Exercise 2

a) Since $\sigma_S = 0$, we have:

$$V(U) = \sigma_U^2, \quad V(R) = \sigma_R^2, \quad Cov(U,R) = 0$$

$$\Rightarrow C = \begin{pmatrix} 0{,}09\,V^2 & 0 \\ 0 & 0{,}04\,k\Omega^2 \end{pmatrix}$$

b) In general: given random variables x_i, Taylor expanding any function $f(\underline{x})$ around the means μ_i yields

$$f(\underline{x}) \approx f(\underline{\mu}) + \sum_i \left.\frac{\partial f}{\partial x_i}\right|_{\underline{x}=\underline{\mu}} (x_i - \mu_i)$$

$$\Rightarrow V(f(\underline{x})) \approx \sum_{i,j} \left.\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}\right|_{\underline{x}=\underline{\mu}} \underbrace{\langle (x_i-\mu_i)(x_j-\mu_j)\rangle}_{=C_{ij}}$$

should μ_u be corrected?

$$= (\underline{\nabla} f)^T C (\underline{\nabla} f)\Big|_{\underline{x}=\underline{\mu}}$$

Here: $\underline{x} = (U, R)$, $f(\underline{x}) = I = U/R$, so

$$V(I) = \sigma_I^2 \approx \left[\left(\frac{\partial I}{\partial U}\right)^2 \sigma_U^2 + \left(\frac{\partial I}{\partial R}\right)^2 \sigma_R^2\right]_{U=\mu_U,\, R=\mu_R}$$

$$= \frac{1}{\mu_R^2}\sigma_U^2 + \frac{\mu_U^2}{\mu_R^4}\sigma_R^2 \approx 1{,}9 \cdot 10^{-9}\,A^2$$

$$\Rightarrow I = \frac{\mu_U}{\mu_R} \pm \sigma_I \approx (1200 \pm 50)\,\mu A$$

c) The malfunction introduces systematic errors, so the new measurements are

$$\tilde{U} = U + S_U, \quad \tilde{R} = R + S_R$$

$$\Rightarrow Cov(\tilde{U}, \tilde{R}) = \rho(\tilde{U}, \tilde{R})\,\sigma_U \sigma_R = -\frac{1}{2}\sigma_U \sigma_R$$

$$\Rightarrow \tilde{C} = \begin{pmatrix} \sigma_U^2 & -\frac{1}{2}\sigma_U\sigma_R \\ -\frac{1}{2}\sigma_U\sigma_R & \sigma_R^2 \end{pmatrix}$$

New uncertainty^2: $1{,}9 \cdot 10^{-9}\,A^2 + \frac{\mu_U}{\mu_R^3}\sigma_U\sigma_R \approx 2{,}6 \cdot 10^{-9}\,A^2$