Exercise Sheet 6

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Exercise 1

a) $f(t,\tau_{1/2}) = \frac{\ln 2}{\tau_{1/2}} \cdot e^{-t\frac{\ln 2}{\tau_{1/2}}}$; $L = \prod_{i=1}^{n} P(x_i, \theta)$

here: $L = \prod_{i=1}^{n} f(t_i, \tau_{1/2}) = \prod_{i=1}^{n} \frac{\ln 2}{\tau_{1/2}} \cdot e^{-t_i \frac{\ln 2}{\tau_{1/2}}}$

$$\ln L = \ln \prod_{i=1}^{n} \frac{\ln 2}{\tau_{1/2}} e^{-t_i \frac{\ln 2}{\tau_{1/2}}} = \sum_{i=1}^{n} \ln\left(\frac{\ln 2}{\tau_{1/2}} \cdot e^{-t_i \frac{\ln 2}{\tau_{1/2}}}\right)$$

$$= \sum_{i=1}^{n}\left(\ln\left(\frac{\ln 2}{\tau_{1/2}}\right) - t_i \frac{\ln 2}{\tau_{1/2}}\right) = n \cdot \ln\left(\frac{\ln 2}{\tau_{1/2}}\right) - \sum_{i=1}^{n} t_i \frac{\ln 2}{\tau_{1/2}}$$

$$= n \cdot \ln\left(\frac{\ln 2}{\tau_{1/2}}\right) - \frac{\ln 2}{\tau_{1/2}} \sum_{i=1}^{n} t_i = n \cdot \ln\left(\frac{\ln 2}{\tau_{1/2}}\right) - \frac{\ln 2}{\tau_{1/2}} \sum_{i=1}^{n} t_i$$

b) $\frac{\partial \ln L}{\partial \tau} = -\frac{n}{\tau_{1/2}} + \frac{\ln 2}{\tau_{1/2}^2} \sum_{i=1}^{n} t_i \overset{!}{=} 0 \iff \frac{\ln 2}{\tau_{1/2}^2} \sum_{i=1}^{n} t_i = \frac{n}{\tau_{1/2}} \quad | \cdot \tau_{1/2}$

$$\frac{\ln 2}{\tau_{1/2}} \sum_{i=1}^{n} t_i = n \Rightarrow \hat{\tau}_{1/2} = \ln 2 \cdot \frac{1}{n} \sum_{i=1}^{n} t_i = \ln 2 \langle t \rangle$$

Bias: $b = \langle \hat{\tau}_{1/2} \rangle - \tau_{1/2}$; Integration over all times $\Rightarrow \int_0^\infty$

$$\langle \hat{\tau}_{1/2} \rangle = \int_0^\infty \hat{\tau}_{1/2} \cdot L \, dt = \int_0^\infty \cdots \int_0^\infty \ln 2 \frac{1}{n} \sum_{i=1}^{n} t_i \cdot \prod_{k=1}^{n} \frac{\ln 2}{\tau_{1/2}} e^{-t_k \frac{\ln 2}{\tau_{1/2}}} dt_1 \cdots dt_n$$

$$= \frac{1}{n} \cdot \frac{(\ln 2)^{n+1}}{\tau_{1/2}^n} \int_0^\infty \cdots \int_0^\infty \sum_{i=1}^{n} t_i \cdot \prod_{k=1}^{n} e^{-t_k \frac{\ln 2}{\tau_{1/2}}} dt_1 \cdots dt_n = \frac{1}{n} \frac{(\ln 2)^{n+1}}{\tau_{1/2}^n} \int_0^\infty \cdots \int_0^\infty \sum_{i=1}^{n} t_i \cdot e^{-t_i \frac{\ln 2}{\tau_{1/2}}} \cdot \prod_{k \neq i}^{n} e^{-t_k \frac{\ln 2}{\tau_{1/2}}} dt_1 \cdots dt_n$$

$$= \frac{1}{n} \cdot \frac{(\ln 2)^{n+1}}{\tau_{1/2}^n} \sum_{i=1}^{n}\left(\int_0^\infty t_i \cdot e^{-t_i \frac{\ln 2}{\tau_{1/2}}} dt_i \cdot \prod_{k \neq i}^{n} \int_0^\infty e^{-t_k \frac{\ln 2}{\tau_{1/2}}} dt_k\right)$$ (all integration bounds are the same)

upper limit to 0

$$= \frac{1}{n} \cdot \frac{(\ln 2)^{n+1}}{\tau_{1/2}^n} \sum_{i=1}^{n}\left(\left[-\frac{\tau_{1/2}^2}{(\ln 2)^2} \cdot \left(\frac{\ln 2}{\tau_{1/2}} \cdot t_i + 1\right) e^{-t_i \frac{\ln 2}{\tau_{1/2}}}\right]_0^\infty \cdot \prod_{k \neq i}^{n} \left[-\frac{\tau_{1/2}}{\ln 2} \cdot e^{-t_k \frac{\ln 2}{\tau_{1/2}}}\right]_0^\infty\right)$$

$$= \frac{1}{n} \frac{(\ln 2)^{n+1}}{\tau_{1/2}^{n}} \sum_{i=1}^{n}\left(\frac{\tau_{1/2}^2}{(\ln 2)^2} \cdot \prod_{k \neq i}^{n} \frac{\tau_{1/2}}{\ln 2}\right) = \frac{1}{n} \frac{(\ln 2)^{n+1}}{\tau_{1/2}^{n}} \sum_{i=1}^{n} \frac{\tau_{1/2}^2}{(\ln 2)^2} \cdot \frac{\tau_{1/2}^{n-1}}{(\ln 2)^{n-1}} = \frac{1}{n} \frac{(\ln 2)^{n+1}}{\tau_{1/2}^{n}} \sum_{i=1}^{n} \frac{(\tau_{1/2})^{n+1}}{(\ln 2)^{n+1}}$$

$$= \frac{1}{n} \cdot \frac{(\ln 2)^{n+1}}{\tau_{1/2}^{n}} \cdot n \cdot \frac{(\tau_{1/2})^{n+1}}{(\ln 2)^{n+1}} = \tau_{1/2}$$ $\Rightarrow$ estimator is unbiased