BIALGEBRAS OF RECURSIVE SEQUENCES AND COMBINATORIAL IDENTITIES

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ABSTRACT. A recursive sequence is an infinite sequence of elements of some fixed ground field which satisfies a recursion relation of finite order. We shall investigate certain bialgebra structures on linear spaces of recursive sequences. By choosing appropriate bases for these bialgebras we show how an explicit formula for the coproduct can imply interesting combinatorial identities.

A recursive sequence is an infinite sequence of elements of some fixed ground field which satisfies a recursion relation of finite order. We shall investigate certain bialgebra structures on linear spaces of recursive sequences. By choosing appropriate bases for these bialgebras we show how an explicit formula for the coproduct can imply interesting combinatorial identities.

The simplest kind of recursive sequence is one defined by a recursion with constant coefficients. Such sequences are called *linearly recursive* and they are the subject of an extensive literature (see for example [LT], [PT], [Poort] and [Taft]). A linearly recursive sequence can be characterized by the condition that its coordinates are the coefficients of the formal power series expansion of a rational function. Such sequences arise naturally in many areas of mathematics. For example, the Hilbert function of a graded module over a polynomial ring is a linearly recursive sequence. In combinatorics linearly recursive sequences are the coefficient sequences for the class of rational generating functions (see below) which are the solutions to many interesting enumeration problems [St1, chapter 4].

Recursive sequences of a more general sort are ubiquitous in the theory of special functions [A, chapter 2]. The coefficients of any hypergeometric series satisfy a first order recursion with polynomial coefficents and the classical hypergeometric functions are particular instances of hypergeometric series.

Recursive sequences which satisfy higher order polynomial recursions play an important role in enumerative combinatorics [St2, chapter 6].

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It is often possible to solve an enumeration problem by deriving an expression for a generating function $f(x)$. This is a function whose power series expansion $f(x) = \sum f_n x^n$ has the property that the coefficient f_n is the number of objects being enumerated which have "size" n. One wants to know whether there is some finite procedure for computing f_n. The theory of *differentiably finite (D-finite) power series* [St3] [St2, chapter 6] investigates this question. The coefficients of a D-finite power series satisfy a finite order recursion with polynomial coefficients and hence each coefficient can be computed in a finite number of steps. Such sequences are called *polynomially recursive*.

The theory of D-finite power series is also concerned with determining the class of operations on power series which preserve the property of being D-finite [L][St2][St3]. For example, one knows that the D-finite power series form a subring of the ring of formal power series [St2, theorem 6.4.9] but that the multiplicative inverse of a D-finite series need not be D-finite [St3, example 2.5]. The problem of characterizing the units in the ring of D-finite power series remains unsolved. On the other hand we can provide the set of D-finite series with a different product, namely the coordinate-wise or Hadamard product, and under this product the D-finite series again constitute an algebra (see [St2, theorem 6.4.12] and section 2 below). In this situation one knows how to characterize the units [Put, proposition 4.5]: they are precisely those sequences which are eventually the "interleaving" of a finite number of hypergeometric sequences (a hypergeometric sequence is one which satisfies a first order recursion with polynomial coefficients).

The research described below was originally motivated by the search for a Hopf-algebraic proof of this last result characterizing the units in the (Hadamard) algebra of polynomially recursive (i.e. D-finite) sequences. We had hoped to generalize the Hopf-algebraic argument [LT] which allows one to characterize the units in the algebra of linearly recursive sequences. This argument proceeds by first embedding the bialgebra of linearly recursive sequences into the Hopf algebra which is the continuous dual of the group algebra of the integers. It soon became obvious that there is no analogous embedding of the bialgebra of polynomially recursive sequences (see [St3, section 3] for a detailed discussion of this point) and so our attempt foundered.

However, in the process of defining and exploring the bialgebra structure on the set of polynomially recursive sequences we noticed the interesting phenomenon which is the subject of this paper. The coproduct of a polynomially recursive sequence, if expressed in terms of what we now call a *sequence-specific basis* for the bialgebra, implies combinatorial identities among the elements of the sequence. In simple cases these identities are well known. In general one wonders whether the identities obtained from the coproduct in this way exhaust the set of all possible combinatorial identities among coordinates of the sequence. We don't know the answer to this last question.

We shall now describe the contents of this paper.

Section 1 contains necessary background material on linearly recursive sequences. We recall how the coproduct of a sequence in the bialgebra of linearly recursive sequences implies identities which must hold among the coordinates of the sequence. Subsequent developments proceed by analogy with the linearly recursive case.

In section 2 we establish a simple but reasonably general framework for defining recursive sequences and for studying their algebraic properties. A set of recursive sequences is specified by first identifying the skew polynomial ring A which consists of the recursions of interest. A *weakly A recursive sequence* is a sequence which satisfies a recursion in A. For example A might consist solely of the constant coefficient recursions. The weakly A recursive sequences then are the linearly recursive sequences. On the other hand A might consist of all finite order recursions with polynomial coefficients. The weakly A recursive sequences in this case are just the polynomially recursive sequences, i.e. the coefficient sequences of D-finite power series. Our framework also encompasses other kinds of recursive sequences such as the Gaussian recursives. These relate to the polynomial recursives as the so-called q-binomial coefficients relate to the ordinary binomial coefficients. We show that in our framework the set of weakly A recursive sequences, S^A, is always an algebra under the Hadamard product and is often an A module as well.

In section 3 we find that the algebra S^A of weakly A recursive sequences has another interesting structure. It supports a coproduct compatible with its algebra structure. However, the target for this coproduct is not the tensor product, $S^A \otimes S^A$, but rather the completed tensor product $S^A \widehat{\otimes} S^A$. We show how to make sense of this by providing S^A with a linear topology. We then explain how S^A thus becomes a topological bialgebra. For our purposes the essential difference between an ordinary bialgebra and a topological bialgebra is that the topology allows one to define a coproduct taking values in a completed tensor product, thus defining the structure of a topological coalgebra in the sense of Takeuchi [Tak].

The existence of the coproduct on S^A is in some sense a purely formal consequence of the definitions. Of much more interest is the problem of expressing the coproduct of a given recursive sequence explicitly in terms of an appropriately chosen basis for S^A. In section 3 we define two apparently distinct concepts of what might plausibly be meant by a topological basis for our topological bialgebras and show that these alternatives are in fact equivalent.

In section 4 we develop methods for constructing topological bases for the topological bialgebra S^A. These bialgebras always have a trivial topological basis consisting of the standard dual basis of unit sequences. However, if a coproduct of a sequence is expressed in terms of a trivial basis no interesting identities result. To get interesting identities one must first find a basis consisting of sequences whose coordinates can be expressed as simple functions of the coordinates of the original sequence. We call such a topological basis a *sequence-specific basis*. The sequence-specific bases of most interest are the *hypergeometric bases*. We show how to construct a hypergeometric basis corresponding to any recursive sequence by using the Casorati determinant which arises in the theory of linear difference equations. (The Casorati is the discrete analog of the Wronskian in the theory of linear differential equations.) We shall derive an explicit formula for the coproduct with respect to this basis by using a generalized version of the binomial inversion method of combinatorics.

In section 5 we illustrate these results by deriving combinatorial identities from the coproducts of some simple sequences. In section 6 we explain the "calculus of coproducts". The coproduct of S^A is an algebra morphism and also a morphism of modules over the ring of constant coefficient recursions. These facts allow one to derive new identities for sequences which are algebraic combinations of sequences whose coproducts are already known.

Finally, we conclude in section 7 by discussing a direction for future research.

1. Linearly Recursive Sequences

We begin by recalling some background material from the theory of linearly recursive sequences. The development in later sections will be modeled on the linearly recursive case.

Let k be a field of characteristic 0 and denote by $k[D]$ the polynomial ring in the indeterminate D over k. We denote by S the linear dual of $k[D]$. Thus S is the vector space of infinite sequences of elements of k. We let $\langle , \rangle$ denote the bilinear form which pairs $k[D]$ with S. Thus if $p(D) = \sum p_i D^i$ and $s = (s_0, s_1, s_2, ...)$ then $\langle p, s\rangle = \sum p_i s_i$.

Note that the formula $\langle p, Ds\rangle \equiv \langle Dp, s\rangle$ defines on S the structure of a left $k[D]$ module; D acts on S by shifting sequences one position to the left, i.e. $Ds = (s_1, s_2, s_3, ...)$. We can use this module structure to focus attention on a special subset of S. Let S^0 denote the subset of S consisting of those sequences which are annihilated by some non-zero element of $k[D]$. The subset S^0 consists of the *linearly recursive* sequences, those sequences which satisfy a finite order recursion equation with constant coefficients. Since $k[D]$ is a commutative ring S^0 is a vector subspace of S.

The vector space of linearly recursive sequences has additional algebraic structure. First of all it is a coalgebra whose coproduct is the transpose of the multiplication on $k[D]$. This fact is not immediately obvious but is a consequence of the observation that if $s \in S^0$ then s annihilates an ideal of finite codimension in $k[D]$. Letting m^* denote the transpose of the multiplication map for $k[D]$ it then follows that the restriction of m^* to S^0 takes its values in $S^0 \otimes S^0$, not just in $(k[D] \otimes k[D])^*$ (compare [Sw, chapter 6]). The coalgebra S^0 is an instance of what is known as the finite or the continuous dual of the algebra $k[D]$.

It is also possible to impose an algebra structure on S^0 in such a way that the coproduct on S^0 becomes an algebra morphism and the product on S^0 becomes a coalgebra morphism, thus turning S^0 into a bialgebra (the required conditions on the unit and counit being trivially satisfied). The algebra structure we shall study is given by the Hadamard, or coordinate-wise product of sequences, i.e.$(st)_i = s_i t_i$. The fact that $k[D]$ is Noetherian easily implies that S^0 is closed under this product. (See theorem 1 and remark 4 in section 2 for the argument.)

Note also that the entire vector space S itself carries the structure of an algebra under the Hadamard product (although it is not a coalgebra in any obvious way). In section 2 we shall use this fact to define recursion equations with variable coefficients.

In [PT] the coproduct for S^0 is studied in detail. We recall these results here because they will be useful in section 6 and because they also offer a model of how to proceed in the more general case of variable coefficient recursion equations.

The fundamental theorem of coalgebras over a field asserts that each element is contained in a finite dimensional subcoalgebra [Mont, theorem 5.1.1]. Applied to the coalgebra S^0 this result takes a particularly nice form. If s is a linearly recursive sequence there exists an integer l with the following property: the subcoalgebra generated by s, C_s, has as a basis the sequences $s, Ds, D^2s, ..., D^{l-1}s$. In fact one can explicitly calculate the coproduct of s with respect to this basis as follows. Let $H(s)$ denote the $l \times l$ matrix whose ij entry is $(D^i s)_j$ for $i, j = 0, ..., l-1$. This is called the Hankel matrix associated with s. It is clearly symmetric and non-singular. Let t_{ij} denote the ij entry of $H(s)^{-1}$. Then $\Delta(s) = \sum t_{ij}(D^i s) \otimes (D^j s)$.

Note that this formula for the coproduct enables us to express each coordinate of the sequence s as a weighted sum of products of other coordinates of s. To see this first note that by definition the coproduct Δ is the transpose of the product in $k[D]$. Thinking of $\Delta(s)$ as a linear form on $k[D] \otimes k[D]$ we find that, for any fixed integers $0 \leq k \leq n$, $s_n = s(D^n) = \Delta(s)(D^k \otimes D^{n-k}) = (\sum t_{ij}(D^i s) \otimes (D^j s))(D^k \otimes D^{n-k}) = \sum t_{ij} s_{k+i} s_{n-k+j}$. Observe that this is really several distinct expressions for a coordinate s_n, one for each choice of the integer k, $0 \leq k \leq n$.

We illustrate this formula by applying it to the Fibonacci sequence $s = (1, 1, 2, 3, 5, 8, 13, ...)$ which satisfies the second order recursion $(D^2 - D - 1)s = 0$. After inverting the Hankel matrix one finds that $\Delta s = 2(s \otimes s) - s \otimes Ds - Ds \otimes s + Ds \otimes Ds$. Fix an integer $k \leq n$. We conclude that $s_n = 2s_k s_{n-k} - s_k s_{n-k+1} - s_{k+1} s_{n-k} + s_{k+1} s_{n-k+1}$.

2. Variable Coefficient Recursions

The structure of the bialgebra of linearly recursive sequences is well understood. There are however many sequences which arise in combinatorics which are not linearly recursive. The simplest example is the sequence $(n!) = (1, 1, 2, 6, 24, ...)$. We shall develop a framework within which one can study the various algebraic structures which sets of recursive sequences arising in combinatorial applications possess. We shall proceed by analogy to the linearly recursive case discussed in section 1.

Linearly recursive sequences are defined as those sequences which are annihilated by some non-zero polynomial in $k[D]$. The algebra $k[D]$ can be regarded as the commutative ring of constant coefficient recursions. Our plan now is to exhibit S as a left module over various rings of skew polynomials in the variable D each of which contains $k[D]$ as a subring. We shall call these *rings of variable coefficient recursions*. A recursive sequence will then be an element of S annihilated by some non-zero element of a ring of variable coefficient recursions.

We next define the skew polynomial rings of interest to us. Let $R \subseteq k(x)$ denote a Noetherian subring of the field of rational functions over k. Let $r \in R$ and let σ denote the algebra endomorphism of $k(x)$ defined by sending x to $r(x)$.

Definition 1. *R is called a* ring of variable coefficients *if:*

1. *$\sigma R \subseteq R$ and $k[x] \subseteq R$*
2. *$\frac{f}{g} \in R$ with f, g polynomials having no common factor implies $\frac{1}{g} \in R$.*

We then let $R\langle D;r\rangle$ denote the skew polynomial ring in D over R with twist defined by $Df = (f \circ r)D$, i.e. the commutation relations between elements of R and powers of D are all consequences of the relations $Df(x) = f(r(x))D$.

Definition 2. *The skew polynomial ring $R\langle D;r\rangle$ is called a* ring of variable coefficient recursions *if the vector space of sequences S can be given the structure of a left $R\langle D;r\rangle$ module with the following two properties:*

1. *D acts on S as the shift to the left; i.e. $(Ds)_i = s_{i+1}$*
2. *x acts on S as Hadamard multiplication by a sequence $q = (q_0, q_1, q_2, ...)$ of elements from k.*

Remark 1. *Notice that if S is a left $R\langle D;r\rangle$ module then the entire sequence q is determined by the constant q_0 and the rational function $r(x)$. To see this observe that in $R\langle D;r\rangle$ we have the relation $Dx = r(x)D$. Applying both sides of this last equation to the sequence $1 = (1, 1, 1, ...)$ we find that $q_{i+1} = r(q_i)$. Moreover, if this equation is to hold the poles of the rational function r must not lie among the coordinates of q. Conversely, it is easy to see that if left Hadamard multiplication by q gives S an R module structure then this condition relating q and r is also sufficient to define an $R\langle D;r\rangle$ module structure on S.*

Remark 2. *The coefficient ring R and the sequence q cannot be chosen independently. For example, if $\frac{1}{x} \in R$ then no coordinate of q can vanish.*

Example 1. *Polynomial Recursions. Let $R = k[x]$, $q = (0, 1, 2, 3, ...)$ and $r(x) = x + 1$. Clearly the action of R on S via $x \cdot s \equiv qs$ (Hadamard product) is well defined. The automorphism of $k(x)$ defined by $x \longmapsto x + 1 = r(x)$ leaves R invariant. Finally $r(i) = i + 1$. Therefore S has the structure of a left $R\langle D;r\rangle$ module. Note that in this example the k algebra $R\langle D;r\rangle$ can be identified with the universal enveloping algebra of the two dimensional, solvable Lie algebra.*

Example 2. *Gaussian Recursions. Let $R = k[x]$. Let $q_1 \in k$ be a non zero element which is not a root of unity. Define the sequence $q \equiv (1, q_1, q_1^2, q_1^3, ...)$. Let $r(x) = q_1 x$. The action of R on S is well defined as in the previous example. The automorphism of $k(x)$ defined by r clearly leaves $k[x]$ invariant. Finally $r(q_i) \equiv r(q_1^i) = q_1 q_1^i \equiv q_{i+1}$. Again S has the structure of a left $R\langle D;r\rangle$ module. Here the k algebra $R\langle D;r\rangle$ can be identified with the algebra of non-commutative polynomials on the quantum plane with global coordinates x and y and commutation relation $xy = q_1 yx$.*

Example 3. *Generalized q Recursions. Let $r \in k(x)$ and let $q_0 \in k$ be a constant such that the constants $q_{i+1} \equiv r(q_i)$ are all well defined and distinct . Denote by $T \subseteq k[x]$ the multiplicative subset consisting of all polynomials which have no roots in the set $\{q_i\}$. Define $R = T^{-1}k[x]$, the localization of $k[x]$ at T. R is a Noetherian subring of $k(x)$ and also satisfies condition 2 in the definition of a coefficient ring. Clearly the action of the coefficient ring R on S is well defined. It remains to check that the endomorphism of $k(x)$ defined by $x \longmapsto r(x)$ maps R into itself. This fact is a consequence of the following simple observation. Let $r(x) \equiv \frac{p_1(x)}{p_0(x)}$ with these two polynomials having no common factor. If $f \in T$ then $f(r(x)) = \frac{h(x)}{p_0(x)^N}$ for some sufficiently large integer N and moreover $h(x) \in T$.*

Now let us fix a ring $R\langle D; r\rangle$ of variable coefficient recursions. Denote this ring by A. The sequence space S is a left A module.

Definition 3. *A sequence $s \in S$ is called* weakly A recursive *if it is annihilated by some non-zero element of A. The sequence is called* A recursive *if it is annihilated by a* monic *element of A. The subset of S consisting of the weakly A recursive sequences will be denoted by S^A while the subset of A recursive sequences will be denoted by S_1^A.*

When the elements of the set $\{q_i\}$ are pairwise distinct any weakly A recursive sequence will be A' recursive for a suitable extension of the ring A. To see this suppose that s is weakly A recursive. Let R' denote the localization of R at the multiplicative subset consisting of all elements of $k[x]$ which do not vanish on the set $\{q_i\}$. As in example 3 the endomorphism of $k(x)$ defined by the rational function r which defines A must leave R' invariant. Then define $A' = R'\langle D; r\rangle$. Now let $a(x)$ denote the rational function which is the lead coefficient of the element of A which annihilates s. Because the $\{q_i\}$ are pairwise distinct there is a sufficiently large integer N such that $a\left(r^N(x)\right)$ will not vanish on $\{q_i\}$; thus this function will be a unit in R. Hence $D^N s$ (and therefore s) will be A' recursive.

Note also that since $k[D] \subseteq A$ every linearly recursive sequence is A recursive.

Theorem 1. *S_1^A is a linear subspace of S and also a subalgebra under the Hadamard product.*

Proof. Let f and g be A recursive sequences. Let M_f and M_g denote the cyclic A submodules of S they generate. (Warning: these are submodules of S, not of S^A; at this point we don't know that S^A is an A module.) Since f and g are annihilated by monic recursions in A

we conclude that M_f and M_g are actually finitely generated modules over R and also that $M_f + M_g$ and $M_f M_g$ (the latter consisting of all Hadamard products of an element from each module) are finitely generated R modules. Now note that $M_{f+g} \subseteq M_f + M_g$ and $M_{fg} \subseteq M_f M_g$. Since R is Noetherian we conclude that M_{f+g} and M_{fg} are finitely generated R modules. It follows immediately that $f + g$ and fg are annihilated by monic elements of A. ∎

Corollary 1. *S^A is a linear subspace of S and also a subalgebra under the Hadamard product.*

Proof. We first consider the case in which the $\{q_i\}$ are pairwise distinct.

We prove closure under the Hadamard product; an almost identical argument proves closure under addition. Define R' to be the localization of $k[x]$ at the multiplicative subset of polynomials which have no zeroes in the set $\{q_i\}$ and let A' denote the associated ring of variable coefficient recursions obtained by extending the ring of variable coefficients in this way. Clearly $R \subseteq R'$ and $A \subseteq A'$. Now let f and g be weakly A recursive sequences and hence weakly A' recursive sequences. The lead coefficients of the recursions which define these two sequences will not in general be units in R'. However, there is a sufficiently large integer N with the property that the sequences $D^N f$ and $D^N g$ are weakly A' recursive and killed by recursions whose lead coefficients are in fact units in R'. To see this, left multiply the recursions which kill f and g by D^N and pass the operator D^N all the way to the right side of the recursion using the commutation relations in A'. Note that the zeroes of the lead coefficients in the original recursion are moved off of the set $\{q_i\}$ in this process. Thus the lead coefficients of the recursions annihilating $D^N f$ and $D^N g$ are now units in the ring R'. Dividing each recursion by its lead coefficient we find that both $D^N f$ and $D^N g$ are elements of $S_1^{A'}$. By theorem 1 the Hadamard product $(D^N f)(D^N g) \in S_1^{A'}$. But this product is just $D^N(fg)$. By replacing D^i by D^{i+N} in this monic recursion annihilating $D^N(fg)$ we obtain a monic recursion annihilating (fg). Then clearing denominators in this monic recursion in A' gives us a (non-monic) recursion in A which kills (fg).

Next we consider the case in which there exist integers $n \geq 0, m \geq 1$ such that $q_n = q_{n+m}$. We claim that in this case $S^A = S$, i.e., that every sequence in S is weakly A recursive. Granting the claim we conclude that S^A is an algebra because S is one. To prove the claim note that because $r(q_i) = q_{i+1}$ for all i we have the equality $q_k = q_{k+jm}$ for all $k \geq n$ and $j \geq 0$. Now consider the rational function $p(x) \equiv (r^n(x) - q_n) \ldots (r^n(x) - q_{n+m-1})$. This is a non-zero element of $R \subseteq A$ which vanishes when evaluated on any element of $\{q_i\}$ by our last observation. Therefore $p(x) \cdot s = 0$ for any sequence s, showing that s is weakly A recursive and thus completing the proof of the corollary. ∎

Remark 3. *The proofs of theorem 1 and its corollary only use the assumption that R is a Noetherian subalgebra of $k(x)$ which contains the ground field k and which is invariant under the algebra endomorphism σ. For any such R we can then conclude that $R\langle D; r\rangle$ recursives and the weakly $R\langle D; r\rangle$ recursives form subalgebras of S.*

Remark 4. *Note that every linearly recursive sequence is annihilated by a monic recursion. Thus from theorem 1 and the preceding remark we conclude that the linearly recursives, i.e. the $k[D]$ recursives, form a subalgebra of S.*

The last paragraph of the proof of corollary 1 shows that if the constants $\{q_i\}$ fail to be pairwise distinct then all sequences are weakly A recursive. This result follows from the observation that one can then construct a rational function which vanishes on $\{q_i\}$ and which therefore annihilates all of S. In light of this it is natural to ask what happens if we demand that the lead coefficient of the annihilating recursion be non-zero when evaluated on the "periodic part" of $\{q_i\}$.

Theorem 2. *Suppose that there are integers $n \geq 0, m \geq 1$ such that $q_n = q_{n+m}$. Suppose given a sequence s which is annihilated by a recursion $\left(\sum_{i=0}^{I} p_i(x) D^i\right)$ in which the rational function p_I has no zeroes in the set $\{q_n, q_{n+1}, ..., q_{n+m-1}\}$. Then s is linearly recursive, i.e. it is annihilated by a non-zero element of $k[D] \subseteq A$.*

Proof. We begin the proof by first reducing to the case in which $q_0 = q_m$.

Define a ring of variable coefficient recursions $\widetilde{A} = \widetilde{R}\langle D; \widetilde{r}\rangle$ (where we take $\widetilde{R} \subseteq k(\widetilde{x})$) by letting $\widetilde{x}$ act on S by Hadamard multiplication by the sequence $\widetilde{q}$ where $\widetilde{q}_j \equiv q_{n+j}$ and by setting $\widetilde{r} \equiv r$. From the hypothesis of the theorem we have that $\widetilde{q}_0 = \widetilde{q}_m$. Note that if a sequence s is annihilated by $\left(\sum_{i=0}^{I} p_i(x) D^i\right)$ then $D^n s$ is annihilated by $\left(\sum_{i=0}^{I} p_i(\widetilde{x}) D^i\right)$. This last fact follows immediately from the observation that for any sequence t and any rational function $p \in R$ we have the equality $D^n p(x) \cdot t = p(\widetilde{x}) D^n \cdot t$. Assuming the theorem is true if $\widetilde{q}_0 = \widetilde{q}_m$ we can conclude that $D^n s$ (and therefore s) is linearly recursive.

We now prove the theorem for the case $q_0 = q_m$.

We may assume that the sequence s is A recursive, i.e. is annihilated by a monic recursion in A. For let $T \subseteq R$ denote the multiplicative subset of polynomials which do not vanish on the set $\{q_0, q_1, ..., q_{m-1}\}$. Extend the ring of variable coefficients R by replacing it by its localization $T^{-1}R$. Note that the rational function p_I is a unit in this latter ring and so the sequence s is in $S_1^{T^{-1}A}$, i.e. s is annihilated by a monic recursion in $T^{-1}A$, the ring of variable coefficient recursions obtained by extending R to its localization $T^{-1}R$. In other words, s is $T^{-1}A$ recursive. Theorem 2 will follow if we can show that s must then be linearly recursive.

Next note that the A recursive sequence s can be written as a sum of m sequences $s = \sum_{k=0}^{m-1} \lambda_k\, s$ (Hadamard product) where the sequence λ_k has a 1 as its i^{th} coordinate whenever $i = k + mj$ for some non-negative j and zeroes elsewhere. The sequences λ_k are A recursive (in fact linearly recursive) and thus by Theorem 1 the sequences $\lambda_k\, s$ are also A recursive. If we can show that the sequences $\lambda_k\, s$ are actually linearly recursive the theorem will then follow from the elementary observation that the linearly recursives are a vector subspace of S.

We can thus fix an integer $0 \leq k \leq m-1$ and assume that the i^{th} coordinate of s is 0 unless $i = k + jm$ for some non-negative integer j. Let S_k denote the subspace of S that consists of all sequences of this form. Because s is A recursive we know that s is annihilated by a monic, variable coefficient recursion. In other words we have an equation

$$\left(D^J + \sum_{j=0}^{J-1} g_j(x) D^j \right) \cdot s = 0$$

Left multiplying this last equation by a suitable power of D we may assume that J is a multiple of m.

Next we observe that the action of x on S_k is identical to the action given by Hadamard multiplication by a constant sequence all of whose coordinates equal q_k. From this and the assumption that $s \in S_k$ we easily deduce the following equality

$$\left(D^J + \sum_{\substack{j=0 \\ m\,|\,j}}^{J-1} g_j(q_k) D^j \right) \cdot s = 0$$

(where the notation $m\,|\,j$ means that m divides j). This constant coefficient recursion is non-zero and thus s is linearly recursive as desired. ∎

Theorem 2 and the discussion which precedes it show that when the constants q_i fail to be pairwise distinct the algebra of weak A recursives is identical to S and the algebra of A recursives concides with the algebra of linearly recursives. For this reason **we shall henceforth assume that the constants q_i are pairwise distinct.**

We next ask under what conditions the algebras S_1^A and S^A are left modules for the ring of variable coefficient recursions A. Note that the Hadamard subalgebra of linearly recursives S^0 is always a $k[D]$ submodule of S.

Theorem 3. *Let A be a ring of variable coefficient recursions. If $q \equiv x \cdot 1$ is an element of S_1^A (respectively S^A) then S_1^A (respectively S^A) is an A submodule of S.*

Proof. We prove the result for S_1^A; the proof for S^A is identical. If $s \in S_1^A$ then clearly so is Ds. It remains to show that $R \cdot S_1^A \subseteq S_1^A$. By theorem 1 and the hypothesis on q we know that $k[x] \cdot S_1^A \subseteq S_1^A$. Let $\frac{f(x)}{g(x)} \in R$ and assume that f and g have no common factors. We know that the sequence $f(x) \cdot 1$ is in S_1^A so again by theorem 1 it will suffice to show that the sequence $\frac{1}{g(x)} \cdot 1 \equiv \frac{1}{g(q)}$ is in S_1^A. (Note that this latter sequence is well defined because S by hypothesis is an R module and $\frac{1}{g(x)} \in R$ because R is assumed to be a ring of coefficients.) Observe that $g(x) \cdot \frac{1}{g(q)} = 1$ (both sides being sequences in S). Since the right side of this last equation is fixed by D so is the left. Thus we have the equation $(Dg(x) - g(x)) \cdot \frac{1}{g(q)} = 0$. Applying the commutation rule to the leading term of the skew polynomial in parenthesis we obtain $(g(r(x))D - g(x)) \cdot \frac{1}{g(q)} = 0$. Since R is a ring of variable coefficients we know that not only is $\frac{1}{g(x)}$ in R but so is $\frac{1}{g(r(x))}$. Thus after multiplying both sides of the last equation by $\frac{1}{g(r(x))}$ we can write $(D - \frac{g(x)}{g(r(x))}) \cdot \frac{1}{g(q)} = 0$ and conclude that $\frac{1}{g(q)}$ is in S_1^A. ∎

Let us next check the hypothesis of theorem 3 for the three examples discussed earlier in this section.

1. Polynomial recursions. The sequence $q = (0, 1, 2, 3, ...)$ is actually linearly recursive and satisfies the monic recursion $(D-1)^2 \cdot q = 0$.
2. Gaussian recursions. The sequence $q \equiv (1, q_1, q_1^2, q_1^3, ...)$ is also linearly recursive and satisfies the monic recursion $(D - q_1) \cdot q = 0$.
3. Generalized q recursions. Since the q_i are by hypothesis all distinct there is at most one index m such that $q_m = 0$. If there is no such index set $m = -1$. Let $r^l(x)$ denote the l fold composition of the rational function r with itself. Then the sequence $q =$

$(q_0, q_1, q_2, ...)$ satisfies a monic recursion which we derive as follows. First observe that the sequence q satisfies the recursion $(xD - r(x)) \cdot q = 0$. We can derive a monic recursion from this only if the coefficient x of D is a unit in R. If $q_m = 0$ then x will not be a unit. However, this is the only zero coordinate in q so by multiplying this equation by D^{m+1} and applying the commutation rule we obtain $(r^{m+1}(x)D^{m+2} - r^{m+2}(x)D^{m+1}) \cdot q = 0$. But now $r^{m+1}(x)$ is a unit in R and we obtain the desired monic recursion $(D^{m+2} - \frac{r^{m+1}(x)}{r^{m+2}(x)}D^{m+1}) \cdot q = 0$.

Thus in all three cases S^A and S_1^A are left A modules.

3. S^A as a Topological Bialgebra

In the previous section we saw that the vector space S^A of weak A recursives is an algebra under the Hadamard product and is also a left A module in the cases of interest to us. We next investigate the coalgebraic structure of S^A and show that S^A can be given the structure of a topological bialgebra.

Let V be an arbitrary vector space over k. By choosing a basis $\{b^\alpha, \alpha \in I\}$ for V we obtain a linear isomorphism $V^* \backsimeq k^{|I|}$. Give the field k the discrete topology and topologize V^* with the product topology transported from $k^{|I|}$ via this isomorphism. Thus a sequence v_n of elements of V^* converges to v if and only if, for any finite subset $I' \subseteq I$ there exists an integer $N_{I'}$ such that $v_n(b^\alpha) = v(b^\alpha)$ for all $\alpha \in I'$ and all $n \geq N_{I'}$. One can show that the resulting topology on V^* does not depend upon the basis chosen for V. We shall call this the *linearly compact topology* on the dual vector space V^*.

One knows [B, section 24] that the open subspaces of V^* in this topology are precisely the annihilators $X^\perp$ of finite dimensional subspaces X of V. The closed subspaces are all of the form $X^\perp$ for X an arbitrary subspace of V. If $Y \subseteq V^*$ is a subspace then its closure is clearly $Y^{\perp\perp}$.

Let W be another vector space and give W^* the linearly compact topology. One topologizes the tensor product $V^* \otimes W^*$ with the strongest topology making the tensor multiplication map $V^* \times W^* \to V^* \otimes W^*$ continuous (where the Cartesian product is endowed with the product topology). Upon completing the tensor product with respect to this topology one can deduce an isomorphism $(V \otimes W)^* \backsimeq V^* \widehat{\otimes} W^*$ of linearly compact vector spaces [Tak, corollary 1.14]. Note that an element of the completed tensor product is an infinite sum of tensors with the property that all but a finite number of elements in this sum vanish on any finite dimensional subspace of $V \otimes W$ .The reader should consult

[B, page 104 ff.] for an especially lucid discussion of linear topologies on tensor products and their completions.

It is important to observe that if $L : V \to W$ is a linear map then its transpose $L^* : W^* \to V^*$ is continuous when the dual spaces are topologized with the linearly compact topology; this follows immediately from the fact that L maps finite dimensional subspaces to finite dimensional subspaces and hence that L^*U is an open subspace if U is. In fact more is true. The functor that sends a vector space to its dual and a linear map to its transpose establishes an equivalence between the opposite category of the category of k vector spaces and linear maps and the category of linearly compact k vector spaces and continuous linear maps [B, proposition 24.8].

Now suppose V^* is a linearly compact vector space.

Definition 4. *The vector space V^* is a* topological bialgebra *if the predual V of V^* is a bialgebra with structure maps $m : V \otimes V \to V, \Delta : V \to V \otimes V$ and unit η and counit ε which all satisfy the standard bialgebra identities. In this case the product, coproduct, unit and counit of V^* are respectively the transposed maps $\Delta^* \circ i$, m^*, ϵ^* and η^* where the map i is the canonical inclusion $V^* \otimes V^* \to (V \otimes V)^*$.*

We justify this definition as follows. From the discussion above we know that the transpose of a linear map is a continuous map between duals with their linearly compact topologies. Moreover, this is a functorial correspondence. Since V is a bialgebra we conclude from functoriality that the transposed maps $m^* : V^* \to (V \otimes V)^* \backsimeq V^* \widehat{\otimes} V^*, \Delta^* \circ i : V^* \otimes V^* \to V^*, \eta^*, \epsilon^*$ satisfy all the usual bialgebra identities for the coproduct, product, counit and unit repectively. Moreover, these maps are all continuous maps of linearly compact vector spaces. Thus V^* deserves to be called a topological bialgebra; indeed, it is only the fact that the coproduct for V^* takes its values in the completion of a tensor product instead of the tensor product itself which prevents V^* from being an "honest" bialgebra with a topology.

If U is a subspace of the topological bialgebra V^* then it is a *subbialgebra* if it contains ε and moreover $\Delta^*(U \otimes U) \subseteq U$ and $m^*(U) \subseteq U \widehat{\otimes} U$.

Thus for our purposes a topological bialgebra is either the dual of an ordinary bialgebra together with the linear compact topology on the dual, or is a subbialgebra of such a dual. The principal example of interest to us is $V \equiv k[D]$ the ordinary polynomial ring over k with coproduct defined by $\Delta D = D \otimes D$. Then $V^* = S$, the space of infinite sequences over k and Δ^*, the product on S, is Hadamard multiplication. Note that the linearly compact topology on S also appears in a more familiar context. Identify S via a k linear isomorphism with the vector space underlying the ring of formal power series in one variable over k. Then the topology on S has as a neighborhood basis of 0 the subspaces which correspond to the powers of the maximal ideal of the power series ring under this linear isomorphism.

At this juncture we wish to adopt more standard notation for the coproduct m^* on S^A. **Henceforth this coproduct will be denoted by Δ.**

We shall be especially concerned with deriving explicit formulae for the coproduct Δ of V^* in terms of a basis. To do this we must define what we shall mean by a basis for a linearly compact vector space.

Definition 5. *Let V^* be a linearly compact vector space. A* topological basis *for V^* is a subset $\{b^\alpha | \alpha \in I\}$ having the following properties:*

1. *$\langle b^\alpha | \alpha \in I \rangle^\perp = 0$, i.e. the only subspace of V which annihilates the linear span of all the basis elements is* 0
2. *if $W \subseteq V$ is a finite dimensional subspace then $W^\perp \cap \{b^\alpha | \alpha \in I\}$ is a cofinite subset of the basis*
3. *if $0 = \sum a_\alpha b^\alpha, a_\alpha \in k$ then $a_\alpha = 0$ for all $\alpha \in I$. (Note that the sum is well defined by 2.)*

Condition 1. of this definition implies that $\langle b^\alpha | \alpha \in I \rangle^{\perp\perp} = V^*$, i.e. that every element of V^* lies in the closure of the subspace spanned by finite linear combinations of the basis elements.

Definition 6. *Let W be a dense subspace of a linear compact vector space V^*. A* topological basis *for W is a subset of W which is a topological basis for V^*.*

There is another notion of a basis for the linearly compact vector space V^* (compare [D, page 8]).

Definition 7. *A subset $\{b^\alpha | \alpha \in I\}$ of V^* is called a* pseudobasis *for V^* if there is a basis $\{e^\alpha | \alpha \in I\}$ (called the* predual basis*) for the vector space V such that $b^\alpha(e^\beta) = \delta_{\alpha\beta}$ (Kronecker delta).*

A pseudobasis is a particularly handy gadget, at least if one has an explicit handle on its predual basis. For if $f \in V^*$ then $f = \sum f(e^\alpha) b^\alpha$ and thus one obtains explicitly the coordinates of f with respect to the pseudobasis. Clearly every pseudobasis for V^* is a topological basis.

Theorem 4. *Every topological basis for V^* is a pseudobasis.*

The proof of this theorem will rest upon the following lemma.

Lemma 1. *Let $\{b^\alpha | \alpha \in I\}$ and $\{c^\alpha | \alpha \in I\}$ be two topological bases for V^*. Then the linear map L^* defined by sending $b^\alpha \mapsto c^\alpha$ is a continuous automorphism of V^*.*

Proof. (of lemma) The map in question is clearly linear and is well defined on all of V^* because it relates two topological bases; thus it is a linear automorphism. It remains to show that it is continuous. Let $U \subseteq V^*$ be an open subspace. We must show that $X \equiv L^{*-1}(U)$ is open in V^*. We shall do this by showing that X is closed and that $X^\perp$ is finite dimensional; because X is closed we know $X^{\perp\perp} = X$ and so X will be open too because it will be the annihilator of a finite dimensional subspace of V.

Consider the subspace $U \subseteq V^*$. Since it is open we know that there is a finite dimensional subspace $W \subseteq V$ such that $W^\perp = U$. Since $\{c^\alpha | \alpha \in I\}$ is a topological basis we know that there is a cofinite set $I' \subseteq I$ such that the closure, Y, of the subspace $\langle c^\alpha | \alpha \in I' \rangle$ is contained in U. Let Z be the finite dimensional subspace $U \cap \langle c^\alpha | \alpha \in I \diagup I' \rangle$. Then clearly $U = Y + Z$ is the sum of a closed subspace and a finite dimensional one.

Since L^* is a linear isomorphism we know that $L^{*-1}Z$ is finite dimensional. Moreover $L^{*-1}Y$ is the closure of the subspace spanned by $\{b^\alpha | \alpha \in I'\}$ by the construction of L. Therefore $L^{*-1}U = L^{*-1}Y + L^{*-1}Z$ is the sum of a closed subspace and a finite dimensional subspace, hence closed. Finally, condition 1. in the definition of a topological basis ensures that $(L^{*-1}Y)^\perp$ and hence $(L^{*-1}U)^\perp$ is finite dimensional. ∎

Proof. (of theorem) Let $\{f^\alpha | \alpha \in I\}$ be a basis for V and let $\{c^\alpha | \alpha \in I\}$ be the pseudobasis of V^* defined by the relation $\left\langle f^\beta, c^\alpha \right\rangle = \delta_{\alpha\beta}$ (Kronecker delta). Let $\{b^\alpha | \alpha \in I\}$ be the topological basis for V^* which we want to show is also a pseudobasis. By the lemma we have a continuous, linear automorphism of V^* defined by $b^\alpha \mapsto c^\alpha$ which we again denote by L^*. Now by [B, proposition 24.8] we know that there is a categorical equivalence between the opposite category of k vector spaces and the category of linearly compact k vector spaces. Thus we conclude that there is a linear isomorphism L of V whose transpose is L^*, i.e.$\langle v, L^*w \rangle = \langle Lv, w \rangle$ for all $v \in V$, $w \in V^*$. Define a basis for V via the equations $e^\alpha \equiv Lf^\alpha$. Then the calculation $\delta_{\alpha\beta} = \left\langle f^\alpha, c^\beta \right\rangle = \left\langle f^\alpha, L^* b^\beta \right\rangle = \left\langle Lf^\alpha, b^\beta \right\rangle = \left\langle e^\alpha, b^\beta \right\rangle$ shows that $\{b^\alpha\}$ is a pseudobasis for V^*. ∎

Theorem 5. *Both S_1^A and S^A are topological subbialgebras of S.*

Proof. Both of these subspaces are subalgebras under the Hadamard product and contain the unit $1 = (1, 1, 1...)$. The counit in each case is the functional which on $s \in S^A$ assumes the value $\langle 1, s \rangle$. We claim that $\Delta : S^A \to S^A \widehat{\otimes} S^A$ and similarly for S_1^A. There is less to this claim than meets the eye. For it is well known [Tak] that for any topological vector spaces W_1 and W_2 equipped with linear topologies one has the equality $\widehat{W_1}\widehat{\otimes}\widehat{W_2} = W_1\widehat{\otimes}W_2$ where $\widehat{W_i}$ denotes the completion of the topological vector space W_i. The linearly compact vector space S is clearly complete. Since S^A and S_1^A are subspaces of a complete vector space their completions coincide with their closures in S. Recall that the closure of a subspace X of S is just $X^{\perp\perp}$. We assert that $(S^A)^{\perp\perp} = S$ and similarly for S_1^A. To see this let f be a sequence in S^A and let $C_f \equiv \langle f, Df, D^2 f, ...\rangle \subseteq S^A$ denote the k-linear span of f and all of its shifts. Suppose $(C_f)^\perp$ contains a non-zero polynomial p. Then the sequence f is by definition a linear recursive sequence. But S^A always contains sequences which are *not* linear recursive. For such a sequence we have $(C_f)^\perp = 0$ and hence $(C_f)^{\perp\perp} = S$, i.e. C_f is dense in S. It follows that $(S^A)^\perp = 0$ and so $(S^A)^{\perp\perp} = S$, i.e. $\widehat{S}^A = S = \widehat{S}$. ∎

4. Sequence-Specific Bases for S^A

In the last section we found that in one sense the coproduct Δ on S^A and S_1^A is trivial to compute. One simply chooses as a topological basis for $S\widehat{\otimes}S$ the set $\{e^i \otimes e^j\}$ where the sequence e^i has 1 as its i^{th} coordinate and zeroes elsewhere. Then for $s \in S^A$ we have $\Delta(s) = \sum s_{i+j} e^i \otimes e^j$. If we then proceed by analogy with the linear recursive case we find that this coproduct expansion implies only the trivial identity $s_n = s_n$. Therefore, if we wish to obtain non-trivial identities we must express the coproduct with respect to a special kind of topological basis for $S\widehat{\otimes}S$. The basis we employ must consist of sequences which themselves are closely related to the sequence s. We shall loosely term such a basis a *sequence-specific basis.*

The first candidate for a sequence-specific basis is suggested by direct analogy with the linearly recursive case. If s is linearly recursive we know the subspace C_s that is the linear span of $s, Ds, D^2 s, ...$ is l dimensional for some l and that $\Delta(s) = \sum t_{ij} D^i s \otimes D^j s$ where $i, j = 1, .., l$. We also have a very explicit formula for the coefficients t_{ij} in terms of the inverse of the Hankel matrix associated with s (see section 1). Thus in this situation we can express the coproduct of s explicitly in terms of a basis for C_s which consists entirely of the shifts of s. If s is no longer linearly recursive a similar but weaker result is still true.

Theorem 6. *Suppose $s \in S^A$ is not linearly recursive. Then there is a topological basis $\{b^i\}$ for S^A and integers $m_0 < m_1 < m_2 < ...$ such that for each i, b^i is a linear combination of the sequences $D^{m_0}s, D^{m_1}s, ..., D^{m_i}s$.*

Proof. It will suffice to construct integers $m_i,\ i=0,1,2,...$ with the property that the matrix $M_i \equiv (s_{m_k+j}),\ k,j=0,...,i$ is invertible. For then the row vector $(0,0,...,0,1) \in k^{i+1}$ will be expressible as a unique linear combination of the rows of M_i. If we let $\lambda_{i0},\lambda_{i1},...,\lambda_{ii}$ denote the corresponding scalar coefficients of this linear combination then the sequence $b^i \equiv \sum_{j=0}^{i} \lambda_{ij} D^{m_j} s$ will be the i^{th} element of the basis. To see that the set $\{b^i\}$ thus defined is indeed a topological basis observe that by construction we will have the equalities $b^i_k = \delta_{ik}$ for $k \leq i$.

It remains to construct the integers m_i. We do this by induction. Since the 0 sequence is linearly recursive we know that there is an integer m_0 such that $s_{m_0} \neq 0$. Now assume that the integers m_i have been constructed for $i \leq n$.

By induction we know that the column vectors $(s_{m_o+j},...,s_{m_n+j})^T, j= 0,...,n$ are a basis for k^{n+1}. In particular the vector $(s_{m_0+n+1},...,s_{m_n+n+1})^T$ is a linear combination of these column vectors. If we let γ_j denote the unique scalars defining this linear combination we have the equation

$$s_{k+n+1} = \sum_{j=0}^{n} \gamma_j s_{k+j}, \qquad k=m_0,m_1,...,m_n \tag{4.1}$$

While this equation holds for the specified values of k it must fail to hold for infinitely many other integers. Otherwise the sequence s would be linearly recursive. Therefore there is an integer $m_{n+1} > m_n$ such that

$$a \equiv s_{m_{n+1}+n+1} - \sum_{j=0}^{n} \gamma_j s_{m_{n+1}+j} \neq 0$$

Now consider the matrix

$$\begin{bmatrix} s_{m_0} & ..... & s_{m_0+n} & s_{m_0+n+1} \\ .... & .... & .... & .... \\ s_{m_n} & ... & s_{m_n+n} & s_{m_n+n+1} \\ s_{m_{n+1}} & ... & s_{m_{n+1}+n} & s_{m_{n+1}+n+1} \end{bmatrix}$$

If we subtract γ_j times the j^{th} column of this matrix from the last column for $j=0,1,..,n$ and then apply the equations (4.1) we obtain the matrix

$$\begin{bmatrix} s_{m_0} & ..... & s_{m_0+n} & 0 \\ .... & .... & .... & .... \\ s_{m_n} & ... & s_{m_n+n} & 0 \\ s_{m_{n+1}} & ... & s_{m_{n+1}+n} & a \end{bmatrix}$$

Expanding the determinant of this matrix along its last column, noting that $a \neq 0$ and applying the inductive hypothesis shows that this determinant is non-zero as desired. ∎

This last result shows that there is an algorithmic procedure for constructing a sequence specific basis for S^A, one consisting entirely of linear combinations of shifts of s. Unfortunately such a basis will not make it easy to derive interesting combinatorial identities. The sequence (m_i) of integers will not in general show any regular behavior, and even when it does the specific linear combination of $D^{m_0}s, D^{m_1}s, ..., D^{m_i}s$ which defines the basis element b^i will not in general be any well behaved function of i. These irregularities will make it difficult to compute a predual basis (see definition 7) for $\{b^i\}$ which must exist by theorem 4. As a practical matter a sequence specific basis is only useful for deriving combinatorial identities if its predual basis can be computed explicitly; only then can the coproduct of s be computed explicitly and it is this last computation which exhibits the desired identities.

We next present a general method for constructing sequence specific, topological bases for S for which predual bases can be computed explicitly. We shall then show how to construct such a basis for any sequence $s \in S_1^A$ which satisfies a mild, non-degeneracy condition.

Let $f \in S$. We say that f is *never zero* if $f_i \neq 0$ for all i. Recall that S is a left $R\langle D; r\rangle$ module and that $x \cdot 1$ is just the sequence $(q_0, q_1, q_2, ...)$. Next define the polynomials $h_0(x) \equiv 1$ and for $i = 1, 2, 3, ...$

$$h_i(x) \equiv \frac{(x-q_0)(x-q_1)...(x-q_{i-1})}{(q_i-q_0)(q_i-q_1)...(q_i-q_{i-1})}$$

Since f is never zero the collection of sequences $\{h_i(x) \cdot f,\ i = 0, 1, 2, ...\}$ is a topological basis for S. We shall compute the predual basis for this topological basis. To do this we need to define constants $\left\{ {n \atop i} \right\}$ as follows:

$$\left\{ {n \atop i} \right\} = \begin{cases} 0 & i > n \\ 1 & i = n \\ \frac{(q_n-q_0)...(q_n-q_{n-1})}{(q_i-q_0)...(q_i-q_{i-1})(q_i-q_{i+1})...(q_i-q_n)} & 0 \leq i < n \end{cases}$$

Finally we define polynomials

$$g_i(D) \equiv \sum_{j=0}^{i} \frac{1}{f_j} \left\{ {i \atop j} \right\} D^j$$

Theorem 7. *The polynomials $\{g_i(D),\ i = 0, 1, 2, ...\}$ are the predual basis for the topological basis $\{h_i(x) \cdot f,\ i = 0, 1, 2, ...\}$.*

Proof. We begin by defining three upper triangular, infinite matrices:

$$\begin{aligned} H &\equiv (h_i(q_j)) \\ G &\equiv (\left\{ \begin{matrix} j \\ i \end{matrix} \right\}) \\ F &\equiv diag(f_0, f_1, f_2 ...) \end{aligned}$$

Note that the j^{th}column of the matrix $F^{-1}G$ is just the coefficient sequence of the polynomial $g_j(D)$. Moreover, the i^{th} row of the matrix HF is just the basis sequence $h_i(x) \cdot f$. We wish to prove the identity $HFF^{-1}G = HG = I$. In other words we wish to prove

$$\sum_{k=0}^{\infty} h_i(q_k) \left\{ \begin{matrix} j \\ k \end{matrix} \right\} = \delta_{ij}$$

First notice that the sum is finite since any term in which $k > i$ or $k > j$ vanishes. Moreover if $i > j$ then each term in the sum vanishes while if $i = j$ only the term with $k = i$ is non-zero and this term is 1. Thus it remains to show that the sum vanishes for $i < j$.

To do this we first rewrite this sum as

$$\begin{aligned} & h_i(q_j) \left[1 + \sum_{k=i}^{j-1} \frac{(q_j - q_i)(q_j - q_{i+1}) \dots (q_j - q_{j-1})}{(q_k - q_i)(q_k - q_{i+1}) \dots . (q_k - q_{k-1})(q_k - q_{k+1}) \dots . (q_k - q_{j-1})} \right] \\ \equiv \; & h_i(q_j) \left[1 + \sum_{k=i}^{j-1} \theta_k(q_j) \right] \end{aligned}$$

Now observe that the term in brackets can be regarded as a polynomial of degree $j - i - 1$ in the variable q_j (note that for each k the last term in the denominator of $\theta_k(q_j)$ cancels one term in the numerator). From the fact that $\theta_k(q_l) = -\delta_{kl}$ for $k, l = i, ..., j - 1$ we conclude that this polynomial has $j - i$ zeroes and hence must itself be the zero polynomial as desired. ∎

Remark 5. *In the polynomially recursive case where $q_i = i$ this argument is a proof of the well known binomial inversion formula* [Aig, corollary 3.38]. *The binomial inversion formula can be proven in a simpler and more conceptual way by noting that the matrices H and G are then the transition matrices between two bases for the algebra $k[D]$, the first being the basis consisting of the powers D^i and the second consisting of the powers $(D-1)^i$. One naturally wonders whether in the general case considered in theorem 7 there are two "simple" bases for $k[D]$ whose transition matrices are the matrices H and G as above. This would give a conceptually neater proof of this result.*

Corollary 2. *Let* $s \in S$. *Then* $\Delta(s) = \sum_{i,j=0}^{\infty} \langle g_i g_j, s\rangle (h_i(x) \cdot f) \otimes (h_j(x) \cdot f) \in S \widehat{\otimes} S$

Proof. We have the equation

$$\begin{aligned}\langle g_i \otimes g_j, \Delta(s)\rangle &\equiv \left\langle g_i \otimes g_j, \sum_{k,l=0}^{\infty} \alpha_{kl} (h_k(x) \cdot f) \otimes (h_l(x) \cdot f)\right\rangle \\ &= \sum_{k,l=0}^{\infty} \alpha_{kl} \langle g_i, (h_k(x) \cdot f)\rangle \langle g_j, (h_l(x) \cdot f)\rangle \\ &= \alpha_{ij}\end{aligned}$$

because the $\{g_i\}$ is the predual basis corresponding to the topological basis $\{h_i(x) \cdot f\}$ by the theorem. Since Δ is the transpose of multiplication on $k[D]$ we have the equalities $\langle g_i \otimes g_j, \Delta(s)\rangle = \langle g_i g_j, s\rangle = \alpha_{ij}$ as desired. ∎

Note that in the cases of interest to us S_1^A (respectively S^A) is an A module by theorem 2. Thus if $f \in S_1^A$ (resp. S^A) the basis elements $h_i(x) \cdot f$ will all be elements of S_1^A (resp. S^A) and hence will constitute a topological basis for the topological bialgebra S_1^A (resp. S^A) by definition 6. Hence in this situation corollary 2 expresses the coproduct of any element of S_1^A (resp. S^A) in terms of a topological basis for that topological bialgebra.

We would like to use the formula of corollary 2 to derive combinatorial identities by evaluating each side of the coproduct equality on the monomial $D^i \otimes D^j$ and then using the fact that $\langle D^i \otimes D^j, \Delta(s)\rangle = \langle D^{i+j}, s\rangle = s_{i+j}$. For this to work we certainly need to choose the sequence f so that its coordinates are of a form similar to those of s.

Suppose $s \in S_1^A$ is hypergeometric, i.e. is s a non-trivial solution to a first order recursion $(D + a(x)) \cdot s = 0$. If the rational function $a(x)$ never vanishes on the set $\{q_i,\ i = 0, 1, 2...\}$ then we can conclude that s is a never-zero sequence. Then $\{h_i(x) \cdot s\}$ is a topological basis for S_1^A to which we can apply theorem 7 and its corollary. In the next section we shall work out some explicit examples of identities which arise from hypergeometric sequences in this way.

If f is a never-zero hypergeometric sequence we shall call the associated topological basis $\{h_i(x) \cdot f\}$ a *hypergeometric basis*. Now suppose $s \in S_1^A$ satisfies a recursion of order $n > 1$. Can one still find a hypergeometric basis for S_1^A whose associated hypergeometric sequence f bears a close relation to s? An affirmative answer to this question is provided by the Casorati determinant from the theory of linear difference equations.

To state this result let us suppose that $s \in S_1^A$ satisfies a recursion of order n i.e. that $(D^n + a_1(x) D^{n-1} + ... + a_n(x)) \cdot s = 0$. Let $t_1, ...t_n$ be a set of n linearly independent solutions to this recursion. Consider the $n \times n$ matrix of sequences $(D^{i-1} t_j)$ for $i, j = 1, ..., n$ and let v denote the sequence that is the determinant of this matrix.

Theorem 8. *The sequence v is hypergeometric and satisfies the recursion*

$$\left(D + (-1)^{n+1} a_n(x)\right) \cdot v = 0.$$

Proof. For the reader's convenience we record the standard proof [M, page 39]. Consider the sequence $Dv = \det(D^i t_j)$ (here we are using the fact that the shift of a Hadamard product of sequences is the Hadamard product of the shifts of the sequences). Now the last row of the matrix $(D^i t_j)$ can be rewritten as a sum of n row vectors by applying the relations (for each j)

$$D^n t_j = (-1)\left(a_1(x) D^{n-1} t_j + ... + a_n(x) \cdot t_j\right)$$

Thus $\det(D^i t_j)$ can be expressed as the sum of n determinants all of which vanish **except** the one corresponding to the terms $a_n(x) \cdot t_j$. This single non-zero determinant is easily seen to be $(-1)^n a_n(x) \cdot v$ thus showing that v satisfies the desired recursion relation and so completing the proof. ∎

Corollary 3. *If the rational function $a_n(x)$ never vanishes on the set $\{q_i,\ i = 0, 1, 2...\}$ then $\{h_i(x) \cdot v\}$ is a hypergeometric basis for S_1^A*

Proof. For the sequence v is a hypergeometric sequence which, under the hypothesis of the corollary, is never zero. ∎

5. Examples

In this section we illustrate the results of section 4 by showing how the coproduct expansions of the hypergeometric sequence $s \equiv (n!)$ and related sequences imply some well known combinatorial identities. We end with an example which shows how this implication may be reversed by deriving a coproduct expansion from a combinatorial identity.

We shall begin by considering the ring of variable coefficient recursions defined by $R\langle D; r\rangle \equiv k[x]\langle D; x+1\rangle \equiv A$ where we let x act on S as Hadamard multiplication by the sequence $(0, 1, 2, ...)$. Thus we shall be considering the polynomially recursive sequences S^A.

Our sequence $(n!)$ satisfies the recursion $(D-(x+1))\cdot(n!)=0$. The hypergeometric basis associated with $(n!)$ is

$$\left\{1, x\cdot(n!), \frac{x(x-1)}{2}\cdot(n!), \frac{x(x-1)(x-2)}{3!}\cdot(n!), \ldots\right\}$$

Note that the n^{th} coordinate of the k^{th} basis sequence is just $\binom{n}{k}n!$. Continuing with the notational conventions of the previous section let us denote the k^{th} basis sequence by $h_k(x)\cdot(n!)$. Note that the sequence $h_k(x)\cdot(n!)$ is also hypergeometric and weakly polynomially recursive in that it satisfies the recursion

$$(h_k(x)D-(x+1)(h_k(x+1)))\,h_k(x)\cdot(n!)=0$$

Note that for $k\geq 1$ this sequence has an initial segment of 0 coordinates and hence has no associated hypergeometric basis. Thus we shall compute the coproduct expansion of each sequence $h_k(x)\cdot(n!)$ with respect to the hypergeometric basis $\{h_l(x)\cdot(n!)\}$ which is associated with the never-zero sequence $(n!)$. To state the result let us define the polynomial $\rho(u,v,w)\equiv 1-uv-w(u+v+2uv)$.

Theorem 9. $\Delta(h_k(x)\cdot(n!))=\sum_{i,j=0}^{\infty}\alpha_{ijk}\,(h_i(x)\cdot(n!))\otimes(h_j(x)\cdot(n!))$ *where α_{ijk} is the coefficient of $u^iv^jw^k$ in the formal power series expansion of the rational function $\frac{1}{\rho}$.*

Proof. (Sketch) The proof is an extended but straightforward computation using the standard combinatorial tricks of formal power series summation. We consider the formal power series

$$\lambda(u,v,w)\equiv\sum_{i,j,k=0}^{\infty}\alpha_{ijk}u^iv^jw^k$$

The results of section 4 give us an explicit formula for the coefficients α_{ijk}. To see this first note that by theorem 7 the predual basis which corresponds to the topological basis $\{h_k(x)\cdot(n!)\}$ is $\{g_i(D)\}$where the latter polynomials are given by the formula

$$g_i(D)=\sum_{j=0}^{i}\frac{(-1)^{i-j}}{j!}\binom{i}{j}D^j$$

From Corollary 2 we conclude that

$$\begin{aligned}\alpha_{ijk} &= \langle g_ig_j,\ h_k(x)\cdot(n!)\rangle\\ &= \sum_{n,m:n+m=0}^{n+m=i+j}(-1)^{i+j-n-m}\binom{i}{m}\binom{j}{n}\binom{n+m}{k}\frac{(n+m)!}{n!\,m!}\end{aligned}$$

If we substitute this into the formula for λ above we obtain an unpleasant expression involving a summation over five separate indices i, j, k, m, n. The trick now is to reverse the order of summation so that we hold m and n fixed and sum first over i, j and k. One obtains closed form expressions for this sum over i, j and k by repeatedly applying the standard identity [W, p. 53]

$$\sum_n \binom{n+k}{k} z^n = \frac{z^k}{(1-z)^{k+1}}$$

Next one sums over the remaining indices m and n by applying the binomial identity and finally the standard identity $\frac{1}{1-z} = \sum z^i$. At the end of this long series of manipulations one finds that $\lambda = \frac{1}{\rho}$. ∎

Example 4. *Let $k = 0$ and consider our original sequence $(n!)$. We find that $\frac{1}{\rho(u,v,0)} = \sum_{i=0}^{\infty} (uv)^i$. Thus by theorem 9*

$$\Delta((n!)) = \sum_{j=0}^{\infty} (h_j(x) \cdot (n!)) \otimes (h_j(x) \cdot (n!))$$

Use the fact that Δ is the transpose of multiplication on $k[D]$ to evaluate the left side of this expression on $D^p \otimes D^q$, thus obtaining $(p+q)!$ Next evaluate the right side of the expression on $D^p \otimes D^q$ to obtain $\sum_j \binom{p}{j} p! \binom{q}{j} q!$. Divide the left and the right side by $(p!\,q!)$ and thus obtain the well known Chu-Vandermonde identity

$$\binom{p+q}{p} = \sum_j \binom{p}{j} \binom{q}{j}$$

Example 5. *Let $k = 1$. A simple calculation shows that the coefficient of w in the power series expansion of $\frac{1}{\rho(u,v,w)}$ is given by the expression*

$$\sum_{j=1}^{\infty} j (uv)^{j-1} (u + v + 2uv)$$

The sequence $h_1(x)(n!)$ *is the sequence* $(n(n!))$. *Thus by theorem 9 we have*

$$\begin{aligned}\Delta(n(n!)) &= \sum_{j=1}^{\infty} 2j\,(h_j(x)\cdot(n!))\otimes(h_j(x)\cdot(n!)) \\ &\quad+\sum_{j=1}^{\infty} j\,(h_{j-1}(x)\cdot(n!))\otimes(h_j(x)\cdot(n!)) \\ &\quad+\sum_{j=1}^{\infty} j\,(h_j(x)\cdot(n!))\otimes(h_{j-1}(x)\cdot(n!))\end{aligned}$$

Evaluating each side of this expression on $D^p\otimes D^q$ *as in the previous example and then dividing each side by* $p!\,q!$ *we obtain the identity*

$$(p+q)\binom{p+q}{q}=\sum_{j=1}^{\infty}\left\{2j\binom{p}{j}\binom{q}{j}+j\binom{p}{j}\binom{q}{j-1}+j\binom{p}{j-1}\binom{q}{j}\right\}$$

We end this section with a simple example which illustrates in a different way the intimate connection between coproducts and combinatorial identities. Instead of using the coproduct formula to derive a combinatorial identity we shall derive the coproduct formula from a combinatorial identity.

Here we shall work in the ring of Gaussian recursions (example 2 in section 2).

We first recall some standard "q" notation. For any integer $n>0$ define $(n)_q\equiv\frac{q^n-1}{q-1}$ and $(n)!_q\equiv(1)_q(2)_q\ldots(n)_q$. Define the q-binomial coefficient $\binom{n}{k}_q\equiv\frac{(n)!_q}{(k)!_q(n-k)!_q}$. Now let the sequence s have as its n^{th} coordinate $(n)!_q$. Define sequences t_k , $k=0,1,2...$ by setting the n^{th} coordinate of t_k to be $\binom{n}{k}_q(n)!_q$. Note that the sequences s and t_k are the Gaussian analogues of the polynomially recursive sequences $(n!)$ and $h_k(x)\cdot(n!)$ which were discussed earlier in this section.

Example 6. *We wish to compute* $\Delta((n)!_q)$. *Example 4 leads us to conjecture the formula*

$$\Delta((n)!_q)=\sum_{j=0}^{\infty}q^{j^2}t_j\otimes t_j$$

To check whether this guess is correct evaluate each side on $D^m \otimes D^l$. We obtain

$$(m+l)!_q = \sum_{j=0}^{\infty} q^{j^2} \binom{m}{j}_q (m)!_q \binom{l}{j}_q (l)!_q$$

Upon dividing each side by $(m)!_q\,(l)!_q$ we find that

$$\binom{m+l}{l}_q = \sum_{j=0}^{\infty} q^{j^2} \binom{m}{j}_q \binom{l}{j}_q$$

This is a form of the q Chu-Vandermonde identity. We can prove this identity directly by a simple calculation in the ring of variable coefficient Gaussian recursions. Recall first the q binomial identity

$$(x+D)^n = \sum_{k=0}^{n} \binom{n}{k}_q x^k D^{n-k}$$

The desired result now follows upon applying this to the equation

$$(x+D)^{m+l} = (x+D)^m\,(x+D)^l$$

and equating the coefficients of $x^m D^l$ on the right and left hand sides using the commutation relation $Dx = qxD$.

6. The Calculus of Coproducts

In this section we offer some simple observations which can aid the calculation of coproduct expansions.

We begin by observing that since S^A is a bialgebra its coproduct Δ is an algebra morphism. Thus if $s, t \in S^A$ we have the equality $\Delta(st) = \Delta(s)\,\Delta(t)$.One should note that each of these three coproducts is in general an infinite sum of tensors which converges in the linearly compact topology on $S^A \widehat{\otimes} S^A$. Since S^A is a topological bialgebra the product of the two infinite sums on the right side of the last equation is well defined, i.e. it is again a convergent, infinite sum of tensors. This is true even though this product will not in general express $\Delta(st)$ in terms of a topological basis for $S^A \widehat{\otimes} S^A$.

An important special case of this last observation arises when attempting to calculate the coproduct of a sequence of the form $a(x) \cdot s$ where $a(x) \in k[x]$. For concreteness sake lets suppose that we are studying polynomial recursions so that $A = k[x]\langle D; x+1\rangle$ and $q_i = i$. Now $a(x) \cdot s$ is just the Hadamard product of the sequence s and the sequence $(a(0), a(1), a(2), ...) \equiv (a(i))$. This latter sequence is actually linearly recursive because the sequence $(0, 1, 2, ...)$ is linearly recursive and the linearly recursives form a subalgebra of S^A. In section 1 we explained the procedure for calculating the coproduct of a linearly recursive. Upon applying it to the sequence (i) we obtain

$$\Delta\left((i)\right) = -2\,(i)\otimes(i) + (i+1)\otimes(i) + (i)\otimes(i+1)$$

From this we conclude that

$$\begin{aligned}\Delta\left(a\left(x\right)\cdot s\right) &= \Delta\left(a\left(i\right)\right)\Delta\left(s\right)\\ &= a\left(\Delta\left(i\right)\right)\Delta\left(s\right)\end{aligned}$$

where the expression $a\left(\Delta\left(i\right)\right)$ has the obvious meaning as a finite sum of products in the algebra $S^A\otimes S^A$.

Note that we could have used this procedure to calculate the coproduct of $x\cdot(n!)$, a calculation which was done using a different method in example 5.

Next we observe that not only is Δ an algebra morphism but it is also a morphism of $k\left[D\right]$ modules where $S^A\widehat{\otimes}S^A$ is a $k\left[D\right]$ module by letting D act as the shift on the left factor, i.e. $D\cdot(s\otimes t)\equiv(Ds)\otimes t$. To see this simply note the equalities

$$\begin{aligned}\left\langle D^i\otimes D^j,\Delta\left(Ds\right)\right\rangle &= \left\langle D^{i+j},Ds\right\rangle\\ &= \left\langle D^{i+j+1},s\right\rangle\\ &= \left\langle D^{i+1}\otimes D^j,\Delta\left(s\right)\right\rangle\\ &= \left\langle D^i\otimes D^j,D\cdot\Delta\left(s\right)\right\rangle\end{aligned}$$

This gives us a procedure for calculating the coproduct of Ds once we know the coproduct of s.

For example, suppose we have an expression of the form

$$\Delta\left(s\right)=\sum_{i,j}\alpha_{ij}\left(h_i(x)\cdot f\right)\otimes\left(h_j\left(x\right)\cdot f\right)$$

where f is itself a hypergeometric sequence which satisfies the recursion $\left(D-a\left(x\right)\right)\cdot f=0$. Then we find that

$$\Delta\left(Ds\right)=\sum_{i,j}\alpha_{ij}\left(h_i(x+1)\,a\left(x\right)\cdot f\right)\otimes\left(h_j\left(x\right)\cdot f\right)$$

If we specialize to the polynomial recursive case in which $h_i\left(x\right)=\frac{(x)(x-1)\ldots(x-i+1)}{i!}$ we see that $h_i\left(x+1\right)=\frac{(x+1)}{i}h_{i-1}\left(x\right)$. Thus in this case we have the equality

$$\Delta\left(Ds\right)=\sum_{i,j}\alpha_{ij}\left(\frac{(x+1)}{i}a\left(x\right)h_{i-1}\left(x\right)\cdot f\right)\otimes\left(h_j\left(x\right)\cdot f\right)$$

7. Concluding Remarks

We have argued in this paper that there is a close connection between the bialgebra structure on S^A and combinatorial identities. The fact that such identities arise naturally as consequences of coproduct equalities suggests an obvious question. Do "all" combinatorial identities arise in this way?

To make sense of this question one must first make precise the notion of combinatorial identity. The work of Zeilberger on combinatorial identities defined by holonomic functions provides a way of doing this.

In [Z] Zeilberger considers identities of the form

$$\sum_{k_1,...,k_m} F(n, k_1, ..., k_m) = p(n)$$

where both sides of this equation are required to be holonomic functions of discrete variables. In the case of the function F this means essentially that F satisfies $m+1$ independent recursions with polynomial coefficients and is thus uniquely determined by a finite number of initial conditions. Identities of this type are known as holonomic function identities. If in addition both side of the identity are hypergeometric functions of discrete variables (for F this means that holding any m coordinates constant and allowing the remaining one to vary defines a hypergeometric sequence) we say that the identity is a combinatorial identity.

We can now pose our problem more precisely. Suppose we are given a combinatorial identity in the sense just described. We ask whether we can find an integer k, a polynomially recursive sequence s and a topological basis $\{b^i\}$ for the bialgebra of polynomial recursive sequences with the following property. If we express the iterated coproduct $\Delta^k s$ in terms of (tensor products of elements of) the basis $\{b^i\}$ and then evaluate $\Delta^k s$ on $(k+1)$ fold tensor products of powers of D we obtain the given combinatorial identity.

We don't have a solution to this problem but we think that results along these lines would shed light on the algebraic origins of combinatorial identities.

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