Estimate A Simple Dynamic Discrete Choice Model*

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1 Introduction

This project solves a basic dynamic discrete choice problem using the nested fixed point MLE algorithm.

2 The general model

2.1 Setup

I consider a single agent dynamic discrete choice problem. At each (discrete) time $t \in \{0, 1, \cdots, \infty\}$, an agent observes a vector of state variables $s_t = (X_t, \varepsilon_t)$ and makes a choice $a_t \in A = \{0, 1\}$. The state vector has an observed component $X_t \in \mathcal{X}$ that has finite support and an unobserved component ε_t to the econometrician (but observed by the firm). The agent chooses $a_t \in A$ to maximize the expected future payoffs

$$\max_{\{a_t\}_t} \mathbb{E}_t \left[\sum_{t=1}^{\infty} \beta^{t-1} U(X_t, a_t, \varepsilon_t; \boldsymbol{\theta}) \right]$$

where U_t is the one-period utility and $\beta \in (0, 1)$ is the discount factor. The goal is to identify the parameters $(\boldsymbol{\theta}, \boldsymbol{\varphi})$, where $\boldsymbol{\theta}$ is in the utility function shown above and $\boldsymbol{\varphi}$ is in the state transition probability $Pr(X_{t+1}, \varepsilon_{t+1} | X_t, \varepsilon_t, a_t; \boldsymbol{\varphi})$.

Assumption 1 (Additive separability). The utility of choosing action $a_t = a$ in period t given state $s_t = (X_t, \varepsilon_t)$ is $U(X_t, \varepsilon_t, a_t = a; \boldsymbol{\theta}) = \bar{u}(X_t; \boldsymbol{\theta}) + \varepsilon_t$.

Assumption 2 (Conditional independence). Assume that ε_t is i.i.d over t, $Pr(\varepsilon_{t+1}|\varepsilon_t) = Pr(\varepsilon_{t+1})$, and $X_{t+1} \perp (\varepsilon_t, \varepsilon_{t+1})$, given (X_t, a_t),

$$Pr(X_{t+1}, \varepsilon_{t+1} | X_t, \varepsilon_t, a_t) = Pr(\varepsilon_{t+1}) Pr(X_{t+1} | X_t, a_t)$$

Assumption 3. Given an initial value X_0, X_t follows a first-order Markov chain with a transition probability matrix Π, where $\Pi_{ij} = Pr(X_{t+1} = x^j | X_t = x^i)$, independent of the agent's choice.

Assumption 4. $\varepsilon_{a,t}$ follows a mean-zero i.i.d. type I extreme value distribution.

The problem can be represented by a Bellman equation,

$$V(X_t, \varepsilon_t; \boldsymbol{\theta}, \boldsymbol{\varphi}) = \max_{a_t \in A} U(X_t, a_t, \varepsilon_t; \boldsymbol{\theta}) + \beta \mathbb{E} \left[V(X_{t+1}, \varepsilon_{t+1}; \boldsymbol{\theta}, \boldsymbol{\varphi}) | a_t, X_t, \varepsilon_t; \boldsymbol{\varphi} \right]$$

*The notes in Section 2 are borrowed from the IO notes by Ali Hortacsu and Joonhwi Joo. The data simulation and estimation part uses the same procedure as Jaap Abbring and Tobias Klein (2020). Both are excellent resources to learn about the dynamic discrete choice model. I am grateful that they generously share their notes online.

By assumption 2,

$$\mathbb{E}\left[V(X_{t+1},\varepsilon_{t+1};\boldsymbol{\theta},\boldsymbol{\varphi})|a_t,X_t,\varepsilon_t;\boldsymbol{\varphi}\right]=\int\int V(X_{t+1},\varepsilon_{t+1};\boldsymbol{\theta},\boldsymbol{\varphi})dPr(\varepsilon_{t+1})dPr(X_{t+1}|X_t,a_t;\boldsymbol{\varphi})$$

Using assumption 1, the Bellman equation can be written as

$$\begin{aligned}V(X_t,\varepsilon_t;\boldsymbol{\theta},\boldsymbol{\varphi})&=\max_{a\in A}\bar{u}_a(X_t;\boldsymbol{\theta})+\beta\int\int V(X_{t+1},\varepsilon_{t+1};\boldsymbol{\theta},\boldsymbol{\varphi})dPr(\varepsilon_{t+1})dPr(X_{t+1}|X_t,a_t=a;\boldsymbol{\varphi})+\varepsilon_{a,t}\\&=\max_{a\in A}\bar{V}_a(X_t;\boldsymbol{\theta},\boldsymbol{\varphi})+\varepsilon_{a,t}\end{aligned}$$

where the term $\bar{V}_a(X_t;\boldsymbol{\theta},\boldsymbol{\varphi})$ represents the deterministic part of the value function for choice a and $\varepsilon_{a,t}$ is the random part specific to choice a. Furthermore, the distribution $\varepsilon_{a,t}$ allows us to derive a closed form solution for $\bar{V}_a(X_t;\boldsymbol{\theta},\boldsymbol{\varphi})$,

$$\bar{V}_a(X_t;\boldsymbol{\theta},\boldsymbol{\varphi})=\bar{u}_a(X_t;\boldsymbol{\theta})+\beta\int\ln\left(\sum_{a\in A}\exp\bar{V}_a(X_{t+1};\boldsymbol{\theta},\boldsymbol{\varphi})\right)dPr(X_{t+1}|X_t,a_t=a;\boldsymbol{\varphi})$$

We observe a time series data of choices and observed states $(a_t,X_t)_{t=\{1,2,\cdots,T\}}$, the likelihood of the data given parameters $(\boldsymbol{\theta},\boldsymbol{\varphi})$is

$$L(\boldsymbol{\theta},\boldsymbol{\varphi})=\prod_{t=2}^{T}\left(\prod_{a\in A}Pr(a_t=a|X_t;\boldsymbol{\theta},\boldsymbol{\varphi})^{1(a_t=a)}\right)Pr(X_t|X_{t-1},a_{t-1};\boldsymbol{\varphi})$$

where $Pr(X_t|X_{t-1},a_{t-1};\boldsymbol{\varphi})$ is the transition probability of state variables. The MLE estimator satisfies

$$(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\varphi}})=\arg\max\sum_{t=2}^{T}\sum_{a\in A}\mathbf{1}(a_t=a)\ln Pr(a_t=a|X_t;\boldsymbol{\theta},\boldsymbol{\varphi})+\sum_{t=2}^{T}\ln Pr(X_t|X_{t-1},a_{t-1};\boldsymbol{\varphi})$$

where the conditional choice probability (CCP) is

$$Pr(a_t=a|X_t;\boldsymbol{\theta},\boldsymbol{\varphi})=\frac{\exp\left(\bar{u}_a(X_t;\boldsymbol{\theta})+\beta\mathbb{E}[V(X_{t+1},\varepsilon_{t+1};\boldsymbol{\theta},\boldsymbol{\varphi})|a_t=a,X_t,\varepsilon_t;\boldsymbol{\varphi}]\right)}{\sum_{a\in A}\exp\left(\bar{u}_a(X_t;\boldsymbol{\theta})+\beta\mathbb{E}[V(X_{t+1},\varepsilon_{t+1};\boldsymbol{\theta},\boldsymbol{\varphi})|a_t=a,X_t,\varepsilon_t;\boldsymbol{\varphi}]\right)}$$

2.2 Estimation

I use the nested fixed point (NFP) algorithm to estimate the model. The NFP consists of two parts.

- (Inner loop). Given each guess $(\boldsymbol{\theta},\boldsymbol{\varphi})$, solve the fixed point equation to get the value function V.

$$\bar{V}_a(X_t;\boldsymbol{\theta},\boldsymbol{\varphi})=\bar{u}_a(X_t;\boldsymbol{\theta})+\beta\int\ln\left(\sum_{a\in A}\exp\bar{V}_a(X_{t+1};\boldsymbol{\theta},\boldsymbol{\varphi})\right)dPr(X_{t+1}|X_t,a_t=a;\boldsymbol{\varphi})$$

- (Outer loop). Given the guess $(\boldsymbol{\theta},\boldsymbol{\varphi})$ and V, compute the CCP and solve the (partial) maximum likelihood problem.

$$(\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\varphi}})=\arg\max\sum_{t=2}^{T}\sum_{a\in A}\mathbf{1}(a_t=a)\ln Pr(a_t=a|X_t;\boldsymbol{\theta},\boldsymbol{\varphi})$$

3 A firm entry and exit model

3.1 Setup

I follow the specific setup of Abbring and Klein (2020) in this exercise. A firm decides whether to enter or exit a market, that is, $a_t \in A = \{0, 1\}$, where $a_t = 0$ indicates exit and $a_t = 1$ indicates entry into market. The flow payoffs of these decisions at t are

$$U(X_t, a_{t-1}, \varepsilon_t; \boldsymbol{\theta}) = \begin{cases} -a_{t-1}\delta_0 + \varepsilon_t(0) & a_t = 0 \\ \beta_0 + \beta_1 X_t - (1 - a_{t-1})\delta_1 + \varepsilon_t(1) & a_t = 1 \end{cases}$$

where $\boldsymbol{\theta} = (\beta_0, \beta_1, \delta_1)$ and δ_0 is fixed at zero (and thus need not to be estimated). The Bellman equation is

$$\bar{V}_a(X_t, a_{t-1}; \boldsymbol{\theta}, \boldsymbol{\varphi}) = \bar{u}_a(X_t, a_{t-1}; \boldsymbol{\theta}) + \rho \int \ln\left(\exp \bar{V}_0(X_{t+1}, a_t; \boldsymbol{\theta}, \boldsymbol{\varphi}) + \exp \bar{V}_1(X_{t+1}, a_t; \boldsymbol{\theta}, \boldsymbol{\varphi})\right) dPr(X_{t+1}|X_t, a_t = a; \boldsymbol{\varphi})$$

The conditional choice probability is

$$Pr(a_t|X_t, a_{t-1}) = a_t + \frac{1 - 2a_t}{1 + \exp(V_1(X_t, a_{t-1}) - V_0(X_t, a_{t-0}))}$$

3.2 Data simulation and estimation

Assume that there are $N = 1000$ firms, $T = 100$ periods, $\chi = [1, 2, 3, 4, 5]'$, and $\rho = 0.95$. The state transition matrix is

$$\Pi = \begin{bmatrix} 0.44 & 0.22 & 0.15 & 0.11 & 0.09 \\ 0.19 & 0.39 & 0.19 & 0.13 & 0.10 \\ 0.13 & 0.19 & 0.38 & 0.19 & 0.13 \\ 0.10 & 0.13 & 0.19 & 0.39 & 0.19 \\ 0.09 & 0.11 & 0.15 & 0.22 & 0.44 \end{bmatrix}$$

The true parameters are

$$\beta_0 = -0.5, \beta_1 = 0.2$$
$$\delta_0 = 0, \delta_1 = 1$$

I simulate the data of choices and observed state variables $(a_t, X_t)_{t \in T}$ using these parameters. Then I use the following algorithm to estimate $(\beta_0, \beta_1, \delta_1)$.

1. Guess $(\beta_0, \beta_1, \delta_1)$.
2. (Inner loop) Compute flow payoffs $(\bar{u}_1, \bar{u}_0)$ and estimate the state transition matrix $\hat{\Pi}$ using the data. Guess the deterministic part of the value function $(\bar{V}_0, \bar{V}_1)$, generate new values of $(\bar{V}_0, \bar{V}_1)$ using the Bellman equation. Repeat the process until the difference between the old and new values converges to zero, that is, solving the fixed point equation.
3. (Outer loop). Given $(\bar{V}_0, \bar{V}_1)$, compute the conditional choice probability $\hat{Pr}(a_t|X_t, a_{t-1})$ and the objective function of (partial log) maximum likelihood function. Go back to step 1 and repeat the process until the maximum is reached.

The estimated parameters are in Table 1. It shows that this algorithm yields consistent estimators.

Table 1: Estimated and True Parameters

	True Parameters	Estimated Parameters
β_0	-0.5	-0.4955
β_1	0.2	0.1982
δ_1	1	0.9991