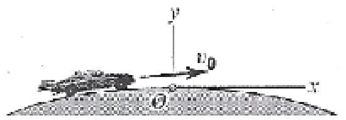
Problem 1

A car is traveling at a constant speed over a hill. If, using a Cartesian coordinate system with origin O at the top of the hill, the hill's profile is described by the function $y = -(0.003\,\mathrm{m}^{-1})x^2$, where x and y are in meters, determine the minimum speed at which the car would lose contact with the ground at the top of the hill. Express the answer in km/h.



> Notice: $y = -0.003x^2$, the unit m^{-1} in the monomial is to align the dimensions of x and y

Recall and Analysis:

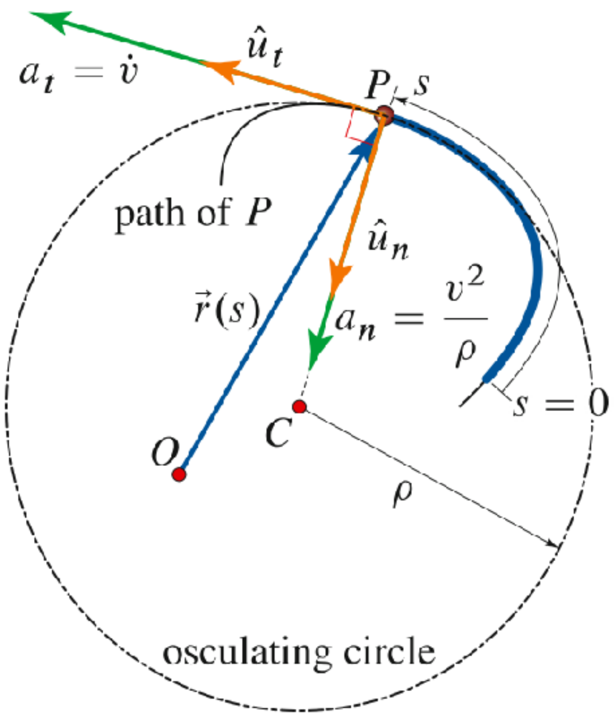
- A path described by Cartesian Coordinates, $y = y(x)$, has the radius of curvature:

$$\rho = \frac{(1 + \frac{dy}{dx}^2)^{3/2}}{|\frac{d^2y}{dx^2}|}$$

  The acceleration in N-T coordinate is:

$$\vec{a} = \dot{v}\hat{u_t} + \frac{v^2}{\rho}\hat{u_n}$$

- At the top of the hill, the tangent to the path is horizontal, so the normal direction coincides with the gravity direction.
- The question "loss contact of the ground" means the normal force equals to gravity, i.e. $a_n = g = \frac{v^2}{\rho}$. Because gravity has two functions: to provide the normal force and to pull the object down. If gravity doesn't pull object down, it means *all* gravity is for normal force.



Solution:

The condition "the car lose contact at top of the hill" indicates that the ground contact force to the car is zero, which means at the minimum speed which meets the condition:

$$a_n = g = \frac{v^2}{\rho}$$

The question is about the minimum velocity, so: $\frac{v_{min}^2}{\rho|_{x=0}} = g \Rightarrow v_{min} = \sqrt{g\rho|_{x=0}}$

For $y = -0.003x^2$, we have:

$$\begin{cases} \dfrac{dy}{dx} = -0.006x \\ \dfrac{d^2y}{dx^2} = -0.006 \end{cases}$$

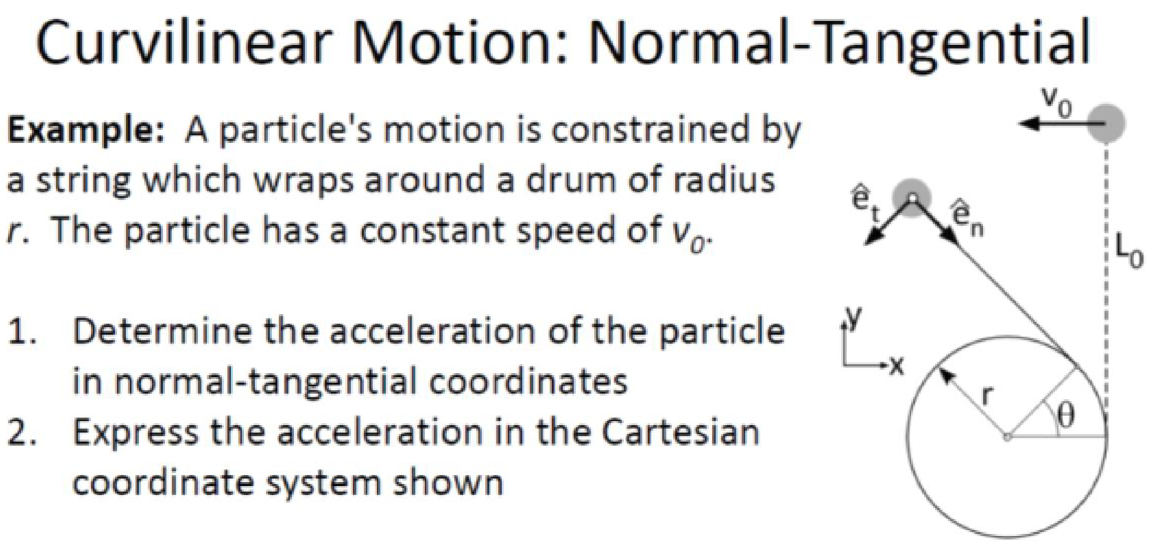
Therefore, for $x = 0$, $\frac{dy}{dx}|_{x=0} = 0$ and $\frac{d^2y}{dx^2}|_{x=0} = -0.006$.

By plugging these values to our equation, we have:

$$\rho|_{x=0} = \frac{(1 + (\frac{dy}{dx}|_{x=0})^2)^{3/2}}{|\frac{d^2y}{dx^2}|_{x=0}|} = \frac{(1+0^2)^{3/2}}{|-0.006|} = \frac{1}{0.006} = 166.67m$$

And $v_{min} = \sqrt{g\rho|_{x=0}} = \sqrt{9.8 \times 166.67} = 42.4m/s = 145.6km/h$.

Problem 2



Recall and Analysis:

- The acceleration in N-T is:
  $\vec{a} = \dot{v}\hat{u_t} + \frac{v^2}{\rho}\hat{u_n}$
- "constant speed" means $\dot{v} = 0$ and $v = v_0$.
- Given conditions: r, θ, v_0, L_0

Solution:

1.

$\rho = L_0 - r\theta$. So, $\vec{a} = \frac{v_0^2}{L_0 - r\theta}\hat{u_n}$.

2.

Because $\dot{v} = 0$, there is no tangential acceleration, i.e. $\hat{u_t}$ part.

Therefore, we only have the normal acceleration, i.e. $\hat{u_n}$ part. To convert the $\hat{u_t}$ in N-T coordinate into $\hat{\imath}, \hat{\jmath}$ in Cartesian coordinate, we just need to find the components in Cartesian Coordinates, by observing the direction of the rope, we have:

$$\hat{u_n} = \sin\theta\hat{\imath} - \cos\theta\hat{\jmath}$$

Then we can derive the acceleration $\vec{a}$:

$$\vec{a} = \frac{v_0^2}{L_0 - r\theta}\hat{u_n} = \frac{v_0^2}{L_0 - r\theta}(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath})$$

To check if the answer is correct, we can calculate the magnitude of the acceleration:

$$|\vec{a}| = \frac{v_0^2}{L_0 - r\theta}\sqrt{\sin^2\theta + \cos^2\theta} = \frac{v_0^2}{L_0 - r\theta}$$

Which equals to the answer we derive in 1.