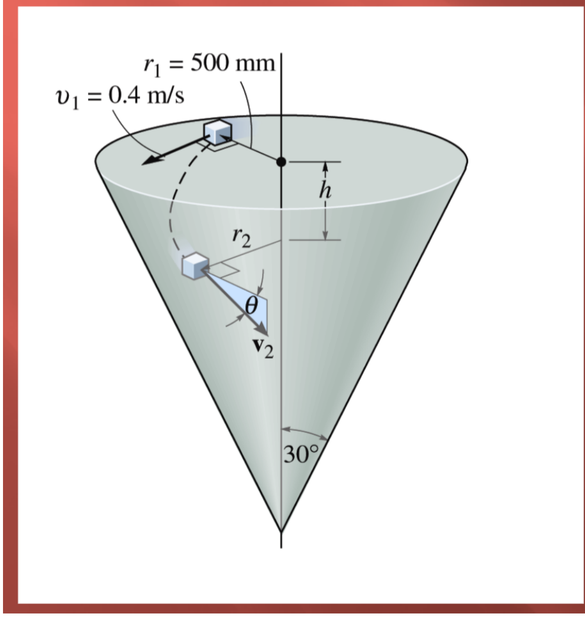
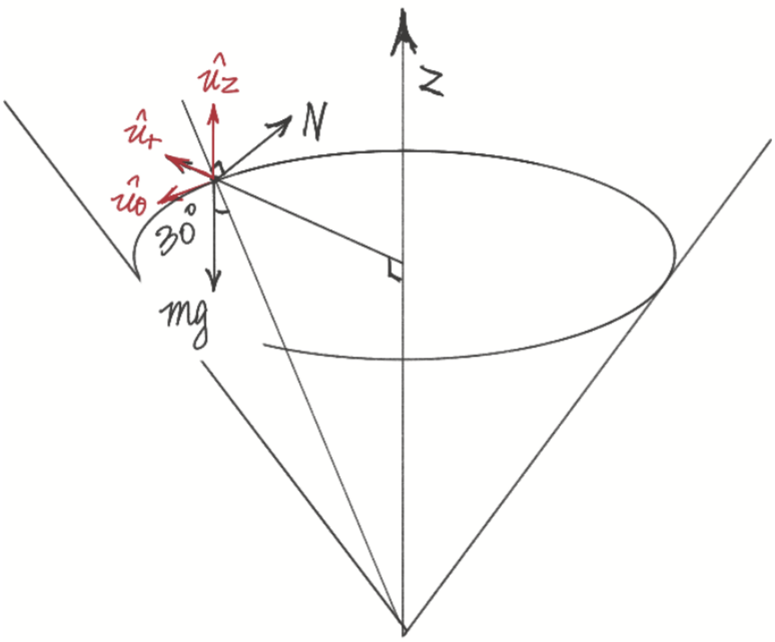
Problem 1

Example 3: A small block having a mass of 0.1 kg is given a horizontal velocity $v_1 = 0.4$ m/s when $r_1 = 500$ mm. It slides along the smooth conical surface. When it descends to $h = 100$ mm, determine its speed and the angle of descent, θ; that is, determine the angle measured from the horizontal to the tangent of the path.



Analysis



To find the equations, we can first draw a FBD of the block, then try to find what equations can we use to solve the unknowns.

First, we try to consider energy: we can easily notice that there are 2 forces, gravity mg, and normal force N from the cone. Gravity is conservative force, and normal force is perpendicular to the cone surface. So we can use conservative of energy for this problem (with $U_{nc} = 0$).

$$T_1 + V_1 = T_2 + V_2$$

Then we look into momentum-impulse. $\overrightarrow{p_1} + F_{1-2} = \overrightarrow{p_2}$

$$\vec{F} = -Ncos\theta \cdot \hat{u}_r + (Nsin\theta - mg) \cdot \hat{u}_z$$

we can see that direction of N is consistently changing with the position of the block, making $\int \vec{F}dt$ hard to solve, and making momentum in all directions not conserved.

We then analyze angular momentum-impulse. Where $\vec{M} = \vec{r} \times \vec{F}$:

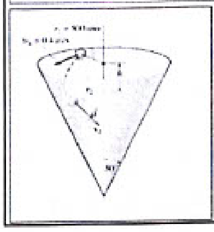
$$\begin{aligned} \vec{M} &= r\hat{u}_r \times (-Ncos\theta \cdot \hat{u}_r + (Nsin\theta - mg) \cdot \hat{u}_z) \\ &= (Nsin\theta - mg)r \cdot \hat{u}_\theta \end{aligned}$$

$\vec{M}$ is along polar direction, which means that on direction $\hat{u}_z$, angular momentum is conserved. Where $\hat{h}_z = r\hat{u}_r \times m(\dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta) = mr^2\dot{\theta}\hat{u}_z$. So

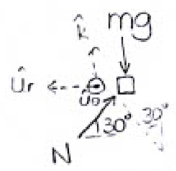
$$mr_1^2\dot{\theta}_1 = mr_2^2\dot{\theta}_2$$

Solution

Example 3. A small block having a mass of 0.1 kg is given a horizontal velocity $v_1 = 0.4$ m/s when $r_1 = 500$ mm. It slides along the smooth conical surface. When it descends to $h = 100$ mm, determine its speed and the angle of descent, θ; that is, determine the angle measured from the horizontal to the tangent of the path.



FBD of particle at position 1:



All forces conservative or $\perp$, so can find speed from work-energy / conservation of energy:

$$T_1 + V_1 = T_2 + \cancel{V_2}^{\,0,\text{ weight datum}}$$

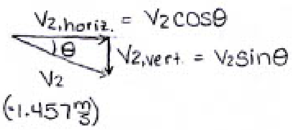
$$\frac{1}{2}\cancel{m}v_1^2 + \cancel{m}gh = \frac{1}{2}\cancel{m}v_2^2$$

$$v_2 = \sqrt{v_1^2 + 2gh}$$

$$= \sqrt{\left(0.4\,\frac{\text{m}}{\text{s}}\right)^2 + 2\left(9.81\,\frac{\text{m}}{\text{s}^2}\right)(100\,\text{mm})\left(\frac{1\,\text{m}}{1000\,\text{mm}}\right)}$$

$$= 1.457\ \frac{\text{m}}{\text{s}}$$

What is angle of descent?



$\Rightarrow$ Need vector info about $\vec{V}$.

Impulse-momentum methods are vector methods.

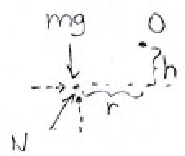
(Work-energy is a scalar method)

Have linear momentum and angular momentum. Is either conserved? (Conserved is helpful because impulse-momentum integrates forces and moments over time, and we don't have time information.)

Angular : $\vec{h}_{O1} + \int_{t_1}^{t_2} \vec{M}\,dt = \vec{h}_{O2}$ $\Rightarrow$ don't know t_1, t_2

maybe ang. mom. conserved in some direction?

Forces	Moment wrt O
mg (↓)	purely horizontal
N (↗)	horizontal from ↑ component
	" " → "



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$\Rightarrow$ no vertical moments

$\Rightarrow$ angular momentum conserved in vertical direction = "about z axis" (If $v_1 = 0$, would slide straight down)

$\vec{h}_{O1} = r_1 \hat{u}_{r1} \times m v_1 \hat{u}_{\theta 1} = r_1 m v_1 \hat{u}_z$ ($\hat{u}_z$, $\hat{u}_{r1}$, $\hat{u}_{\theta 1}$ toward me)

$\vec{h}_{O2} = (r_2 \hat{u}_{r2} - h\hat{u}_z) \times (m v_2)(\cos\theta\, \hat{u}_{\theta 2} - \sin\theta\, \hat{u}_z)$ ($\hat{u}_z$, $\hat{u}_{r2}$, $\hat{u}_{\theta 2}$ toward me)

$= r_2 m v_2 \cos\theta\, \hat{u}_z - r_2 m v_2 \sin\theta\,(-\hat{u}_{\theta 2}) - h m v_2 \cos\theta\,(-\hat{u}_{r2}) + \vec{0}$

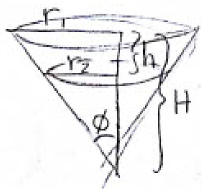
$h_{O2z} = r_2 m v_2 \cos\theta = h_{O1z} = r_1 m v_1$

$$\cos\theta = \frac{r_1 \cancel{m} v_1}{r_2 \cancel{m} v_2}$$

Need r_2

$\Rightarrow$ cone geometry

$$\tan\phi = \frac{r_1}{H} = \frac{r_2}{H-h}$$



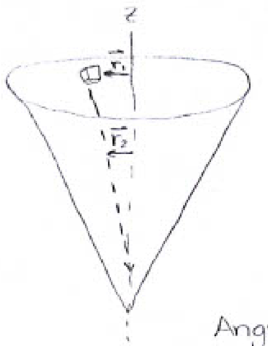
$\phi = 30°$, $r_1 = 500$ mm $\Rightarrow H = 866$ mm

$r_2 = 442.3$ mm

$$\cos\theta = \frac{\overset{r_1}{(500)}\ \overset{v_1}{(0.4)}}{\underset{r_2}{(442.3)}\ \underset{v_2}{(1.457)}}$$

$\theta = 71.9°$

Consider what would happen if v_1 were 0 instead of $0.4\frac{m}{s}$:



Block would slide straight down the cone.

Linear momentum would change from $m\vec{v}_1 = \vec{0}$ to $m\vec{v}_2 \neq \vec{0}$.

$\Rightarrow$ Linear momentum not conserved.

Angular momentum about z-axis <u>would</u> be conserved.

$\vec{r}_1 \times m\vec{v}_1 = \vec{0}$ because $v_1 = 0$

$\vec{r}_2 \times m\vec{v}_2 \neq \vec{0}$, but $\vec{r}_2 \times m\vec{v}_2 = (\text{something})\hat{u}_\theta$, nothing in the $\hat{k}$ direction because $\vec{v}_2 = V_{2r}\hat{u}_r + V_{2z}\hat{k}$ and $\hat{u}_r \times \hat{u}_r = 0$, $\hat{u}_r \times \hat{k} = -\hat{u}_\theta$

Conservation of angular momentum about the z-axis means the forces don't have moments about the z-axis.

Same forces whether $v_1 = 0$ or $v_1 = 0.4\frac{m}{s}$ ($m\vec{g}$, $\vec{N}$ - see FBD)

Can also prove $\vec{N}$, $m\vec{g}$ don't have moments about the z-axis using vectors:

$$\vec{M}_z = \vec{r} \times \vec{F}$$

↑ perpendicular to z-axis, $\vec{r} = r\hat{u}_r$

$$\vec{N} = -N_r\hat{u}_r + N_z\hat{k}$$

$$\vec{M}_{z,\vec{N}} = (r\hat{u}_r) \times (-N_r\hat{u}_r + N_z\hat{k}) = rN_z(\hat{u}_r \times \hat{k}) = rN_z(-\hat{u}_\theta)$$

$\Rightarrow$ no $\hat{k}$ component

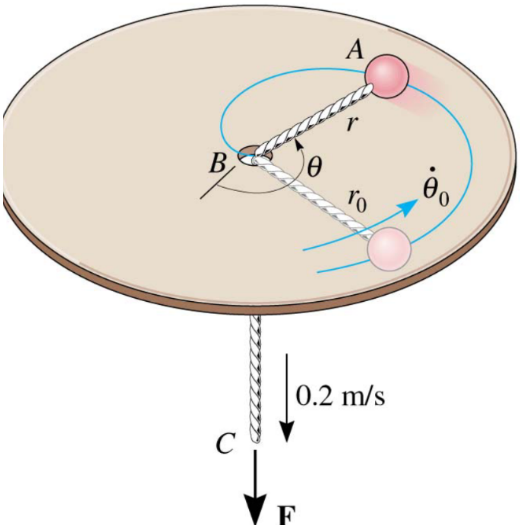
$$m\vec{g} = -mg\hat{k}$$

$$\vec{M}_{z,m\vec{g}} = (r\hat{u}_r) \times (-mg\hat{k}) = -rmg(-\hat{u}_\theta) \Rightarrow \text{no } \hat{k} \text{ component}$$

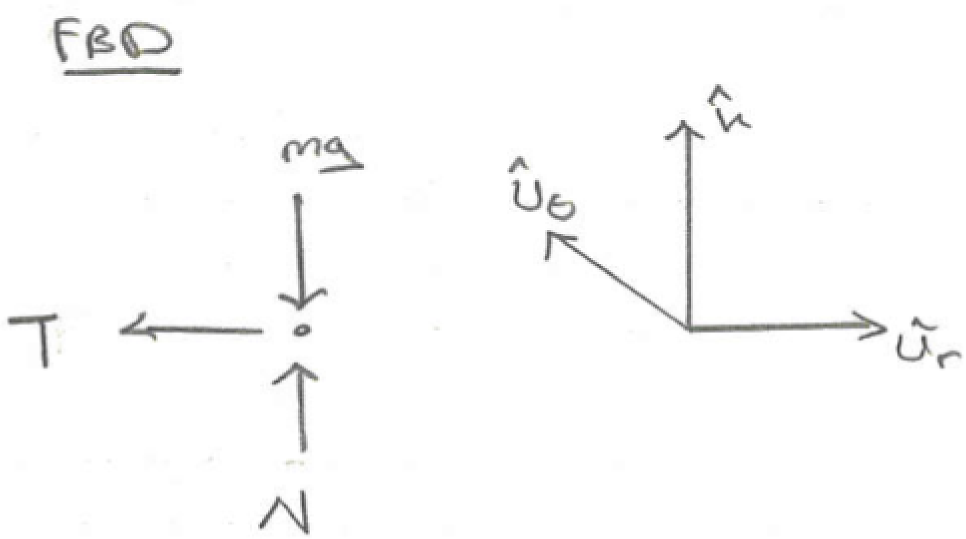
Problem 2

Example 3 (Lec. 11, Ex. 2 revisited): The 2-kg particle is initially traveling around a ***horizontal*** circular path of radius r_0 = 0.5 m such that the angular velocity is 1 rad/sec. If the attached cord *ABC* is drawn down through the hole at a constant speed of 0.2 m/s,

determine the tension the cord exerts on the particle at the instant r = 0.25 m. Also compute the angular velocity at this instant. Neglect friction.



Recall & Analysis



- Angular momentum: $\vec{H} = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$
- Conservation of angular momentum: if the torque $\vec{M}$ is zero, then $\vec{H}$ is constant. Because $\vec{M} = \frac{d\vec{H}}{dt}$, if $\vec{M} = 0$, then $\frac{d\vec{H}}{dt} = 0$, which means $\vec{H}$ is constant.
- In this problem, the $\vec{H} = m\vec{r} \times \vec{v} = m(r\hat{u}_r) \times (v\hat{u}_\theta) = mvr\hat{u}_k$, which is always perpendicular to the z-axis. Therefore, if there is no torque on the $\hat{u}_k$ direction, the angular momentum is conserved.
- In this problem, the torque $\vec{M} = \vec{r} \times \sum \vec{F} = (r\hat{u}_r) \times (N\hat{u}_k - mg\hat{u}_k - T\hat{u}_r) = (-rN + rmg)\hat{u}_\theta$. Therefore, there is no torque on the $\hat{u}_k$ direction.

Solution

Since the angular momentum is conserved, we can use the angular momentum conservation to solve this problem: $H_1 = H_2$.

$$\begin{aligned}
H_1 &= H_2 \\
m r_1 v_1 &= m r_2 v_2 \\
m r_1 (\dot{\theta}_1 r_1) &= m r_2 (\dot{\theta}_2 r_2) \\
\dot{\theta}_1 r_1^2 &= \dot{\theta}_2 r_2^2 \\
\frac{\dot{\theta}_1}{\dot{\theta}_2} &= \frac{r_2^2}{r_1^2} \\
\frac{\dot{\theta}_1}{\dot{\theta}_2} &= \frac{0.25^2}{0.5^2} \\
\frac{\dot{\theta}_1}{\dot{\theta}_2} &= \frac{1}{4} \\
\dot{\theta}_2 &= 4\dot{\theta}_1 \\
\dot{\theta}_2 &= 4 \text{ rad/s}
\end{aligned}$$

From the FBD:

$$\sum F_r = m\ddot{r} - m r \dot{\theta}^2 = -T$$

where $\ddot{r} = 0$ because the string is always drawn down at a constant speed, i.e. $T = m r \dot{\theta}^2$.
So, at $r_2 = 0.25$ m, $\dot{\theta}_2 = 4$ rad/s, $T = 2 \times 0.25 \times 4^2 = 8$ N.