Estimate A Basic Dynamic Discrete Choice Model*

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1 Introduction

This project uses the simulation-based estimation method to solve a single agent dynamic discrete choice problem. I use the nested fixed point algorithm developed in Rust (1987) to solve the simulated maximum likelihood problem. I also test the robustness of the assumption that the unobserved state variable follows a type I extreme value distribution. I simulate three sets of data from three distributions and use the same estimation procedure to estimate the parameters. In practice, researchers almost always adopt the type I extreme value assumption, this exercise sheds light on the question that what if the true data generating process is not extreme value distributed.

2 The general model

2.1 Setup

I consider a single agent dynamic discrete choice problem. At each (discrete) time $t \in \{0, 1, \cdots, \infty\}$, an agent observes a vector of state variables $s_t = (X_t, \varepsilon_t)$ and makes a choice $a_t \in A = \{0, 1\}$. The state vector has an observed component $X_t \in \mathcal{X}$ that has finite support and an unobserved component ε_t to the econometrician (but observed by the firm). The agent chooses $a_t \in A$ to maximize the expected future payoffs

$$\max_{\{a_t\}_t} \mathbb{E}_t \left[\sum_{t=1}^{\infty} \beta^{t-1} U(X_t, a_t, \varepsilon_t; \boldsymbol{\theta}) \right]$$

where U_t is the one-period utility and $\beta \in (0, 1)$ is the discount factor. The goal is to identify the parameters $(\boldsymbol{\theta}, \boldsymbol{\varphi})$, where $\boldsymbol{\theta}$ is in the utility function shown above and $\boldsymbol{\varphi}$ is in the state transition probability $Pr(X_{t+1}, \varepsilon_{t+1} | X_t, \varepsilon_t, a_t; \boldsymbol{\varphi})$.

Assumption 1 (Additive separability). The utility of choosing action $a_t = a$ in period t given state $s_t = (X_t, \varepsilon_t)$ is $U(X_t, \varepsilon_t, a_t = a; \boldsymbol{\theta}) = \bar{u}(X_t; \boldsymbol{\theta}) + \varepsilon_t$.

Assumption 2 (Conditional independence). Assume that ε_t is i.i.d over t, $Pr(\varepsilon_{t+1} | \varepsilon_t) = Pr(\varepsilon_{t+1})$, and $X_{t+1} \perp (\varepsilon_t, \varepsilon_{t+1})$, given (X_t, a_t),

$$Pr(X_{t+1}, \varepsilon_{t+1} | X_t, \varepsilon_t, a_t) = Pr(\varepsilon_{t+1}) Pr(X_{t+1} | X_t, a_t)$$

*The notes in Section 2 are borrowed from the IO notes by Ali Hortacsu and Joonhwi Joo. The data simulation and estimation part uses the same procedure as Jaap Abbring and Tobias Klein (2020). Both are excellent resources to learn about the dynamic discrete choice model. I am grateful that they generously share their notes online.

Assumption 3. Given an initial value X_0, X_t follows a first-order Markov chain with a transition probability matrix Π, where $\Pi_{ij} = Pr(X_{t+1} = x^j | X_t = x^i)$, independent of the agent's choice.

Assumption 4. $\varepsilon_{a,t}$ follows a mean-zero i.i.d. type I extreme value distribution.

The problem can be represented by a Bellman equation,

$$V(X_t, \varepsilon_t; \boldsymbol{\theta}, \boldsymbol{\varphi}) = \max_{a_t \in A} U(X_t, a_t, \varepsilon_t; \boldsymbol{\theta}) + \beta \mathbb{E}\left[V(X_{t+1}, \varepsilon_{t+1}; \boldsymbol{\theta}, \boldsymbol{\varphi}) | a_t, X_t, \varepsilon_t; \boldsymbol{\varphi}\right]$$

By assumption 2,

$$\mathbb{E}\left[V(X_{t+1}, \varepsilon_{t+1}; \boldsymbol{\theta}, \boldsymbol{\varphi}) | a_t, X_t, \varepsilon_t; \boldsymbol{\varphi}\right] = \int \int V(X_{t+1}, \varepsilon_{t+1}; \boldsymbol{\theta}, \boldsymbol{\varphi}) dPr(\varepsilon_{t+1}) dPr(X_{t+1} | X_t, a_t; \boldsymbol{\varphi})$$

Using assumption 1, the Bellman equation can be written as

$$\begin{aligned} V(X_t, \varepsilon_t; \boldsymbol{\theta}, \boldsymbol{\varphi}) &= \max_{a \in A} \bar{u}_a(X_t; \boldsymbol{\theta}) + \beta \int \int V(X_{t+1}, \varepsilon_{t+1}; \boldsymbol{\theta}, \boldsymbol{\varphi}) dPr(\varepsilon_{t+1}) dPr(X_{t+1} | X_t, a_t = a; \boldsymbol{\varphi}) + \varepsilon_{a,t} \\ &= \max_{a \in A} \bar{V}_a(X_t; \boldsymbol{\theta}, \boldsymbol{\varphi}) + \varepsilon_{a,t} \end{aligned}$$

where the term $\bar{V}_a(X_t; \boldsymbol{\theta}, \boldsymbol{\varphi})$ represents the deterministic part of the value function for choice a and $\varepsilon_{a,t}$ is the random part specific to choice a. Furthermore, the distribution $\varepsilon_{a,t}$ allows us to derive a closed form solution for $\bar{V}_a(X_t; \boldsymbol{\theta}, \boldsymbol{\varphi})$,

$$\bar{V}_a(X_t; \boldsymbol{\theta}, \boldsymbol{\varphi}) = \bar{u}_a(X_t; \boldsymbol{\theta}) + \beta \int \ln \left(\sum_{a \in A} \exp \bar{V}_a(X_{t+1}; \boldsymbol{\theta}, \boldsymbol{\varphi}) \right) dPr(X_{t+1} | X_t, a_t = a; \boldsymbol{\varphi})$$

We observe a time series data of choices and observed states $(a_t, X_t)_{t=\{1,2,\cdots,T\}}$, the likelihood of the data given parameters $(\boldsymbol{\theta}, \boldsymbol{\varphi})$is

$$L(\boldsymbol{\theta}, \boldsymbol{\varphi}) = \prod_{t=2}^{T} \left(\prod_{a \in A} Pr(a_t = a | X_t; \boldsymbol{\theta}, \boldsymbol{\varphi})^{1(a_t = a)} \right) Pr(X_t | X_{t-1}, a_{t-1}; \boldsymbol{\varphi})$$

where $Pr(X_t | X_{t-1}, a_{t-1}; \boldsymbol{\varphi})$ is the transition probability of state variables. The MLE estimator satisfies

$$(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}) = \arg\max \sum_{t=2}^{T} \sum_{a \in A} \mathbf{1}(a_t = a) \ln Pr(a_t = a | X_t; \boldsymbol{\theta}, \boldsymbol{\varphi}) + \sum_{t=2}^{T} \ln Pr(X_t | X_{t-1}, a_{t-1}; \boldsymbol{\varphi})$$

where the conditional choice probability (CCP) is

$$\begin{aligned} Pr(a_t = a | X_t; \boldsymbol{\theta}, \boldsymbol{\varphi}) &= \frac{\exp(\bar{V}_a(X_t; \boldsymbol{\theta}, \boldsymbol{\varphi})}{\sum_{a \in A} \exp(\bar{V}_a(X_t; \boldsymbol{\theta}, \boldsymbol{\varphi})} \\ &= \frac{\exp\left(\bar{u}_a(X_t; \boldsymbol{\theta}) + \beta \mathbb{E}[V(X_{t+1}, \varepsilon_{t+1}; \boldsymbol{\theta}, \boldsymbol{\varphi}) | a_t = a, X_t, \varepsilon_t; \boldsymbol{\varphi}]\right)}{\sum_{a \in A} \exp\left(\bar{u}_a(X_t; \boldsymbol{\theta}) + \beta \mathbb{E}[V(X_{t+1}, \varepsilon_{t+1}; \boldsymbol{\theta}, \boldsymbol{\varphi}) | a_t = a, X_t, \varepsilon_t; \boldsymbol{\varphi}]\right)} \end{aligned}$$

2.2 Estimation

I use the nested fixed point (NFP) algorithm to estimate the model. The NFP consists of two parts.

- (Inner loop). Given each guess $(\boldsymbol{\theta}, \boldsymbol{\varphi})$, solve the fixed point equation to get the value function V.

$$\bar{V}_a(X_t; \boldsymbol{\theta}, \boldsymbol{\varphi}) = \bar{u}_a(X_t; \boldsymbol{\theta}) + \beta \int \ln \left(\sum_{a \in A} \exp \bar{V}_a(X_{t+1}; \boldsymbol{\theta}, \boldsymbol{\varphi}) \right) dPr(X_{t+1} | X_t, a_t = a; \boldsymbol{\varphi})$$

- (Outer loop). Given the guess $(\boldsymbol{\theta}, \boldsymbol{\varphi})$ and V, compute the CCP and solve the (partial) maximum likelihood problem.

$$(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}) = \arg\max \sum_{t=2}^{T} \sum_{a \in A} \mathbf{1}(a_t = a) \ln Pr(a_t = a | X_t; \boldsymbol{\theta}, \boldsymbol{\varphi})$$

3 A firm entry and exit model

3.1 Setup

I follow the specific setup of Abbring and Klein (2020) in this exercise. A firm decides whether to enter or exit a market, that is, $a_t \in A = \{0, 1\}$, where $a_t = 0$ indicates exit and $a_t = 1$ indicates entry into market. The flow payoffs of these decisions at t are

$$U(X_t, a_{t-1}, \varepsilon_t; \boldsymbol{\theta}) = \begin{cases} -a_{t-1}\delta_0 + \varepsilon_t(0) & a_t = 0 \\ \beta_0 + \beta_1 X_t - (1 - a_{t-1})\delta_1 + \varepsilon_t(1) & a_t = 1 \end{cases}$$

where $\boldsymbol{\theta} = (\beta_0, \beta_1, \delta_1)$ and δ_0 is fixed at zero (and thus need not to be estimated). The Bellman equation is

$$\bar{V}_a(X_t, a_{t-1}; \boldsymbol{\theta}, \boldsymbol{\varphi}) = \bar{u}_a(X_t, a_{t-1}; \boldsymbol{\theta}) + \rho \int \ln\left(\exp \bar{V}_0(X_{t+1}, a_t; \boldsymbol{\theta}, \boldsymbol{\varphi}) + \exp \bar{V}_1(X_{t+1}, a_t; \boldsymbol{\theta}, \boldsymbol{\varphi})\right) dPr(X_{t+1} | X_t, a_t = a; \boldsymbol{\varphi})$$

The conditional choice probability is

$$Pr(a_t | X_t, a_{t-1}) = a_t + \frac{1 - 2a_t}{1 + \exp(V_1(X_t, a_{t-1}) - V_0(X_t, a_{t-0}))}$$

3.2 Data simulation

Assume that there are $N = 1000$ firms, $T = 100$ periods, $\succ = [1, 2, 3, 4, 5]'$, and $\rho = 0.95$. The state transition matrix is

$$\Pi = \begin{bmatrix} 0.44 & 0.22 & 0.15 & 0.11 & 0.09 \\ 0.19 & 0.39 & 0.19 & 0.13 & 0.10 \\ 0.13 & 0.19 & 0.38 & 0.19 & 0.13 \\ 0.10 & 0.13 & 0.19 & 0.39 & 0.19 \\ 0.09 & 0.11 & 0.15 & 0.22 & 0.44 \end{bmatrix}$$

The true parameters are

$$\beta_0 = -0.5, \beta_1 = 0.2$$
$$\delta_0 = 0, \delta_1 = 1$$

I simulate the data of choices and observed state variables $(a_t, X_t)_{t \in T}$ using these parameters. In additional to these parameters, one important assumption in the dynamic discrete choice model is that ε_{at} follows a type I extreme value distribution to give closed-form expressions for choice probabilities. However, this strong assumption may not always be true in the data. In this exercise, I simulate three sets of data and use the same procedure to estimate the parameters. The goal is to check the robustness of estimated parameters under misspecification. The three sets of simulated data are

1. (EV) Assume ε follows a type I extreme value distribution.

2. (Normal) Assume ε follows a standard normal distribution.

3. (Bimodal) Assume $p(\varepsilon) \propto 0.5\exp(-0.7(\varepsilon-2)^2) + 0.5\exp(-0.7(\varepsilon+2)^2)$. I use the Metropolis-Hastings algorithm to generate this bimodal distribution. The algorithm is

   (a) Initialize a vector $\varepsilon^{(0)}$

   (b) For $i = 1 : N$

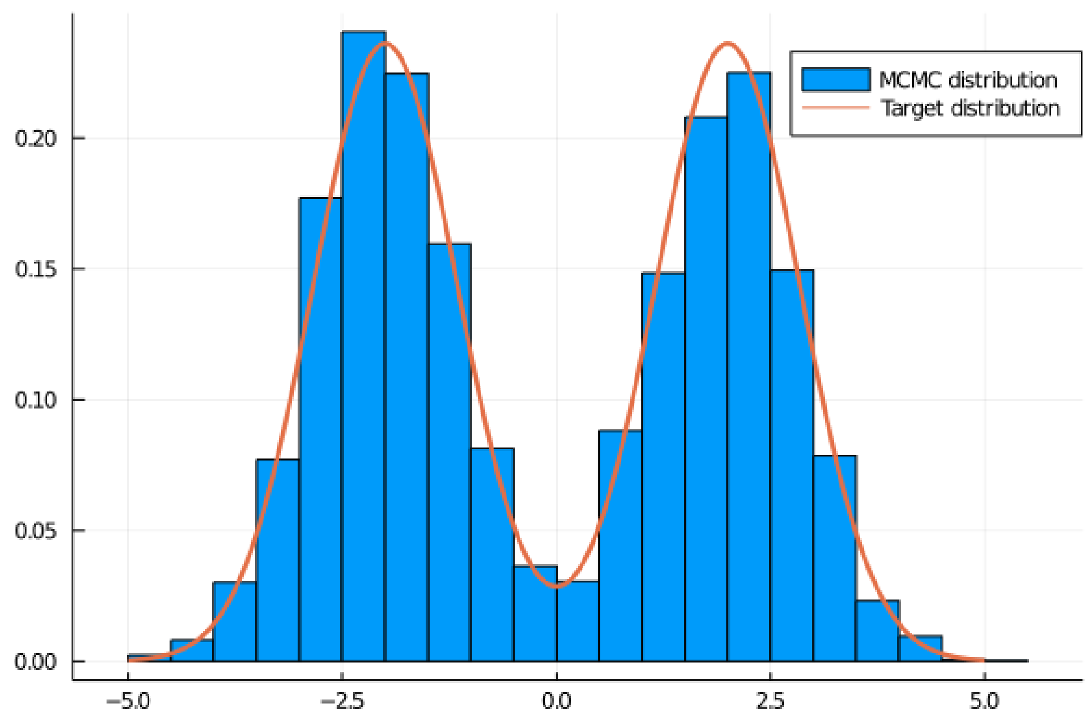
      i. Generate a candidate from the proposal distribution, $\varepsilon^* \sim q(\varepsilon^*|\varepsilon^{(i-1)}) = \mathcal{N}(x^{(i-1)}, 25)$

      ii. Compute the acceptance probability $\mathcal{A}(\varepsilon^{(i-1)}, \varepsilon^*) = \min\{1, \frac{p(\varepsilon^*)}{p(\varepsilon^{(i-1)})}\}$

      iii. Generate $u_i \sim U([0,1])$, set (accept) $\varepsilon^{(i)} = \varepsilon^*$ if $u_i \leq A(\varepsilon^{(i-1)}, \varepsilon^*)$, and set (reject) $x^{(i)} = x^{(i-1)}$ otherwise.

The distribution of the bimodal data using the Metropolis-Hastings algorithm is shown in Figure 1. I am able to generate data from the target distribution.

Figure 1: Distribution of ε in misspecified data



3.3 Estimation

I use the following algorithm to estimate $(\beta_0, \beta_1, \delta_1)$ from the two sets of simulated data. I estimate the state transition matrix $\hat{\Pi}$ using the data before the algorithm.

1. Guess $(\beta_0, \beta_1, \delta_1)$.

2. (Inner loop) Compute flow payoffs $(\bar{u}_1, \bar{u}_0)$. Guess the deterministic part of the value function $(\bar{V}_0, \bar{V}_1)$, generate new values of $(\bar{V}_0, \bar{V}_1)$ using the Bellman equation. Repeat the process until the difference between the old and new values converges to zero, that is, solving the fixed point equation.

3. (Outer loop). Given $(\bar{V}_0, \bar{V}_1)$, compute the conditional choice probability $\hat{Pr}(a_t|X_t, a_{t-1})$ and the objective function of (partial log) maximum likelihood function. Go back to step 1 and repeat the process until the maximum is reached.

The estimated parameters are in Table 1. It shows that this algorithm yields consistent estimators using the true data generating process (original data). The estimated parameters from the normally distributed errors are biased, but not very far from the true parameters. This is because the standard normal distribution is quite close to the type I extreme value distribution. The bimodal distributed data generates very biased estimates.

Table 1: Estimated and True Parameters

	True Parameters	Estimated Parameters		
		EV	Normal	Bimodal
β_0	-0.5	-0.5018	-0.5596	-0.3166
β_1	0.2	0.1981	0.2249	0.1270
δ_1	1	1.0032	1.1254	0.6355

References

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