Optimal Meshing for Inverse Scattering Problems

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Abstract

In this paper, we are concerned with the optimal mesh for contrast reconstruction without SNR in the inverse scattering problems. We employ the far-field data and the near-field data to study the optimal mesh separately. Our results are expected to have important applications in geophysics, nondestructive testing, biological studies, evaluation and medicine.

Keywords: optimal mesh, inverse scattering, recovery.

1 Introduction

In this paper, we are concerned with the optimal mesh of the numerical reconstruction methods to identify the inhomogeneous medium scatterers by scattered fields. The inverse scattering problems has been studied for the past few decades, which can find wide applications in geophysics, nondestructive testing, biological studies, evaluation and medicine. A large variety of numerical reconstruction methods have been developed and applied in practice, such as the linear sampling or probing methods (LSM) , the time-reversal multiple signal classification (MUSIC) method , the direct sampling methods (DSM), the multilevel sampling methods (MSM) , the contrast source inversion (CSI) method, the subspace-based optimization method (SOM), etc. The LSM, the MUSIC, the DSM and the MSM are implemented to recover the unknown scatterer by the values of indicator functions at each sampling point, so an optimal mesh of sampling region can help to reduce the computational burden of these methods in practice. In addition, the CSI and the SOM are applied to reconstruct the unknown scatterers by the optimization methods to minimize the objective functions. As a comparable large sampling domain is considered in the reconstruction process, an optimal mesh is considerable important for the CSI and the SOM. We will show that the optimal mesh can provide the best identifications of the unknown obstacles. In other words, the reconstructions would not be better with the mesh that finer than the optimal one.

In this work we consider the following inverse scattering problem in $\mathbb{R}^d$ for $d = 2, 3$,

$$\Big(\Delta + k^2\Big(1 + q(x)\Big)\Big)u = 0, \tag{1.1}$$

where u is the total field, $q(x) > 0$ is the contrast of the medium and k is the wave number. We consider an inhomogeneous medium Ω contained inside a homogeneous background medium, and assume that Ω is an open bounded connected domain with a $\mathcal{C}^{1,\alpha}$ boundary for some $0 < \alpha < 1$. Suppose that $q \in L^\infty$ and is supported in Ω. We shall complement the system (1.1) by the physical outgoing Sommerfeld radiation condition:

$$\frac{\partial}{\partial r}u^s - iku^s = O(|x|^{-\frac{1+d}{2}}) \quad \text{as} \quad |x| \to \infty\,, \tag{1.2}$$

where $u^s := u - u^i$ is the scattered field and u^i is the incident field. The solution u to the system (1.1) represents the total field due to the scattering from the inclusion Ω corresponding to the incident field u^i.

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2 Optimal Mesh size for contrast reconstruction without SNR

2.1 Far Field Data

In what follows we hope to obtain an optimal mesh size for reconstruction of inhomogeneous medium from the far field data. Actually from Born approximation and Far-field approximation, we have

$$A_\infty(\theta, \tilde{\theta}, k) = \int_\Omega q(y) e^{ik\left(d_\theta - d_{\tilde{\theta}}\right)\cdot y} dy + O(||q||^2)\,. \tag{2.1}$$

Therefore by Shannon sampling theorem, we have

Theorem 2.1. $N = \frac{4|\Omega|}{\lambda^2}$ *is an optimal mesh for far-field reconstruction without noise.*

Proof. Denote the fourier transform $\mathfrak{F}(q)(\xi)$ of $q(x)$ be as follows

$$\mathfrak{F}(q)(\xi) := \int_\Omega q(y) e^{i\xi\cdot y} dy\,, \quad \xi \in \mathbb{R}^d\,. \tag{2.2}$$

Now we observe that

$$A_\infty(\theta, \tilde{\theta}, k) = \mathfrak{F}(q)(\xi) + O(||q||^2), \tag{2.3}$$

where $\xi := k(d_\theta - d_{\tilde{\theta}}) \in \{\xi : |\xi| \le 2k\}$. From the fact that $\mathbb{S}^{d-1} + \mathbb{S}^{d-1} = \overline{B_0(2)} \subset \mathbb{R}^d$, where $B_0(2)$ denotes the open ball of radius 2, and for any $A, B \subset \mathbb{R}^d$ the notation $A + B$ denotes the Minkowski sum of A and B, $A + B := \{a + b : a \in A, b \in B\}$, we are able to collect all the values of $\mathfrak{F}(q)(\xi)$ for all ξ such that $|\xi| \le 2k$ from the far-field data. An application of the Nyquist-Shannon sampling theorem [] yields the result. □

From the fact that unique continuation is exponentially ill-posed, we cannot reconstruct $\mathfrak{F}(q)(\xi)$ outside. This forces us to do a zero padding, i.e. setting $\mathfrak{F}(q)(\xi) = 0$ outside.

2.2 Near Field Data

Next, we wish to find an optimal mesh for contrast reconstruction from the near field data. From the Born approximation, we have the following approximation of MSR operator when the potential q is of the form $q(x) = \sum_{l=1}^{N} q(z_l)\delta_{z_l}$,

$$[\mathrm{MSR}(q)](x, y) = \sum_{l=1}^{N} q(z_l) H_0^{(1)}(k|z_l - x|) H_0^{(1)}(k|z_l - y|) + O(Q^2) \tag{2.4}$$

where $x, y \in \Gamma$ with Γ being a given measurement set (either a set of points or a smooth measurement surface) and $Q = \sup_l q(z_l)$. To investigate the optimal mesh size with respect to the near field data, we would like to investigate the number of observable singular values for the following linearized MSR operator A when z_l are close to each other:

$$A := [L\,\mathrm{MSR}(q)](x, y) = \sum_{l=1}^{N} q(z_l) H_0^{(1)}(k|z_l - x|) H_0^{(1)}(k|z_l - y|) \tag{2.5}$$

with $x, y \in \Gamma$.

For this purpose, we first define a conjugation map on $L^2(\Gamma)$ as follows,

$$C : L^2(\Gamma) \to L^2(\Gamma) \quad f \mapsto \bar{f}\,. \tag{2.6}$$

We shall also often write the $L^2(\Gamma)$ inner product on $L^2(\Gamma)$ as

$$\langle f, g\rangle_\Gamma = \int_\Gamma \overline{f(y)} g(y)\, d\sigma_y\,, \quad \forall f, g \in L^2(\Gamma)\,. \tag{2.7}$$

where $d\sigma_y$ is either the surface measure along Γ if Γ is a surface or the counting measure if Γ is a set of points. For sake of exposition, we shall also define the following bilinear form on $L^2(\Gamma)$ as

$$[f, g]_\Gamma = \langle Cf, g\rangle_\Gamma\,, \quad \forall f, g \in L^2(\Gamma)\,. \tag{2.8}$$

For simplicity, we shall sometimes also denote $\langle\cdot,\cdot\rangle_\Gamma$ and $[\cdot,\cdot]_\Gamma$ as $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]$ respectively when the domain of integration Γ is known in the context. For notational sake, we also define the following two dual maps:

$$(\cdot)^* : L^2(\Gamma) \to [L^2(\Gamma)]^*\,, \quad f \mapsto \langle f, \cdot\rangle\,, \tag{2.9}$$

$$(\cdot)^T : L^2(\Gamma) \to [L^2(\Gamma)]^*\,, \quad f \mapsto [f, \cdot]\,. \tag{2.10}$$

3 Optimal Mesh size in terms of SNR

3.1 Far Field Data and Near field

In this section, we shall derive a formula for the step size (or the resolution) in terms of the signal-to-noise ratio and the distances between the point and the source and the receiver points. Let $L_r = |y-x|$ and $L_s = |z-y|$. Assume that the measurements are corrupted with a Gaussian noise with variance σ and mean zero. Introduce the signal-to-noise ratio as,

$$SNR = \left(\frac{\delta}{\sigma}\right)^2,$$

where δ is the order of magnitude of the perturbations in the permittivity we are looking to reconstruct (the permittivity which we consider is of the form of $1+\delta q(x)$ where $||q||_{L^\infty} = 1$). Let $\omega_{\max}$ be the highest frequency of q we can recover at y. We have

$$e^{-2\sqrt{\omega_{\max}^2-\omega^2}(L_s+L_r)} \geq SNR^{-1} \tag{3.1}$$

and therefore, if we write $\omega_{\max}^2 = (1+\alpha^2)\omega^2$, then

$$\alpha \leq \frac{|\log SNR|}{2\omega(L_r+L_s)}\,. \tag{3.2}$$

From this, it follows from the Nyquist-Shannon sampling theorem that the optimal mesh size is given by

$$H(SNR) = \frac{\lambda}{2\sqrt{1+(\lambda|\log SNR|)^2/(2\pi(L_r+L_s))^2}}. \tag{3.3}$$

This formula shows that if $\lambda/(L_s+L_r) \ll 1$ (far-field), then the step size is $\lambda/2$. In the near field $\lambda/(L_s+L_r) = O(1)$, one needs a high SNR in order to enhance significantly the resolution. It also gives a continuous formula of the step size in both Far-field and Near-field.

4 Multilevel Reconstruction using a Sampling method.

We may use any sampling type algorithm, for example the MUSIC type algorithm [2]. Assume $q = \chi_D$, where D is an open set. Assume the sampling method gives an index function I such that the approximate reconstruction contrast is given by either $Q = CI$ for some given C or $Q = C_1\chi_{\{I>C_2\}}$ for some given constant C, C_1, C_2. Let Q^{SNR} be the function reconstructed from noisy data with SNR. Assume that the method is ε accuracy when no noise is present, i.e. $d(\frac{q}{||q||_1}, \frac{Q^\infty}{||Q^\infty||_1}) < \varepsilon$ for some metric, e.g. d is chosen as d_W the Wasserstein 2-metric, which was proposed by Engquist *et. al.* to compare the distribution in inverse problems. We assume a series of mesh/triangulation $\mathcal{T}_n$, where the set of nodes in $\mathcal{T}_{n+1}$ is contained in the set of nodes of $\mathcal{T}_{n+1}$ and that $h < H(n)$. Let Q_n^{SNR} is given by Q^{SNR} on nodes of $\mathcal{T}_n$ and then piecewise linear interpolation. Then we shall be able to infer that

Theorem 4.1.

$$d(\frac{q}{||q||_1}, \frac{Q_n^{SNR}}{||Q_n^{SNR}||_1}) < G(\min\{n, SNR\}) + \varepsilon \tag{4.1}$$

where G shall be a decreasing function w.r.t. the argument.

OR

Try to shown convergence for adaptive refinement scheme, for example [5] for convergence up to optimal meshsize. The localization operator in Theorem (3.1) of [5] is a factorization method which can also be considered as an index described above.

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