Worst case analysis of random circuits using Taylor models and Bernstein polynomials

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***Abstract*—This paper proposes a novel paradigm to optimize the Taylor models based worst-case analysis results of random circuits responses in time domain. The methodology leverages the so-called Bernstein polynomials to yield a conservative, yet tight, estimation of the worst case bounds. The proposed framework is then applied to the time-domain analysis of a linear circuit, which demonstrated its feasibility and strength.**

Keywords—worst case analysis; Taylor models; Bernstein polynomials; reliability; uncertainty

I. INTRODUCTION

Circuits and system variability has long been impacting the stability and reliability of electrical and electronic equipment, and only in the last century or so have scientists and engineers been able to investigate the responses of electronic devices affected by unknown uncertainties via numerical techniques [1]. This pursuit was made possible through advancements in computer techniques [2-3]. Among these numerical methods, Taylor models (TM), as a rigorous approach combining the strength of the traditional interval analysis (IA) method, offers reasonable inherent worst case (WC) responses of circuits and systems in a timely fashion [4-6]. It is based on the representation of random parameters in terms of truncated Taylor expansions plus an interval remainder accounting for the truncation errors, and the corresponding TM operations abide both polynomial and IA rules. The final range of the response is provided by both the polynomial and the IA-remainder, hence it is necessary to determine the bound of a polynomial, which is non-trivial, especially for high-order multivariate polynomials [6]. A rough approximation of the polynomial bound will possibly lead to unnecessarily strict margins, which are very unlikely to occur [3].

In order to improve the TM based simulation results, one idea is to adopt more sophisticated bound functions for the polynomial part. Bernstein polynomials play an extremal position in some classes of operations [7]. It is capable of determining tight bounds on the range of a multivariate polynomial over a closed box, and is therefore widely used for global optimization [8] and numerical approximation [9]. The aim of this paper is to outline how Bernstein basis helps improve the TM-based WC analysis of passive linear circuits in time domain.

II. TAYLOR MODELS

The idea of the so called TM approach is to represent a nonlinear function f of an interval variable $\overline{x}=[a,b]$ as Taylor polynomials plus an interval remainder. The nth-order TM of $f(\overline{x})$ around the center of the interval x_0 is defined as

$$f(\overline{x})=P_f(x-x_0)+I_f \tag{1}$$

where

$$P_f(x-x_0)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \tag{2}$$

is the nth-order Taylor expansion of $f(x)$ around x_0, and I_f being an interval variable, which is commonly defined as the Lagrange remainder of the Taylor expansion (2), see [4-5] for detailed discussion on the calculation of IA-remainder.

Denote $B(\cdot)$ as the bound operator of a function over its domain, and the overall bound of the TM form function is represented as

$$B(f(\overline{x}))=B(P_f)+I_f \tag{3}$$

This overall bound is obtained from the IA-sum of two sub-intervals, and it should enclose $f(x)$ between two curves for $\forall x\in[a,b]$. Ideally, $I_f=[0,0]$ implies that $P_f(x)$ provides an exact parametric representation of $f(x)$, however in reality, the TM form bound of (3) always provides over-estimated results.

Given two TM form functions $f(\overline{x})=P_f(x)+I_f$ and $g(\overline{x})=P_g(x)+I_g$, the algebraic operations between them consists of a suitable combination of standard and IA calculations. For instance, the product between them yields

$$\begin{aligned} u(\overline{x}) &= f(\overline{x})\,g(\overline{x})=(P_f(x)+I_f)(P_g(x)+I_g) \\ &= P_f(x)P_g(x)+P_f(x)I_g+P_g(x)I_f+I_fI_g \end{aligned} \tag{4}$$

The first term is expressed as

$$P_f(x)\,P_g(x)=P_u(x)+P_e(x) \tag{5}$$

where $P_u(x)$ is up to order n and corresponds to the polynomial part of the TM of $u(x)$, whilst $P_e(x)$ is the remaining higher-order contribution. The remainder of the product includes all the remaining terms and is computed as

$$I_u = B(P_e) + B(P_f)I_g + B(P_g)I_f + I_f I_g \tag{6}$$

It is worth noting that the use of Taylor models requires a fast and accurate bound operator $B(\cdot)$ of the polynomials. Since IA has been included in each step of the TM operations, a rough bound of the polynomial part may lead to a large over-estimation in a long chain of realistic operations. In the univariate case, the bounds of a polynomial can be readily obtained with analytical approaches, while in the multivariate case it is non-trivial and requires suitable solutions. One possible solution is introduced in the next part.

III. Multivariate Polynomial Bounds

As has been stated in the previous section, at the end of a series of TM operations, we have obtained a TM form function, the polynomial part of which is commonly represented by the so called *power basis*

$$P(x) = \sum_{i=0}^{l} a_i x^i, \quad x^i = \prod_{\mu=1}^{n} x_\mu^{i_\mu} \tag{7}$$

in n variables, $x_1,...,x_n$, of degree $l = (l_1,...,l_n)$, and a box

$$X = [\underline{x}_1, \overline{x}_1] \times \cdots \times [\underline{x}_n, \overline{x}_n] \tag{8}$$

This is only one of an infinite number of bases for the space of polynomials. In order to determine a tight outer approximation for $P(x)$ over X, one method is to convert the power basis form multivariate polynomial (7) into the Bernstein form, which reads:

$$P(x) = \sum_{i=0}^{l} b_i B_i(x) \tag{9}$$

where

$$B_i(x) = \binom{l}{i} x^i (1-x)^{l-i} \tag{10}$$

form a basis for the vector space of polynomials of degree l, and

$$b_i = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} a_j, \quad 0 \le i \le l \tag{11}$$

are the so-called Bernstein coefficients. It is relevant to point out that the evaluation of b_i in (11) is effective only for *unit* box, namely $X = [0,1]^n$. For any arbitrary non-empty box, the general form of Bernstein coefficients are given by

$$b_i = \sum_{j=0}^{i} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x} - \underline{x})^j \sum_{k=j}^{l} \binom{k}{j} \underline{x}^{k-j} a_k \tag{12}$$

Once we obtain the coefficients of the Bernstein expansion of a given polynomial over a specified box, the range of the polynomial is determined simultaneously, which is tightly bounded by these coefficients, as the property states: the range of a polynomial P over X is contained within the interval spanned by the minimum and maximum Bernstein coefficients [6].

$$\min_i \{b_i\} \le P(x) \le \max_i \{b_i\}, \quad x \in X \tag{13}$$

Obviously, one has to compute all of the Bernstein coefficients to determine the minimum and maximum, and this can be extremely time-consuming in high-order operations consisting of a relatively large number of variables. Smith proposed a new method for the representation and computation of Bernstein coefficients of multivariate polynomials in [6], where he begins with multivariate monomials. Consider the following polynomial comprising a single term

$$q(x) = a_k x^k, \quad x = (x_1,...,x_n), 0 \le k \le l \tag{14}$$

The Bernstein coefficients for monomial can be modified as

$$b_i = a_k \sum_{j=0}^{\min\{i,k\}} \frac{\binom{i}{j}}{\binom{l}{j}} (\overline{x} - \underline{x})^j \binom{k}{j} \underline{x}^{k-j} \tag{15}$$

If a polynomial P consists of m terms, then the overall Bernstein coefficients equal to the sum of the corresponding coefficients of each monomial. In this case, the computational complexity is only linear to the number of terms in the polynomial.

IV. Numerical Results

This section collects the results on the application of the proposed technique to the time-domain simulation of the passive circuits in Fig. 1, where all the component values R, L, and C are defined by an interval value with given bounds.

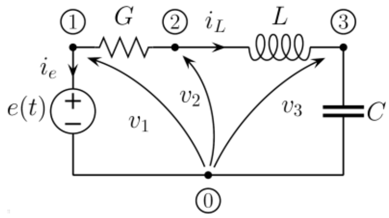

Fig. 1. Illustrative RLC example used to demonstrate the performance of the proposed tools for the worst-case circuit analysis.

The nominal values of the circuit parameters are $R_0 = 1\Omega$, $L_0 = 1\text{H}$, and $C_0 = 1$ F. with all variables a ±10% tolerance. The circuit equation in form of matrix writes as follows:

$$\mathbf{Y_d}\frac{d}{dt}\mathbf{w}(t)+\mathbf{Y_g}\mathbf{w}(t)=\mathbf{J} \qquad (16)$$

where $\mathbf{w}=[v_1(t),v_2(t),v_3(t),i_L(t),i_e(t)]^T$ is the vector of unknown variables, $\mathbf{J}=[0,0,0,0,e(t)]^T$, and

$$\mathbf{Y_g}=\begin{bmatrix} G & -G & 0 & 0 & 1 \\ -G & G & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix},\ \mathbf{Y_d}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -L \end{bmatrix}$$

are the interval valued parameter matrices. To solve the differential equation (16), we resort to the custom Trapezoidal rule, namely, if $\frac{dw}{dt}=f(t,w)$, then

$$\frac{w_{n+1}-w_n}{h}=\frac{1}{2}\left[f(t_n,w_n)+f(t_{n+1},x_{n+1})\right] \qquad (17)$$

where $h=t_{n+1}-t_n$ is the time step. The final time-domain analysis using sampled-data iterative map is performed as

$$\left(2\mathbf{Y_d}+h\mathbf{Y_g}\right)\mathbf{w}_{n+1}=\left(2\mathbf{Y_d}-h\mathbf{Y_g}\right)\mathbf{w}_n+h\left(\mathbf{a}_{n+1}+\mathbf{a}_n\right) \qquad (18)$$

Unfortunately, a divergence is observed for the time-domain responses, as can be seen in Fig. 2, due to the unlimited growth of the IA-remainder.

If and only if, we use a higher-order Bernstein bound method, the over-estimation is beat successfully, and the optimized responses of capacitance voltage for the Fig. 1 circuit can be observed as shown in Fig. 3.

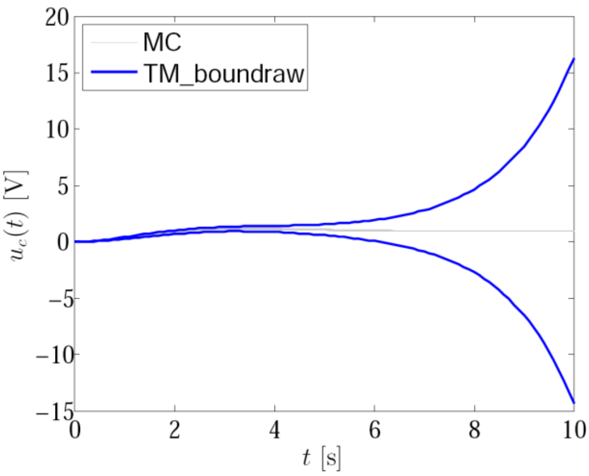

Fig. 2. Error explosion due to the divergence of IA-remainder

V. Conclusion

This paper presents a TM-based approach combing the strength of Bernstein polynomials to jointly obtain the time-domain responses of a circuit with bounded uncertain parameters. The simulated results demonstrated the accuracy and feasibility of the technique, and highlighted that the proposed framework allows for achieving a substantial improvement with respect to plain TM calculations.

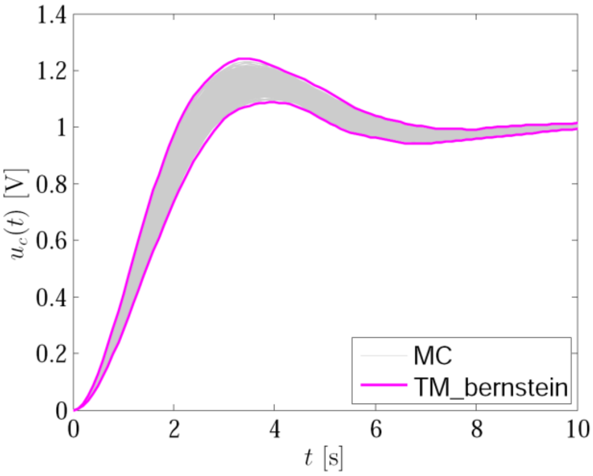

Fig. 3. Time-domain solution of the capacitor voltage. The upper and lower bounds are obtained via TMs combining Bernstein bound solution (magenta lines), and are compared against the spread given by MC samples (gray area).

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