Statistical Methods, Exercise 6

Problem 1

a) $$L(\{t_i\}, \tau_{1/2}) = \prod_{i=1}^{n} f(t_i;\tau_{1/2}) = \left(\frac{\ln 2}{\tau_{1/2}}\right)^n \exp\left(-\frac{\ln 2}{\tau_{1/2}}\sum_{i=1}^{n} t_i\right)$$

$$\ln L = n \ln\left(\frac{\ln 2}{\tau_{1/2}}\right) - \frac{\ln 2}{\tau_{1/2}}\sum_{i=1}^{n} t_i$$

b) $$\frac{d}{d\tau_{1/2}} \ln L = n\frac{\tau_{1/2}}{\ln 2}\left(-\frac{\ln 2}{\tau_{1/2}^2}\right) + \frac{\ln 2}{\tau_{1/2}^2}\sum_{i=1}^{n} t_i$$

$$= -\frac{n}{\tau_{1/2}} + \frac{\ln 2}{\tau_{1/2}^2}\sum_{i=1}^{n} t_i \overset{!}{=} 0$$

$$\Rightarrow \hat{\tau}_{1/2} = \ln 2 \frac{1}{n}\sum_{i=1}^{n} t_i = \underline{\underline{\ln 2 \langle t \rangle}}$$

Bias?

$$\langle \hat{\tau}_{1/2} \rangle = \int_0^\infty \hat{\tau}_{1/2} L(\{t_i\}, \tau_{1/2})\, dt_1 \dots dt_n$$

$$= \left(\frac{\ln 2}{\tau_{1/2}}\right)^n \frac{\ln 2}{n} \int_0^\infty \sum_{i=1}^{n} t_i \exp\left(-\frac{\ln 2}{\tau_{1/2}}\sum_{k=1}^{n} t_k\right) dt_1 \dots dt_n$$

$$= \left(\frac{\ln 2}{\tau_{1/2}}\right)^n \frac{\ln 2}{n} \int_0^\infty \sum_{i=1}^{n} t_i \prod_{k=1}^{n} \exp\left(-\frac{\ln 2}{\tau_{1/2}} t_k\right) dt_1 \dots dt_n$$

$$= \left(\frac{\ln 2}{\tau_{1/2}}\right)^n \frac{\ln 2}{n} \sum_{i=1}^{n} \left(\int_0^\infty t \exp\left(-\frac{\ln 2}{\tau_{1/2}} t\right) dt\right)\left(\int_0^\infty \exp\left(-\frac{\ln 2}{\tau_{1/2}} \tilde{t}\right) d\tilde{t}\right)^{n-1}$$

$$= \left(\frac{\ln 2}{\tau_{1/2}}\right)^n \frac{\ln 2}{n}\, n \left(\frac{\tau_{1/2}}{\ln 2}\right)^2 \left(\frac{\tau_{1/2}}{\ln 2}\right)^{n-1} = \frac{(\ln 2)^{n+1}}{\tau_{1/2}^n} \frac{(\tau_{1/2})^{n+1}}{(\ln 2)^{n+1}} = \tau_{1/2}$$

$\Rightarrow$ Unbiased!