Exercise Sheet 3

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Exercise 1

Poisson: $P(r,\lambda) = e^{-\lambda}\frac{\lambda^r}{r!}$

Variance $V = \langle r^2\rangle - \langle r\rangle^2 = \sum_{r=0}^{\infty} r^2 e^{-\lambda}\frac{\lambda^r}{r!} - \left(\sum_{r=0}^{\infty} r e^{-\lambda}\frac{\lambda^r}{r!}\right)^2$

$$= \sum_{r=1}^{\infty} r^2 e^{-\lambda}\frac{\lambda^r}{r!} - \left(\sum_{r=1}^{\infty} r e^{-\lambda}\frac{\lambda^r}{r!}\right)^2 = e^{-\lambda}\sum_{r=1}^{\infty} r^2\frac{\lambda^r}{r!} - e^{-2\lambda}\left(\sum_{r=1}^{\infty} r\frac{\lambda^r}{r!}\right)^2$$

$$= e^{-\lambda}\sum_{r=1}^{\infty} r\frac{\lambda^r}{(r-1)!} - e^{-2\lambda}\left(\sum_{r=1}^{\infty}\frac{\lambda^r}{(r-1)!}\right)^2$$

$$= \lambda\cdot e^{-\lambda}\sum_{r=1}^{\infty} r\frac{\lambda^{r-1}}{(r-1)!} - \lambda^2 e^{-2\lambda}\left(\sum_{r=1}^{\infty}\frac{\lambda^{r-1}}{(r-1)!}\right)^2$$

$$= \lambda e^{-\lambda}\sum_{r=0}^{\infty}(r+1)\frac{\lambda^r}{r!} - \lambda^2 e^{-2\lambda}\underbrace{\left(\sum_{r=0}^{\infty}\frac{\lambda^r}{r!}\right)^2}_{=e^{2\lambda}}$$

$$= \lambda e^{-\lambda}\left(\sum_{r=0}^{\infty} r\frac{\lambda^r}{r!} + \frac{\lambda^r}{r!}\right) - \lambda^2 = \lambda e^{-\lambda}\sum_{r=0}^{\infty} r\frac{\lambda^r}{r!} + \lambda e^{-\lambda}\underbrace{\sum_{r=0}^{\infty}\frac{\lambda^r}{r!}}_{=e^{\lambda}} - \lambda^2$$

$$= \lambda e^{-\lambda}\sum_{r=1}^{\infty} r\frac{\lambda^r}{r!} + \lambda - \lambda^2 = \lambda e^{-\lambda}\sum_{r=1}^{\infty}\frac{\lambda^r}{(r-1)!} + \lambda - \lambda^2$$

$$= \lambda^2 e^{-\lambda}\sum_{r=1}^{\infty}\frac{\lambda^{r-1}}{(r-1)!} + \lambda - \lambda^2 = \lambda^2 e^{-\lambda}\underbrace{\sum_{r=0}^{\infty}\frac{\lambda^r}{r!}}_{=e^{\lambda}} + \lambda - \lambda^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \underline{\underline{\lambda}}$$

Uniform: $P(x; \alpha, \beta) = \begin{cases} \frac{1}{\beta-\alpha} & , \alpha < x < \beta \\ 0 & , \text{otherwise} \end{cases}$

$$V(x) = \int (x - \langle x \rangle)^2 P(x)\, dx = \int_\alpha^\beta (x - \langle x \rangle)^2 \cdot \frac{1}{\beta-\alpha}\, dx + 0$$

$\langle x \rangle = \frac{1}{2}(\alpha+\beta)$

$$\overset{\downarrow}{=} \int_\alpha^\beta (x - \tfrac{1}{2}(\alpha+\beta))^2 \cdot \frac{1}{\beta-\alpha}\, dx = \frac{1}{\beta-\alpha} \int_\alpha^\beta (x - \tfrac{1}{2}(\alpha+\beta))^2\, dx$$

$$= \frac{1}{\beta-\alpha} \left[\tfrac{1}{3} (x - \tfrac{1}{2}(\alpha+\beta))^3 \right]_\alpha^\beta = \frac{1}{\beta-\alpha} \left(\tfrac{1}{3} (\beta - \tfrac{1}{2}(\alpha+\beta))^3 - \tfrac{1}{3} (\alpha - \tfrac{1}{2}(\alpha+\beta))^3 \right)$$

$$= \frac{1}{\beta-\alpha} \left(\tfrac{1}{3} (\tfrac{1}{2}(\beta-\alpha))^3 - \tfrac{1}{3} (\tfrac{1}{2}(\alpha-\beta))^3 \right) = \frac{1}{\beta-\alpha} \left[\tfrac{1}{24} (\beta-\alpha)^3 - \tfrac{1}{24} (\alpha-\beta)^3 \right]$$

$$= \frac{(\beta-\alpha)^{\cancel{3}2}}{24\cancel{(\beta-\alpha)}} - \frac{(\alpha-\beta)^3}{24(\beta-\alpha)} = \frac{(\beta-\alpha)^2}{24} + \frac{(\alpha-\beta)^{\cancel{3}2}}{24\cancel{(\alpha-\beta)}} = \frac{(\beta-\alpha)^2 + (\alpha-\beta)^2}{24}$$

$$= \frac{\beta^2 - 2\alpha\beta + \alpha^2 + \alpha^2 - 2\alpha\beta + \beta^2}{24} = \frac{2\alpha^2 - 4\alpha\beta + 2\beta^2}{24} = \frac{\alpha^2 - 2\alpha\beta + \beta^2}{12}$$

$$= \frac{\beta^2 - 2\alpha\beta + \alpha^2}{12} = \frac{1}{12}(\beta-\alpha)^2$$