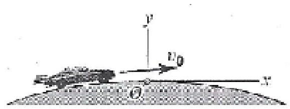
Problem 1

A car is traveling at a constant speed over a hill. If, using a Cartesian coordinate system with origin O at the top of the hill, the hill's profile is described by the function $y = -(0.003\,\mathrm{m}^{-1})x^2$, where x and y are in meters, determine the minimum speed at which the car would lose contact with the ground at the top of the hill. Express the answer in km/h.



> Notice: $y = -0.003x^2$, the m^{-1} is just about the unit.

Recall and Analysis:

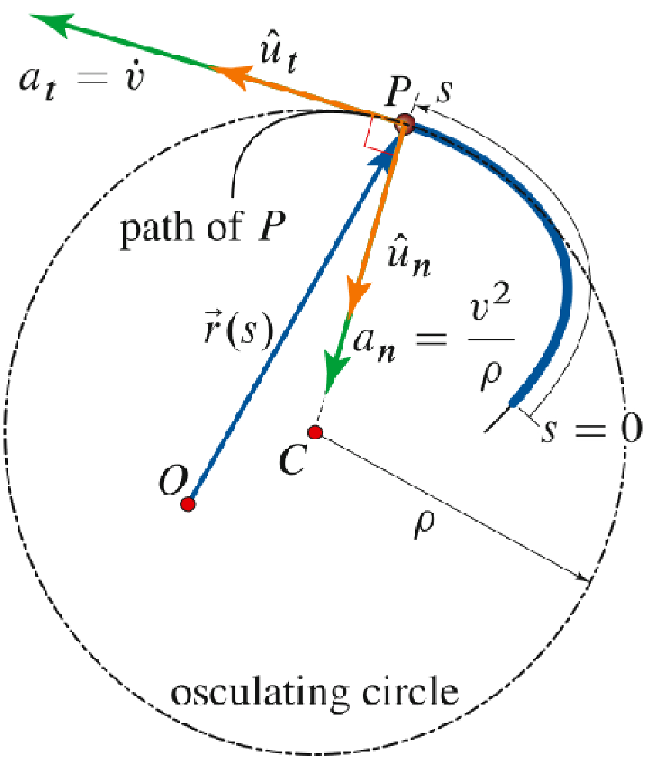
- A path described by Cartesian Coordinates, $y = y(x)$, has the radius of curvature:

$$\rho = \frac{(1 + \frac{dy}{dx}^2)^{3/2}}{|\frac{d^2y}{dx^2}|}$$

- The acceleration in N-T coordinate is:

$$\vec{a} = \dot{v}\hat{u}_t + \frac{v^2}{\rho}\hat{u}_n$$

- At the top of the hill, the tangent to the path is horizontal, so the normal direction coincides with the gravity direction.
- The question "loss contact of the ground" means the normal force equals to gravity, i.e. $a_n = g = \frac{v^2}{\rho}$. Because gravity has two functions: to provide the normal force and to pull the object down. If gravity doesn't pull object down, it means *all* gravity is for normal force.



Solution:

$a_n = g = \frac{v^2}{\rho}$ means $\frac{v^2}{\rho} = g$.

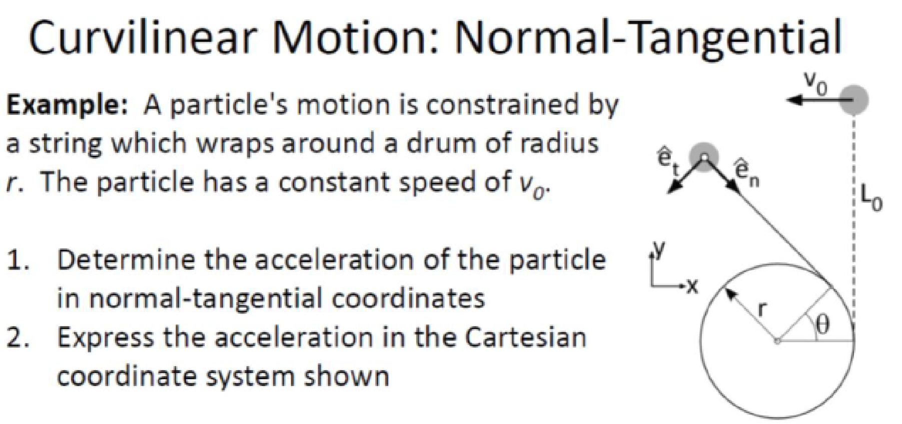
The question is about the minimum velocity, so: $\frac{v_{min}^2}{\rho|_{x=0}} = g \Rightarrow v_{min} = \sqrt{g\rho|_{x=0}}$

For $y = -0.003x^2$, we have:

$$\frac{dy}{dx} = -0.006x$$
$$\frac{d^2y}{dx^2} = -0.006$$

So, for $x = 0$, $\frac{dy}{dx}|_{x=0} = 0$ and $\frac{d^2y}{dx^2}|_{x=0} = -0.006$.
So, $\rho|_{x=0} = \frac{(1+(\frac{dy}{dx}|_{x=0})^2)^{3/2}}{|\frac{d^2y}{dx^2}|_{x=0}|} = \frac{(1+0^2)^{3/2}}{|-0.006|} = \frac{1}{0.006} = 166.67m$.
And $v_{min} = \sqrt{g\rho|_{x=0}} = \sqrt{9.8 \times 166.67} = 42.4m/s = 145.6km/h$.

Problem 2



Recall and Analysis:

- The acceleration in N-T is:

$$\vec{a} = \dot{v}\hat{u}_t + \frac{v^2}{\rho}\hat{u}_n$$

- "constant speed" means $\dot{v} = 0$ and $v = v_0$.

Solution:

1.

$\rho = L_0 - r\theta$. So, $\vec{a} = \frac{v_0^2}{L_0 - r\theta}\hat{u}_n$.

2.

Since $\dot{v} = 0$, so there is no tangential acceleration, i.e. $\hat{u}_t$ part.
So, we only have the normal acceleration, i.e. $\hat{u}_n$ part. To convert the $\hat{u}_t$ in N-T coordiante into $\hat{i}, \hat{j}$ in Cartesian coordinate, we need:

$$\hat{u}_t = \sin\theta\hat{i} - \cos\theta\hat{j}$$

$\vec{a} = \frac{v_0^2}{L_0 - r\theta}\hat{u_n} = \frac{v_0^2}{L_0 - r\theta}(\sin\theta\hat{i} - \cos\theta\hat{j})$

To check it, we can calculate the magnitude of the acceleration:

$$|\vec{a}| = \frac{v_0^2}{L_0 - r\theta}\sqrt{\sin^2\theta + \cos^2\theta} = \frac{v_0^2}{L_0 - r\theta}$$