# Knowledge clip on a way to visualize the magnetic vector potential

r.serra@tue.nl

June 2022

## 1 Introduction

This document is a brief summary on an idea to co-create a video which should help visualize and get a more physically-consistent intuition about the magnetic vector potential. The goal of this document is to communicate the intention and the main ideas behind the video. I do not propose this document as a story line for a video.

The magnetic vector potential $\mathbf{A}$ is normally introduced as an *assumption*:

$$\mathbf{B} = \nabla \times \mathbf{A}\,, \tag{1}$$

with $\mathbf{B}$ being the magnetic induction field. This assumption is normally justified by the fact that the magnetic induction field is solenoidal, and therefore, $\nabla \cdot \mathbf{B} = 0$. Since the divergence of the curl of *any* vector field is always equal to zero, hence the assumption in (1).

If we accept (1), then solving some electromagnetic radiation problems become much easier. For instance, consider the in-homogeneous Helmholtz equation for $\mathbf{H} = \frac{\mathbf{B}}{\mu}$, in the presence of a current source $\mathbf{J}$:

$$\nabla^2 \mathbf{H} + k^2 \mathbf{H} = -\nabla \times \mathbf{J}\,, \tag{2}$$

with $\mu$ the magnetic permeability of the medium where the radiation occurs, $\mathbf{H}$ the magnetic field vector and $k$ the wave-number.

Equation (1) is difficult to solve, mainly due to the rotor operation over the current density $\mathbf{J}$. When the substitution in (1) is applied to Maxwell's equations, (2), reduces to:

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}\,, \tag{3}$$

and we got rid of $\nabla \times \mathbf{J}$.

*Note: there are a few mathematical steps which are not made explicit here for the sake of brevity. The main one is the application of the Lorenz gauge, an interesting aspect of the whole discussion. I want to focus, for the moment, on* $\mathbf{A}$ *mainly.*

In my course, for instance, we solve (3) for the case of a point dipole. The calculations are straightforward.

## 2 The motivation

I am personally motivated when we can have an intuitive visualization and interpretation of the phenomena we are studying. As a teacher, I put quite some efforts in that aspect, with various degrees of success. I think that establishing relations, visualizations, interpretations, etc. are a sign of clarity and can certainly add significantly to the motivation to study a certain topic. Your (Grant's, that is) videos are a prominent example of this.

Anyway, though I manage to provide such intuitions for most parts of my course, for years I have been in trouble to assign such an interpretation to the vector potential. Every book or explanation I saw, treats this vector as just a mathematical aid. A trick. What is the vector potential? It's just an abstract and nonphysical mathematical object, whose curl is the magnetic field. For most (of us), the magnetic field itself is something which takes time to imagine and build an intuition for. Imagine this one other vector, whose rotation gives the magnetic field...

So, even though for years I "followed the herd" in the explanation and introduction of $\mathbf{A}$, there was always something in the back of my head bothering me: how can I provide a physically-consistent interpretation of the magnetic vector potential?

The answer came across after many years of trying. However, it did not from me, but from the "Maestro" himself: James Clerk Maxwell.

## 3 On physical lines of force...

Maxwell needed many years to come up with the compendium of equations which now represents the whole classical EM theory. His first efforts, published in a series of papers, were directed to try to explain electricity and magnetism using equivalent fluid-dynamic analogies. He would, for instance, describe the electrical potential as fluid pressure, and so on. In hindsight, one would wonder why did he need to come up with such analogies, but what Maxwell was trying to do was actually brilliant. The story is long and complicated, but in a nutshell it encompasses the following components: Faraday threw a bomb by hypothesizing this idea of "*lines of force*"[1]. In essence, Faraday's lines of force are what we now call "fields" and is at the basis of most of modern physics. But... Faraday was mathematically-challenged, to put it in a certain way. The idea of lines of force competed with the more Newtonian-like type of physics of "*action at a distance*". Many, like Weber, for instance, tried to propose models

[1] The circumstances and story on how do we know Faraday's idea of lines of forces is fascinating, highly entertaining and full of twists. For another talk...

for electricity and magnetism in this Newtonian way. Two different views contradicting each other. Positions, debates, etc. So, Maxwell took the challenge to provide Faraday's ideas with the necessary mathematical toolkit. At least one that mathematicians would not have difficulty to accept.

And here's his model. The paper can be accessed here.

Some interesting excerpts from the introduction of this paper:

"*My object in this paper is to clear the way for speculation in this direction, by investigating the **mechanical results** of certain stages of tension and motion in a medium, and comparing these with the observed phenomena of magnetism and electricity.*" [the bold font is mine].

Another part:

"*I propose now to examine magnetic phenomena from a **mechanical** point of view, and to determine what tensions in, or motions of, a medium are capable of producing the mechanical phenomena observed. If, by the same hypothesis, we can connect the phenomena of magnetic attraction with electromagnetic phenomena and with those of induced currents, we shall have found a theory which, if not true, can only be proved to be erroneous by experiments which will greatly enlarge our knowledge of this part of physics.*"

In his "On physical lines of force" paper, he tackles several experimental laws (like for instance that like charges repel each other, etc.). One of these laws is the one which interests us the most in this study, and it is what we now know as Faraday's law of induction: if a loop of conducting wires is immersed in a time-changing magnetic field, the rate of change of the magnetic flux through the cross-section area of that loop, induces an electric current in the loop.

## 4 Spinning, tiny, closely-packed spheres.

Maxwell thought of a medium comprised of these tiny spheres cells which could spin. As they spun, these spherical cells would change their shape, tending to become an oblate spheroid, flattening in their poles and expanding in their equators. These "equators" of course are determined orthogonal to the direction of the spin. This simple, yet powerful, analogy is able to describe two main characteristics of Faraday's lines of force: some sort of "tension" (the field) along the rotating axis of the sphere, as well as some kind of " pressure" which would act sideways to the spheres, transmitted to nearby spheres, and could be a model for this kind of repulsive force which helps the lines of force "propagate" through the medium. The inverse square law was also embedded in this model.

The model does not end there. Maxwell completed it with two key elements (without them you wouldn't really have a good analogy). 1) how to prevent that neighbouring cells rub against each other and, 2) how do you set these cells to spin?

His solution was to add smaller particles between the cells. These particles act like ball bearings in a two-gear mechanism. In this way, two nearby spheres can rotate in the same direction, solving the first issue above. But there's more, he called these in-between particles *particles of electricity*, which, if the material

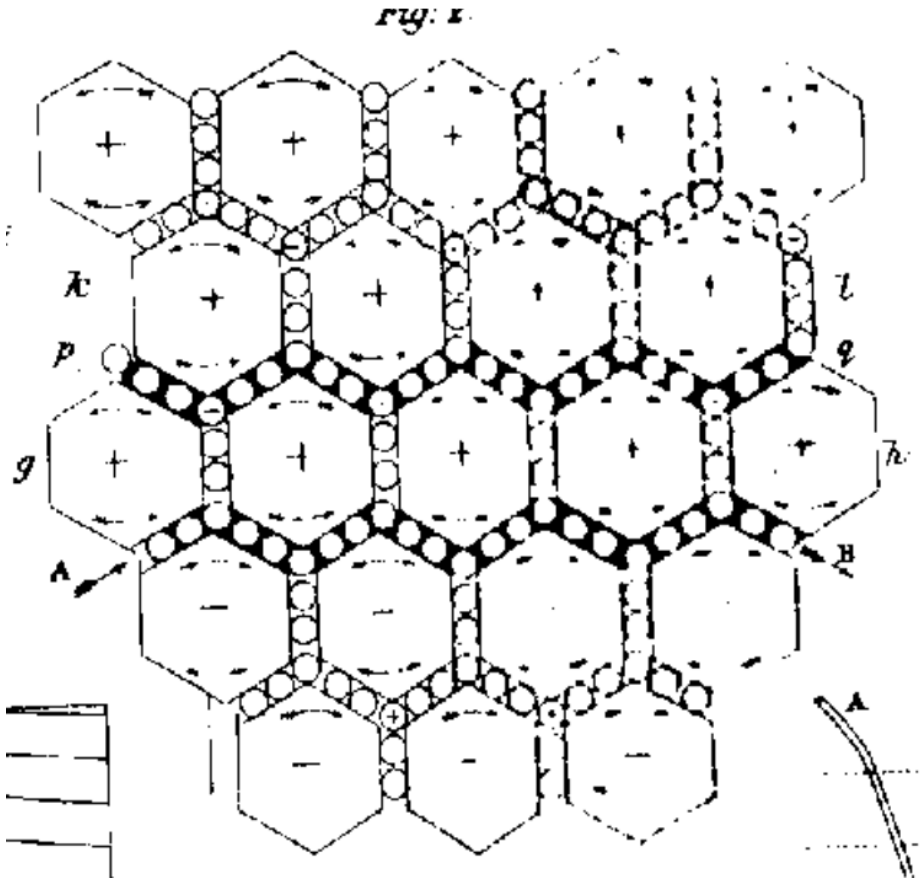


Figure 1: A schematic diagram of Maxwell's model for the magnetic field.

present in the space they were in allowed it, they would move in the channels formed by the spaces in-between the spheres. They could be moved by an electromotive force (in the case where we have what we now call an impressed current) or they could move due to the propagation of the rotation of the spheres through space (induction). If a particle occupied a space where there were no conductors, then this sphere was only allowed to rotate, not to move.

In Fig. 1 it can be seen a snapshot of Maxwell's imagined system of spherical cells and particles of electricity (taken form his 'On physical lines of force' paper)

The hexagons represent the cells. Not sure why Maxwell drew them as hexagons, but many sources indicate he did it "for artistic reasons" (see some references at the end). The black line of particles of electricity connecting point A to B represent a conductor with an impressed electric current running. These particles move from left to right (west to east, in Maxwell's description), and thus make the spheres rotate counter-clock wise (seen from our side) in the region between AB and PQ, while they rotate clock-wise in the region below (south) of AB.

The spheres rotate and they propagate the disturbance through space by exerting pressure on one another, until they reach the particles connecting points P and Q (a free conductor). If this conductor is connected at both ends, then these particles will be allowed to move as well as to rotate. If the conductor is not connected as a circuit, then the particles will only rotate. Due to the nature of elongation and elasticity of these spheres, the propagation of the motion will happen provided that the current in AB is time-changing.

# 5 But... what is the magnetic vector potential then?

The interpretation of the magnetic vector potential is not explicitly described in Maxwell's work, or in any of the references cited below, but it follows from Maxwell's model described above quite parsimoniously.

**We can interpret the magnetic vector potential as the velocity field of all the spinning spheres.**

By recalling that $\mathbf{B} = \nabla \times \mathbf{A}$, and thinking of $\mathbf{A}$ as a velocity field, then $\mathbf{B}$ becomes the *angular velocity* of a point/sphere in this velocity field $\mathbf{A}$. Maxwell called $\mathbf{B}$ the *angular momentum* (something proportional to angular speed). Somehow similar to vortexes in a fluid flow (Maxwell's first attempt). The problem with the fluid model is that it can't model well the fact that there could be different pressures in different directions.

Look back, for instance, to (3). Isn't now more straightforward to interpret this equation? Electric currents are the *direct* sources of wave 'motions' in the velocity field described by the spheres... If one is more interested in knowing the magnetic field, instead of the magnetic vector potential, then one must calculate the rotation (find the vortexes) of the velocity field.

Can we transmit these interpretation efficiently in a video? Explaining it without visual aid takes too much time and it is very hard, in my experience. A good video can, on the other hand, be more clear and convincing.

# 6 Some further considerations

There are several other 'neighbouring' aspects to exploit, in my opinion, regarding this topic.

## 6.1 On the physical consistence of equivalent models.

Looking back, Maxwell's spheres seem rough and primitive. He was able to mature later on and to leave aside any mechanical interpretation of the electromagnetic phenomena. However, I believe that this kind of analogies are not only a necessary step in the learning process, they are also of great value on their own. Provided that one is fully aware of their limitations. Maxwell seem to be fully aware of this, as you can read in p. 346 of his paper:

"*These particles, in our theory, play the part of electricity. Their motion of translation constitutes an electric current, their rotation serves to transmit the motion of the vortices from one part of the field to another, and the tangential pressures thus called into play constitute electromotive force. The conception of a particle having its motion connected with that of a vortex by perfect rolling contact may appear somewhat awkward. I do not bring it forward as a mode of connexion existing in nature, or even as that which I would willingly assent to as an electrical hypothesis. It is, however, a mode of connexion which is mechanically conceivable, and easily investigated, and it serves to bring out the actual mechanical connexions between the known electro-magnetic phenomena; so that I venture to say that any one who understands the provisional and temporary character of this hypothesis, will find himself rather helped than hindered by it in his search after the true interpretation of the phenomena,*"

Many reflections are possible, but leaving them due to lack of time.

## 6.2 On the perceived importance of physical quantities.

Also, I'd just mention very briefly this point without elaboration, due to time constraints. The main point is to argue that for Maxwell, the potentials (magnetic vector and electric scalar) were the **real** physical quantities and the fields were abstractions. We are now taught fields, but that's not exactly how Maxwell thought about the phenomena. We owe mainly to Heaviside (and others) the formulation of the theory in terms of fields, not potentials. For Heaviside, on the contrary, fields were the real physical quantity to consider.

## 6.3 On the importance of scientific conversation.

EM theory came about through a myriad of debates, conversations and rebuttals. How important is that for the advance of knowledge! In my field, we hardly ever see serious debates. I have recently published an article, together with some colleagues, on a debate that I organized in a conference. Apart from reporting on the debate (interesting only to those really in the topic), I also reflected on their importance. If you are curious, have a look here (in case you do not have access to this journal, let me know and I can send you a preprint).

## 6.4 Lorenz gauge.

One of the issues of assuming $\mathbf{B} = \nabla \times \mathbf{A}$ is that the magnetic field is then not uniquely determined by a vector potential. In fact, the same result would be found if we add any arbitrary vector field to $\mathbf{A}$ which is the gradient of a scalar field, i.e. $\mathbf{B} = \nabla \times (\mathbf{A} + \nabla\Phi)$, gives the same magnetic field.

(Maybe this description is trivial to you, sorry... just for the sake of completeness). This extra degrees of freedom is what normally justifies the aplication of the so-called Lorenz gauge:

$$\nabla \cdot \mathbf{A} + \mathrm{j}\omega\varepsilon\mu\Psi = 0\,, \tag{4}$$

with $\Psi$ being the electric scalar potential.

### 6.5 The electric scalar potential.

Two (to me) interesting considerations regard the electric scalar potential. The first one, is how it is more easily accepted and perceived by people (students, teachers, engineers...) as a physical quantity. This potential is something you can measure directly with a voltmeter (it's just the line integral of the electric field between two points). It has a clear analogy with the gravitation potential, and its field, etc. However, it remains as intriguing as the vector potential, in essence... We just have a closer way to visualize it. The second consideration comes when looking at the expression of the electric field, expressed in terms of potentials:

$$\mathbf{E} = -\nabla\Psi - \mathrm{j}\omega\mathbf{A} \tag{5}$$

The expression in (5) is the complete expression for the electric field. However, in many (too many) applications, the electric field is simply considered to be equal to the gradient of the scalar potential, neglecting the magnetic vector potential. For instance, the whole circuit theory is one crucial example of such assumption.

Neglecting the $\mathrm{j}\omega\mathbf{A}$ term, implicitly means assuming that the electric field is irrotational (under this assumption, $\nabla \times \mathbf{E} = \mathbf{0}$) and therefore, it is also conservative. This is not true at all.

In engineering, neglecting the vector potential (thus using purely circuit theory for our designs) means that a myriad of electric and magnetic phenomena can't be modeled, and will appear in the final design, probably making it fail. That's why I mention the billion dollar loss for companies.

## References

[1] Maxwell, J. C. (1861). "On physical lines of force". *Philosophical Magazine.* 90: 11–23.

[2] B. Mahon, *The Man Who Changed Everything: The Life of James Clerk Maxwell*, Wiley, 2004.

[3] N. Forbes, B. Mahon, *Faraday, Maxwell, and the Electromagnetic Field: How Two Men Revolutionized Physics*, Prometheus, 2019.