# A Generalized Eigenvalue Equation Directly Associated with Radiation Power

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***Abstract*—In this paper, a generalized eigenvalue equation associated with the resistance matrix and the Gram matrix is investigated, which provides an alternative way to generate eigen modes for a scatterer. The radiation capability of an eigenmode can be directly predicted by its eigenvalue. Numerical examples are shown to demonstrate the feature of the proposed generalized eigenvalue equation.**

***Keywords—Eigenmode, eigenvalue, electromagnetic scattering, Gram matrix***

## I. INTRODUCTION

The theory of characteristic mode (TCM) for perfectly electrically conducting (PEC) bodies has been applied successfully in scattering and radiation problems. In TCM, the real and imaginary parts of the impedance matrix are selected to form a generalized eigenvalue equation, from which a set of real and orthogonal characteristic currents on the surface of conducting bodies can be obtained [1-2]. Modal significance (MS) is introduced to indicate the relationship between radiation power and eigenvalues. However, MS basically describes the ratio of the amplitude of the complex power to the radiation power created by a modal current, and larger MS does not always mean higher radiation capability.

In this paper, a generalized eigenvalue equation directly associated with the radiation power is proposed and investigated, which is composed with the resistance matrix and the corresponding Gram matrix defined by the inner product of basis functions. The eigenvalues and the associated eigenvectors solved from the new generalized eigenvalue equation are all real. It can be found that the radiation power is directly in proportional to the eigenvalues.

The formulation of the two generalized eigenvalue equation is briefly introduced in section II. Two numerical examples are presented in section III, and conclusions are drawn in section IV.

## II. FORMULATION

### *A. Generalized Eigenvalue Equation in TCM*

In TCM, the eigenvalues and modal currents can be calculated by solving the following generalized eigenvalue equation [3]:

$$\mathbf{X}\cdot\mathbf{J}_n = \lambda_n \mathbf{R}\cdot\mathbf{J}_n \tag{1}$$

where X and R denote the imaginary and real parts of the impedance matrix Z, respectively. Since both the matrix are real and symmetrical, the eigenvectors $\mathbf{J}_n$ and the eigenvalues $\lambda_n$ are all real. Besides, $\mathbf{J}_n$ form an orthogonal base for surface currents.

The modal significance MS is introduced to describe the contribution of each mode, which is defined as:

$$MS=\left|\frac{1}{1+j\lambda_n}\right| \tag{2}$$

When TCM is applied for electrically small antenna designs, the characteristic modes are usually sorted in descending order according to their MS. Some researchers tend to use MS as a reference parameter for the radiation capability of the corresponding modal current. Recall that

$$\mathbf{Z}\cdot\mathbf{J}_n = \left(\mathbf{R}+j\mathbf{X}\right)\cdot\mathbf{J}_n = \left(1+j\lambda_n\right)\mathbf{R}\cdot\mathbf{J}_n \tag{3}$$

From which it can be derived

$$MS = \frac{\mathbf{J}_n\cdot\mathbf{R}\cdot\mathbf{J}_n}{\left|\mathbf{J}_n\cdot\mathbf{R}\cdot\mathbf{J}_n + j\mathbf{J}_n\cdot\mathbf{X}\cdot\mathbf{J}_n\right|} \tag{4}$$

where $\mathbf{J}_n\cdot\mathbf{X}\cdot\mathbf{J}_n$ is related to the stored energy.

### *B. Generalized Eigenvalue Equation Based On Gram Matrix*

The entries of the Gram matrix $\mathbf{G}$ are inner products of the basis functions,

$$G(m,n) = \left\langle \mathbf{f}_n,\mathbf{f}_m\right\rangle \tag{5}$$

where in the case of a PEC scatterer, $\mathbf{f}_n$ and $\mathbf{f}_m$ are two basis functions on the PEC surface, like RWGs. As can be observed, Gram matrix is real and symmetrical. Consider the following generalized eigenvalue equation,

$$\mathbf{R}\bullet\mathbf{J}_n=\sigma_n\mathbf{G}\cdot\mathbf{J}_n \tag{6}$$

$\sigma_n$ is an eigenvalue, and $\mathbf{J}_n$ is the corresponding eigenvector. Both of them are real. The radiation power can then be expressed as

$$P = \mathbf{J}_n \cdot \mathbf{R} \cdot \mathbf{J}_n = \sigma_n \mathbf{J}_n \cdot \mathbf{G} \cdot \mathbf{J}_n = \sigma_n \int_S \left|\vec{J}_n\right|^2 dS \tag{7}$$

When all modal currents are normalized on the surface of the scatterer, it can be seen from (7) that the modal current with larger $\sigma_n$ will certainly radiate higher power.

## III. Numerical Examples

In this section, the relationship between eigenvalue and radiation power is analyzed and verified by two numerical examples. In order to compare the radiation power obtained from the two generalized eigenvalue equations, eigenvectors are all normalized to 1. The conducting bodies are all incident by a plane wave travelling along –z direction. The amplitude of the incident electric field is 1V/m.

### *A. PEC Sphere*

A PEC sphere with radius 1.0m is analyzed. The frequency of the incident plane wave is $f = 128\text{MHz}$. The surface of the sphere is meshed into 806 triangles and 1209 RWGs. The density of meshes is about $\lambda/12$ ($\lambda$ is the wavelength of the incident plane wave).

The 3D gain patterns and characteristic currents on the

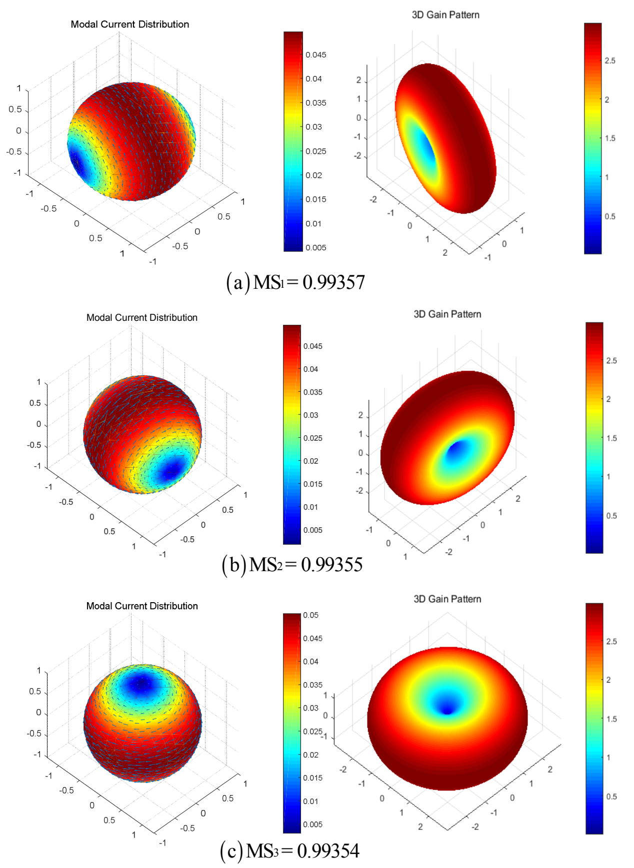


Fig. 1. Characteristic currents and gain patterns of the first three modes associated with the first three largest MS. Solved by the generalized eigenvalue equation in TCM.

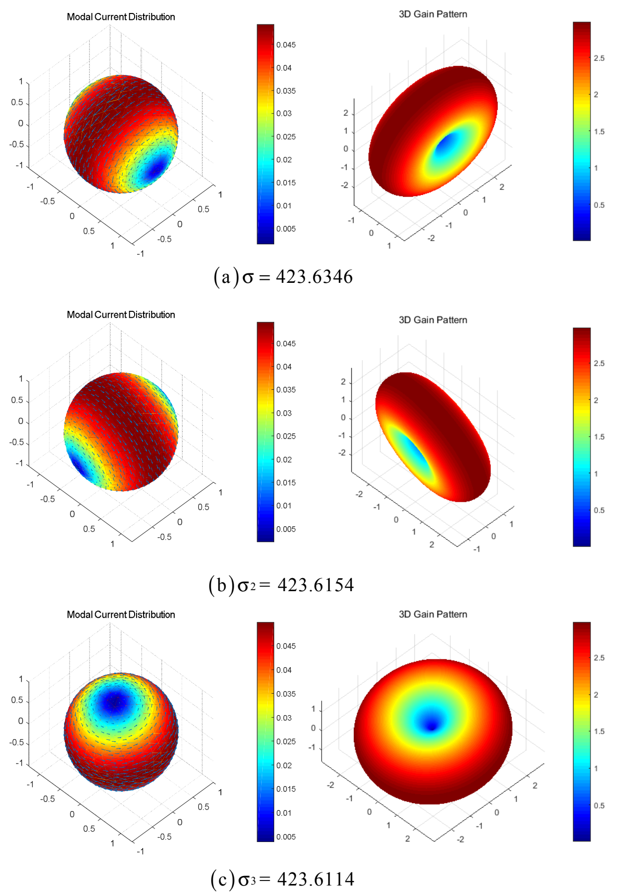


Fig. 2. The modal currents and gain patterns of the first three modes associated with first three largest eigenvalues. Solved by using the new generalized eigenvalue equation.

surface of conducting bodies are calculated for comparison and confirming the effectiveness of the new generalized eigenvalue equation. The characteristic currents and 3D gain patterns of the first three modes obtained from solving the conventional and the proposed generalized eigenvalue equation are listed in Fig.1 and Fig.2, respectively. Obviously, the three modes obtained using the two generalized eigenvalue equations are respectively consistent with each other.

The radiation powers of the first 40 eigenmodes are given in Fig.3. The Eigenvalues associated with the conventional eigenvalue equation are sorted according to the MS in descending order, while those associated with the new one are sorted according to their amplitude. As can be seen in Fig.3, the radiation power calculated by the traditional generalized eigenvalue equation does not vary monotonically, while the radiation power almost monotonically decreases with the eigenvalues calculated with the new generalized eigenvalue equation. The slight increase at mode3 and mode7 may be caused by the numerical truncation errors.

### *B. Rectangular PEC Patch*

A rectangular PEC patch with 100mm×40mm is illuminated by a plane wave with frequency $f = 3GHz$. The surface is meshed into 358 triangles and 509 RWGs.

The variations of radiation power with the eigenvalues solved by the two generalized eigenvalue equations are showed in Fig.4, respectively.

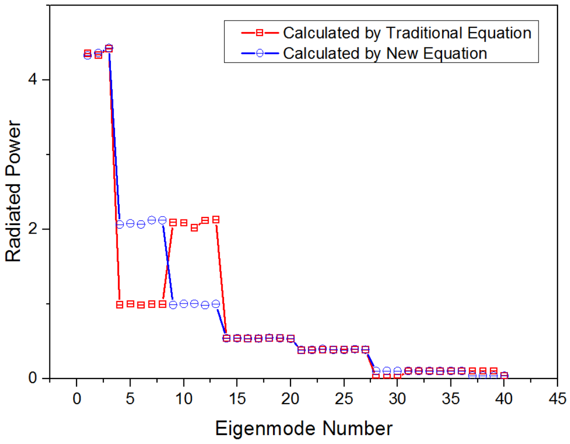


Fig. 3. Radiation power solved by traditional and new generalized eigenvalue equation of PEC sphere.

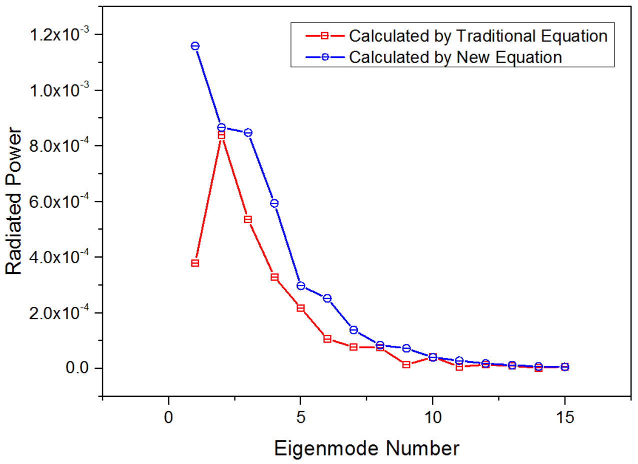


Fig. 4. Radiation power solved from traditional and new generalized eigenvalue equation of rectangular PEC patch.

It can be clearly found that the radiation power decreases with the decrease of the amplitude of eigenvalues solved by using the new generalized eigenvalue equation, while the radiation power may be smaller for larger MS.

## IV. Conclusion

A new generalized eigenvalue equation associated directly with radiation power is proposed in this paper. The numerical examples show that the resultant eigenvalues can accurately reflect the radiation capability of a modal current.

## Acknowledgment

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