Consumption Strikes Back? Measuring Long-Run Risk

Hansen, Heaton & Li, 2008, JPE
[Hansen et al., 2008]

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Introduction

Introduction

This Literature formalize

- How Long-run growth variation in consumption and asset cash flows contributes to valuation
- Isolate model primitives that have important consequences for long-run valuation and heterogeneity across cash flows
- Long-run cash flow risk and limiting contribution to the one-period returns coming from cash flows in distant future

Empirical Application

- Growth portfolio grow much more slowly than value portfolio, then why high pd ratio?
- Long-run cash flow growth of growth portfolio has negligible cov with consumption, but value portfolio has positive cov

General Framework

General Setup

State dynamics

- State vector X_t follow. stationary and ergodic Markov process
- Law of motion: $X_{t+1} = \psi\left(X_t, W_{t+1}\right)$

Multiplicative functionals M

- Definition: $\log M_{t+1} - \log M_t = \kappa\left(X_t, W_{t+1}\right)$
  - Used to model Cash Flow (**G**), SDF(**S**) and their Multiple(**GS**)

Factorization

- Any multiplicative functional M^i can be decomposed into:

$$\frac{M_t^i}{M_0^i} = \exp(\eta^i t)\frac{e^i(X_0)}{e^i(X_t)}\frac{\widetilde{M}_t^i}{\widetilde{M}_0^i}$$

$\rightarrow$ 1. η^i: deterministic growth/decay rate. 2.$e^i(x) > 0$: func of Markov state. 3. $\widetilde{M}^i$: multiplicative martingale

General Setup

Long-term risk return tradeoff

- Form

$$\frac{S_t}{S_0} = \exp\left(t\eta^s\right)\left(\frac{\widetilde{M}_t^s}{\widetilde{M}_0^s}\right)\left[\frac{e^s\left(X_0\right)}{e^s\left(X_t\right)}\right]$$

$$\frac{G_t}{G_0} = \exp\left(t\eta^g\right)\left(\frac{\widetilde{M}_t^g}{\widetilde{M}_0^g}\right)\left[\frac{e^g\left(X_0\right)}{e^g\left(X_t\right)}\right]$$

$$\frac{S_t G_t}{S_0 G_0} = \exp\left(t\eta^{sg}\right)\left(\frac{\widetilde{M}_t^{sg}}{\widetilde{M}_0^{sg}}\right)\left[\frac{e^{sg}\left(X_0\right)}{e^{sg}\left(X_t\right)}\right]$$

- Then, limiting risk premium follow

$$\lim_{t\to\infty}\left[\frac{1}{t}\log E\left[G_t \mid X_0\right] - \frac{1}{t}\log E\left[S_t G_t \mid X_0\right] + \frac{1}{t}\log E\left[S_t \mid X_0\right]\right]$$

$$= \eta^g + \eta^s - \eta^{sg}$$

- Limiting risk-free rate $= -\eta^s$

Model Setup

Preview of IRF of consumption

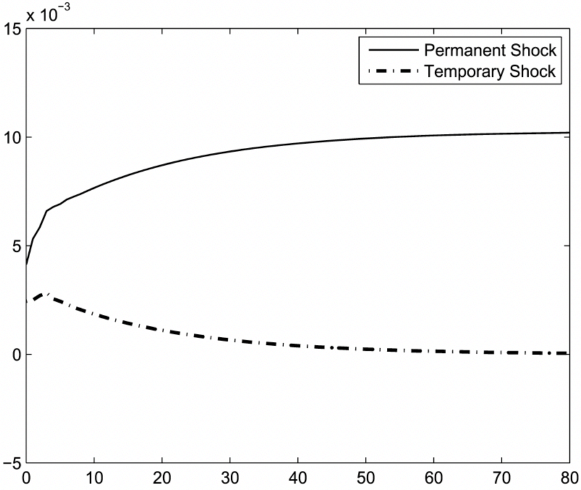


FIG. 3.—Impulse responses of consumption to permanent and temporary shocks implied by bivariate VARs where consumption and earnings are assumed to be cointegrated. Each shock is given a unit impulse. Responses are given at quarterly intervals.

Model Setup

State dynamics

- State vector x_t follows VAR(1):

$$x_{t+1} = Gx_t + Hw_{t+1}; \quad w_{t+1} \sim \text{i.i.d.}\mathcal{N}(0, I)$$

SDF

- Log consumption process: $c_{t+1} - c_t = \mu_c + U_c x_t + \lambda_0 w_{t+1}$
- Then, Log SDF with recursive utility can be written as:

$$s_{t,t+1} = \mu_s + U_s x_t + \xi_0 w_{t+1}$$

  - With <u>unit IES</u>,

$$\xi_0 = -\lambda_0 + (1-\gamma)\lambda(\beta)$$

    where

$$\lambda(\beta) = \lambda_0 + U_c \sum_{j=1}^{\infty} \beta^j G^{j-1} H = \lambda_0 + \beta U_c (I - \beta G)^{-1} H$$

    $\rightarrow$ discounted impulse response of consumption to w_{t+1}

Model Setup

Dividend (cash flow) Process

- $D_t = D_t^* f(x_t)$
  - Log D_t^* process:
  $$\log D_{t+1}^* - \log D_t^* = \zeta + \pi w_{t+1}$$
  - Transitive and stationary component $f(x_t)$:
  $$\log D_{t+1} - \log D_t = \zeta + \pi w_{t+1} + \log(f(x_{t+1})) - \log(f(x_t))$$

Valuation

- Define a time t valuation operator of D_{t+1}:
$$\mathcal{P} f(x) = E\left[\exp\left(s_{t,t+1} + \zeta + \pi w_{t+1}\right) f\left(x_{t+1}\right) \mid x_t = x\right]$$
- Time t value of date $t+j$ cash flow D_{t+j} follow

$$D_t^*\left[\mathcal{P}^j f\left(x_t\right)\right] = D_t^* E\left[\exp\left[\sum_{\tau=1}^{j}\left(s_{t+\tau-1,t+\tau} + \pi w_{t+\tau}\right) + j\zeta\right] f\left(x_{t+j}\right) \mid x_t = x\right]$$

Special case: $f(x) = \exp(\omega x + \kappa)$

- Using property of lognormal,

$$\mathcal{P}f(x) = \exp\left(\omega^* x + \kappa^*\right)$$

where

$$\omega^* = \omega G + U_s \tag{1}$$

$$\kappa^* = \kappa + \mu_s + \zeta + \frac{\left|\omega H + \xi_0 + \pi\right|^2}{2} \tag{2}$$

- Iterating (1) gives

$$\bar{\omega} = U_s(I - G)^{-1}$$

$(\kappa^* - \kappa)$ from (2) converges to

$$-\nu \equiv \mu_s + \zeta + \frac{|\bar{\omega}H + \xi_0 + \pi|^2}{2}$$

- $\mathcal{P}[\exp(\bar{\omega}x + a)] = \exp(-\nu + a)\exp(\bar{\omega}x)$

Stationary f(x); SDF; Cash Flow; State dynamics

Model Setup

Result 1:

The equation

$$\mathcal{P}e(x) = \exp(-\nu)e(x)$$

has a strictly positive solution e given by

$$e(x) = \exp\left[\bar{\omega}x\right]$$

The corresponding value of $-\nu$ is

$$-\nu = \mu_s + \zeta + \frac{|\bar{\omega}H + \xi_0 + \pi|^2}{2}$$

- In general, it is an **eigenvalue problem** and the **principal eigenfunction** $e(x)$ is the unique (up to scale) solution that is strictly positive and satisfies a stability condition.

Model Setup

Growth operator

- Define a one period growth operator of D_{t+1}

$$\mathcal{G}f(x) = E\left[\exp\left(\zeta + \pi w_{z+1}\right) f\left(x_{t+1}\right) \mid x_t = x\right]$$

  - Eigenfunction is unit function: $\mathcal{G}[1] = \exp(\zeta + \frac{1}{2}\pi \cdot \pi) * 1$
  - Eigenvalue is $\exp(\eta)$ where $\eta = \zeta + \frac{1}{2}\pi \cdot \pi$

Asymptotic rate of return

$$\eta + \nu = \iota^* + \pi^* \cdot \pi$$

$$\text{where } \iota^* \equiv -\mu_s - \frac{\pi^* \cdot \pi^*}{2}; \quad \pi^* \equiv -\xi_0 - U_s(I - G)^{-1} H$$

- In power utility, $\pi^* = \gamma\lambda(1)$
- Recursive utility with unit IES, $\pi^* = \lambda(1) + (\gamma - 1)\lambda(\beta)$

Risk Premia over Alternative Horizons

Two definitions of expected return of dividend strip D_{t+j}

1. Expected return of holding it until maturity:

$$\frac{1}{j} E_t \left[\log \mathcal{G}^j f\left(x_t\right) - \log \mathcal{P}^j f\left(x_t\right) \right]$$

   - Depends on transitory cash flow $f(x_{t+j})$
   - Risk premium: Subtract it by horizon-matching risk-free return
   - For $j \to +\infty$: the asymptotic return is $\eta + \nu$

2. One-period returns:

$$R^j_{t,t+1} = \exp\left(\zeta + \pi w_{t+1}\right) \frac{\mathcal{P}^{j-1} f(x_{t+1})}{\mathcal{P}^j f(x_t)}$$

   - For $j \to +\infty$:

$$R^d_{t,t+1} = \exp(\nu(\pi)) \exp\left(\zeta + \pi w_{t+1}\right) \frac{e(x_{t+1})}{e(x_t)}$$

   - Recall: $e(x) = \exp[\bar{\omega} x]$

Discussion on the long bond and long dividend

Under the setup of the paper,

- Eigenfunction e(x) is the same for the pricing operator on long bond and long dividend
- $\log(\frac{R^d_{t,t+1}}{R^b_{t,t+1}}) = \frac{|\bar{\omega}H+\xi_0|^2}{2} - \frac{|\bar{\omega}H+\xi_0+\pi|^2}{2} + \pi w_{t+1}$
- Yield of long bond: $\iota^* \equiv -\mu_s - \frac{\pi^* \cdot \pi^*}{2}; \quad \pi^* \equiv -\xi_0 - \bar{\omega}H$
- Unconditional mean of short rate gives $-\mu_s$, so we can recover π^*
- Mean of $\log(\frac{R^d_{t,t+1}}{R^b_{t,t+1}})$ recovers π
- Variance of $\log(\frac{R^d_{t,t+1}}{R^b_{t,t+1}})$ recovers $\pi^2\sigma_w$

Estimation

Estimating consumption dynamics (VAR)

- $y_t = \{\text{log consumption, log corporate earnings}\}$
- $\{y_t\}$ is assumed to evolve as a VAR of order l:

$$A_0 y_t + A_1 y_{t-1} + A_2 y_{t-2} + \cdots + A_l y_{t-l} + B_0 = \omega_t$$

  - ω_t is the shock vector with mean zero and covariance matrix I
- Lag polynominal: $A(z) = A_0 + A_1 z + A_2 z^2 + ... + A_l z^l$
  - $A(1)$ is singular: unit root components
  - $A(z)$ is non-singular for $|z| < 1$
- As required by the model, the **geometric average response of consumption** can be derived out as

$$\lambda(\beta) = (1 - \beta) u_c A(\beta)^{-1}$$

  - u_c selects the first row (consumption responses)

Two restrictions on the matrix A(1)

1. Consumption and earnings have a unit root and they are co-integrated
   - $A(1) = \alpha[1 - 1]$ with column vector α freely estimated
   - Estimate it as a VAR in first diff of the log consumption and diff between log earnings and log consumption
2. Normalize the shocks so that only one shock has long-run effects
   - First, construct a temporary shock to consumption that has no long-run impact on consumption and earnings
   - Construct the second so that it is uncorrelated with first one and has equal permanent effects on consumption and earnings

Long-Run Cash Flow Risk (Martingale Extraction)

Log-linear model of cash flow growth

$$d_{t+1} - d_t = \mu_d + U_d x_t + \iota_0 w_{t+1} = \mu_d + \iota(L) w_{t+1}$$

where $\iota(z) = \sum_{j=0}^{\infty} \iota_j z^j$ with $\iota_j = \begin{cases} \iota_0 & \text{if } j = 0 \\ U_d G^{j-1} H & \text{if } j > 0 \end{cases}$

Martingale Extraction (decomposition)

$$d_{t+1} - d_t = \mu_d + \iota(1) w_{t+1} - U_d^* x_{t+1} + U_d^* x_t$$

where $\iota(1) = \iota_0 + U_d (I - G)^{-1} H$ and $U_d^* = U_d (I - G)^{-1}$

- Linking back to the model:
  $\underline{1.\iota(1) = \pi, \quad 2.\mu_d = \zeta}$ $\quad 3.f(x_t) = exp(-U_d^* x_t)$

Empirical Specification of Dividend process

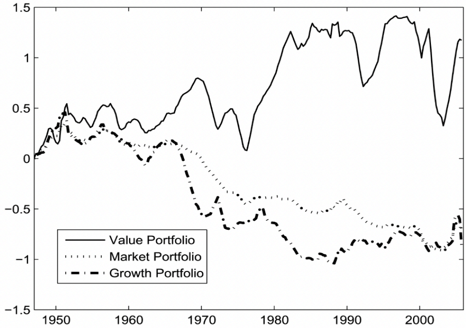


FIG. 1.—Cash flows relative to consumption for two portfolios. Quarterly logarithms of the ratios of portfolio cash flows to consumption are depicted.

- Cash flows of growth portfolio grow much more slowly than value portfolio, suggesting they have different ζ and/or π

Empirical Specification of Dividend process

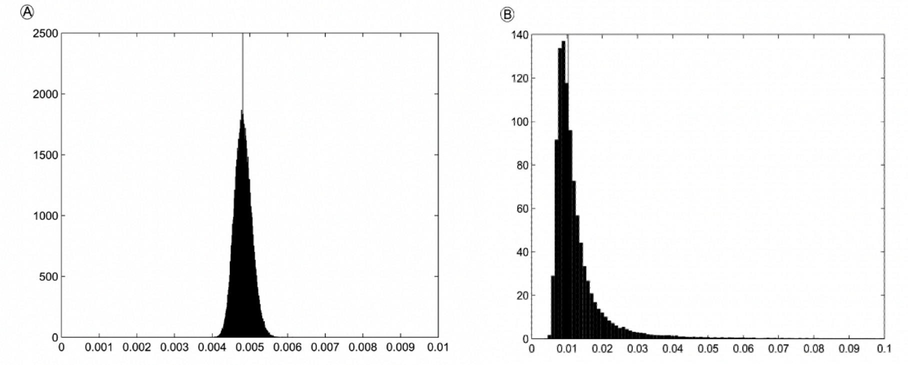
Appending a dividend equation

$$A_0^* y_t^* + A_1^* y_{t-1}^* + A_2^* y_{t-2}^* + \cdots + A_l^* y_{t-l}^* + B_0^* = w_t^*$$

- $y_t^* = [y_t; d_t]$ where d_t is a vector including dividend from five book-to-market portfolios +the market
- Dividend shock w_t^* is uncorrelated with shock w_t
- Assume w_t^* has a permanent impact on dividends: $A^*(1) = \begin{bmatrix} \alpha^* & -\alpha^* & 0 \end{bmatrix}$
- Estimate VAR with $(c_t - c_{t-1})$, $(e_t - c_t)$ and $(d_t - d_{t-1})$

Results

Posterior histogram



- A: $|\lambda_0|$: Immediate response of consumption to shocks
- B: $|\lambda(1)|$: Long-run response of consumption to the permanent shock

Pricing Implications for Aggregate Consumption

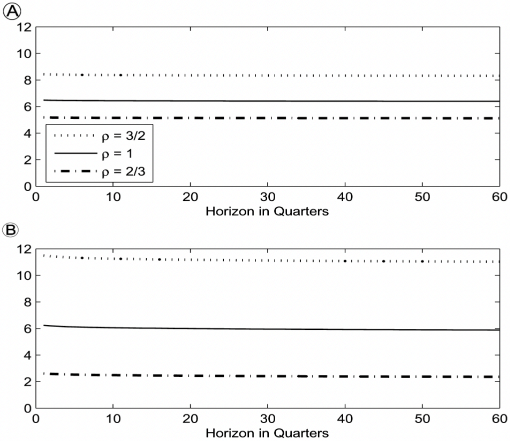
- π = long-run exposure of consumption to the two shocks: $\lambda(1)$
- Asymptotic excess return of consumption claim:

$$\lambda(1) \cdot [\lambda(1) + (\gamma - 1)\lambda(\beta)]$$

  - Estimates implies $\lambda(1) \cdot \lambda(1) = 0.0001$
  - When $\beta \approx 1$, increasing γ has small impact on expected excess return
  - E.g. when = 10, annual expected excess return is 0.4 percent

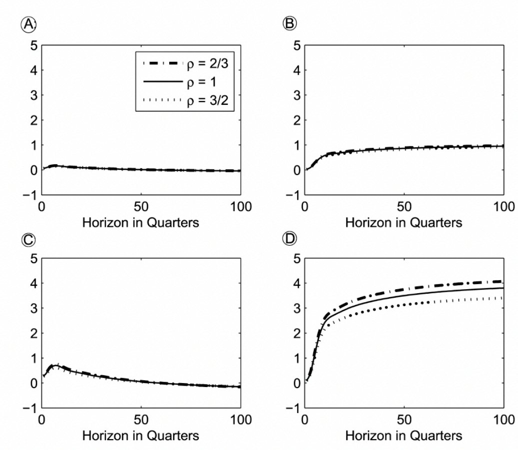
Log risk-free returns for alternative horizons

Figure A: $\gamma = 5$ and Figure B: $\gamma = 10$



Log excess returns for alternative horizons

A+B: $\gamma = 5$; C+D: $\gamma = 20$;
A+C: Portfolio 1 (growth); B+D: Portfolio 5 (value)



Continue...

- Portfolio 1 (growth; low book-to-market) has low cash flow covariance with consumption at all horizons
- Portfolio 5 (value)'s cash flow have much different exposure to consumption risk across horizons (much higher long-run exposure).
- It potentially provides an explanation for value premium: portfolio 5 has cash flows with substantial exposure to long-run consumption risk

Value-Based Measures of Duration

Contributions of cash flows in the distant future to current period values

Gordon growth model

$$\frac{\text{price}}{\text{dividend}} = \frac{\exp\ (\text{growth rate})}{\exp\ (\text{return rate})\ - \exp\ (\text{growth rate})}$$

- ν : The difference between long-term rates of return and growth
  - Limiting measure of the duration
  - When ν is small, cash flows in distant future remain important contributors to current-period values
- Growth firms (low book-to-market) have low limiting growth rates, but have high pd ratios. How can it happen?
  - Their model-implied limiting rate of returns are substantially lower

Limiting cash flow discount and growth rates

Portfolio	Rate of Return	Derivative	Rate of Growth
		$\gamma = 5$	
1	6.27	3.81	2.11
2	6.42	3.81	1.94
3	7.03	3.76	4.32
4	7.16	3.77	4.02
5	7.42	3.75	7.02
		$\gamma = 20$	
1	5.39	10.45	2.11
2	5.98	10.35	1.94
3	8.37	9.64	4.32
4	8.89	9.75	4.02
5	9.92	9.51	7.02

NOTE.—Limiting expected rates of return and growth rates for the cash flows of portfolios 1–5. The derivative entries are computed with respect to ρ and evaluated at $\rho = 1$. Returns and growth rates are reported in annualized percentages.

- 1.Sizeable γ is required
- 2.Changing ρ has important impact , but is same across portfolios

Alternative specification of cash flow

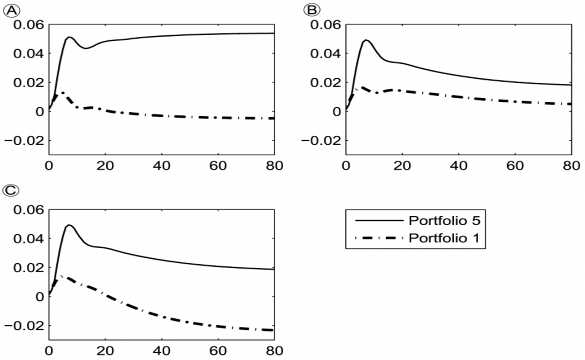


FIG. 8.—Impulse responses to a permanent shock to consumption of the cash flows to portfolios 1 and 5: *A*, from the first-difference specification used as our baseline model; *B*, from the level specification without time trends; *C*, from the level specification with time trends.

- Various models have different implications for long-run returns, but all have the gap of limiting exposure to long-run risks between portfolios

References I

Hansen, L. P., Heaton, J. C., and Li, N. (2008).
Consumption strikes back? measuring long-run risk.
Journal of Political economy, 116(2):260–302.