Calculus Basics for Greeks

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Outline

Definition and Notations

- The first-order derivative can be defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  - Second-order derivative $f''(x)$ is the derivative of $f'(x)$

- The partial derivative of $f(x_1, x_2)$ w.r.t x_1 can be defined by

$$f_1(x_1, x_2) = \lim_{h \to 0} \frac{f(x_1 + h, x_2) - f(x_1, x_2)}{h}$$

  - Second-order derivatives: $f_{11}(x_1, x_2), f_{22}(x_1, x_2)$ and $f_{12}(x_1, x_2) = f_{21}(x_1, x_2)$
  - E.g.
    $f_{12}(x_1, x_2)$ can be obtained by taking partial derivative of $f_1(x_1, x_2)$ w.r.t x_2

Derivatives of common functions

- Power function: $(x^a)' = ax^{a-1}$
  - E.g. $(x^5)' = 5x^4; \qquad (x)' = 1$
- Exponential function: $(e^{cx})' = ce^{cx}$
  - E.g. $(e^{2x})' = 2e^{2x}$
- Logarithm function: $(\ln x)' = \frac{1}{x}$
- The derivative of CDF is its PDF

$$F(x) = \int_{-\infty}^{x} f(t)dt \quad \text{then} \quad F'(x) = f(x)$$

  - E.g. Standard Normal: $\Phi'(x) = \phi(x)$

Some Basic Rules

- Linearity: if $h(x) = c_1 f(x) + c_2 g(x)$, then $\frac{dh^n(x)}{dx^n} = c_1 \frac{df^n(x)}{dx^n} + c_2 \frac{dg^n(x)}{dx^n}$ for any order of derivatives n if exists
  - E.g. $W(S) = \sum_i n_i C_i(S) + \sum_j n_j P_j(S)$
    then $W'(S) = \sum_i n_i C_i'(S) + \sum_j n_j P_j'(S)$
- Multiplication Rule: if $h(x) = f(x)g(x)$, then $h'(x) = f'(x)g(x) + f(x)g'(x)$
  - E.g. $h(x) = x^3\Phi(x)$, then $h'(x) = 3x^2\Phi(x) + x^3\phi(x)$
- Quotient Rule: if $h(x) = \frac{f(x)}{g(x)}$, then $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
  - E.g. $h(x) = \frac{1}{x}$, then $h'(x) = -\frac{1}{x^2}$
- Chain Rule: if $h(x) = f(g(x))$, then $h'(x) = f'(g(x))g'(x)$
  - E.g. $h(x) = \Phi(ln(x))$, then $h'(x) = \phi(ln(x))\frac{1}{x}$

Taylor Approximation

Second-order Talyor expansion for univariate function:

$$f(x) \approx f\left(x^*\right) + \frac{df\left(x^*\right)}{dx}\left(x - x^*\right) + \frac{1}{2}\frac{d^2 f\left(x^*\right)}{dx^2}\left(x - x^*\right)^2$$

$$\Delta f(x^*) \approx \frac{df\left(x^*\right)}{dx}\Delta x^* + \frac{1}{2}\frac{d^2 f\left(x^*\right)}{dx^2}\Delta x^{*2}$$

Second-order Taylor expansion for multivariate function:

$$\begin{aligned}\Delta f\left(x_1^*, x_2^*\right) \approx\ & f_1\left(x_1^*, x_2^*\right)\Delta x_1^* + f_2\left(x_1^*, x_2^*\right)\Delta x_2^* \\ & + \frac{1}{2}\left[f_{11}\left(x_1^*, x_2^*\right)\Delta x_1^{*2} + 2 f_{12}\left(x_1^*, x_2^*\right)\Delta x_1^* \Delta x_2^* + f_{22}\left(x_1^*, x_2^*\right)\Delta x_2^{*2}\right]\end{aligned}$$

For the first-order approximation, drop all second-order terms