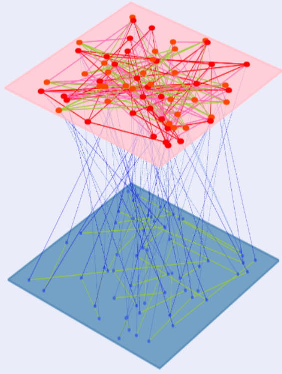
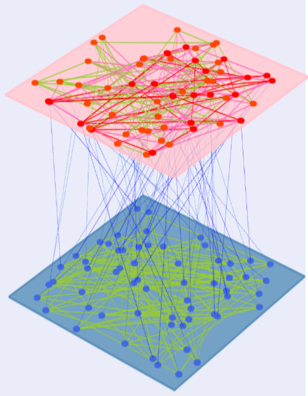
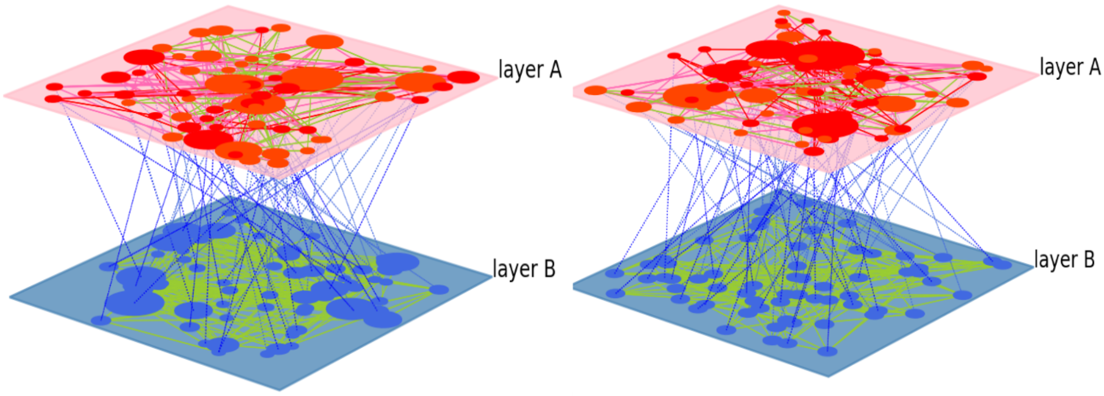
state	negative dominant	positive dominant	coexistence dominant
figure			
internal degrees on layer A	small	large	large
internal degrees on layer B	large or small	small	large

Figure 3–12 Categorizing network state according to internal degrees on two layers



(a) BA-BA (b) BA-RR

Figure 3–13 Competition on interconnected networks with different structures

3.4 Competition on networks with different structures

So far, each layer of the interconnected network consisted of *RR(random regular networks)* that has the same number of edges for each node. Now, the simulation would be implemented on different network type. Here, we use *Barabasi-Albert network(BA)* structure as introduced in [34]. *Barabasi-Albert(BA)* network has N nodes with attaching new nodes each with K edges that are preferentially attached to existing nodes with high degrees. But, there is no change on external degree, which would be fixed to only 1.

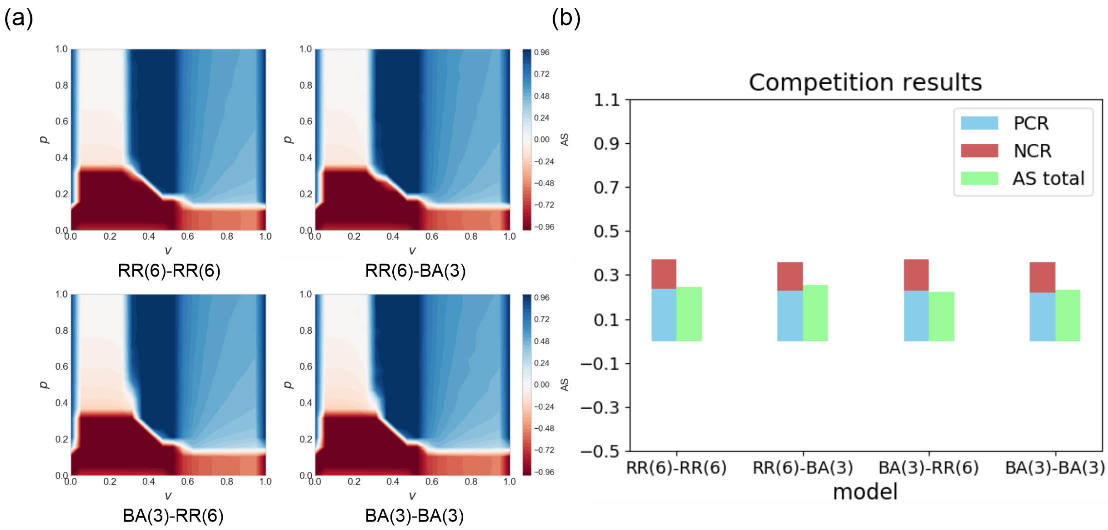


Figure 3–14 Simulation results with different network types

To evaluate the influence of network structure, 4 simulations are implemented with switching network structures. The *BA* or *RR* network is applied for both layers or switched on each layer. To restrain the influence of internal degree number, the number of internal degrees in *BA* is set up to be similar with the number of internal degrees in *RR*. The number of internal degrees in *BA* is $6,135$, and the number of internal degrees in *RR* is $6,144$. The simulation results are shown in Fig. 3–14. The results of all simulations have almost the same features. The gap of *PCR, NCR* and *CR* is less than 0.02. The structure of network make no obvious difference of consensus results. Next, the number of internal degrees would be increased on the network, where consists of two *BA*. It would be found out that how the number of internal degree work on different network type. 2 models, BA(3)-BA(3) and BA(5)-BA(5) would be simulated. BA(3)-BA(3) model has $6,135$ internal degrees on each layer, and BA(5)-BA(5) model has $10,215$ internal degrees on each layer.

As shown in Fig. 3–15, BA(5)-BA(5) has more coexistence area because of too many internal degrees. The influence of internal and external degrees number is more important to make network state and consensus than the influence of network type. However, If there are stubborn nodes on networks, the simulation results would be changed because node centrality of stubborn nodes would be changed according to network type. Finding

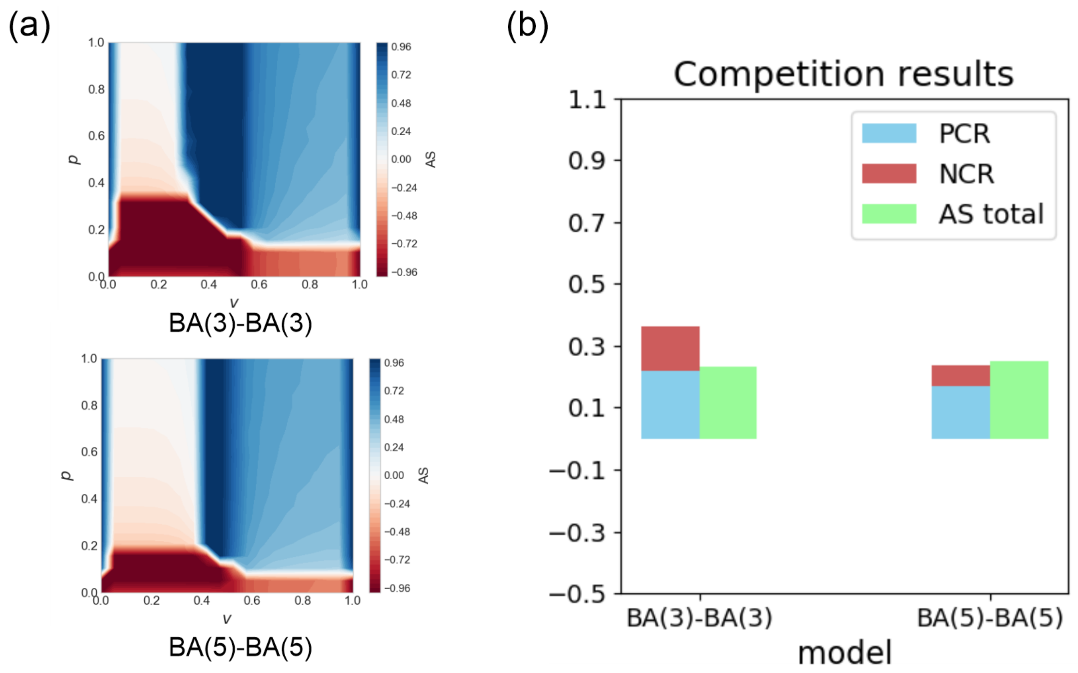


Figure 3–15 BA-BA networks with changing internal degrees

key nodes on *BA* network would be simulated and analyzed in chapter. 5.

3.5 Conclusion

Various simulations have been simulated to find out the role of internal and external degrees and the influence of network types. All results of simulations are shown in Table. 3–1. Through the simulation results, several facts would be arranged like the followings. If there are no stubborn nodes, network types do not make different result for network state and consensus. But, we can provide three conclusions about the roles of internal and external degrees. First, hierarchical models show that it is easy to make consensus on both layers when the number of external edges in decision making is more than opinion layer. Second, the number of internal degrees on layer A has the tendency to keep positive state and to change negative state into positive state. Third, the number of internal degrees on layer B has the tendency to hinder positive state. Forth, too many internal edges on each layer can cause inner conflict, and that makes it hard to have consensus state. We could apply these facts to make network structures or organization in real world.

Table 3–1 Consensus properties of Simulation Models

Div	A nodes	B nodes	A edges	B edges	AS total	PCR	NCR	CR
RR(2)-RR(5)	2,048	2,048	2,048	5,120	-0.3186	0.0550	0.3175	0.3725
RR(3)-RR(5)	2,048	2,048	3,072	5,120	-0.1368	0.1400	0.2925	0.4325
RR(4)-RR(5)	2,048	2,048	4,096	5,120	0.0804	0.2250	0.2275	0.4525
RR(5)-RR(2)	2,048	2,048	5,120	2,048	0.4927	0.6725	0.1725	0.8450
RR(5)-RR(3)	2,048	2,048	5,120	3,072	0.3555	0.3800	0.1525	0.5325
RR(5)-RR(4)	2,048	2,048	5,120	4,096	0.2633	0.2850	0.1525	0.4375
RR(2)-RR(2)	2,048	2,048	2,048	2,048	-0.1412	0.1475	0.4050	0.5525
RR(3)-RR(3)	2,048	2,048	3,072	3,072	0.0084	0.2275	0.2825	0.5100
RR(4)-RR(4)	2,048	2,048	4,096	4,096	0.1448	0.2525	0.2075	0.4600
RR(5)-RR(5)	2,048	2,048	5,120	5,120	0.2034	0.2475	0.1575	0.4050
RR(6)-RR(6)	2,048	2,048	6,144	6,144	0.2444	0.2350	0.1375	0.3725
RR(6)-BA(3)	2,048	2,048	6,144	6,135	0.2541	0.2275	0.1300	0.3575
BA(3)-RR(6)	2,048	2,048	6,135	6,144	0.2242	0.2300	0.1425	0.3725
BA(3)-BA(3)	2,048	2,048	6,135	6,135	0.2329	0.2200	0.1400	0.3600
BA(5)-BA(5)	2,048	2,048	10,215	10,215	0.2496	0.1675	0.0675	0.2350
HM(2)	2,048	1,024	5,120	2,560	0.3073	0.3275	0.1425	0.4700
HM(4)	2,048	512	5,120	1,280	0.4128	0.4125	0.1275	0.5400
HM(8)	2,048	256	5,120	640	0.4846	0.4925	0.1150	0.6075
HM(16)	2,048	128	5,120	320	0.5610	0.5800	0.1100	0.6900
HM(32)	2,048	64	5,120	160	0.5959	0.6275	0.1025	0.7300
HM(64)	2,048	32	5,120	80	0.6185	0.6775	0.1025	0.7800
HM(128)	2,048	16	5,120	40	0.6379	0.7350	0.0900	0.8250
HM(256)	2,048	8	5,120	20	0.6454	0.7675	0.0900	0.8575

Chapter 4 Competition on two layer with time-related updating rules

Here, we would control dynamics orders between layers and time-related updating rules of nodes states. With changing dynamics orders and time-related updating rules, it would be investigated how the state of network is changed. In this chapter, each layer consists of *Barabasi-Albert(BA)* network that has N nodes with attaching new nodes each with K edges that are preferentially attached to existing nodes with high degree as introduced in [34]. Each node of one layer is connected with a random node on the other layer. This means each node has only 1 external un-directed edge. Simulations are preformed on network with $N = 2048$, and $K = 3$.

When considering dynamics order on two-layer networks, there are many ways to update the state of nodes. Dynamics order of two layer can be considered whether layer A works first or layer B works first or both layers work together. And, nodes can be thought about whether the states of nodes are changed simultaneously or sequentially or randomly. Links connected with a node also can be deliberated whether links are activated on a node sequentially or simultaneously or randomly. But, in layer B dynamics, order of edges in one node is always for simultaneous updating rule, because dynamics formula already considers states of all connected neighbor nodes simultaneously. To sum up, as shown in Table.4–1, 25 updating rules would be considered according to layers, nodes and edges.

In table remarks, 'O(o, o) $\rightarrow$ D(s)' means Opinion layer(node : sequential order updating, edges : sequential order updating) $\rightarrow$ Decision Making layer(node : simultaneous updating). And 'O(o, o) $\Leftrightarrow$ D(o)' means that one node in Opinion layer is updated, and then one node in Decision Making layer is updated, this rule is repeated until all nodes are updated. Dynamics with 25 updating rules are simulated with parameter $p = 0.4$ and $v = 0.4$. Simulation results are divided by order of layers, nodes and edges.

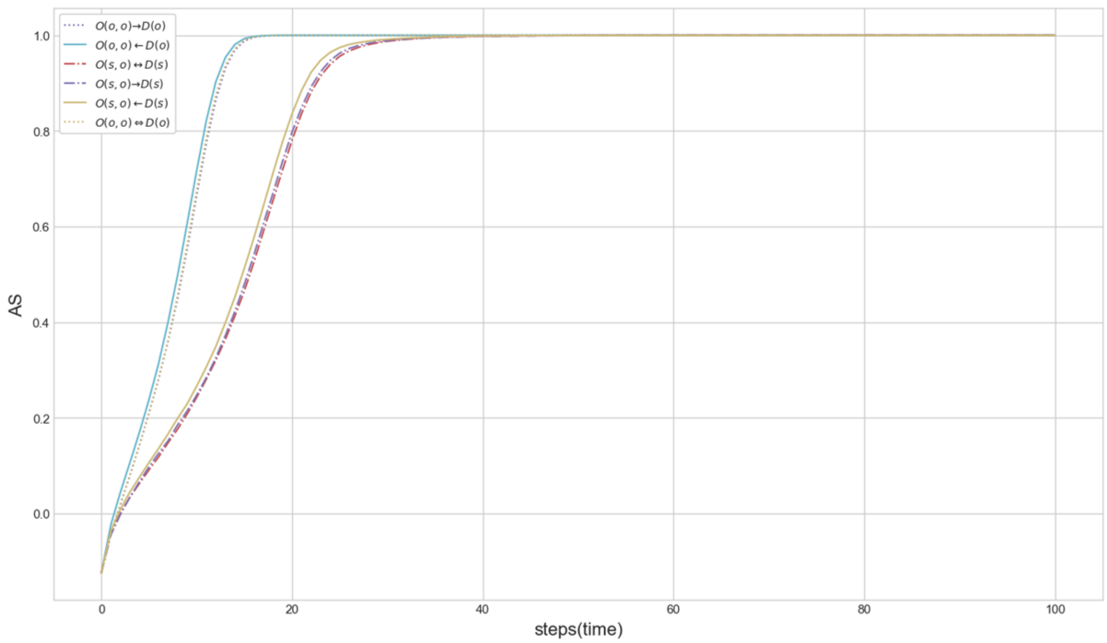
4.1 Order of layers

There exist two layers on interconnected network. And each layer have its own dynamics, such as *M-Model* and *AS-Model*. Two dynamics can be operated simultaneously or sequentially. If they act sequentially, dynamics of layer A can act first or dynamics

Order of layers	Layer A		Layer B	remarks
	Order of nodes	Order of edges	Order of nodes	
Layer A → Layer B	Sequential	Sequential	Sequential	O(o, o) → D(o)
			Simultaneous	O(o, o) → D(s)
		Simultaneous	Sequential	O(o, s) → D(o)
			Simultaneous	O(o, s) → D(s)
	Simultaneous	Sequential	Sequential	O(s, o) → D(o)
			Simultaneous	O(s, o) → D(s)
		Simultaneous	Sequential	O(s, s) → D(o)
			Simultaneous	O(s, s) → D(s)
	Random	Random	Sequential	O(r, r) → D(o)
			Simultaneous	O(r, r) → D(s)
Layer A ← Layer B	Sequential	Sequential	Sequential	O(o, o) ← D(o)
			Simultaneous	O(o, o) ← D(s)
		Simultaneous	Sequential	O(o, s) ← D(o)
			Simultaneous	O(o, s) ← D(s)
	Simultaneous	Sequential	Sequential	O(s, o) ← D(o)
			Simultaneous	O(s, o) ← D(s)
		Simultaneous	Sequential	O(s, s) ← D(o)
			Simultaneous	O(s, s) ← D(s)
	Random	Random	Sequential	O(r, r) ← D(o)
			Simultaneous	O(r, r) ← D(s)
Layer A ↔ Layer B	Simultaneous	Sequential	Simultaneous	O(s, o) ↔ D(s)
		Simultaneous	Simultaneous	O(s, s) ↔ D(s)
Layer A ⇔ Layer B	Sequential	Sequential	Sequential	O(o, o) ⇔ D(o)
		Simultaneous	Sequential	O(o, s) ⇔ D(o)
	Random	Random	Random	O(r, r) ⇔ D(r)

Table 4–1 25 updating rules according to order of layers, nodes, and edges

of layer B can work previously. Otherwise, regardless of layers order, nodes of two layers can interact mutually, i.e. one node in layer A are updated and then one node in layer B are updated until all nodes are updated. Considering all situations, there are 4 ways in order of two layers, *Layer A → Layer B, Layer A ← Layer B, Layer A ↔ Layer B(simultaneous), Layer A ⇔ Layer B(interaction regardless of layers)*. Fig. 4–1 shows 4 simulation results related to orders of layers. As seen in Fig. **??**, it is shown that there is little difference between orders of layers. Consensus time and result are almost same, though dynamics order is different. Regardless of dynamics directions, when other conditions, such as order of nodes and edges are same, the dynamics results are also very similar. Dynamics order of layers does not have an significant influence on the network state.



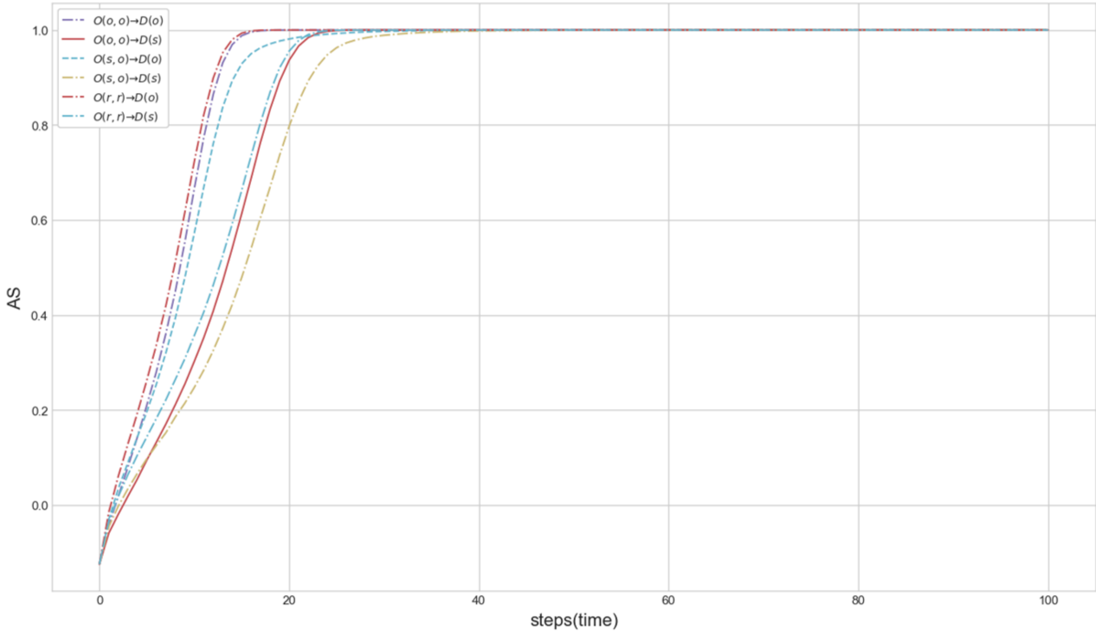
Div	Fast Consensus	Slow Consensus
Orders	① O(o, o) ← D(o) ② O(o, o) ⇔ D(o) ③ O(o, o) → D(o)	④ O(s, o) ← D(s) ⑤ O(s, o) → D(s) ⑥ O(s, o) ↔ D(s)

Figure 4–1 Simulation results according to orders of layers

4.2 Order of nodes

In the simulation model, each layer has 2048 nodes, and each node has interaction with other nodes. Now, interaction order of nodes would be considered. One node can be updated sequentially after neighbor nodes are updated. Otherwise, every node can be updated simultaneously. Simulation results would be different according to interaction order of nodes. In addition, random order between nodes is also simulated. In random order, one edge is selected randomly and updated until all edges are considered regardless of orders in nodes or links. Interaction order of nodes have meaning related to time. If networks have short time to change states, networks follow simultaneous updating rule. However, if networks have enough time to update states, networks follow sequential updating rules. In real world, discussion or conversation with enough time means sequential updating rule of nodes, and vote or election means simultaneous updating rule of nodes.

Fig. 4–2 shows simulation results. The results are classified to two categories, fast consensus and slow consensus. It is shown that simultaneous interaction between nodes



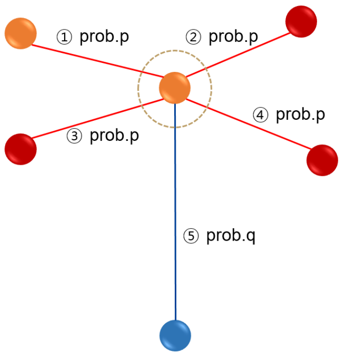
Div	Fast Consensus	Slow Consensus
Orders	① O(r, r) → D(o) ② O(o, o) → D(o) ③ O(s, o) → D(o)	④ O(r, r) → D(s) ⑤ O(o, o) → D(s) ⑥ O(s, o) → D(s)

Figure 4–2 Simulation results according to orders of nodes: comparison between order of nodes under same conditions such as order of layers and edges.

makes slow consensus. Simultaneous order in layer A does not make large difference, but it make consensus slightly slow. Simultaneous interaction between nodes in layer B have more influence on consensus time than in layer A. Random order has similar results with sequential order and does not make different states. For quick social consensus, both opinion layer and decision making layer need sequential updating rule, such as conversation and discussion.

4.3 Order of edges

Each node has several edges connected with other nodes. Simulation results can be different according to that edges are activated sequentially or simultaneously. If edges of each node work sequentially, a state of node is changed whenever each edge is activated. However, If edges of a node are activated simultaneously, a state of node would be changed considering all connected nodes. In real world, order of edges in one node can be analyzed as characteristics of nodes. If order of edges is sequential, the node would be rash. If order of edges is simultaneous, the node would be considerate. For example,



Order of edges	Next state
Sequential	Hard to calculate (According to order of activated edges, final state is changed)
Simultaneous	Unchanged: prob.p(activated) = prob.q(activated) Persuasion : prob.p(activated) > prob.q(activated) Compromise: prob.p(activated) < prob.q(activated)

Figure 4–3 one node connected with other nodes changes its state with sequential or simultaneous order of edges

considering the case that one node is connected with other nodes as shown in Fig. 4–3, we can think how the state of node change. If the edges follow sequential updating rule, it is hard to calculate the probabilities, because the states can change according to sequential order of edges. Therefore, we can get next states of nodes by using computer simulation

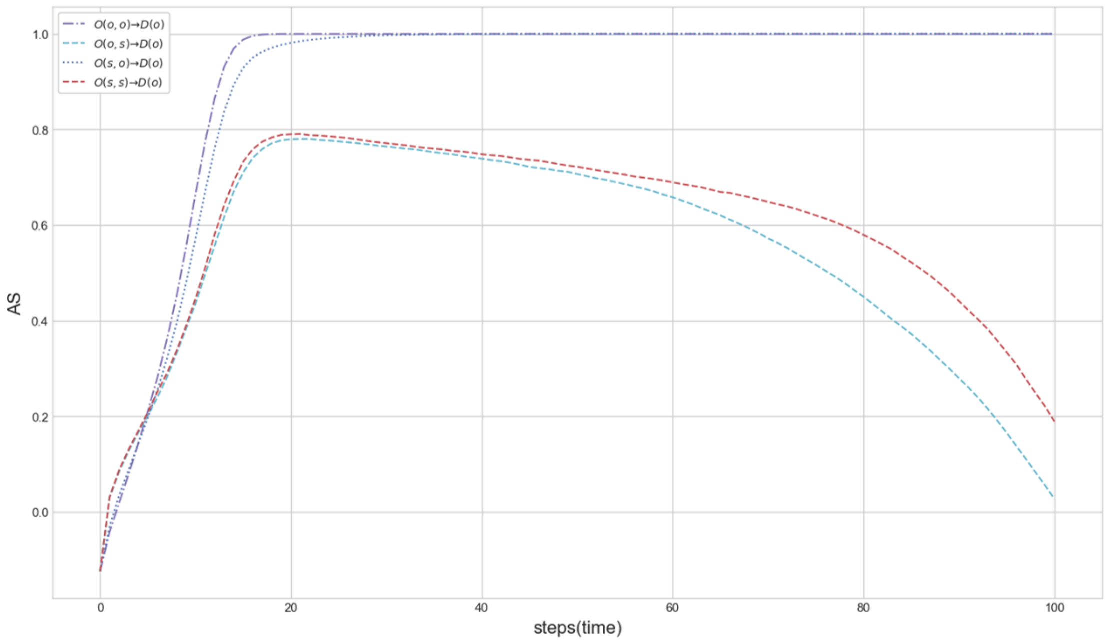
If the edges follow simultaneous updating rule, it needs some assumptions:

1. If the number of activated *prob.p* is more than the number of activated *prob.q*, persuasion dynamics would work.
2. If the number of activated *prob.p* is same with the number of activated *prob.q*, the state would be unchanged.
3. If the number of activated *prob.p* is less than the number of activated *prob.q*, compromise dynamics would work.

Through these assumptions, we can calculate probabilities of changing state in layer by considering all cases like these formula.

$$K = \{k \quad | \quad 0, \cdots, n^{-S_i}\}, \quad L = \{l \quad | \quad 0, \cdots, n^{S_i}\}, \quad M = \{m \quad | \quad k - l\},$$
$$P_A(S_i \mapsto S'_i) = \begin{cases} \text{unchanged}(k = l) : \sum p^{n^{-S_i}+m} \cdot (1-p)^{n^{S_i}-m} \cdot {}_{n^{S_i}}C_k \cdot {}_{n^{-S_i}}C_l \\ \text{persuasion}(k > l) : \sum p^{n^{-S_i}+m} \cdot (1-p)^{n^{S_i}-m} \cdot {}_{n^{S_i}}C_k \cdot {}_{n^{-S_i}}C_l \\ \text{compromise}(k < l) : \sum p^{n^{-S_i}+m} \cdot (1-p)^{n^{S_i}-m} \cdot {}_{n^{S_i}}C_k \cdot {}_{n^{-S_i}}C_l \end{cases} \tag{4–1}$$

Fig. 4–4 shows the simulation result according to order of edges. The results are categorized to consensus and coexistence(not reaching consensus) due to order of edges.



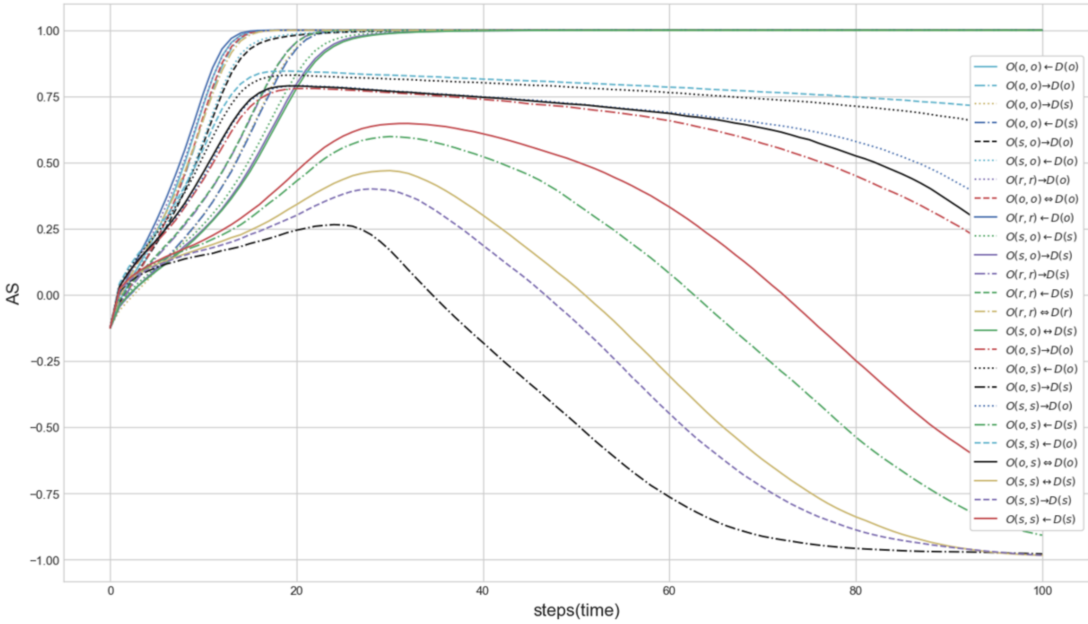
Div	Consensus	Not reaching consensus
Orders	① O(o, o) → D(o) ② O(s, o) → D(o)	③ O(o, s) → D(o) ④ O(s, s) → D(o)

Figure 4–4 Simulation results according to orders of edges: comparison between order of edges under same conditions such as order of layers and nodes

Sequential updating rule of edges makes consensus, i.e. rash nodes make consensus. But simultaneous updating rule of edges makes it hard to reach consensus, i.e. considerate nodes do not make consensus. It can be analyzed that rash node is easy to be extreme and make consensus, but considerate node is very moderate and hard to reach consensus.

4.4 Comparison and Analysis

It is found out that there are different simulation results according to orders of layers, nodes, and edges. To sum up all updating rules, they can be categorized into 3 parts, positive consensus, coexistence, and negative consensus as shown in Fig. 4–5. To clearly classify the state of two-layers, the results can be analyzed by using *CI* as shown in Fig. 4–6. There are three branch points. In the first branch point, the results are divided according to whether order of nodes in layer B is sequential or simultaneous. In the second and third branch point, the results are divided according to whether order of edges



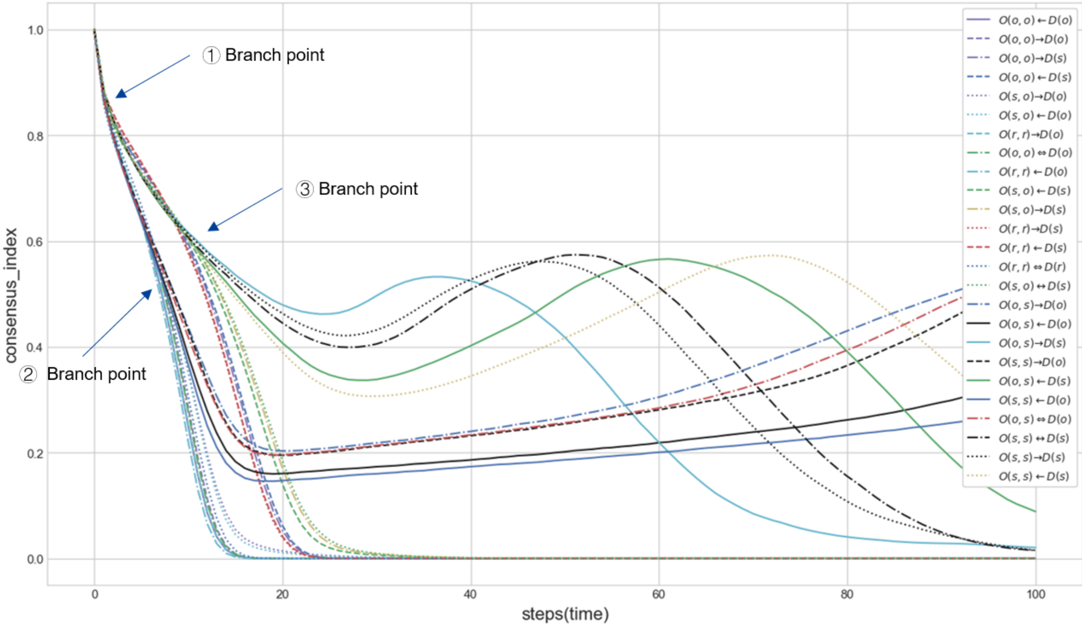
Div	Positive Consensus (close to positive)	Coexistence	Negative Consensus (close to negative)
Orders	O(r, r) ← D(o) O(r, r) → D(o) O(o, o) ← D(o) O(o, o) ⇔ D(o) O(o, o) → D(o) O(r, r) ⇔ D(r) O(r, r) → D(s) O(r, r) ← D(s) O(o, o) → D(s) O(o, o) ← D(s) O(s, o) ← D(o) O(s, o) → D(o) O(s, o) ← D(s) O(s, o) → D(s) O(s, o) ↔ D(s)	O(s, s) ← D(o) O(o, s) ← D(o) O(s, s) → D(o) O(o, s) ⇔ D(o) O(o, s) → D(o)	O(s, s) ← D(s) O(o, s) ← D(s) O(o, s) → D(s) O(s, s) → D(s) O(s, s) ↔ D(s)

Figure 4–5 Total results of 25 updating rules with *AS*

in layer A is sequential or simultaneous. As the results, there are 4 categories such as fast positive consensus, slow positive consensus, coexistence and slow negative consensus.

4.5 Conclusion

Through these results, several important facts can be arranged. First, networks with more simultaneous updating rules make slow consensus or coexistence, sometimes make transition to opposite orientation. On the other hands, networks with more sequential



Div.		States	Orders
① Branch point	② Branch point : Sequential order of node in layer B	Fast positive consensus : Sequential order of edge	O(r, r) ← D(o) O(r, r) → D(o) O(o, o) ← D(o) O(o, o) ⇔ D(o) O(o, o) → D(o) O(r, r) ⇔ D(r) O(s, o) ← D(o) O(s, o) → D(o)
		Coexistence : Simultaneous order of edge	O(s, s) ← D(o) O(o, s) ← D(o) O(s, s) → D(o) O(o, s) ⇔ D(o) O(o, s) → D(o)
	③ Branch point : Simultaneous order of node in layer B	Slow positive consensus : Sequential order of edge	O(r, r) → D(s) O(r, r) ← D(s) O(o, o) → D(s) O(o, o) ← D(s) O(s, o) ← D(s) O(s, o) → D(s) O(s, o) ↔ D(s)
		Slow negative consensus : Simultaneous order of edge	O(s, s) ← D(s) O(o, s) ← D(s) O(o, s) → D(s) O(s, s) → D(s) O(s, s) ↔ D(s)

Figure 4–6 Total results of 25 updating rules with *CI*

updating rules make fast consensus. In other words, if opinion layer has more rash nodes, more time to have some conversation and decision making layer has more time to discuss topics, the network have more probabilities to make consensus for opinion layer. Second, dynamics order between layers does not have an influence for network state, though there exists tiny consensus time gap. Third, order of nodes in layer B has more influence for network states than order of nodes in layer A. order of nodes in layer B makes the first branch point. But order of nodes in layer A does not make any branch point, though there exists tiny consensus time gap. It means that the communication method is very important in decision making layer. Forth, order of edges in layer A is very influential so that it makes different network states. It can be analyzed as that characteristics of nodes in layer A, such as rash and considerate, affects consensus time and sometimes makes transition to coexistence or opposite orientation.

Chapter 5 Finding key nodes on two layer networks

In this chapter, it would be investigated that what nodes are important to keep or change their orientation on two-layer networks. There exist many methods to find key nodes, such as pagerank, degree centrality, and eigenvector centrality. And, in [42, 44], it has been proved that multiple indicators are useful to identify key nodes and prevent the slow way to find important nodes. Based on these methods such as single node centrality and combined node centrality, it would be researched that which method is the most effective and the most influential for changing state on two layers.

5.1 Method for finding key nodes

As initial condition for finding key nodes, each layer is made of *BA* network with 512 nodes, $K = 3$, and 1 external edge. Each simulation takes 100 steps, and 100 simulations are considered for average results. To demonstrate the difference of network state clearly, for finding key nodes on layer A, the parameters would be set to be negative consensus state. Then, as the stubborn nodes on layer A are increased, the network state would be gradually changed into positive state. Inversely for finding key nodes on layer B, the parameters would be set to be positive consensus state. Then, as the stubborn nodes on layer B are increased, the network state would be gradually changed into negative state. Here is the way to find key nodes by using single node centrality.

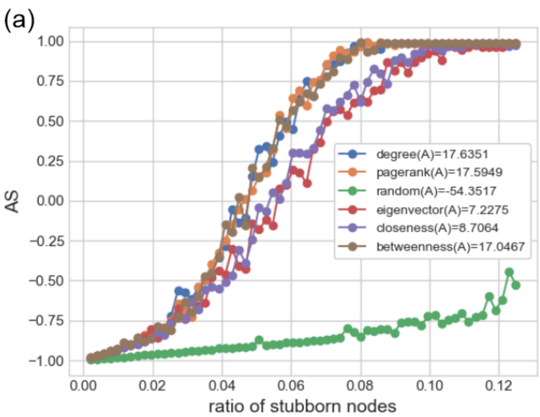
1. All nodes are ranked by 6 node centralities(pagerank, degree, eigenvector, closeness, betweenness).
2. The nodes would be deactivated from high ranked order until the state of network has significant difference, i.e. the ratio of stubborn node would be increased according to high ranked order.
3. The results would be compared according to node centralities. If the least ratio of stubborn node makes the largest difference of network state, its node centrality is the most influential for competition of the interconnected network
4. To clarify which method is the most effective, each single indicator is calculated with summation of all AS according to the ratio of stubborn nodes. That means the summation of graph line. It could be recognized that the larger the value is

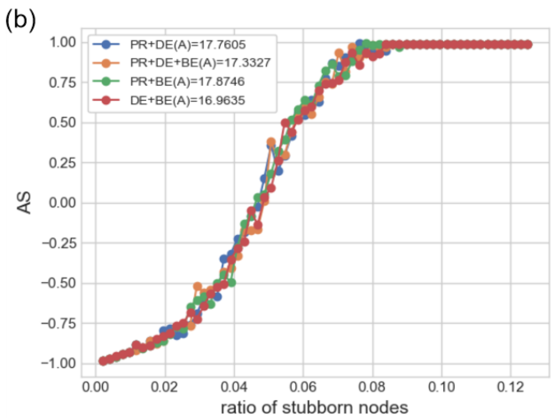
on layer A, the more influential that indicator is, inversely the smaller the value is on layer B, the more influential that indicator is.

And, we would research the way to recognize important node by using multiple indicator such as combined node centrality. Combined node centrality is made up with several selected node centralities. When it is proven that a node centrality is effective to find key nodes through the simulations, it would be selected as a factor of combined node centrality. 2 or 3 node centralities would be selected. The way to recognize key nodes by using combined node centrality follow like this steps.

1. All nodes are ranked by each selected node centralities. All nodes has the ranks as the number of selected node centralities.
2. Combined node centrality is the summation of all ranks which a node has.
3. All nodes are ranked again by combined node centrality. As the combined node centrality is smaller, a node are ranked higher.
4. The nodes would be deactivated from high ranked order until the state of network has significant difference, i.e. the ratio of stubborn node would be increased according to high ranked order.

5.2 Key nodes on layer A





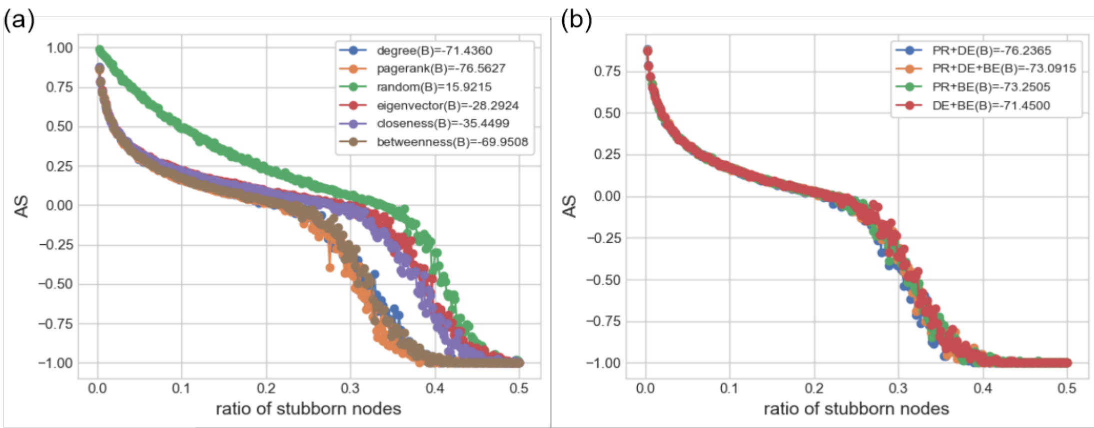
Div(single)	DE	PR	RA	EI	CL	BE
AS	17.6351	17.5949	-54.3517	7.2275	8.7064	17.0467

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	17.7605	17.3327	17.8746	16.9635

Figure 5–1 Key nodes on layer A in BA-BA network($p = 0.2, v = 0.4$) : (a) Single indicator methods, (b) Multiple indicator methods

To find key nodes on layer A, parameters are set to be negative consensus state like $p = 0.2, v = 0.4$. And we would separate single indicators and multiple indicators to identify and compare the simulation results. As single indicators, 5 node centralities(pagerank, degree, eigenvector, closeness, betweenness) are used, and random selected nodes are compared with 5 node centralities. As multiple indicators, 2 or 3 node centralities are combined such as pagerank, degree and betweenness which have good performance as single indicators. Fig. 5–1 shows the simulation result for recognizing keynode on layer A. As single indicator, degree centrality has the best performance. The next rank is pagerank and betweenness. As multiple indicator, *PR+BE* has the most effective result. The next is *PR+DE*. These two methods of multiple indicators work better than degree centrality. Totally, compared with all methods, the best method is PR+BE. It could be found out that some multiple indicators are more effective than single indicators.

5.3 Key nodes on layer B



Div(single)	DE	PR	RA	EI	CL	BE
AS	-71.4360	-76.5627	15.9215	-28.2924	-35.4499	-69.9508

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	-76.2365	-73.0915	-73.2505	-71.4500

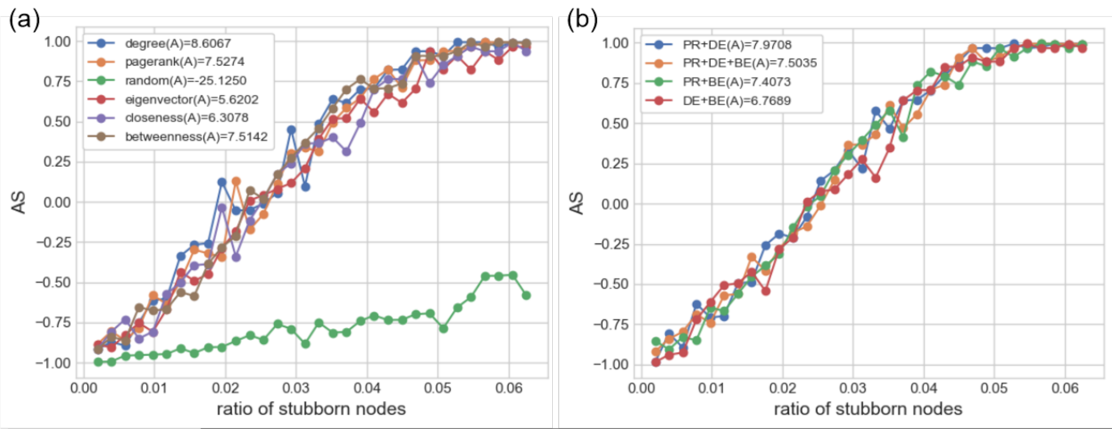
Figure 5–2 Key nodes on layer B in BA-BA network($p = 0.3, v = 0.5$) : (a) Single indicator methods, (b) Multiple indicator methods

To find key nodes on layer B, parameters are set to be positive consensus state like $p = 0.3, v = 0.5$. Fig. 5–2 shows the simulation result for identifying key nodes on layer B. As single indicators, the most effective way to recognize important nodes is pagerank centrality. The next ranks are degree and betweenness. As multiple indicators, *PR+DE* has the best performance. Totally, pagerank is the most effective method for finding key nodes on layer B. However, all multiple indicators work better than degree centrality, the second rank in single indicators. It could be found out that combined node centralities have good performance to find key nodes, though they are not the best.

5.4 Key nodes on two layers with different structures

In this section, we would try to find the key nodes in the various structural networks described in chapter. 3. Node centralities and combined node centralities are also used as the methods for finding key nodes. First, Hierarchical Model would be applied to identify important nodes. Second, we would consider the case that each layer has different network type, such as *BA*-*RR* or *RR*-*BA* networks. Third, it would be taken into account that each layer has different number of edges. Layer A would have more internal links or Layer B would be have more internal links. Both cases would be checked.

5.4.1 Key nodes in Hierarchical Model

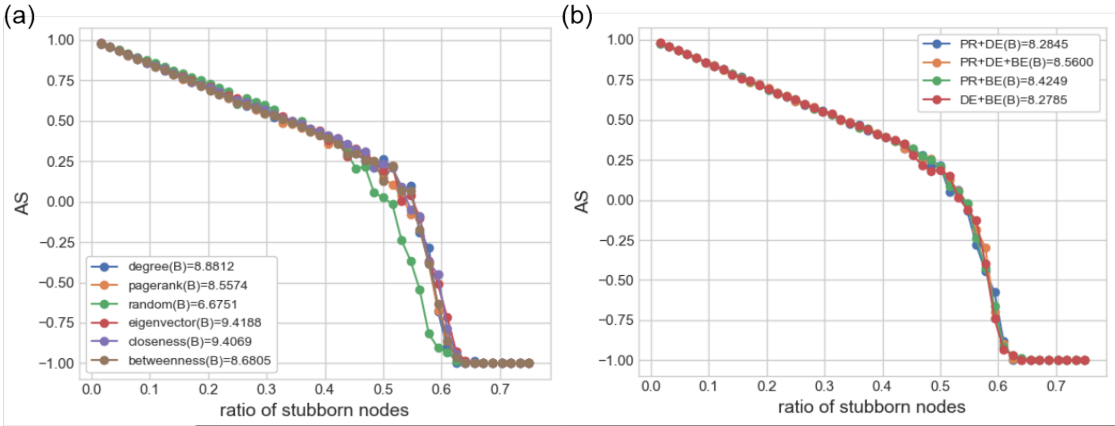


Div(single)	DE	PR	RA	EI	CL	BE
AS	8.6067	7.5274	-25.1250	5.6202	6.3078	7.5142

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	7.9708	7.5035	7.4073	6.7689

Figure 5–3 Key nodes on layer A in Hierarchical Model($p = 0.2, v = 0.2$): (a) Single indicator methods, (b) Multiple indicator methods

Each layer consists of *BA* network with $k = 3$. Layer A has 512 nodes, and layer B has 64 nodes. We denote these models as *HM(8) with BA(3)*. Fig. 5–3 shows the simulation result of key nodes on layer A.



Div(single)	DE	PR	RA	EI	CL	BE
AS	8.8812	8.5574	6.6751	9.4188	9.4069	8.6805

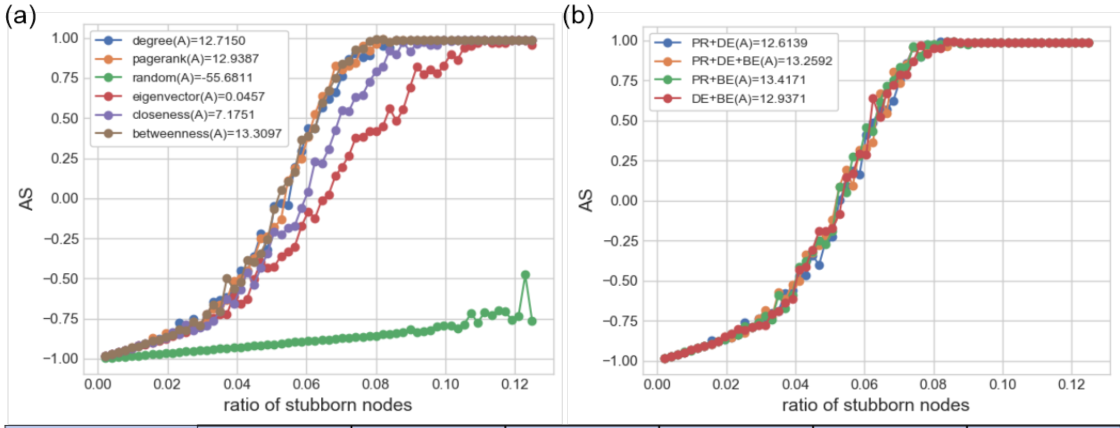
Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	8.2845	8.5600	8.4249	8.2785

Figure 5–4 Key nodes on layer B in Hierarchical Model($p = 0.25, v = 0.3$): (a) Single indicator methods, (b) Multiple indicator methods

Simulations results show that degree centrality is the best method for recognizing key nodes on *HM*. The second method is *PR+DE* as multiple indicator. The curve of changing the network states shown in Fig. 5–3 is more straight than Fig. 5–1. That means the speed of changing network states is faster. Fig. 5–4 shows the simulation result of key nodes on layer B. However, the result is different from other simulation results. The best performance method is random method. That means node centralities do not work on this model. And the curve of changing the network states shown in Fig. 5–4 is also more straight than Fig. 5–2. That means the speed of changing network states is faster. Decreasing the number of nodes in layer B make it hard to recognizing key nodes and make it easy to have consensus of two layers.

5.4.2 Key nodes on two layers with different network types

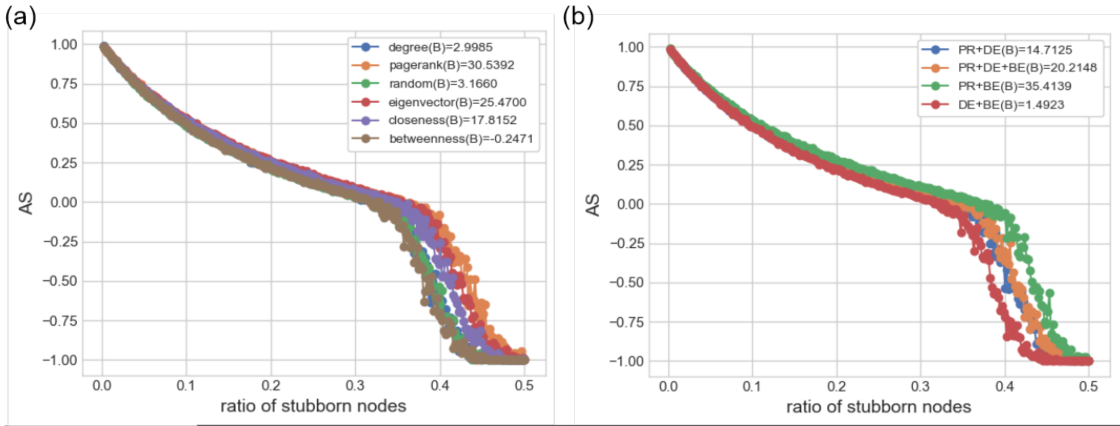
Here, we would consider two types of network, *BA-RR* and *RR-BA*. The number of internal links on each layer would be set up as same or almost same number to exclude the influence of internal links. These models would be compared with *BA-BA* to find out the influence of network types under same conditions, such as p, v, and $ratio of stubborn nodes$. First, *BA-RR* network would be investigated. Fig. 5–5 shows the simulation result of key nodes on layer A. Betweenness is the most influential method



Div(single)	DE	PR	RA	EI	CL	BE
AS	12.7150	12.9387	-55.6811	0.0457	7.1751	13.3097

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	12.6139	13.2592	13.4171	12.9371

Figure 5–5 Key nodes on layer A in BA-RR Model($p = 0.2, v = 0.4$): (a) Single indicator methods, (b) Multiple indicator methods



Div(single)	DE	PR	RA	EI	CL	BE
AS	2.9985	30.5392	3.1660	25.4700	17.8152	-0.2471

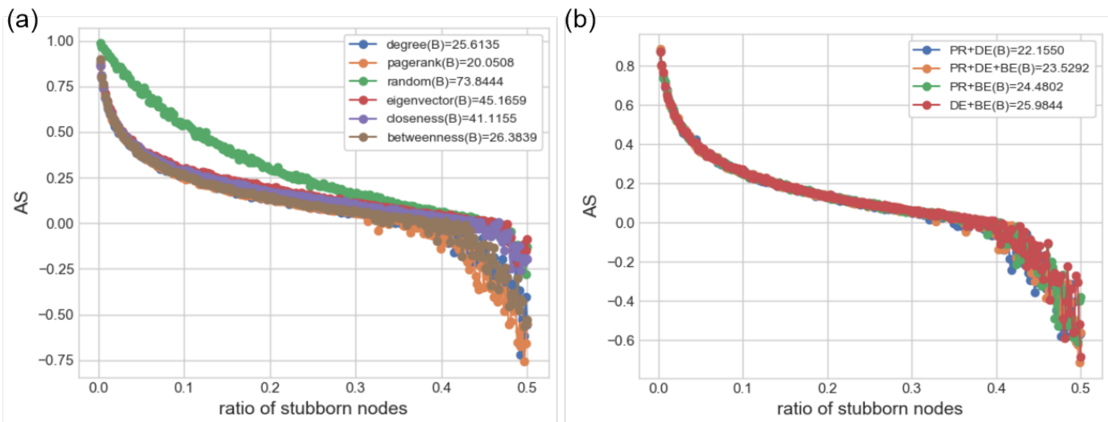
Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	14.7125	20.2148	35.4139	1.4923

Figure 5–6 Key nodes on layer B in BA-RR Model($p = 0.3, v = 0.5$): (a) Single indicator methods, (b) Multiple indicator methods

in the single indicators. However, *PR+BE* has better performance than betweenness. Compared with *BA-BA* shown in Fig. 5–1, *BA-RR* has smaller *AS* values and more gentle curve to change the state of network.

Fig. 5–6 shows the simulation result of key nodes on layer B. Betweenness is the best method for finding key nodes on layer B in *BA*-*RR* network. In this model, degree is not meaningful method because degree of each node is same in *RR* network. However, degree and random method is the third and fourth method for recognizing key nodes. That means other methods except for betweenness do not work for finding key nodes. Compared with *BA*-*BA* shown in Fig. 5–2, *BA*-*RR* has larger *AS* values and more gentle curve to change the state of network.

Next, *RR*-*BA* network would be considered. Fig. 5–7 shows the simulation result of key nodes on layer B. Compared with *BA*-*BA* shown in Fig. 5–2 , *RR*-*BA* also has larger *AS* values and more gentle curve to change the state of network.



Div(single)	DE	PR	RA	EI	CL	BE
AS	25.6135	20.0508	73.8444	45.1659	41.1155	26.3839

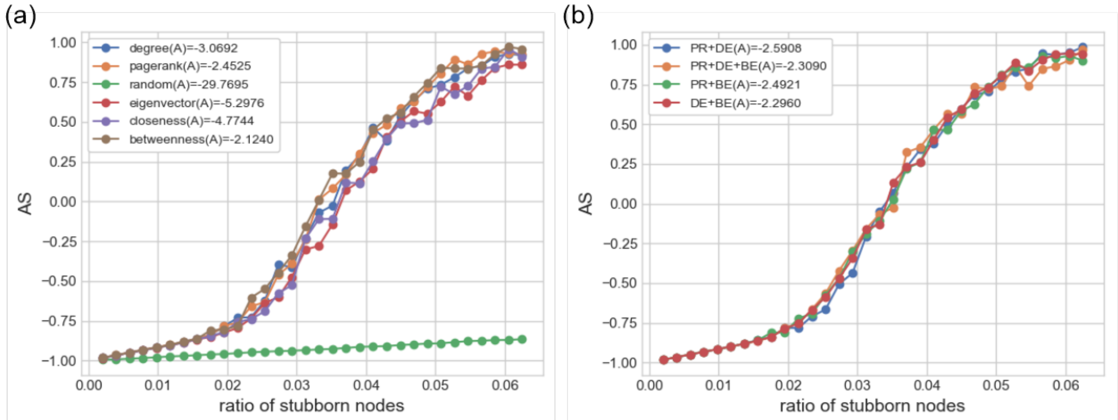
Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	22.1550	23.5292	24.4802	25.9844

Figure 5–7 Key nodes on layer B in RR-BA Model($p = 0.3, v = 0.5$): (a) Single indicator methods, (b) Multiple indicator methods

Totally, compared with *BA*-*BA* network, both *BA*-*RR* and *RR*-*BA* have more gentle curve. It could be analyzed that *RR* network makes it slow to change the state. And, betweenness has good performance for *RR* network, but other methods do not work.

5.4.3 Key nodes on two layers with different number of internal links

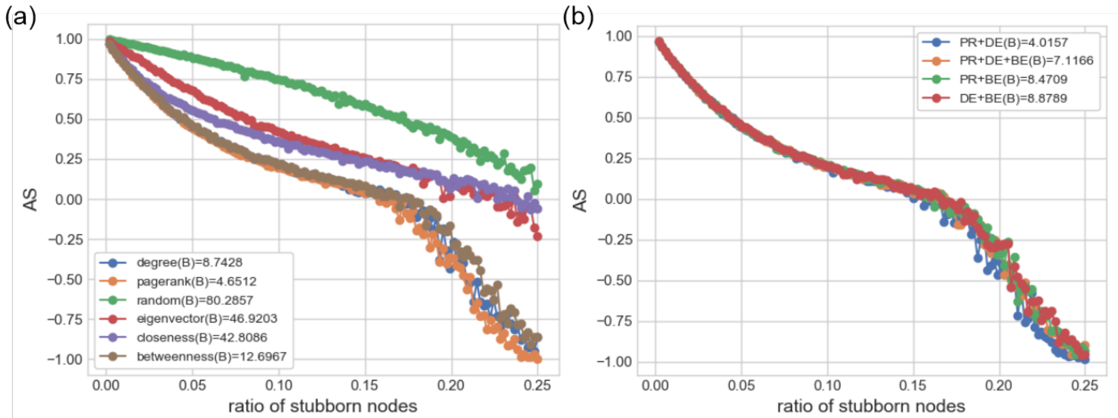
In case that layer A has more internal links, layer A consists of *BA* network with $k = 4$, but Layer B consists of *BA* network with $k = 2$. Inversely, in case that layer B has more internal links, layer B consists of *BA* network with $k = 4$, but Layer A consists of *BA* network with $k = 2$. First, the case of more internal links on layer A than layer



Div(single)	DE	PR	RA	EI	CL	BE
AS	-3.0692	-2.4525	-29.7695	-5.2976	-4.7744	-2.1240

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	-2.5908	-2.3090	-2.4921	-2.2960

Figure 5–8 Key nodes on layer A in BA(4)-BA(2) Model($p = 0.15, v = 0.3$): (a) Single indicator methods, (b) Multiple indicator methods



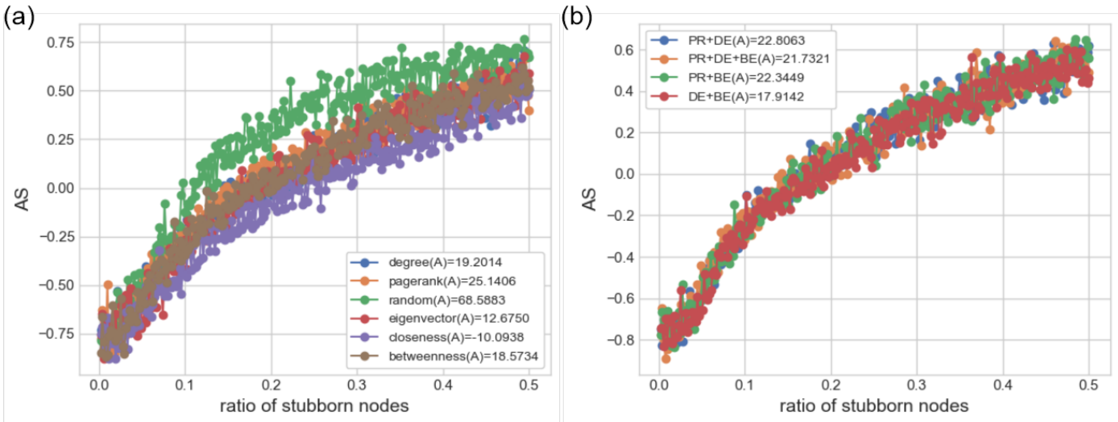
Div(single)	DE	PR	RA	EI	CL	BE
AS	8.7428	4.6512	80.2857	46.9203	42.8086	12.6967

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	4.0157	7.1166	8.4709	8.8789

Figure 5–9 Key nodes on layer B in BA(4)-BA(2) Model($p = 0.2, v = 0.4$): (a) Single indicator methods, (b) Multiple indicator methods

B would be investigated. Fig. 5–8 shows the simulation result of key nodes on layer A in *BA(4)-BA(2)* network. Betweenness has the best performance for changing network states. Next ranks are *DE+BE*, *PR+DE+BE* and pagerank. Betweenness is the best method, but totally multiple indicators work well for changing the network state. Fig. 5–9 shows the simulation result of key nodes on layer B in *BA(4)-BA(2)* network. *PR+DE* is the most influential method. Next ranks are pagerank, *PR+DE+BE* and *PR+BE*. The most effective method on layer B is different from the most influential method on layer A. However, totally, multiple indicators also work well for changing the network states.

Next, the case of more internal links on layer B than layer B would be researched.

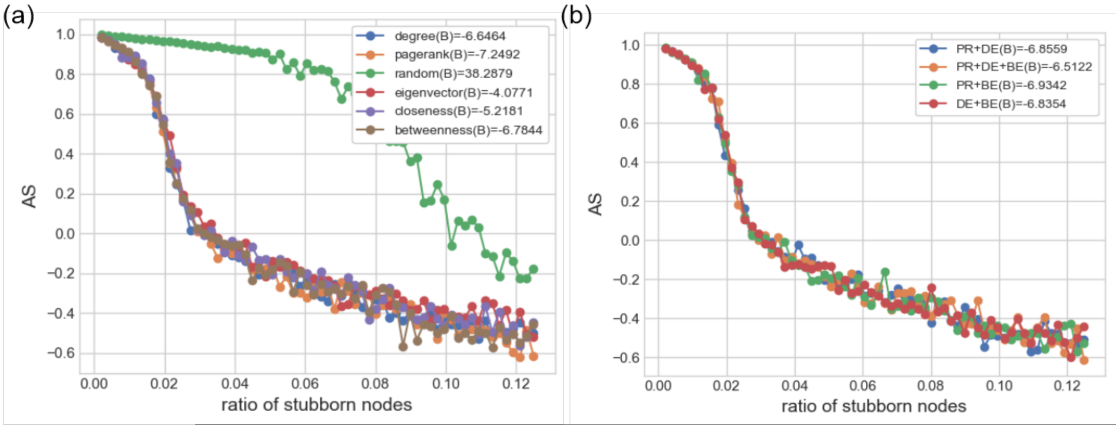


Div(single)	DE	PR	RA	EI	CL	BE
AS	19.2014	25.1406	68.5883	12.6750	-10.0938	18.5734

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	22.8063	21.7321	22.3449	17.9142

Figure 5–10 Key nodes on layer A in BA(2)-BA(4) Model(p = 0.57, v = 0.37): (a) Single indicator methods, (b) Multiple indicator methods

Fig. 5–10 shows the simulation result of key nodes on layer A in *BA(2)-BA(4)* network. However, the simulation results are different from other results, because random method has the best performance. That means node centralities do not work on this model. Compared with *BA(4)-BA(2)* network, the curve of changing the state that is shown in Fig. 5–10 is much slower. Decreasing the number of internal links on layer A makes it hard to find key nodes and to have positive consensus. Fig. 5–11 shows the simulation result of key nodes on layer B in *BA(2)-BA(4)* network. Pagerank has the most effective performance. Next ranks are *PR+BE*, *PR+DE* and *DE+BE*. Compared with *BA(4)-BA(2)* network, the curve of changing the state that is shown in Fig. 5–10 is much faster. But consensus doesn't happen in this model. It could be analyzed that decreasing the number of internal links on layer A have influence on making consensus of two layers hard .



Div(single)	DE	PR	RA	EI	CL	BE
AS	-6.6464	-7.2492	38.2879	-4.0771	-5.2181	-6.7844

Div(multiple)	PR+DE	PR+DE+BE	PR+BE	DE+BE
AS	-6.8559	-6.5122	-6.9342	-6.8354

Figure 5–11 Key nodes on layer B in BA(2)-BA(4) Model($p = 0.6, v = 0.4$): (a) Single indicator methods, (b) Multiple indicator methods

5.5 Conclusion

By using node centrality, key nodes on each layer have been found out on various structural networks. Table. 5–1 shows total simulation results for finding key nodes on various interconnected networks. Here, we could found out several facts from these

Table 5–1 Effective methods for finding key nodes on various networks

Div	A nodes	B nodes	A edges	B edges	layer	1st method	2nd method	3rd method
BA(3)-BA(3)	512	512	1,527	1,527	A	PR+BE	PR+DE	degree
					B	pagerank	PR+DE	PR+BE
BA(3)-RR(6)	512	512	1,527	1,536	A	PR+BE	betweenness	PR+DE+BE
					B	betweenness	DE+BE	degree
RR(6)-BA(3)	512	512	1,536	1,527	A			
					B	pagerank	PR+DE	PR+DE+BE
BA(4)-BA(2)	512	512	2,032	1,020	A	betweenness	DE+BE	PR+DE+BE
					B	PR+DE	pagerank	PR+DE+BE
BA(2)-BA(4)	512	512	1,020	2,032	A	random	pagerank	PR+DE
					B	pagerank	PR+BE	PR+DE
HM(8) with BA(3)	512	64	1,527	183	A	degree	PR+DE	pagerank
					B	random	DE+BE	PR+DE

simulation results. First, it could be found out that the best and most influential method is different according to network structures and layers. Especially, we can see that betweenness has good performance on *RR* network. Second, as single indicators, pagerank, degree and betweenness are good method to find key nodes. Second, as multiple indicators, combined node centrality has good performance to recognize the key nodes on various networks. Combined node centralities are first or second method on every model.(except random method) Third, as the results shown in networks with different internal links, decreasing the number of links on layer A makes it hard to find key nodes and to have consensus by stubborn nodes. Fourth, as the results shown in *HM* network, decreasing the number of nodes on layer B makes it hard to identify key nodes and makes it easy to have consensus by stubborn nodes. Fifth, as the results shown in networks with different network types, network types have the influence on making consensus by stubborn nodes. It is found out that *RR* network makes it slow to have consensus by stubborn nodes.

Chapter 6 Conclusion

We have researched the competition of two layer networks. By changing network structures, switching updating rules and finding key nodes, the features of competition on two layers are found out. We hope that deficiency of this research would be researched forward and developed.

6.1 Summary

So far, many simulations have been carried out. In summary, it can be arranged as follows. To begin with, competing interconnected networks with different dynamics on each layer were introduced to understand the dynamics of competition on interconnected network. And some indexes were provided to measure how the network state is changed and to evaluate the consensus on two layer. Based on this modeling, various simulations were implemented according to 3 main topics as follows.

- Competition on two layer with different structural network
- Competition on two layer with different updating rules
- Finding key nodes on two layer networks

In chapter 3, we have investigated competitions on two layer with various structural network. With changing network structures, it was measured and evaluated that how the interconnected network change its state and make consensus. As the method to revise the network structure, 3 ways were provided such as changing internal degrees, changing external degrees, and switching network types. First, as the result of changing the internal degrees, it could be found out that internal degrees on each layer has different features. The number of internal degrees on layer A has the tendency to keep positive state and to change negative state into positive state. And the number of internal degrees on layer B has the tendency to hinder positive state. Second, as the result of changing the external degrees, hierarchical models were provided. Hierarchical models show that it is easy to make consensus on both layers when the number of external edges in decision making is more than opinion layer. Third, as the result of switching the network type, there is no obvious result of network state change. If there are no stubborn nodes, network types do not matter, but internal degrees have more influential role for changing the state.

In chapter 4, it has been researched that how the updating rules have influence on the competition of two-layers network. Though updating rules are very various, we just have considered time-related updating rules, simultaneous rule or sequential rule. According to where the updating rules are applied, we have implemented the simulations of 3 categories, order of layers, order of nodes and order of links. Through simulation results, several conclusions could be arranged. First, dynamics order between layers does not have an influence for network state. Second, networks with more simultaneous updating rules are easy to make slow consensus or coexistence and to change into opposite state, otherwise networks with more sequential updating rules are easy to make fast consensus. It can be analyzed as that if opinion layer has more rash nodes, more time to have some conversation and decision making layer has more time to discuss topics, the network have more probabilities to make consensus for opinion layer. Third, order of links in layer A is very influential so that it makes different network states. It can be explained as that characteristics of nodes in layer A, such as rash and considerate, affects the state of network. Forth, order of nodes in layer B are more influential for network states than order of nodes in layer A. It can be thought as that the communication method is very important in decision making layer.

In chapter 5, it has been studied that how the key nodes can be found out on the various interconnected network. To find key nodes on the network, we use single indicators and multiple indicators on various networks described in chapter 3. Through simulation results, several conclusions could be arranged. First, the most effective method to identify key nodes is different according to network structures and layers as shown in Table. 5–1. Second, as single indicators, pagerank, degree and betweenness work well for finding key nodes. Second, as multiple indicators, combined node centrality totally has good results to recognize the key nodes on various networks. Third, decreasing the number of links on layer A makes it hard to find key nodes and to have consensus by stubborn nodes. Fourth, decreasing the number of nodes on layer B makes it hard to identify key nodes and makes it easy to have consensus by stubborn nodes. Fifth, network types have the influence on making consensus by stubborn nodes. It is found out that *RR* network makes it slow to have consensus by stubborn nodes.

6.2 Discussion

So far, the competitions of two-layers network has been researched and analyzed under various conditions. It was found out that how network structures have the influence on the consensus of two-layers, how the updating rules affect the state of network, what nodes have more influential to affect the network state, and which method is more effective way to identify important nodes. Through these results, the state of interconnected networks may be controlled by managing the number of degrees and the method of updating rules. And as the best and fastest way to change the state of networks, the important nodes could be recognized and controlled by using the method to find key nodes. In real world, we can find out the phenomenon of these competitions, such as election, legislation, adoption of new policies and making decision on social conflict issues. These competitions of real world may have similar characteristics with our simulation results. Therefore, based on simulation results, these competitions can be applied to solve the social conflict. As future work, it could be very interesting to make generalized competition models with various structures and updating rules and find key nodes on generalized competition models.

Bibliography

[1] KIVELÄ M, ARENAS A, BARTHELEMY M, et al. Multilayer networks[J]. J. Complex Networks, 2014, 2: 203-271.

[2] DANZIGER M M, BONAMASSA I, BOCCALETTI S, et al. Dynamic interdependence and competition in multilayer networks[J]. Nature Physics, 2019, 15: 178-185.

[3] NEWMAN M J. Networks: An Introduction[M]. [S.l.]: Oxford Scholarship Online, 2010.

[4] BOCCALETTI S, BIANCONI G, CRIADO R, et al. The structure and dynamics of multilayer networks[J]. CoRR, 2014, abs/1407.0742.

[5] DE DOMENICO M, SOLÉ-RIBALTA A, COZZO E, et al. Mathematical Formulation of Multilayer Networks[J/OL]. Phys. Rev. X, 2013, 3: 041022. https://link.aps.org/doi/10.1103/PhysRevX.3.041022. DOI: 10.1103/PhysRevX.3.041022.

[6] TOMASINI M. An Introduction to Multilayer Networks[C]//. [S.l. : s.n.], 2015. DOI: 10.13140/RG.2.2.16830.18243.

[7] VU N K. Robustness of Interconnected Complex Networks with Directed Dependency[D]. Yonsei University Department of Computer Science, 2017.

[8] WU F, HUBERMAN B A. Social Structure and Opinion Formation[J]., 2004.

[9] S Z A, Fedyanin. Models of opinion control for agents in social networks[J]. Automation and Remote Control, 2012, 73: 1753-1764.

[10] Et AL M F L. The dynamics of opinion in hierarchical organizations[J]. Physica A: Statistical Mechanics and its Applications, 2005, 351: 580-592.

[11] MASUDA N. Opinion control in complex networks[J/OL]. CoRR, 2014, abs/1412.2170arXiv: 1412.2170. http://arxiv.org/abs/1412.2170.

[12] SMYRNAKIS M, BAUSO D, HAMIDOU T. An evolutionary game perspective on quantised consensus in opinion dynamics[J/OL]. PLOS ONE, 2019, 14(1): 1-17. https://doi.org/10.1371/journal.pone.0209212. DOI: 10.1371/journal.pone.0209212.

[13] BIANCONI G. Multilayer Networks: Structure and Function[M/OL]. [S.l.]: Oxford University Press, 2018. https://books.google.co.kr/books?id=6v5cDwAAQBAJ.

[14] REDNER S. Dynamics of Voter Models on Simple and Complex Networks[J]. ArXiv e-prints, 2017, arXiv:1705.02249: arXiv:1705.02249arXiv: 1705.02249 [physics.soc-ph].

[15] HU H. Competing opinion diffusion on social networks[J/OL]. Royal Society Open Science, 2017, 4(11): 171160eprint: https://royalsocietypublishing.org/doi/pdf/10.1098/rsos.171160. https://royalsocietypublishing.org/doi/abs/10.1098/rsos.171160. DOI: 10.1098/rsos.171160.

[16] AMATO R, KOUVARIS N, SAN MIGUEL M, et al. Opinion competition dynamics on multiplex networks[J]. New Journal of Physics, 2017, 19: 123019. DOI: 10.1088/1367-2630/aa936a.

[17] QUATTROCIOCCHI W, CALDARELLI G, SCALA A. Opinion dynamics on interacting networks: Media competition and social influence[J]. Scientific reports, 2014, 4. DOI: 10.1038/srep04938.

[18] CASEY M. SCHNEIDER-MIZELL L M S. A Generalized Voter Model on Complex Networks[J]. J Stat Phys 136:59-71, DOI : 10.1007/s10955-009-9757-6, 2009.

[19] HUA J, WANG L, WANG X. An information diffusion model based on individual characteristics[C]//. [S.l. : s.n.], 2014: 2866-2871. DOI: 10.1109/ChiCC.2014.6897094.

[20] ZHOU S, SHI S, WANG L. Immunizations of Interacting Diseases[C]//. [S.l. : s.n.], 2018: 9686-9691. DOI: 10.23919/ChiCC.2018.8482798.

[21] ZHOU S, XU S, WANG L, et al. Propagation of interacting diseases on multilayer networks[J/OL]. Phys. Rev. E, 2018, 98: 012303. https://link.aps.org/doi/10.1103/PhysRevE.98.012303. DOI: 10.1103/PhysRevE.98.012303.

[22] ALVAREZ-ZUZEK L G, ROCCA C E L, VAZQUEZ F, et al. Interacting Social Processes on Interconnected Networks[C]//PloS one. [S.l. : s.n.], 2016.

[23] GÓMEZ-GARDEÑES J, DOMENICO M D, GUTIÉRREZ G, et al. Layer-layer competition in multiplex complex networks[J]. Philosophical transactions. Series A, Mathematical, physical, and engineering sciences, 2015, 373 2056.

[24] DIEP(LPTM) H T, KAUFMAN M, KAUFMAN S. Dynamics of two-group conflicts: A statistical physics model[J]. Physica A: Statistical Mechanics and its Applications, Elsevier, 2017, 469: 183-199.

[25] ROCCA C E L, BRAUNSTEIN L A, VAZQUEZ F. The influence of persuasion in opinion formation and polarization[J]. CoRR, 2014, abs/1403.3011.

[26] VELÁSQUEZ-ROJAS F, VAZQUEZ F. Opinion dynamics in two dimensions: Domain coarsening leads to stable bi-polarization and anomalous scaling exponents[J]. Journal of Statistical Mechanics: Theory and Experiment, 2018, 2018. DOI: 10.1088/1742-5468/aab1b4.

[27] ABRAMS D M, STROGATZ S H. Linguistics: Modelling the dynamics of language death[J]. Nature, 2003, 424: 900-900.

[28] VÁZQUEZ F, CASTELLÓ X, MIGUEL M S. Agent Based Models of Language Competition : Macroscopic descriptions and Order-Disorder transitions[J]. Journal of Statistical Mechanics: Theory and Experiment, 2010.

[29] E. BEN-NAIM S R, P. L. Krapivsky. Bifurcations and patterns in compromise processes[J]., 2003, 183: 190-204. DOI: 10.1016/S0167-2789(03)00171-4.

[30] CHAU H F, WONG C Y, CHOW F K, et al. Social judgment theory based model on opinion formation, polarization and evolution[J]., 2014, 415: 133-140. DOI: 10.1016/j.physa.2014.07.082.

[31] PATRIARCA M, CASTELLÓ X, URIARTE J R, et al. Modeling Two-Language Competition Dynamics[J]. Advances in Complex Systems, 2012, 15.

[32] P. ERDÖS A R. On the Evolution of Random Graphs[J]., 5: 17-61.

[33] WATTS D J, STROGATZ S H. Collective dynamics of 'small-world' networks[J/OL]. Nature, 1998, 393(6684): 440-442. http://dx.doi.org/10.1038/30918. DOI: 10.1038/30918.

[34] BARABÁSI A L, ALBERT R. Emergence of Scaling in Random Networks[J/OL]. Science, 1999, 286(5439): 509-512eprint: https://science.sciencemag.org/content/286/5439/509.full.pdf. https://science.sciencemag.org/content/286/5439/509. DOI: 10.1126/science.286.5439.509.

[35] SÍRBU A, LORETO V, SERVEDIO V, et al. Opinion Dynamics: Models, Extensions and External Effects[M]//. [S.l. : s.n.], 2017: 363-401. DOI: 10.1007/978-3-319-25658-0_17.

[36] HOFFMAN R, KAGEL J, LEVIN D. Simultaneous versus Sequential Information Processing[J]. Economics Letters, 2011, 112: 16-18. DOI: 10.1016/j.econlet.2011.03.006.

[37] STAUFFER D. DIFFICULTY FOR CONSENSUS IN SIMULTANEOUS OPINION FORMATION OF SZNAJD MODEL[J/OL]. The Journal of Mathematical Sociology, 2004, 28(1): 25-33eprint: https://doi.org/10.1080/00222500490278531. https://doi.org/10.1080/00222500490278531. DOI: 10.1080/00222500490278531.

[38] FREEMAN L. Centrality in Social Networks Conceptual Clarification[J]., 1979, 1: 215-239. DOI: http://dx.doi.org/10.1016/0378-8733(78)90021-7.

[39] KOSCHÜTZKI D, SCHREIBER F. Centrality analysis methods for biological networks and their application to gene regulatory networks[J]., 2008, 2: 193-201.

[40] EOM Y H, SHEPELYANSKY D L. Opinion formation driven by PageRank node influence on directed networks[J/OL]. Physica A: Statistical Mechanics and its Applications, 2015, 436: 707-715. http://dx.doi.org/10.1016/j.physa.2015.05.095. DOI: 10.1016/j.physa.2015.05.095.

[41] WHITE S, SMYTH P. Algorithms for Estimating Relative Importance in Networks[C/OL]//. Washington, D.C.: ACM, 2003: 266-275. http://doi.acm.org/10.1145/956750.956782. DOI: 10.1145/956750.956782.

[42] MESGARI I, KERMANI A, HANNEMAN R, et al. Identifying Key Nodes in Social Networks Using Multi-Criteria Decision-Making Tools[M]. [S.l. : s.n.], 2015: 137-150. DOI: 10.1007/978-3-319-16619-3_10.

[43] HWANG C, YOON K. Multiple Attribute Decision Making: Methods and Applications[M]. New York, NY, USA: Springer-Verlag, 1981. DOI: http://dx.doi.org/10.1007/978-3-642-48318-9.

[44] HUANG S, LV T, ZHANG X, et al. Identifying Node Role in Social Network Based on Multiple Indicators[J/OL]. PLOS ONE, 2014, 9(8): 1-16. https://doi.org/10.1371/journal.pone.0103733. DOI: 10.1371/journal.pone.0103733.

[45] RODRIGUES F A. Network Centrality: An Introduction[C]//. [S.l. : s.n.], 2019.

[46] LANCHIER N. Stochastic Modeling[M]. [S.l.]: Springer Nature, DOI:1007/978-3-319-50038-6, 2017.

[47] HEGSELMANN R, KRAUSE U. Opinion Dynamics and Bounded Confidence Models, Analysis and Simulation[J]. Journal of Artificial Societies and Social Simulation, 2002, 5.

[48] SANGWOO K. Structure and dynamics of complex networks[D]. Yonsei Univ, Department of physics, 2012.

[49] CHOI W. A Study on the Characteristics and the Transition of the Complex System Network Structure and Dynamics[D]. 2011.

[50] BOLLOBÁS B. Random Graphs[M]. 2nd ed. [S.l.]: Cambridge University Press, 2001. DOI: 10.1017/CBO9780511814068.

Publications

[1] Hyunchel Cho, Arfan Mahmood, Lin Wang. Competition of Social Opinions on Two layer Networks. In Proceeding of the 38th Chinese Control Conference, July 2019, Guangzhou, China, Accepted.