Lecture 6: Log-linearization

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1 Lineariztion v.s. Log-linearization

1.1 Lineariztion and deviation form

For non-linearized equations, linearization means one order approximation. In mathematics, approximation means that we first need to find a point from where we approximate. Typically speaking, this point is the steady state values of the variables. For different variables, the steady state will be different significantly. This is the reason why we need deviation form which is unit-less variables.

The definition for (percentage) deviation form of a variable x_t is:

$$\tilde{x}_t = \frac{x_t - x^*}{x^*}$$

where x^* stands for the steady state value of x_t. The Taylor expansion of $f(x)$ is:

$$f(x) = f(x^*) + f'(x^*)(x - x^*) + \frac{f''(x^*)}{2!}(x - x^*)^2 + ...$$

For multivariate function $f(x, y)$ evaluating at points (x^*, x^*), we have:

$$f(x, y) = f(x^*, y^*) + f_x(x^*, y^*)(x - x^*) + f_y(x^*, y^*)(y - y^*) + ...$$

Approximation techniques via Taylor series approximation is sometimes called perturbation methods.

1.2 Log-linearization

Log-linearization involves 1st order approximation where we manipulate variables to % difference from log-difference:

$$\tilde{x}_t \approx log x_t - log x^*$$

this is actually the method from Uhlig(1999). We can show why above approximation holds since

$$logx_t - logx^* = log(\frac{x_t}{x^*}) = log(\frac{x_t - x^*}{x^*} + 1) \approx \frac{x - x^*}{x^*} = \tilde{x}_t$$

and this is the reason why we call it log-linearization. Log-linearization make the linearization more easier and so we often take natural logarithms at both sides of the equations before make 1st order Taylor expansion(simple linearization). The above reasoning will ensure that log-linearization will have the same results with the one we do not apply logarithms before we linearize. The following is the so-called Cookbook approach for log-linearization:

1. take (natural) logs of both sides of expression;

2. Taylor series approximation: 1st order approximation;

3. simplify and collect terms together so that $\tilde{x}_t \equiv \frac{x-x^*}{x^*} \approx logx_t - logx^*$

2 Log-linearization Example

1. $y = f(x)$, $y^* = f(x^*)$, where x^*, y^* are steady state values of x and y.

2. Cookbook approach: $logy = logf(x)$, $logy^* + \frac{y-y^*}{y^*} \approx logf(x^*) + \frac{f'(x^*)}{f(x^*)}(x - x^*)$, we have $\tilde{y} = \frac{x^* f'(x^*)}{f(x^*)}\tilde{x}$.

3. Or taking 1st order approximation directly at both side, we have $|y^* + y - y^*| = f(x^*) + f(x^*)(x - x^*)$, we get the same results as in step 2.

4. A further example: the Cobb-Douglass production function

$$y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$$

Taking logs at both sides, we have

$$logy_t = logA_t + \alpha logK_t + (1-\alpha)logN_t \tag{1}$$

Since this is true for any time t, it must be true at steady state. Hence, we have:

$$logy^* = logA^* + \alpha logK^* + (1-\alpha)logN^*$$

According to the cookbook approach, we have:

$$logy^* + \frac{y_t - y^*}{y^*} = logA^* + \frac{A_t - A^*}{A^*} + \alpha(logK^* + \frac{K_t - K^*}{K^*}) + (1-\alpha)(logN^* + \frac{N_t - N^*}{N^*})$$

then we have:

$$\tilde{y}_t = \tilde{A}_t + \alpha\tilde{K}_t + (1-\alpha)\tilde{N}_t \tag{2}$$

There is another convenient way to do the log-linearization. We can simply take differentiation at both sides of Equation (1) and evaluating at the steady states:

$$\begin{aligned} dlogy_t &= dlogA_t + \alpha dlogK_t + (1-\alpha)dlogN_t \\ \frac{dy_t}{y^*} &= \frac{dA_t}{A^*} + \alpha\frac{dK_t}{K^*} + (1-\alpha)\frac{dN_t}{N^*} \end{aligned}$$

if we define $dy_t = y_t - y^*$ and imilar for other variables, then we get the same result as in Equation (2).

5. A further example: share-weighted percentage deviation.

$$y_t = c_t + I_t$$

Taking logs at both sides, $logy_t = log(c_t + I_t)$. Then 1st order Taylor expansion, we have:

$$logy^* + \frac{y_t - y^*}{y^*} = log(c^* + I) + \frac{c_t - c^*}{c^* + I^*} + \frac{I_t - I^*}{c^* + I^*} \tag{3}$$

Recall the definition of log-linearized variables as percentage deviations from steady state, hence, we have: $c_t - c^* = c^*\tilde{c}_t$ and $I_t - I^* = I^*\tilde{I}_t$. Substitute into Equation (3), can get:

$$\tilde{y}_t = \frac{c^*}{y^*}\tilde{c}_t + \frac{I^*}{y^*}\tilde{I}_t$$

6. A further example: law of motion for capital (i.e., capital accumulation equation).

$$K_{t+1} = I_t + (1-\delta)K_t \tag{4}$$

At steady state, capital is constant over time, thus: $K_{t+1} = K_t = K^*$, and $I^* = \delta K^*$. Taking logs, we have:

$$\log K_t = \log\left(I_t + (1-\delta)K_t\right)$$

Taking 1st order Taylor expansion, we have:

$$\begin{aligned} \log(I_t + (1-\delta)K_t) \approx &\log(I^* + (1-\delta)K^*) + \frac{\partial}{\partial I_t}\log(I_t + (1-\delta)K_t)\bigg|_* (I_t - I^*) \\ &+ \frac{\partial}{\partial K_t}\log(I_t + (1-\delta)K_t)\bigg|_* (K_t - K^*) \end{aligned}$$

Thus,

$$\log K^* + \frac{K_{t+1} - K^*}{K^*} \approx \log(I^* + (1-\delta)K^*) + \frac{I_t - I^*}{I^* + (1-\delta)K^*} + \frac{(1-\delta)(K_t - K^*)}{I^* + (1-\delta)K^*}$$

At steady state: $I^* = \delta K^* \Rightarrow I^* + (1-\delta)K^* = K^*$ Therefore, we can get:

$$\log K^* + \frac{K_{t+1} - K^*}{K^*} = \log\left(I^* + (1-\delta)K^*\right) + \frac{I_t - I^*}{K^*} + (1-\delta)\frac{K_t - K^*}{K^*}$$

We want everything in terms of tilde variables. Also note: $\frac{I_t - I^*}{K^*} = \frac{I^*}{K^*} \cdot \frac{I_t - I^*}{I^*} = \frac{I^*}{K^*}\tilde{I}_t$, Thus, final expression:

$$\tilde{K}_{t+1} = \frac{I^*}{K^*}\tilde{I}_t + (1-\delta)\tilde{K}_t$$

Since $\frac{I^*}{K^*} = \delta$ (from the steady-state condition), we simply the above eqaution to:

$$\tilde{K}_{t+1} = \frac{I^*}{K^*}\tilde{I}_t + (1-\delta)\tilde{K}_t = \delta\tilde{I}_t + (1-\delta)\tilde{K}_t \tag{5}$$

Alternatively, ff we taking differentiation at both sides of Eq.(4), it seems to be much simpler:

$$dK_{t+1} = dI_t + (1-\delta)\, dK_t$$

evaluating at steady states, it will produce the same result as in Eq.(5).

7. Consumption Euler equation:

$$\left(\frac{C_{t+1}}{C_t}\right)^{\sigma} = \beta(1 + r_t)$$

Taking natural logs of both sides:

$$\sigma \log\left(\frac{C_{t+1}}{C_t}\right) = \log\beta + \log(1 + r_t)$$

Differentiate for both sides:

$$\sigma\left(d\log C_{t+1} - d\log C_t\right) = d\log\beta + d\log(1 + r_t)$$

We know $log(1 + r_t) \approx r_t$ (valid when r_t is small). The discount factor β is related to the steady-state real interest rate r^* by: $\beta = \frac{1}{1+r^*}$ (the intertemporal tradeoff: people discount the future at a rate related to the return on saving/investing). Then the above equation can be written as:

$$\sigma\left(\tilde{C}_{t+1} - \tilde{C}_t\right) = \frac{1}{1 + r^*}(dr_t) = \beta\tilde{r}_t \approx \tilde{r}_t$$

if the discount factor β is large enough. Here we define: $\tilde{r}_t = r_t - r^*$ as the absolute deviation not percentage deviation. In general, for the variables that are already in percentage form, such as interest rate, inflation rate etc., the percentage deviation does not make any sense for these variables, thus we define absolute deviation (i.e., the variable minus its steady state value).

References

Uhlig, H. (1999). A toolkit for analyzing nonlinear dynamic stochastic models easily. Available online at: https://www.sfu.ca/~kkasa/uhlig1.pdf