We find polar coordinates useful in expressing *functional relationships* be-
'een two quantities. We often take the angle θ as the independent variable (some-
nes called the *argument*), while we may take r as the dependent variable (called
: *modulus*). Figure 16-3, for example, shows how the radio-frequency field inten-
y E (indicated by the length of the radius vector) at a certain distance from
antenna might vary as a function of the angle θ.

3-2 PROJECTIONS In the work which follows it will be convenient to
: the idea of the *projection* of one line upon another. In Fig. 16-4, we define
projection of a line segment of length s upon a second line AB as the distance
etween perpendiculars dropped from the ends of s upon AB. Let θ be the
gle between s (extended) and AB. We have $\cos\theta = p/s$, or

$$p = s\cos\theta \tag{16-2}$$

ch enables us to calculate the length of the projection p.

If s is the area of a plane figure, rather than the length of a line, we take
projection p of this area upon another plane A-B as that area marked off on
by perpendiculars dropped from *each point* on the perimeter of the given
re. We can use (16-2) to calculate the projected area p. In Fig. 16-5, for instance,
winding W is situated in a uniform vertical magnetic field. Let the winding

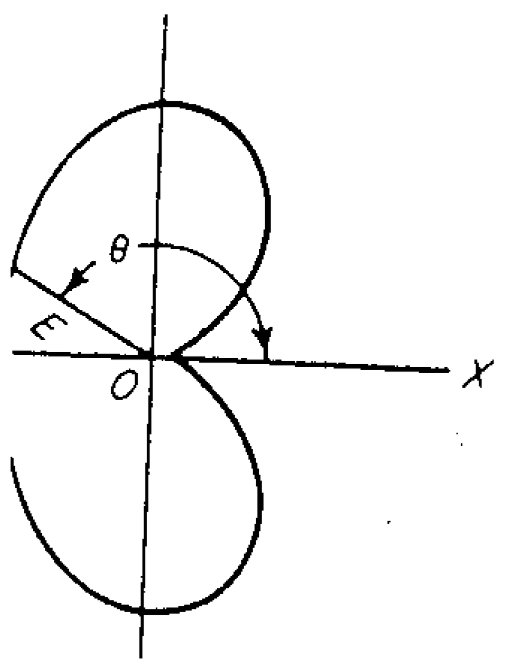

Fig. 16-3 One can show a functional relationship in terms of polar coordinates. In this graph, we see how a particular function E (indicated by the length of the radius vector) varies with the independent variable θ.

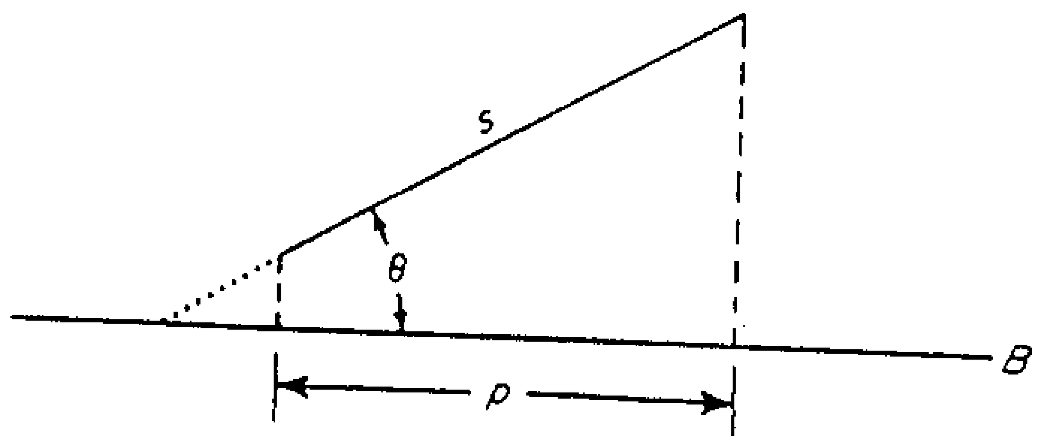

Fig. 16-4 We define the *projection* of a line segment s upon a line AB as the distance p. This distance p is the separation between perpendiculars dropped upon AB from the ends of s. Here $p = s\cos\theta$.

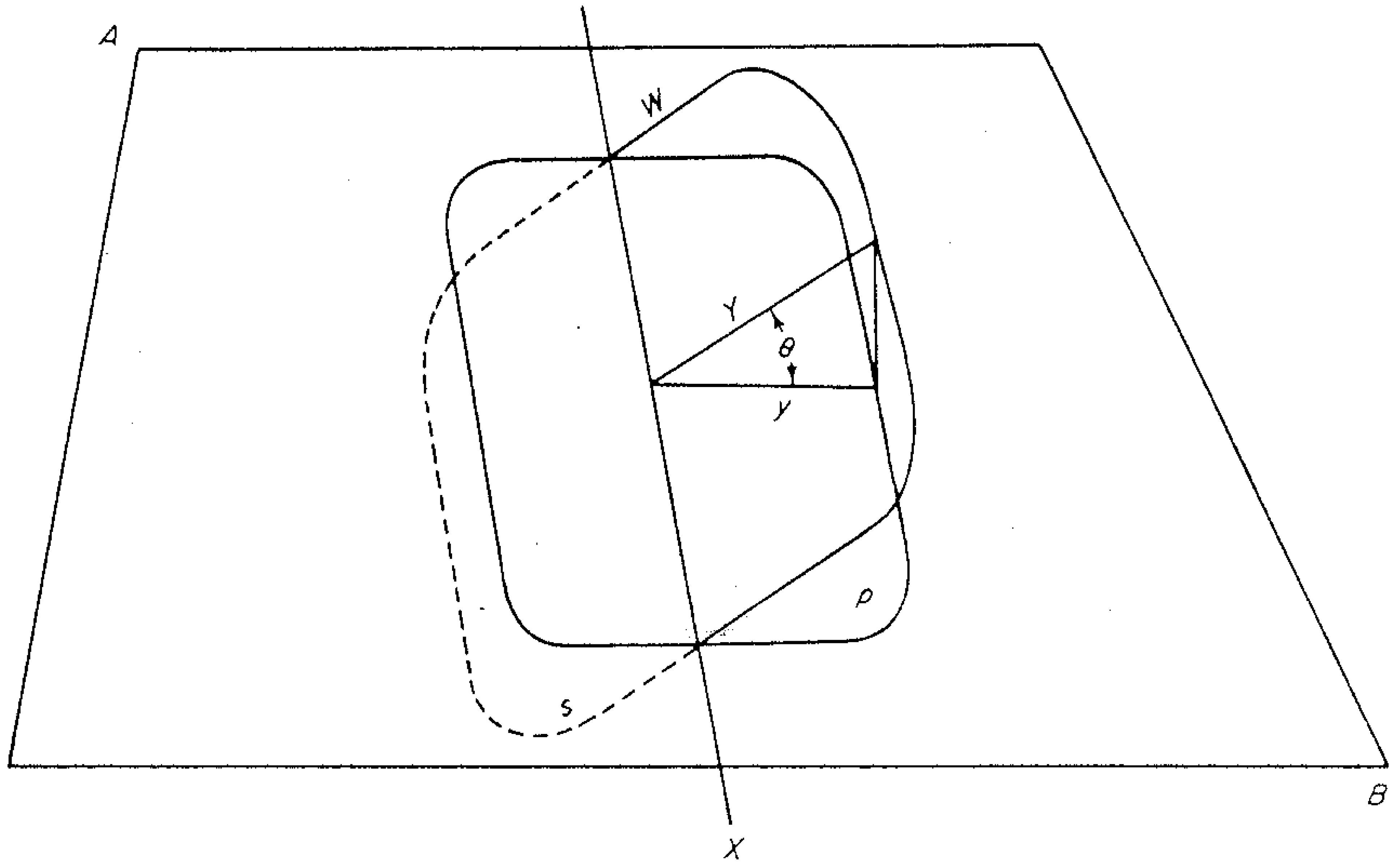

Fig. 16-5 Here s is the area of a plane figure (rather than the length of a line segment). We are interested in the area p, which is the projection of s upon a plane A-B. As explained in the text, we again have the result $p = s\cos\theta$.

cross-sectional area be s. The number of flux lines which pass through W is equal to the number of lines passing through its projected area upon the horizontal plane A-B. To show that (16-2) gives this area, we note that the area of the winding is given by

$$s = 2\int Y\,dx$$

from (15-16). And the area p of the projection of s upon A-B is

$$p = 2\int y\,dx$$

But y is everywhere the projection of some line Y in the original winding area, so that $y = Y\cos\theta$. This gives

$$p = 2\cos\theta\int Y\,dx \tag{16-3}$$

or

$$p = s\cos\theta \tag{16-2}$$

QUESTIONS 16-1

1. What two quantities are needed to fix the position of a point in a plane using rectangular coordinates? using polar coordinates?

2. Given the x and y coordinates of a point in a plane, what formulas would we use to transform this information into polar coordinates?

3. What formulas would transform the position of a point in a plane, expressed in polar coordinates, into rectangular coordinates?

4. Define the *projection* of a line segment upon a second line. What formula gives the length of this projection?

PROBLEMS 16-1

1. In a directional broadcast antenna, tower 2 is located 212 feet from tower 1, in a direction 38° north of east. How far to the north of tower 1 is tower 2 situated?

2. A target-practice object is observed on a radar screen at an airline distance of 18,000 feet and at an elevation angle of 72°. If the object were shot down, at what horizontal distance from the observer would the wreckage fall, assuming a vertical fall?

3. A radar screen shows an object 70 miles from the observer at an angle of 50° east of north. How far to the east of the observer is this object? (The reference axis here is in a "vertical," or northerly, direction on the paper.)

4. A broadcast antenna is 100 feet tall and it is recommended that the guy wires to the top be anchored into the ground at an elevation angle to the horizon of 50°. Find (*a*) the length of each guy wire and (*b*) how far the anchor points are from the antenna base.

5. A microphone diaphragm intercepts 6.75×10^{-9} watt of acoustic power when turned broadside to a sound source. What will be the (theoretical) intercepted power if the diaphragm is turned at an angle of 55° to the source?

6. Light radiations having a plane wavefront strike a photosensitive surface at an angle of 30°. If the surface were turned to face the light directly, by what factor would the amount of received light energy be increased? (The surface is assumed smaller than the cross section of the light beam.)

7. A "curtain" receiving antenna is broadside to a distant transmitter. If it is now turned through an angle of 21°, by what factor will the radiated power impinging upon the curtain be reduced? (This calculation does not include the effect of the antenna directional pattern.)

8. A radar screen shows an object 40° east of north. A second radar screen located 100 miles directly east of the first screen locates the same object as 55° west of north at a distance of 76.9 miles. How far to the east of the first screen is this object?

9. On polar-coordinate paper plot a graph of the function $r = 10 \sin 2\theta$.

10. A half-wave dipole has a radiation pattern given by $\mathbf{E} = K \cos [(\pi/2) \cos \theta]$. Plot this pattern on polar-coordinate paper if $K = 50$.

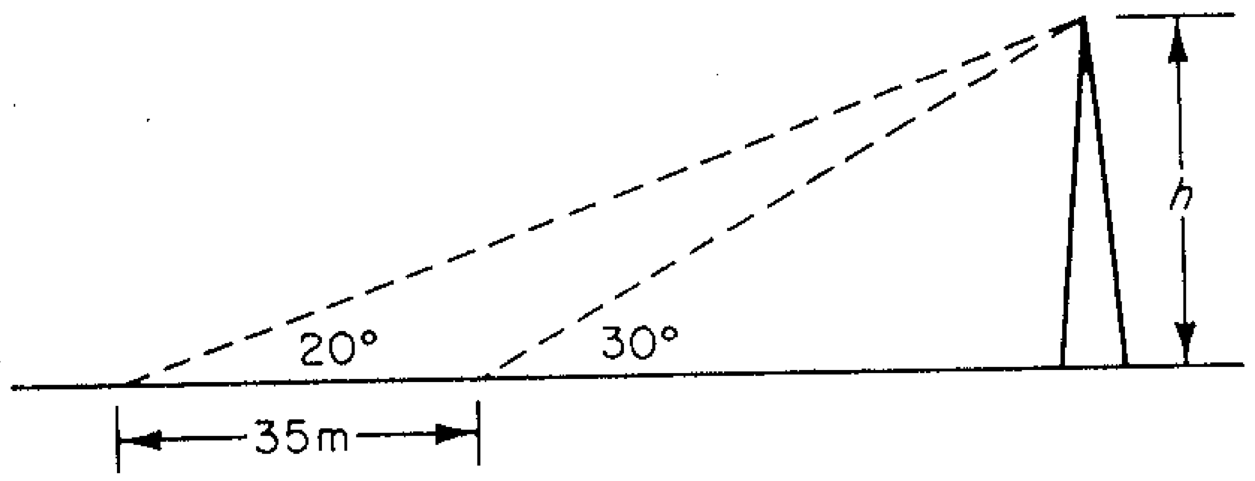

Fig. 16-6 See Prob. 11.

11. An observer standing an unknown distance from the base of a fenced antenna measures the angle between the horizon and the top of the antenna to be 30° (Fig. 16-6). When the observer moves 35 meters further away from the base, the angle becomes 20°. How tall is the antenna?

16-3 RADIAN MEASURE

For *numerical calculations* it is most convenient to consider angles as measured in degrees and decimal parts of a degree. For *calculus operations,* however, a different unit is handier. This unit is called the *radian.* In radian measure, an angle θ is expressed as the ratio of the circular arc s which it intercepts to the radius r of the circle. That is,

$$\theta = \frac{s}{r} \quad \text{radians} \tag{16-4}$$

Accordingly,

> *An* angle of one radian *is that angle subtended (cut off) at the center of a circle by an arc having a length equal to the radius.*

Figure 16-7 shows a circle of radius r with its center at O. The angle θ ($= \angle AOB$) is here of such size that the length of the arc AB is equal to the radius OA. Here, then, $\theta = 1$ radian.

We must note that an angle of 1 radian does *not* cut off a chord equal to r. For example, if we should draw a straight line (chord) connecting A and B in the figure, the length of this line would not be equal to the radius r.

The *relation between radian measure and degree measure* is easily found. Since the circumference of a circle is $C = 2\pi r$, then 2π radians are required to make up one complete rotation of r. Thus 2π radians = 360°, or 1 radian = 57°17′44.8″, approximately. For most purposes, the following approximations are sufficient:

$$1 \text{ radian} = 57.3° \qquad 1° = 0.01745 \text{ radian} \tag{16-5}$$

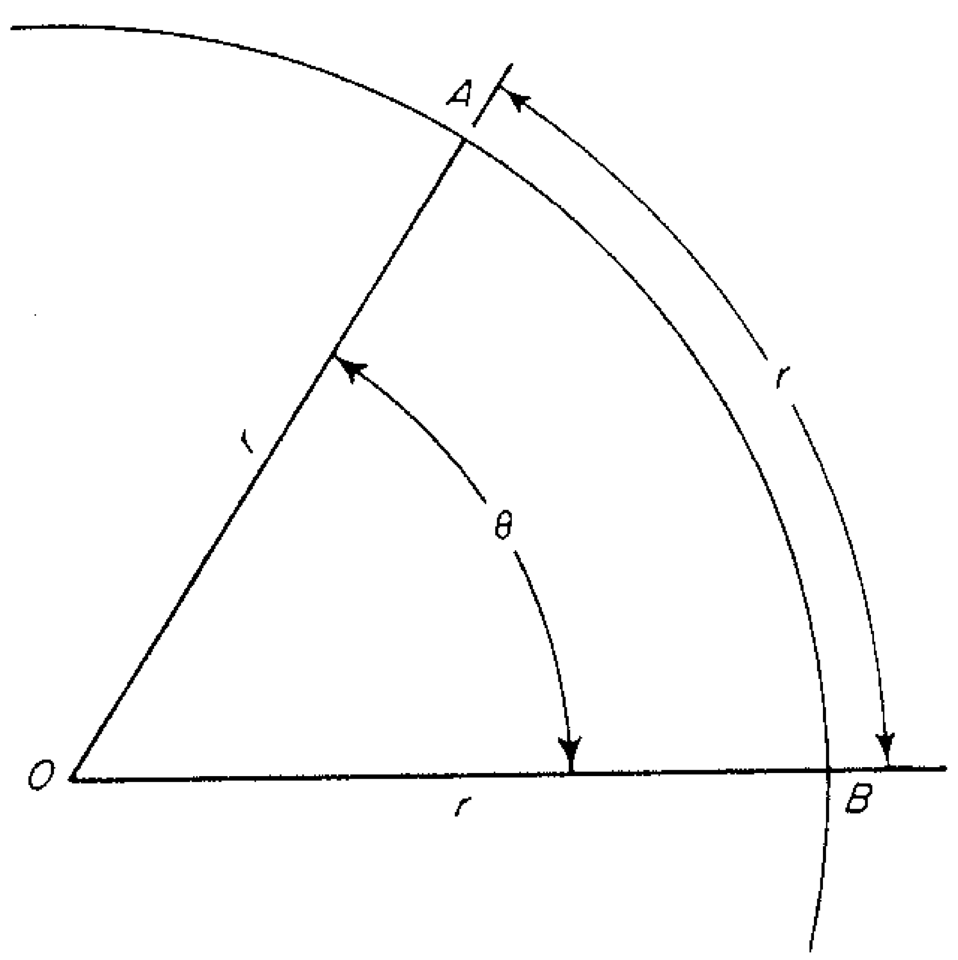

Fig. 16-7 If we wish, we can measure an angle in radians (rather than in degrees). In this figure, the angle θ equals 1 radian, equivalent to 57.3°. In radian measure, an angle $\theta = s/r$, where s is the length of the circular arc intercepted by θ and r is the length of the circular radius.

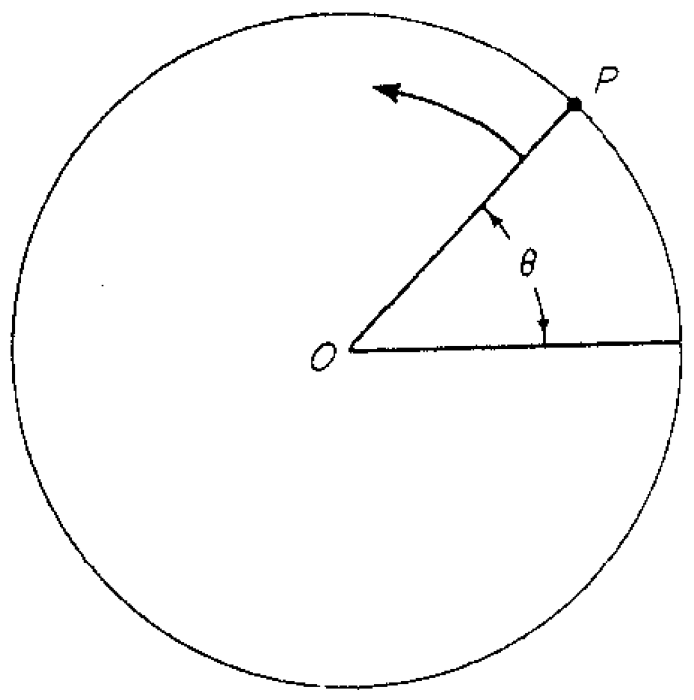

Fig. 16-8 At any instant, the angular speed ω (in radians per second) of a point P is given by $d\theta/dt$. The angular acceleration of the point is $d\omega/dt$ $(= d^2\theta/dt^2)$.

16-4 ANGULAR SPEED AND ACCELERATION In Fig. 16-8 consider the point P as rotating about the center O, so that the angle θ changes with time. The *angular speed* of P at any time t is defined as the rate of change of θ. This quantity is represented by ω, so that

$$\omega = \frac{d\theta}{dt} \tag{16-6}$$

We shall measure θ usually in radians and ω in radians per second.

If ω changes with respect to time, we represent the rate of change of angular speed by the term *angular acceleration*, indicated by α:

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{16-7}$$

16-5 RELATIONS OF ANGULAR TO LINEAR QUANTITIES

Let Fig. 16-9 represent a rotating object, such as an armature. The armature, taken as a unit, possesses *angular* motion, but at the same time, different points

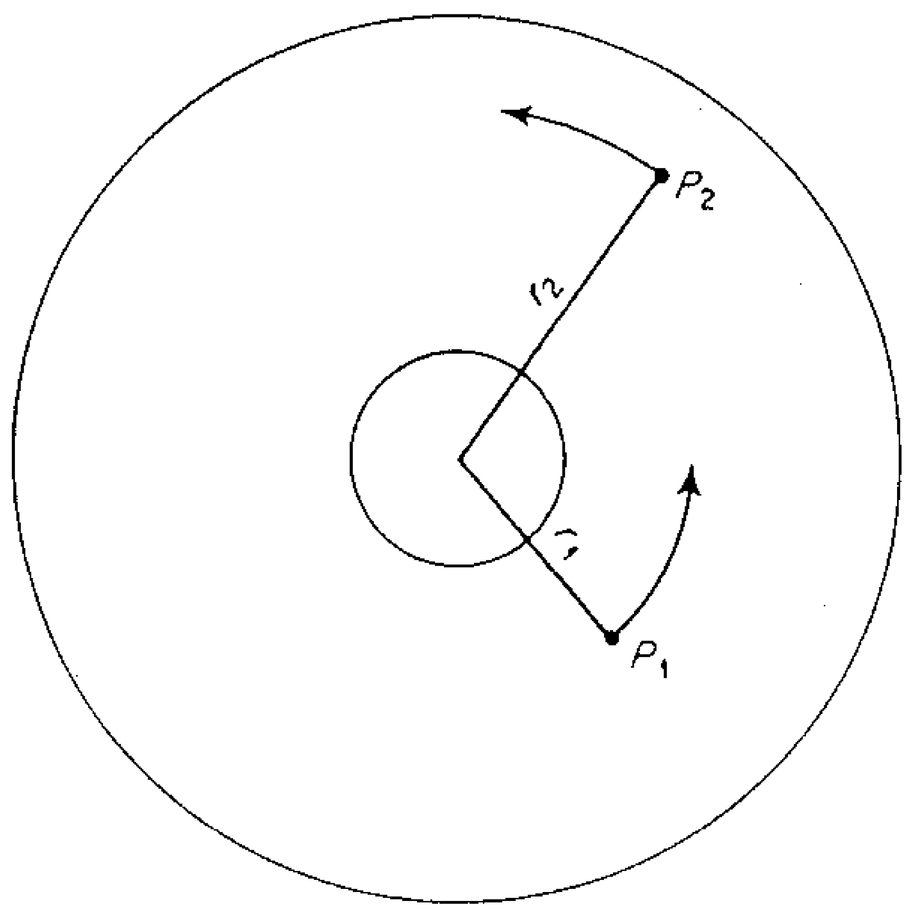

Fig. 16-9 When a point (such as P_1 or P_2) rotates through an angle θ, the point covers a *linear* distance given by $s = r\theta$. The linear speed of the point is $v = r\omega$.

on the armature (such as P_1 and P_2) are moving along circles. Since P_2 is ted closer to the rim than P_1 is, we find P_2 traveling along a larger circle than P_1 does. Thus in a given time P_2 describes a greater linear distance than P_1 does.

To establish the relation between the angle traversed by P_2 and the linear distance through which it travels, we use (16-4), getting $s_2 = r_2\theta$. Thus, in general, any rotating point P covers a linear distance

$$s = r\theta \tag{16-8}$$

By differentiating (16-8), we can get an equation for the linear speed of any point P at any instant, in terms of the angular speed:

$$\frac{ds}{dt} = r\frac{d\theta}{dt} \qquad \text{or} \qquad v = r\omega \tag{16-9}$$

A further differentiation gives the linear acceleration of P:

$$\frac{dv}{dt} = r\frac{d\omega}{dt} \qquad \text{or} \qquad a = r\alpha \tag{16-10}$$

16-6 COMPONENT VELOCITIES When a point P moves along a curved path, its *direction* is, at any instant, taken to be *along a tangent* to the curve.

In Fig. 16-10 for instance, the point P moves along a circle at a fixed distance r from the center O. Let P move in such a way that the rate of change of the angle θ is fixed; that is, let ω be a constant. Let us now find the rates of change of the *rectangular* coordinates x and y of point P.

At the instant depicted, P moves along the tangent line T. Its speed, by (16-9), is $v = r\omega$. Let us indicate the velocity (speed and direction) of P, at this instant, by the vector PQ. This vector can be resolved into horizontal and vertical components $v_x = dx/dt$ and $v_y = dy/dt$, as shown. (That is, the vector sum of v_x and v_y is the actual velocity v.)

Since a tangent to a circle at any point P is perpendicular to the radius at P, we have the fact that T is perpendicular to r. Also, QR is perpendicular to

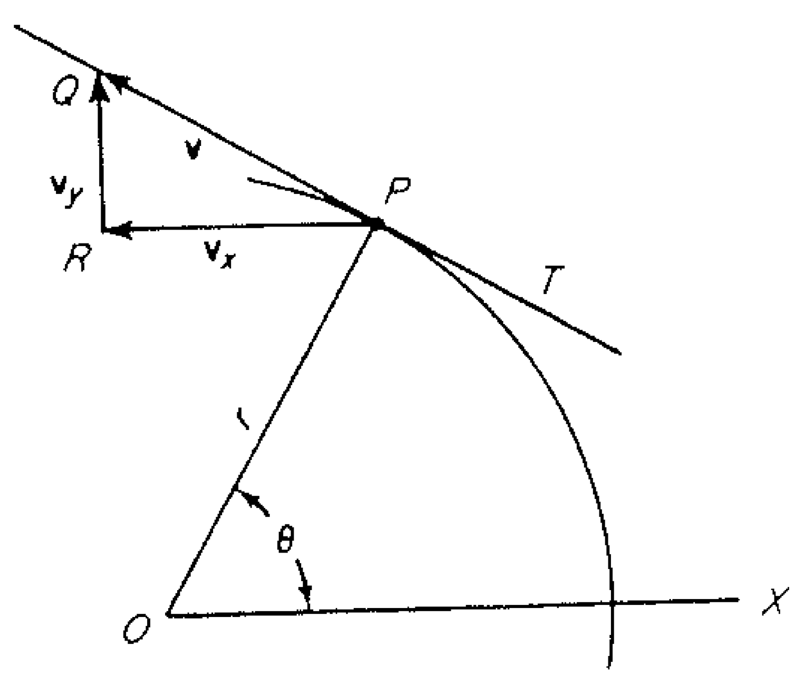

Fig. 16-10 Here point P moves along a curved path. We consider its direction at any instant to be along a tangent line to the curve. As explained in the text, the horizontal speed of P at each instant is $v_x = r\omega \sin\theta$, while the vertical speed is $v_y = r\omega \cos\theta$.

the polar axis OX. Therefore the angle PQR is equal to the central angle θ, for its two sides are perpendicular, respectively, to the sides of θ. From the figure,

$$v_x = \frac{dx}{dt} = r\omega \sin\theta \qquad \text{and} \qquad v_y = \frac{dy}{dt} = r\omega\cos\theta \tag{16-11}$$

Example A directional antenna is mounted on a circular platform whose radius is 5 feet. The platform is turned counterclockwise at 1 revolution per minute. When a point P on the circumference is 60° north of east from the center, find (*a*) the velocity of P toward the west and (*b*) the velocity of P toward the north.

The angular velocity of the platform is ω = 1 revolution per minute = $2\pi/60 = \pi/30$ radian per second. By (16-11), we find (*a*) the velocity to the west is $v_x = 5(\pi/30)\sin 60° = (3.142/6)0.866 = 0.453$ foot per second, and (*b*) the velocity to the north is $v_y = 5(\pi/30)\cos 60° = (3.142/6)0.5 = 0.262$ foot per second.

QUESTIONS 16-2

1. Give a formula expressing an angle θ in radians in terms of an intercepted circular arc and the radius of the arc.

2. How many degrees in 1 radian? How many radians in 1 degree?

3. If a point P rotates about a center O, what formula expresses its angular speed?

4. If a point P rotates about a center O, what formula expresses its angular acceleration?

5. If a point P rotates about a center O, give a formula for the distance s traveled as P traverses an angle θ.

6. As a point P rotates about a center O, what expressions give (*a*) its linear speed and (*b*) its linear acceleration?

7. Give formulas for the components of velocity of a point (*a*) in the horizontal, or x, direction and (*b*) in the vertical, or y, direction as the point describes a growing angle θ about a center O.

PROBLEMS 16-2

1. How many radians correspond to each of the following angles? Express as multiples of π radians.

(*a*) 180°	(*c*) 45°	(*e*) 30°	(*g*) 20°
(*b*) 90°	(*d*) 60°	(*f*) 15°	(*h*) 54°

2. How many degrees correspond to each of the following angles expressed in radians?

(*a*) $\pi/4$	(*c*) $\pi/9$	(*e*) $5\pi/3$	(*g*) $4\pi/3$
(*b*) $3\pi/2$	(*d*) $2\pi/3$	(*f*) $\pi/10$	(*h*) $3\pi/4$

3. An instrument pointer moves through an arc of 270°. To how many radians is this equivalent?

4. The radiation pattern of an antenna has a minimum value in a direction 24° off the antenna axis. Express this angle in radians.

5. An armature turns at 1,800 revolutions per minute. To what value of ω, in radians per second, does this correspond?

6. The coil of an instrument rotates at the rate of 0.005 radian per millisecond. Express this angular speed in degrees per second.

7. A motor accelerates at a rate of 600 revolutions per minute per second. To how many radians per second squared is this equal?

8. An instrument pointer is 2.1 inches long. The tip of the pointer moves over a scale 2.4 inches long. What angle does the pointer describe, in radians?

9. An alternator has a rotating field which is 32 inches in diameter. When the field is turned at 120 revolutions per minute, what is the linear speed of a point on its circumference?

10. If the field assembly of Prob. 9 is accelerated at 12 revolutions per minute per second, what linear acceleration is applied to a point on its circumference?

11. If the field assembly of Prob. 9 turns in a counterclockwise direction, what is the upward component of the velocity at a point P on its circumference when P is at an angle of 45° above the horizontal?

12. An airplane propeller has a radius of 3 feet to the blade tip. It is desired to keep the tip velocity below the speed of sound (769.5 miles per hour). What number of revolutions per minute would correspond to this limit?

13. It can be shown that, when an armature of radius r rotates at ω radians per second, a point on its circumference is given a constant normal acceleration toward the center equal to $\mathbf{a}_n = r\omega^2$. If an armature 0.3 meter in diameter is rotated at 2,000 revolutions per minute, to what normal acceleration will a conductor on the surface be subjected? Using $\mathbf{F} = m\mathbf{a}$, what centrifugal force in newtons will be applied to a conductor of mass 0.04 kilogram located at the circumference?

14. When power is applied to a motor, the shaft speed during initial power-up corresponds to $10t + 4t^2$ revolutions per second. (*a*) Write an equation for the angular speed ω, in radians per second, of the shaft. (*b*) Find an equation for the angular acceleration α of the shaft at any time. (*c*) Find an equation for the angle θ, in radians, of the shaft position at any time. (*d*) Find the angular position θ, in degrees, at $t = 0.1$ second and $t = 0.2$ second. (Assume $\theta = 0°$ at $t = 0$.)

16-7 DERIVATIVE OF THE SINE FUNCTION In Fig. 16-11 we find the point $P(x, y)$ moving along a circle, describing an angle which we shall call u radians at the center of the circle. We proceed to find a formula for the rate of change of the sine of u as u varies. Although this is accomplished by the delta method in more formal treatments, the following presentation illustrates the result. By (16-11),

$$\frac{dy}{dt} = r\omega\cos u \tag{16-12}$$

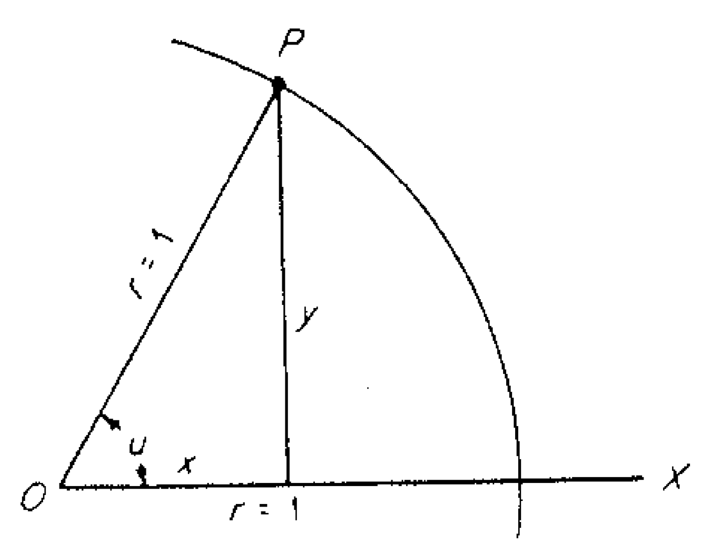

Fig. 16-11 Here a point P moves along a circle, describing a varying angle u radians. As explained in the text, $(d/dx)\sin u = \cos u\,(du/dx)$.

It will be convenient to consider a special case in which r has a length equal to 1 unit. We note that this makes $y = \sin u$, so that in the present case the above equation becomes

$$\frac{d}{dt}\sin u = \omega \cos u \qquad (16\text{-}13)$$

We may write du/dt for ω:

$$\frac{d}{dt}\sin u = \cos u \frac{du}{dt} \qquad (16\text{-}14)$$

We may multiply both sides of (16-14) by dt/dx, getting

$$\frac{d}{dx}\sin u = \cos u \frac{du}{dx} \qquad \mathbf{(16\text{-}15)}$$

We have, then, the interesting and simple result that

The derivative of the sine of an angle is equal to the cosine of that same angle times the derivative of the angle.

In (16-15), x may be *any* variable of which u is a function, and is not limited to the variable x in the figure.

16-8 DERIVATIVE OF THE COSINE FUNCTION

To find the derivative of the cosine of an angle u we make use of the familiar trigonometric identity*

$$\sin^2\theta + \cos^2\theta = 1 \qquad (16\text{-}16)$$

from which

$$\cos^2 u = 1 - \sin^2 u \qquad (16\text{-}17)$$

Differentiating implicitly, we have

$$2\cos u \frac{d}{dx}\cos u = -2\sin u \frac{d}{dx}\sin u \qquad (16\text{-}18)$$

By (16-15), the final factor is equal to $\cos u \, du/dx$. The above equation simplifies immediately to

$$\frac{d}{dx}\cos u = -\sin u \frac{du}{dx} \qquad \mathbf{(16\text{-}19)}$$

* In what follows, it is sometimes necessary to use formulas called *trigonometric identities,* which are ordinarily presented in courses in trigonometry. Certain of these identities are listed in Table 3 of Appendix D. See also Appendix B.

In words,

The derivative of the cosine of an angle is equal to the negative sine of the angle multiplied by the derivative of the angle.

Example 1 The rectangular winding of Fig. 16-12 turns in a uniform magnetic field. Let Φ_{max} represent the amount of flux passing through the winding when the winding is in a horizontal position. When the winding has turned through an angle θ, the flux passing through the winding is smaller, being proportional to the projection of the winding upon the horizontal plane $X'X$. Thus for any angle θ, the flux is

$$\phi = \Phi_{max}\cos\theta \qquad (16\text{-}20)$$

Let the winding turn at a constant rate ω radians per second, so that $\theta = \omega t$. Then

$$\phi = \Phi_{max}\cos\omega t \qquad (16\text{-}21)$$

Differentiating (16-21) according to (16-19), we get

$$\frac{d\phi}{dt} = -\omega\Phi_{max}\sin\omega t \qquad (16\text{-}22)$$

Recalling that the induced emf in a winding is $v_{ind} = -N\, d\phi/dt$, we get from (16-22)

$$v_{ind} = \omega N \Phi_{max}\sin\omega t \qquad \mathbf{(16\text{-}23)}$$

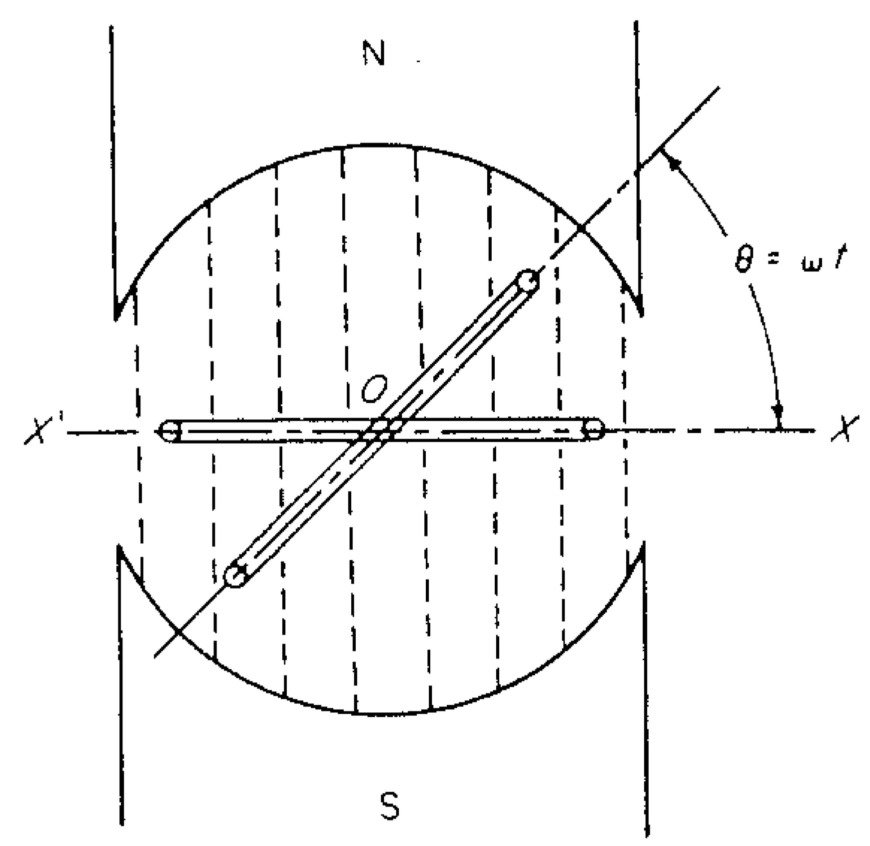

Fig. 16-12 A rectangular winding turns in a uniform magnetic field of intensity Φ_{max}. As explained in Example 1, if the winding consists of N turns and if it turns at a constant angular speed ω, then the voltage induced in the winding at each instant is $v_{ind} = \omega N\Phi_{max}\sin\omega t$.

That is, a *sine wave* of voltage is induced in the winding as a result of its rotation in the magnetic field. We note that the greatest value which can be attained by $\sin \omega t$ is 1; hence we can determine from (16-23) that the greatest value of the induced emf is

$$V_{max} = \omega N \Phi_{max} \tag{16-24}$$

Observe that to make $\sin \omega t$ equal to 1, and thus obtain a maximum of induced emf, the winding must have turned through an angle $\omega t = \pi/2$, or $3\pi/2$, and so on. Interestingly enough, the actual value of ϕ at these points is, by (16-21), equal to zero.

Example 2 Consider the circuit of Fig. 16-13*a*. The impedance of this circuit has a phase angle δ as illustrated in the phasor diagram (*b*) in the figure. Assume that we apply to this circuit a voltage wave of the form $v = V_{max} \sin \omega t$, which is graphed in part (*c*). Find di/dt.

From our knowledge of elementary electricity, we see that the current i differs in phase by an angle $-\delta$ from the applied voltage. The current wave is shown in Fig. 16-13*d*. Had the current been *in phase* with the voltage, its equation would have been $i = I_{max} \sin \omega t$; but we must modify this form to allow for the phase angle δ, so that the actual current equation is $i = I_{max} \sin (\omega t - \delta)$. Differentiating the current wave according to (16-15), we have

$$\frac{di}{dt} = I_{max}[\cos (\omega t - \delta)]\omega = \omega I_{max} \cos(\omega t - \delta) \tag{16-25}$$

Example 3 Differentiate $y = \cos^3 x$.

We treat this function primarily as a power function, since the quantity $\cos^3 x$ is actually $\cos x$ raised to the third power. (Letting $\cos x$ be called u, we could regard the function as being $y = u^3$ and differentiate accordingly.) We have, then,

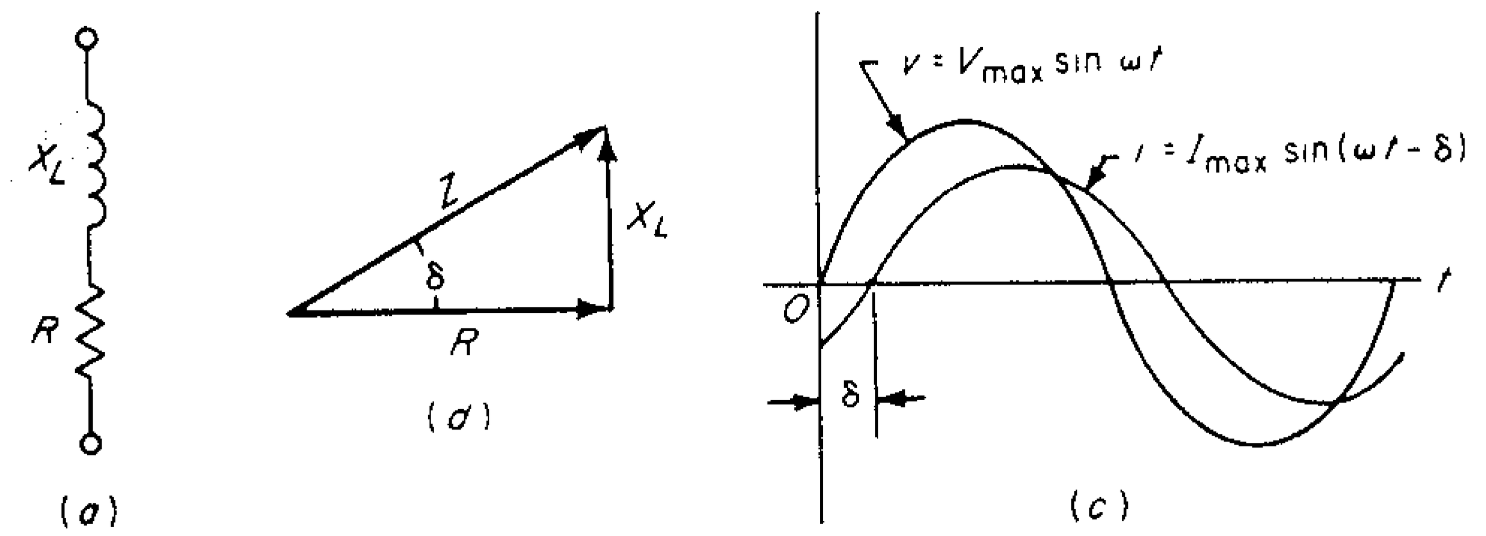

Fig. 16-13 A series circuit, shown in illustration (*a*), consists of an inductive reactance X_L and a resistance R. The impedance of this circuit has a phase angle δ as shown in illustration (*b*). As described in Example 2, we apply to this circuit a voltage $v = V_{max} \sin \omega t$ [illustration (*c*)]. The resulting current waveform is $i = I_{max} \sin (\omega t - \delta)$ [illustration (*d*)]. Example 2 shows that $di/dt = \omega I_{max} \cos (\omega t - \delta)$.

$$\frac{dy}{dx} = 3 \cos^2 x \frac{d}{dx} \cos x = -3 \sin x \cos^2 x \tag{16-26}$$

PROBLEMS 16-3

In Probs. 1 through 12 differentiate with respect to the independent variable x or t.

1. $y = \sin 2x$
2. $y = 3 \sin x$
3. $y = 12 \sin 14t$
4. $y = 10 \sin 10t^{1/2}$
5. $y = \sin t^2$
6. $y = 2 \cos 3t^3$
7. $y = 500 \cos (t^2 - t)^{1/2}$
8. $y = 10t^3 + \cos t$
9. $y = \sin^2 t$
10. $y = -\cos^2 t^{-1}$
11. $y = 2 \sin^2 t^2$
12. $y = \sin (\cos t^2)$

13. Let the primary current in a transformer be $i_1 = I_{max} \sin \omega t$, where I_{max} is the crest value of the current. Write a formula for the induced secondary emf v_2.

14. In Prob. 13, state a formula for the maximum value of v_2. For what value of i_1 will the greatest value of v_2 occur?

15. A voltage $v = 2{,}000 \sin 500t$ is impressed across a 20-microfarad capacitor. Find a formula for the resulting current.

16. In Prob. 15, (*a*) what is the maximum current that ever flows? (*b*) At what time does this occur? (*c*) What instantaneous voltage is across the capacitor at this time?

17. Experiment shows that the current I_{max} at a current loop in a vertical antenna of length l is related to the base current by $I_{max}/I_{base} = 1/\sin (2\pi l/\lambda)$. How fast does this ratio change with respect to the transmitted wavelength λ when l is constant?

18. In televising a poster of area A, a light source of intensity I candelas is used at a distance r from the poster. The luminous flux Φ lumens supplied to the poster is $\Phi = IA(\cos \theta)/r^2$, where θ is the angle between the incident rays and a normal to the poster surface. How fast does Φ change with respect to θ?

19. A rectangular winding of length l and width $2r$ is made up of N turns. When the winding is inserted into a magnetic field of density **B** and a current I is sent through the winding, the resulting torque applied to the winding is $\mathbf{T} = 2\mathbf{B}lINr \cos \theta$, where θ is the angle between the plane of the winding and the direction of the flux. What expression gives the rate of change of **T** as the winding rotates?

20. At a certain point distant from a transmitting antenna the intensity **E** of the electric field associated with the transmitted wave varies with distance y as follows: $\mathbf{E} = \mathbf{E}_{max} \sin (2\pi y/\lambda)$. Here $\mathbf{E}_{max}$ is the crest value of the field, and λ is the wavelength. Find the rate at which **E** varies with y at a given instant.

21. A voltage $v = 10 \sin 377t$ volts is impressed across a 0.1-microfarad capacitor. Find the equation for the rate of change of the power in the capacitor.

22. A single-turn winding of radius r carries a current I. A point P is located on the winding axis at a distance s from the winding. Then a line PQ connecting P with any pont Q on the winding circumference makes an angle θ with the winding axis. It can be shown that the magnetic field intensity at P caused by I is $\mathbf{H} = (2\pi I/r) \sin^3 \theta$. Get a formula for the rate of change of **H** with respect to θ.

PROBLEMS 16-6

In Probs. 1 through 14 differentiate with respect to x.

1. $y = \sin^{-1} 2x$
2. $y = \cos^{-1} x^2$
3. $y = 5 \tan^{-1} x^3$
4. $y = \text{ctn}^{-1} 2x^2$
5. $y = \cos^{-1} x/2$
6. $y = \sec^{-1} (x^2 + 1)^{1/2}$
7. $y = (\sin^{-1} 2x)/\cos^{-1} 2x$
8. $y = (\sin^{-1} x^2)^2$
9. $y = \sin^{-1} (1 - x^2)^{1/2}$
10. $y = (\text{ctn}^{-1} 2x)(\tan^{-1} 2x)$
11. $y = \sin^{-1} x \sin x$
12. $y = \sin^{-1} (\sin x^2)$
13. $y = \text{ctn}^{-1} (\csc x + \text{ctn}\, x)$
14. $y = \sec^{-1} (x^2 + 2x)$
15. Derive (16-51).
16. Derive (16-58).
17. Derive (16-65).
18. The distance s meters from the end of a certain transmitting antenna at which the current is I amperes is given by $s = (\lambda/2\pi) \sin^{-1} (I/I_0)$, where λ is the operating wavelength and I_0 is the maximum value of the current along the antenna. Find ds/dI.
19. The angle of grid-current flow in a triode class C amplifier is $\theta_g = 2 \cos^{-1} (V_c/V_s)$, where V_c is the bias voltage and V_s the peak excitation voltage. Find $d\theta_g/dV_c$.
20. The impedance phase angle of an inductor is related to the Q of the inductor by the relation $\theta = \tan^{-1} Q$. If $Q = 100$, what would be the approximate change in θ resulting from an increase in Q to 101?
21. A simple low-pass four-terminal network uses a series resistor R followed by a shunt capacitor C. The phase shift introduced into a certain circuit by this network is $\phi = \tan^{-1} \omega RC$. Find a formula for the rate of change of ϕ with respect to C.
22. A plane flies at 600 miles per hour on a straight course past a transmitting station, passing the station at a distance of 50 miles. An automatic direction finder (ADF) aboard the plane continuously indicates the bearing ξ of the transmitter relative to the course of the plane ($\theta = 90°$ when plane is 50 miles from transmitter). How fast is the ADF reading changing, in degrees per second, 3 minutes after the plane passes the station?
23. A guy wire is attached to a ring 35 feet above ground on an antenna pole. If we change the length s of the guy wire, we also change the angle between the pole and the wire. Calling this angle ϕ, find $d\phi/ds$ when $s = 52$ feet.
24. Light emerges from a refracting element at an angle $r = \sin^{-1} [(\sin i)/\mu]$, where i is the angle of incidence of the light and μ is the index of refraction of the refracting material. If we change the design of the element to use various materials, find $dr/d\mu$.

16-14 INTEGRALS YIELDING TRIGONOMETRIC FORMS

The differentiation formulas which we have developed enable us to integrate several functions. To integrate $\sin u\, du$ we begin with (16-19):

$$\frac{d}{dx} \cos u = -\sin u \frac{du}{dx} \tag{16-66}$$

or

$$\sin u\, du = -d(\cos u) \tag{16-67}$$

Integrating, we get the formula

$$\int \sin u\, du = -\cos u + K \tag{16-68}$$

A somewhat similar treatment of (16-15) gives

$$\int \cos u\, du = \sin u + K \tag{16-69}$$

If we turn to (16-38), we have

$$\frac{d}{dx} \tan u = \sec^2 u \frac{du}{dx} \tag{16-70}$$

or

$$\sec^2 u\, du = d(\tan u) \tag{16-71}$$

so that

$$\int \sec^2 u\, du = \tan u + K \tag{16-72}$$

Similarly, we can use (16-39) to give

$$\int \csc^2 u\, du = -\text{ctn}\, u + K \tag{16-73}$$

We turn next to (16-41):

$$\frac{d}{dx} \sec u = \sec u \tan u \frac{du}{dx} \tag{16-74}$$

or

$$\sec u \tan u\, du = d(\sec u) \tag{16-75}$$

so that

$$\int \sec u \tan u\, du = \sec u + K \tag{16-76}$$

A parallel treatment of (16-42) in place of (16-41) gives

$$\int \csc u\, \text{ctn}\, u\, du = -\csc u + K \tag{16-77}$$

Later in this book we encounter further trigonometric integrals.

Example 1 If the peak value of a sine wave of current is I_{max}, find the average value I_{av} of the current, taken over an interval of $\frac{1}{2}$ cycle.

Figure 16-17 shows the wave under consideration. Its equation is $i = I_{max} \sin \omega t$. Our problem is to find the average height of the curve over the base interval from $\omega t = 0$ to $\omega t = \pi$. To do this we shall find by integration the area under the curve over this interval; then, dividing this area by the length of the base, we shall get the average height of the curve:

$$I_{av} = \frac{1}{\pi} I_{max} \int_0^{\pi} \sin \omega t\, d(\omega t) = -\frac{1}{\pi} I_{max} \cos \omega t \Big]_{\omega t = 0}^{\pi} \tag{16-78}$$

$$I_{av} = \frac{2}{\pi} I_{max} = 0.637 I_{max} \tag{16-79}$$

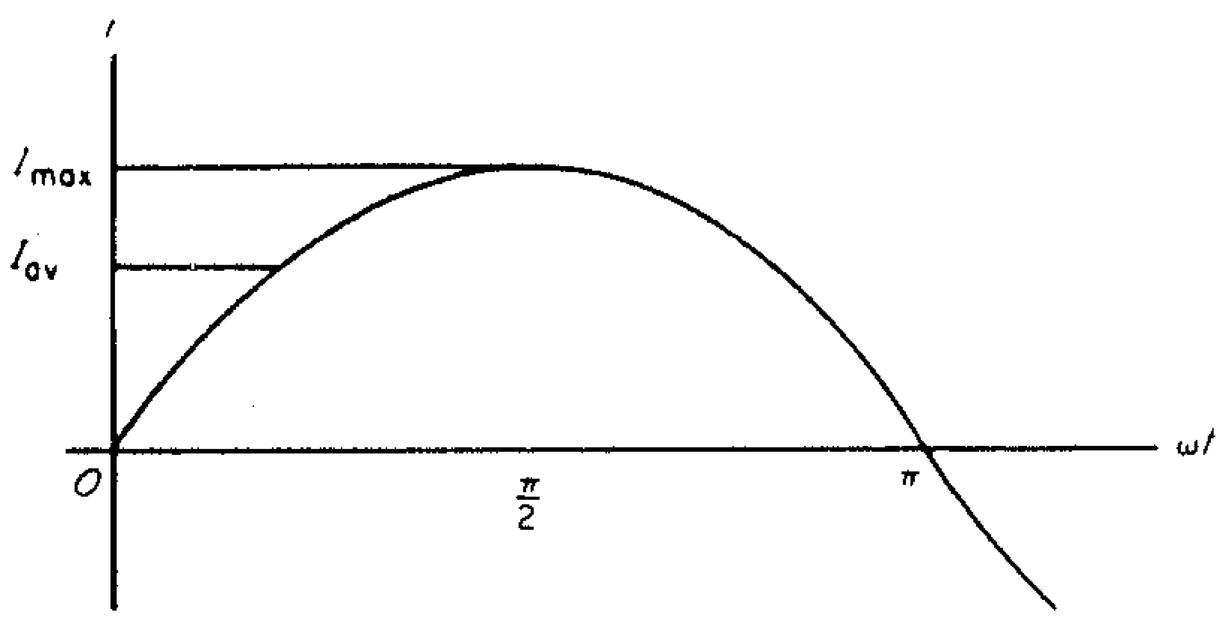

Fig. 16-17 Section of a sine wave of current whose peak value is I_{max}. Example 1 explains that the average current, over $\frac{1}{2}$ cycle, is $I_{av} = 0.637\, I_{max}$.

The latter is the familiar form which is arrived at only approximately, in more elementary courses, by averaging a finite number of values of current. Note that if we average the current over a $\frac{1}{4}$-cycle interval, as from $\omega t = 0$ to $\omega t = \pi/2$, we get a similar result. This comes from the symmetry between succeeding $\frac{1}{4}$ cycles of the wave. On the other hand, if we take the average current over a complete cycle (or a whole number of cycles) the average current is *zero* since for each alternation of current in a positive direction there is an equal alternation in the reverse direction. We likewise find that a *cosine* function, or any other function whose graph has the form of a sine wave, regardless of its phase angle, has a definite integral equal to zero when the interval is a whole number of cycles. And therefore its average value is zero when taken over a whole number of cycles.

Example 2 Show that the average power P in an ac circuit is equal to $IV \cos \theta$, where I and V are the effective current and effective voltage, respectively, and θ is the phase angle between them.

The power at any instant is the product of the instantaneous current i and the instantaneous voltage v: $p = iv$. But the voltage wave is of the form $v = V_{max} \sin \omega t$, while the current wave has the form $i = I_{max} \sin (\omega t + \theta)$. Then

$$p = I_{max} V_{max} \sin \omega t \sin (\omega t + \theta) \tag{16-80}$$

A trigonometric identity (No. 19, Table 3) shows that the product of the sines of two angles is

$$\sin \alpha \sin \beta = \tfrac{1}{2}[\cos (\alpha - \beta) - \cos (\alpha + \beta)] \tag{16-81}$$

Letting $\alpha = \omega t + \theta$ and $\beta = \omega t$, we have

$$p = \tfrac{1}{2} I_{max} V_{max}[\cos \theta - \cos (2\omega t + \theta)] \tag{16-82}$$

To get the average value of this power over $\frac{1}{2}$ cycle we obtain by integration the area under the power curve for an interval of 0 to π radians; dividing the result by π gives the average height of the power curve:

$$P = \frac{I_{max} V_{max}}{2\pi} \int_0^{\pi} [\cos \theta - \cos (2\omega t + \theta)]\, d(\omega t) \tag{16-83}$$

Integrating term by term, we obtain

$$P = \tfrac{1}{2} I_{max} V_{max} \cos \theta \tag{16-84}$$

But it is well known (and will be demonstrated in Chap. 20), that the effective currents and voltages I and V are given respectively by $I_{max}/\sqrt{2}$ and by $V_{max}/\sqrt{2}$. Then

$$P = \frac{I_{max}}{\sqrt{2}} \frac{V_{max}}{\sqrt{2}} \cos \theta = IV \cos \theta \tag{16-85}$$

Example 3 Evaluate $\int \sin \phi \cos \phi \, d\phi$.

Let u represent $\sin \phi$. Then $du = \cos \phi \, d\phi$, so that the given integral is of the form $\int u\, du$. This has the value $u^2/2 + K$, or

$$\int \sin \phi \cos \phi \, d\phi = \tfrac{1}{2} \sin^2 \phi + K \tag{16-86}$$

Example 4[†] Evaluate $\int x \cos x^2 \, dx$.

Let $u = x^2$ and $du = 2x\, dx$. Rewriting the given integral

$$\int x \cos x^2 \, dx = \tfrac{1}{2} \int \cos x^2 (2x\, dx) \tag{16-87}$$

we see that it takes the form $(\int \cos u\, du)/2$, so that

$$\int x \cos x^2 \, dx = \tfrac{1}{2} \sin x^2 + K \tag{16-88}$$

Example 5 We apply a voltage $v = V_{max} \sin \omega t$ across an inductor. Neglecting the resistance of the inductor, calculate the *opposition* offered by the inductor to the current.

We calculate first the current in the inductor. At any instant, this is $i = -(1/L) \int v\, dt$:

$$i = -\frac{1}{L} V_{max} \int \sin \omega t \, dt = -\frac{V_{max}}{\omega L} \int \sin \omega t \, d(\omega t) = \frac{V_{max}}{\omega L} \cos \omega t\, [+\, K (= 0)] \tag{16-89}$$

We see the current is a cosine wave whose peak value is $V_{max}/\omega L$. Using the familiar fact (to be demonstrated in Chap. 20) that the effective value of a sine or cosine wave is equal to the peak value divided by $\sqrt{2}$, we write the effective voltage V and the effective current I in the inductor

$$V = \frac{V_{max}}{\sqrt{2}} \qquad \text{and} \qquad I = \frac{I_{max}}{\sqrt{2}} = \frac{V_{max}}{\sqrt{2}\omega L} \tag{16-90}$$

Consider the *opposition* to the current as being equal to the effective voltage divided by the effective current:

$$\frac{V}{I} = \frac{V_{max}/\sqrt{2}}{V_{max}/\sqrt{2}\omega L} = \omega L \tag{16-91}$$

We refer to this opposition as the *inductive reactance* of the inductor, and we measure its value in ohms:

$$X_L = \omega L \qquad \text{ohms} \tag{16-92}$$

In a parallel way, we find capacitive reactance $X_C = \dfrac{1}{\omega C}$ ohms.

16-15 AREA IN POLAR COORDINATES Figure 16-18 shows a graph of a function $r = f(\theta)$; that is, for each value of the angle θ (between the radius vector r and the polar axis OX) there exists a length r of the radius vector. Let it be desired to get a formula for the area A included within this polar graph between the positions $\theta = a$ and $\theta = b$ of the radius vector.

To do this, consider the region from $\theta = a$ to $\theta = b$ to be broken up into a large number of small equal angles $\Delta\theta$, as indicated. We accomplish this by drawing successive radii r_1, r_2, etc., from the pole O.

Next, we draw small arcs of circles, as shown, each arc having O as its center. The radii of these arcs are r_1, r_2, etc.

The desired area A is seen to be *approximately* equal to the sum of a large number of small *circular sectors* ("pieces of pie"). The error in this approximation is indicated by the sum of the shaded portions of the figure. And we should expect the amount of this error to diminish if we further subdivided the sectors into smaller and smaller sectors.

The area A_n of any sector of Fig. 16-18 (say the nth sector), whose radius is r_n and whose central angle is $\Delta\theta$, is simply its proportionate share of the area of a circle having the same radius. That is, $A_n = (\Delta\theta/2\pi)\pi r^2 = r^2\,\Delta\theta/2$.

Consider the desired area A as being equal to the limit approached by the sum of the sectors as their number n increases without bound and $\Delta\theta$ approaches zero:

$$A = \lim_{n\to\infty} \left[\tfrac{1}{2}(r_1^2\,\Delta\theta + r_2^2\,\Delta\theta + \cdots + r_n^2\,\Delta\theta)\right] \tag{16-93}$$

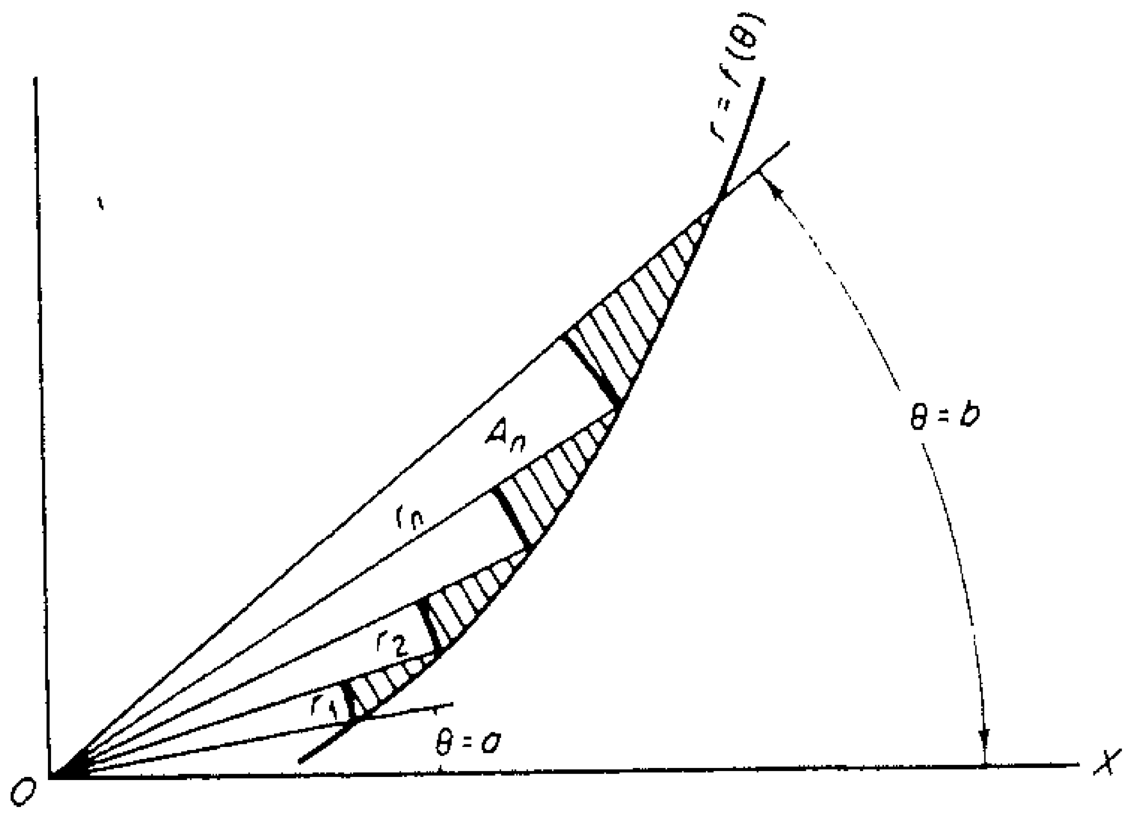

Fig. 16-18 The curve in this graph describes a function $r = f(\theta)$. Section 16-15 explains that, between the limits $\theta = a$ and $\theta = b$, this graph includes an area $A = \int_a^b r^2\,d\theta$.

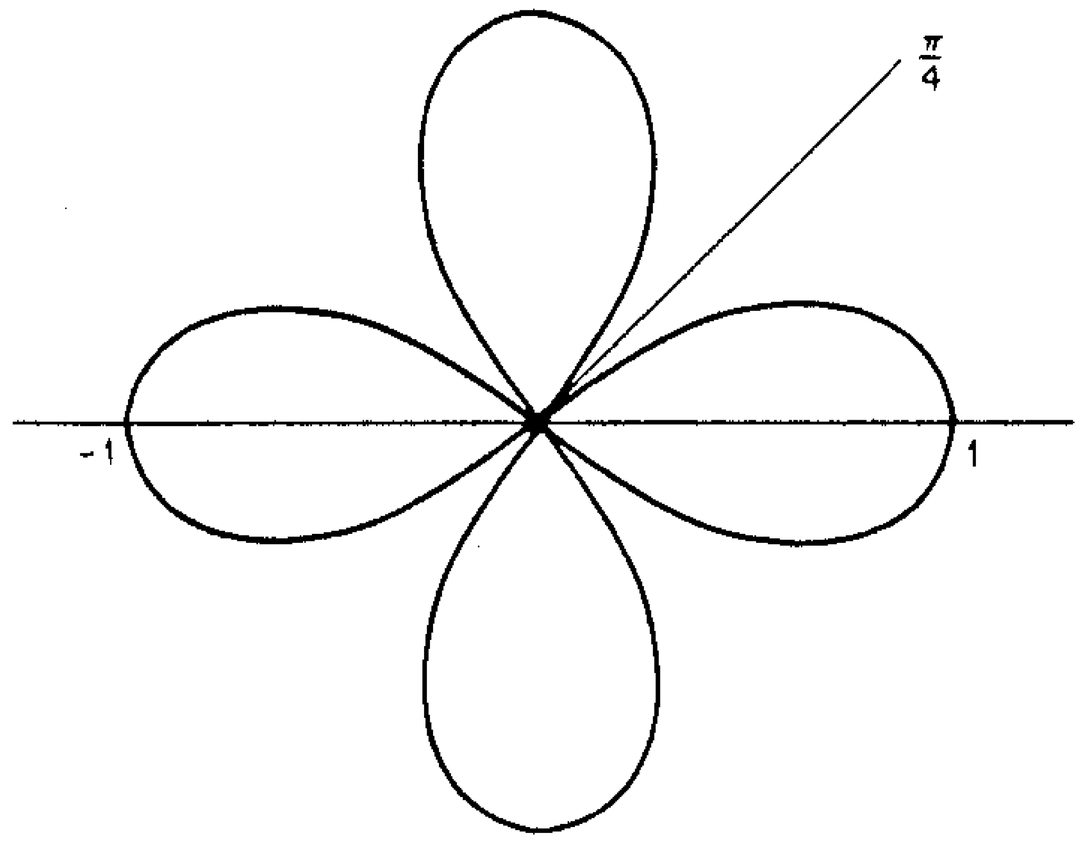

Fig. 16-19 This graph describes the function $r = \cos 2\theta$. The example of Sec. 16-15 shows that the included area is $\pi/2$ area units.

By the fundamental theorem [Eq. (15-34)], this is*

$$A = \frac{1}{2}\int_{\theta=a}^{b} r^2\,d\theta \tag{16-94}$$

Example Find the entire area contained in the curve $r = \cos 2\theta$ (Fig. 16-19).

The total area is eight times that between $\theta = 0$ and $\theta = \pi/4$. Applying (16-94) and identity 9 from Table 3, we get

$$A = 4\int_0^{\pi/4} \cos^2 2\theta\,d\theta = 2\int_0^{\pi/4} (1 + \cos 4\theta)\,d\theta = \frac{\pi}{2} \text{ area units} \tag{16-95}$$

16-16 INTEGRALS YIELDING INVERSE TRIGONOMETRIC FORMS The first of these forms which we consider is

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + K \tag{16-96}$$

We can show the correctness of this formula by differentiating its right-hand member according to (16-50):

$$\frac{d}{dx}\left(\sin^{-1}\frac{u}{a} + K\right) = \frac{1}{\sqrt{1 - u^2/a^2}}\,\frac{1}{a}\,\frac{du}{dx} \tag{16-97}$$

* Note that we do not depend upon (16-94) to obtain the formula for the area of a circle, which formula we used in deriving (16-94). We can calculate the area of a circle in rectangular coordinates in the manner of Sec. 21-2. In applying (16-94), it is advisable for simplicity and accuracy to find the smallest portion of the area from which the entire area can be deduced from considerations of symmetry.

Multiplying both members by dx and clearing fractions, we get

$$d\left(\sin^{-1}\frac{u}{a}+K\right)=\frac{du}{\sqrt{a^2-u^2}} \tag{16-98}$$

The right-hand member of (16-98) is identical with the integrand in (16-96); therefore (16-96) is correct.

Other integrals are

$$\int\frac{du}{a^2+u^2}=\frac{1}{a}\tan^{-1}\frac{u}{a}+K \tag{16-99}$$

and

$$\int\frac{du}{u\sqrt{u^2-a^2}}=\frac{1}{a}\sec^{-1}\frac{u}{a}+K \tag{16-100}$$

which can be proved by differentiation.

Example 1 Evaluate $\int dx/(7+3x^2)$.

We can write the given denominator as the sum of two squares: $7+3x^2=(\sqrt{7})^2+(\sqrt{3}\,x)^2$. Letting $a=\sqrt{7}$, $u=\sqrt{3}\,x$, and $du=\sqrt{3}\,dx$, we apply (16-99):

$$\int\frac{du}{7+3x^2}=\frac{1}{\sqrt{3}}\int\frac{\sqrt{3}\,dx}{(\sqrt{7})^2+(\sqrt{3}\,x)^2}=\frac{1}{\sqrt{3}}\frac{1}{\sqrt{7}}\tan^{-1}\frac{\sqrt{3}\,x}{\sqrt{7}}+K$$

$$=\frac{1}{\sqrt{21}}\tan^{-1}(\sqrt{\tfrac{3}{7}}\,x)+K \tag{16-101}$$

Example 2 Evaluate $\int dx/(x^2+4x+7)$.

We can readily handle this form if we complete the square in the given denominator:

$$x^2+4x+7=x^2+4x+4+3=(x+2)^2+(\sqrt{3})^2 \tag{16-102}$$

Let $u=x+2$, $du=dx$, $a=\sqrt{3}$. Then by (16-99),

$$\int\frac{dx}{x^2+4x+7}=\int\frac{dx}{(x+2)^2+(\sqrt{3})^2}=\frac{1}{\sqrt{3}}\tan^{-1}\frac{x+2}{\sqrt{3}}+K \tag{16-103}$$

Example 3 Evaluate $\int dx/\sqrt{16+24x-16x^2}$.

Completing the square in the quantity under the radical, we can express that quantity as the difference of two squares:

$$16+24x-16x^2=25-(9-24x+16x^2)=5^2-(4x-3)^2 \tag{16-104}$$

Let $a=5$, $u=4x-3$, $du=4dx$. Then use (16-96):

$$\int\frac{dx}{\sqrt{16+24x-16x^2}}=\frac{1}{4}\int\frac{4dx}{\sqrt{5^2-(4x-3)^2}}=\frac{1}{4}\sin^{-1}\frac{4x-3}{5}+K \tag{16-105}$$

QUESTIONS 16-5

1. State the formulas which give the following integrals:

(a) $\int \sin u\,du$ (b) $\int \cos u\,du$ (c) $\int \sec^2 u\,du$ (d) $\int \csc^2 u\,du$ (e) $\int \sec u \tan u\,du$ (f) $\int \csc u \operatorname{ctn} u\,du$

2. Give a formula for the area enclosed within a figure in polar coordinates.

3. State the formulas which give the following integrals:

(a) $\int\frac{du}{\sqrt{a^2-u^2}}$ (b) $\int\frac{du}{a^2+u^2}$ (c) $\int\frac{du}{u\sqrt{u^2-a^2}}$

PROBLEMS 16-7

In Probs. 1 through 19 find y. Check by differentiation.

1. $y=\int \sin 2x\,dx$
2. $y=\int \sec^2 5\theta\,d\theta$
3. $y=\int \csc^2 v\,dv$
4. $y=\int \sin x\cos^2 x\,dx$
5. $y=\int 4\sin^4 x\cos x\,dx$
6. $y=\int \tan 2x\sec^2 2x\,dx$
7. $y=\int 4\csc^4 4x \operatorname{ctn} 4x\,dx$
8. $dy=\csc^2 u \operatorname{ctn} u\,du$
9. $dy=\theta\sin\theta^2\,d\theta$
10. $dy=x^2\cos x^3\,dx$
11. $dy=\theta\sin 4\theta^2\,d\theta$
12. $dy=\phi\cos 3\phi^2\,d\phi$
13. $dy=\frac{dt}{100+t^2}$
14. $dy=\frac{dx}{x\sqrt{4x^2-8}}$
15. $dy=\frac{10x\,dx}{\sqrt{9-25x^4}}$
16. $dy=\frac{dx}{x^2+4x+13}$
17. $dy=\frac{dx}{\sqrt{1+4x-x^2}}$
18. $dy=\frac{dx}{\sqrt{21+12x-9x^2}}$
19. $dy=\frac{dx}{(x+1)\sqrt{2x+x^2}}$

20. Solve Example 3 of Sec. 16-14, using this time $u=\cos\theta$. Compare with answer shown in example. If the results differ, explain (Sec. 13-11).

21. The voltage applied to a 1-henry inductor of negligible resistance is $v_L=10\sin 377t$. Find a formula for the current through the inductor. Show that this circuit operates in accordance with the principle, from elementary electricity, that the alternating current through an inductor of negligible resistance lags the applied voltage by 90°.

22. Find the area described by the radius vector of the curve $r=2\sec 2\theta$ as it rotates between the positions $\theta=0$ and $\theta=\pi/8$.

23. A microphone has a cardioid directional pattern given approximately by $r=k(1-\cos\theta)$. Find the rms value of this pattern (that is, find the radius of a circle having the same area). [*Hint:* In completing the integration, use the identity $\cos^2\theta=(1+\cos 2\theta)/2$ (No. 9, Table 3).]