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Abstract

We investigate the category of Eilenberg-Moore algebras for the Giry monad associated with stochastic relations over Polish spaces with continuous maps as morphisms. The algebras are characterized through convex partitions of the space of all probability measures. Examples are investigated, and it is shown that finite spaces usually do not have algebras at all.

Keywords: Stochastic relations, Giry monad, Eilenberg-Moore algebras, computation through monads, Kleisli construction.

1 Introduction

Modelling a computation through a monad (as suggested e.g. by E. Moggi [12]), one represents state transitions or the transformation from inputs a to outputs b through a morphism $a \to \mathbf{T}b$ with $\mathbf{T}$ as the functor underlying the monad. Working in a probabilistic setting, an input from an input space X is in this way associated with a probability distribution $K(x)$ on the output space Y. Here $K : X \to \mathbf{P}(Y)$ is a morphism for the probability monad, in which the functor assigns a space its probabilities. But we now have only a distribution of the outputs, not the outputs proper. What is needed for this is a map $h : \mathbf{P}(Y) \to Y$ that would transform a distribution into an output. In this way a computation $K : X \to \mathbf{P}(Y)$ may be transformed into an input-output system $h \circ K : X \to Y$.
When h complies with the rest of the monad, the pair $\langle Y, h\rangle$ is called an Eilenberg-Moore algebra for this monad. Structurally, these algebras help to construct an adjunction for which the monad is just the given one [11, Theorem VI.2.1]. In fact, this adjunction and the one constructed through the Kleisli category form in some sense the extreme points in a category of all adjunctions from which the given monad can be recovered [11, Theorem VI.5.3]. Thus it is of algebraic interest to identify these algebras in general, and in particular to the probability functor. It has as a Kleisli construction stochastic relations and is in this sense quite similar to the powerset functor. For the latter functor the algebras are completely characterized,

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but the stochastic side of the analogy is not explored yet: We do not know too much about the algebras for this monad. This paper makes an attempt at improving this situation by providing characterizations for these algebras under the assumption of continuity. We work in the category of Polish spaces (these spaces are explained in Section 2) with continuous maps as morphisms. In this category the algebras for the Giry monad are identified, and the category of all algebras is investigated. The natural approach is to think of these algebras in terms of an equivalence relation which may be thought to identify probability distributions, and to investigate either these relations or the partitions associated with them. Both cases lead to a characterization of the algebras and their morphisms, so that the category of all algebras can be identified.
The world of stochastic relations being somewhat stronger polychromatic than the one of their step-twins, the set-theoretic relations, some examples for these algebras for some well known spaces are used to illustrate this variety. The cases include finite spaces, and some geometrically appealing spaces like the unit ball in higher dimensions. The negative result for the finite case probably comes as a surprise.

Related Work The monad on which the present investigation is based was originally proposed and investigated by M. Giry [9] in an approach to provide a categorical foundation of Probability Theory. The functor on which it is based assigns each measurable space all probabilities defined on its σ-algebra, it is somewhat similar to the functor assigning each set its power set on which the monad investigated by Manes is based. While the Kleisli construction for the latter leads to relations based on sets, the Kleisli construction for the former one leads to stochastic relations as a similar relational construction. This point of view was emphasized by P. Panagaden in [14] when pointing out similarities between set based and probability based relations. It was extended further in [1]. In [3] this aspect is elaborated in depth by showing how a software architecture can be modelled using a monad as the basic computational model [12]; the monad is shown to subsume both the Manes and the Giry monad as special cases. Stochastic relations turned out to be a fruitful field for investigations [1, 13, 4, 5, 6] in particular in such areas as labelled Markov transition systems and modelling stochastically algebraic aspects of modal logic. A development of the semantics of probabilistic programs in terms of the Eilenberg-Moore algebras on probabilistic powerdomains is presented in [10].

Organization We define the objects we are dealing with in Section 2, in particular, the space of all probability measures on a Polish space is introduced together with the weak topology that renders it a Polish space. The Giry monad is also introduced. Section 3 is devoted to the characterization of the algebras for this monad through partitions and smooth equivalence relations. It is shown that the category of algebras is isomorphic to the corresponding categories. Some examples are given in Section 4, indicating that the search for algebras in the – usually easily dealt with — finite case is somewhat hopeless. Section 5 proposes further work along the lines developed here.

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2 The Giry Monad

In this section the constructions underlying the Giry monad are collected. We remind the reader of Polish spaces, of the topology of weak convergence on the space of all probabilities on a Polish space, and finally of the monad investigated by Giry.
Let X be a Polish space, i.e., a separable metric space for which a complete metric exists, and denote by $\mathbf{P}(X)$ the set of all probability measures on the Borel sets $\mathcal{B}(X)$ of X. The *weak topology* on $\mathbf{P}(X)$ is the smallest topology which makes $\tau \mapsto \int_X f \; d\tau$ continuous, whenever $f \in \mathcal{C}(X) := \{g : X \to \mathbb{R} \mid g \text{ is bounded and continuous}\}$. It is well known that the discrete measures are dense, and that $\mathbf{P}(X)$ is a Polish space with this topology [15, Section II.6]. A sequence $(\tau_n)_{n\in\mathbb{N}}$ with $\tau_n \in \mathbf{P}(X)$ converges to $\tau_0 \in \mathbf{P}(X)$ in this topology (indicated by $\tau_n \rightharpoonup_w \tau_0$) iff

$$\forall f \in \mathcal{C}(X) : \int_X f \; d\tau_n \to \int_X f \; d\tau_0$$

holds. We will assume throughout that $\mathbf{P}(X)$ is endowed with the weak topology.
Denote by $\mathfrak{Pol}$ the category of Polish spaces with continuous maps as morphisms. $\mathbf{P}$ assigns to each Polish space X the space of probability measures on X; if $f : X \to Y$ is a morphism in $\mathfrak{Pol}$, its image $\mathbf{P}(f) : \mathbf{P}(X) \to \mathbf{P}(Y)$ is defined through

$$\mathbf{P}(f)(\tau)(B) := \tau\left(f^{-1}[B]\right),$$

where $\tau \in \mathbf{P}(X)$ and $B \in \mathcal{B}(Y)$ is a Borel set. By virtue of the *Change of Variable Formula*

$$\int_Y g \; d\mathbf{P}(f)(\tau) = \int_X g \circ f \; d\tau$$

it is easy to see that $\mathbf{P}(f)$ is continuous. Thus $\mathbf{P} : \mathfrak{Pol} \to \mathfrak{Pol}$ is functor.
Denote by $\mu_X : \mathbf{P}(\mathbf{P}(X)) \to \mathbf{P}(X)$ the map

$$\mu_X(M)(A) := \int_{\mathbf{P}(X)} \tau(A) \; M(d\tau)$$

which assigns to each measure M on the Borel sets of $\mathbf{P}(X)$ a probability measure $\mu_X(M)$ on the Borel sets of X. Thus $\mu_X(M)(A)$ averages over the probabilities for A using measure M. Standard arguments show that

$$\int_X f \; d\mu_X(M) = \int_{ProbX} \left(\int_X f \; f\tau\right) \; M(d\tau)$$

for each measurable and bounded map $f : X \to \mathbb{R}$.
The map μ_X is a morphism in $\mathfrak{Pol}$, as the following Lemma shows.

Lemma 1 $\mu_X : \mathbf{P}(\mathbf{P}(X)) \to \mathbf{P}(X)$ *is continuous.*

Proof Let $(M_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbf{P}(\mathbf{P}(X))$ with $M_n \rightharpoonup_w M_0$, then we get for $f \in \mathcal{C}(X)$ through the Change of Variable Formula, and because

$$\tau \mapsto \int_X f \; d\tau$$

is a member of $\mathcal{C}(\mathbf{P}(X))$, the following chain

$$\begin{aligned}\int_{\mathbf{P}(X)} f \, d\mu_X(M_n) &= \int_{\mathbf{P}(X)} \left(\int_X f \, d\tau\right) \, M_n(d\tau)\\ &\to \int_{\mathbf{P}(X)} \left(\int_X f \, d\tau\right) \, M_0(d\tau)\\ &= \int_{\mathbf{P}(X)} f \, d\mu_X(M_0).\end{aligned}$$

Thus $\mu_X(M_n) \rightharpoonup_w \mu_X(M_0)$ is established, as desired. $\square$

The argumentation in [9] shows that $\mu : \mathbf{P}^2 \stackrel{\bullet}{\to} \mathbf{P}$ is a natural transformation. Together with $\eta_X : X \to \mathbf{P}(X)$, which assigns to each $x \in X$ the Dirac measure δ_x on x, and which is a natural transformation $\eta : \mathbb{1} \stackrel{\bullet}{\to} \mathbf{P}$, the triplet $\langle \mathbf{P}, \eta, \mu\rangle$ forms a monad [9]. It was originally proposed and investigated by Giry and will be referred to as the *Giry monad*. This means that these diagrams commute in the category of endofunctors of $\mathfrak{Pol}$ with natural transformations as morphisms:

$$\begin{tikzcd} \mathbf{P}^3 \arrow[r, "\mathbf{P}\mu"] \arrow[d, "\mu\mathbf{P}"'] & \mathbf{P}^2 \arrow[d, "\mu"] \\ \mathbf{P}^2 \arrow[r, "\mu"'] & \mathbf{P} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{P} \arrow[r, "\eta\mathbf{P}"] \arrow[dr, "id"'] & \mathbf{P}^2 \arrow[d, "\mu"] & \mathbf{P} \arrow[l, "\mathbf{P}\eta"'] \arrow[dl, "id"] \\ & \mathbf{P} & \end{tikzcd}$$

3 Characterizing the Algebras

The Eilenberg-Moore algebras are represented through partitions and through smooth equivalence relations, both on the respective space of probability measures. We first deal with partitions and investigate the partition induced by an algebra. This leads to a necessary and sufficient condition for a partition to be generated from an algebra which in turn can be used for characterizing the category of these algebras by introducing a suitable notion of morphisms for partitions. The second representation capitalizes on the fact that equivalence relations induced by continuous maps (as special cases of Borel measurable maps) have some rather convenient properties in terms of measurability. This is used for an alternative description of the category of all algebras.

3.1 Algebras

An *Eilenberg-Moore algebra* $\langle X, h\rangle$ for the Giry monad is an object X in $\mathfrak{Pol}$ together with a morphism $h : \mathbf{P}(X) \to X$ such that the following diagrams commute

$$\begin{tikzcd} \mathbf{P}(\mathbf{P}(X)) \arrow[r, "\mathbf{P}(h)"] \arrow[d, "\mu_X"'] & \mathbf{P}(X) \arrow[d, "h"] \\ \mathbf{P}(X) \arrow[r, "h"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "id_X"'] & \mathbf{P}(X) \arrow[d, "h"] \\ & X \end{tikzcd}$$

When talking about algebras, we refer always to Eilenberg-Moore algebras for the Giry monad, unless otherwise indicated. An *algebra morphism* $f : \langle X, h\rangle \to \langle X', h'\rangle$ between the algebras $\langle X, h\rangle$ and $\langle X', h'\rangle$ is a continuous map $f : X \to X'$ which makes the diagram

$$\begin{tikzcd} \mathbf{P}(X) \arrow[r, "h"] \arrow[d, "\mathbf{P}(f)"'] & X \arrow[d, "f"] \\ \mathbf{P}(X') \arrow[r, "h'"'] & X' \end{tikzcd}$$

commute. Algebras together with their morphisms form a category $\mathfrak{Alg}$. This construction is discussed for monads in general in [11, Chapter IV.2].

Remark: Looking aside, we mention briefly a well-known monad in the category $\mathfrak{Set}$ of sets with maps as morphisms. The functor $\mathcal{P}$ assigns each set A its power set $\mathcal{P}(A)$, and if $f : A \to B$ is a map, $\mathcal{P}(f) : \mathcal{P}(A) \to \mathcal{P}(B)$ assigns each subset $A_0 \subseteq A$ its image $f[A_0]$, thus $\mathcal{P}(f)(A_0) = f[A_0]$. Define the natural transformation $\mu : \mathcal{P}^2 \stackrel{\bullet}{\to} \mathcal{P}$ through

$$\mu_A : \mathcal{P}(\mathcal{P}(A)) \ni M \mapsto \bigcup M \in \mathcal{P}(A),$$

and $\eta : \mathbb{1} \stackrel{\bullet}{\to} \mathcal{P}$ through $\eta_X : x \mapsto \{x\}$, then the triplet $\langle \mathcal{P}, \eta, \mu\rangle$ forms a monad (the *Manes monad*). It is well known that the algebras for this monad may be identified with the complete sup-semi lattices [11, Exercise VI.2.1]. —

For the rest of this paper each free occurrence of X refers to a Polish space.

We need some elementary properties for later reference. They are collected in this Lemma:

Lemma 2 1. *Let $f : A \to B$ be a map between the Polish spaces A and B, and let $\tau = c \cdot \delta_{a_1} + (1-c)\cdot\delta_{a_2}$ be the convex combination of Dirac measures with $a_1, a_2 \in A$. Then $\mathbf{P}(f)(\tau) = c\cdot\delta_{f(a_1)} + (1-c)\cdot\delta_{f(a_2)}$.*

2. *Let τ_1, τ_2 be probability measures on X, and let $M = c\cdot\delta_{\tau_1} + (1-c)\cdot\delta_{\tau_2}$ be the convex combination of the corresponding Dirac measures in $\mathbf{P}(\mathbf{P}(X))$. Then $\mu_X(M) = c\cdot\tau_1 + (1-c)\cdot\tau_2$.*

Proof The first part follows directly from the observation that $\delta_x(f^{-1}[D]) = \delta_{f(x)}(D)$, and the second one is easily inferred from

$$\begin{aligned} \mu_X(\delta_\tau)(Q) &= \int_{\mathbf{P}(X)} \rho(Q)\ \delta_\tau(d\rho) \\ &= \tau(Q) \end{aligned}$$

for each Borel subset $Q \subseteq X$, and from the linearity of the integral. $\Box$

3.2 Convex Partitions

Assume that the pair $\langle X, h\rangle$ is an algebra, and define for each $x \in X$

$$G_h(x) := \{\tau \in \mathbf{P}(X) \mid h(\tau) = x\} \left(= h^{-1}[\{x\}]\right).$$

Then $G_h(x) \neq \emptyset$ for all $x \in X$ due to h being onto. The algebra h will be characterized through properties of the set-valued map G_h. Define the *weak inverse* $\exists R$ for a set-valued map $R : X \to \mathcal{P}(Y) \setminus \{\emptyset\}$ with non-empty images through

$$\exists R(W) := \{x \in X \mid R(x) \cap W \neq \emptyset\}.$$

for $W \subseteq Y$. If Y is a topological space, if R takes closed values, and if $\exists R(W)$ is compact in X whenever $W \subseteq Y$ is compact, then R is called *k-upper-semicontinuous* (abbreviated as k.u.s.c.). If Y is compact, this is the usual notion of upper-semicontinuity (cf. [16, Section 5.1]).
The importance of being (not Earnest, but) k.u.s.c. becomes clear at once from

Lemma 3 *Let $f : A \to B$ be a surjective map between the Polish spaces A and B, and put $G_f(b) := f^{-1}[\{b\}]$ for $b \in B$. Then f is continuous iff G_f is k.u.s.c.*

Proof A direct calculation for the weak inverse shows $\exists G_f(A_0) = f[A_0]$ for each subset $A_0 \subseteq A$. The assertion now follows from the well-known fact that a map between metric spaces is continuous iff it maps compact sets to compact sets. $\square$
Applying this observation to the set-valued map G_h, we obtain:

Proposition 1 *The set-valued map $x \mapsto G_h(x)$ has the following properties:*

1. *$\delta_x \in G_h(x)$ holds for each $x \in X$.*
2. *$\mathcal{G}_h := \{G_h(x) \mid x \in X\}$ is a partition of $\mathbf{P}(X)$ into closed and convex sets.*
3. *$x \mapsto G_h(x)$ is k.u.s.c.*
4. *Let $\sim_h$ be the equivalence relation on $\mathbf{P}(X)$ induced by the partition $\mathcal{G}_h$. If $\tau_1 \sim_h \tau_1'$ and $\tau_2 \sim_h \tau_2'$, then $(c \cdot \tau_1 + (1-c) \cdot \tau_2) \sim_h (c \cdot \tau_1' + (1-c) \cdot \tau_2')$ for $0 \le c \le 1$.*

Proof Because $\{x\}$ is closed, and h is continuous, $G_h(x) = h^{-1}[\{x\}]$ is a closed subset of $\mathbf{P}(X)$. Because h is onto, every G_h takes non-empty values; it is clear that $\{G_h(x) \mid x \in X\}$ forms a partition of $\mathbf{P}(X)$. Because h is continuous, G_h is k.u.s.c. by Lemma 3. Convexity will follow immediately from part 4.
Assume that $h(\tau_1) = h(\tau_1') = x_1$ and $h(\tau_2) = h(\tau_2') = x_2$, and observe that $h(\delta_x) = x$ holds for all $x \in X$. Using Lemma 2, we get:

$$\begin{aligned} h(c \cdot \tau_1 + (1-c) \cdot \tau_2) &= (h \circ \mu_X)(c \cdot \delta_{\tau_1} + (1-c) \cdot \delta_{\tau_2}) \\ &= (h \circ \mathbf{P}(h))(c \cdot \delta_{\tau_1} + (1-c) \cdot \delta_{\tau_2}) \\ &= h\left(c \cdot \delta_{h(\tau_1)} + (1-c) \cdot \delta_{h(\tau_2)}\right) \\ &= h(c \cdot \delta_{x_1} + (1-c) \cdot \delta_{x_2}) \end{aligned}$$

In a similar way, $h(c \cdot \tau_1' + (1-c) \cdot \tau_2') = h(c \cdot \delta_{x_1} + (1-c) \cdot \delta_{x_2})$ is obtained. This implies the assertion. $\square$
Thus $\mathcal{G}_h$ is invariant under taking convex combinations. It is a convex partition in the sense of the following definition.

Definition 1 *An equivalence relation α on $\mathbf{P}(X)$ is said to be* convex *iff $\tau_1 \,\alpha\, \tau_1'$ and $\tau_2 \,\alpha\, \tau_2'$ together imply $(c \cdot \tau_1 + (1-c) \cdot \tau_2)\ \alpha\ (c \cdot \tau_1 + (1-c) \cdot \tau_2)$ for each c with $0 \le c \le 1$. A partition of $\mathbf{P}(X)$ is called* convex *iff its associated equivalence relation is.*

Note that the elements of a convex partition form convex sets. The converse to Proposition 1 characterizes algebras:

Proposition 2 *Assume $\mathcal{G} = \{G(x) \mid x \in X\}$ is a convex partition of $\mathbf{P}(X)$ into closed sets indexed by X such that $\delta_x \in G(x)$ for each $x \in X$, and such that $x \mapsto G(x)$ is k.u.s.c. Define $h : \mathbf{P}(X) \to X$ through $h(\tau) = x$ iff $\tau \in G(x)$. Then $\langle X, h\rangle$ is an algebra for the Giry monad.*

Proof 1. It is clear that h is well defined and surjective, and that $\exists G(F) = h[F]$ holds for each subset $F \subseteq \mathbf{P}(X)$. Thus $h[K]$ is compact whenever K is compact, because G is k.u.s.c. Thus h is continuous by Lemma 3.
2. An easy induction establishes that h respects convex combinations: if $h(\tau_i) = h(\tau_i')$ for $i = 1, \ldots, n$, and if $c_i \geq 0$ with $\sum_{i=1}^n c_i = 1$, then

$$h(\sum_{i=1}^n c_i \cdot \tau_i) = h(\sum_{i=1}^n c_i \cdot \tau_i').$$

We claim that $(h \circ \mu_X)(M) = (h \circ \mathbf{P}(h))(M)$ holds for each *discrete* $M \in \mathbf{P}(\mathbf{P}(X))$. In fact, let

$$M = \sum_{i=1}^n c_i \cdot \delta_{\tau_i}$$

be such a discrete measure, then Lemma 2 implies that

$$\mu_X(M) = \sum_{i=1}^n c_i \cdot \tau_i,$$

thus

$$(h \circ \mu_X)(M) = h\left(\sum_{i=1}^n c_i \cdot \tau_i\right) = h\left(\sum_{i=1}^n c_i \cdot \delta_{h(\tau_i)}\right) = (h \circ \mathbf{P}(h))(M),$$

because we know also from Lemma 2 that

$$\mathbf{P}(h)(M) = \sum_{i=1}^n c_i \cdot \delta_{h(\tau_i)}$$

holds.
3. Since the discrete measures are dense in the weak topology, we find for $M_0 \in \mathbf{P}(\mathbf{P}(X))$ a sequence $(M_n)_{n\in\mathbb{N}}$ of discrete measures M_n with $M_n \rightharpoonup_w M_0$. Consequently, we get from the continuity of both h and μ_X (Lemma 1) together with the continuity of $\mathbf{P}(h)$

$$(h \circ \mu_X)(M_0) = \lim_{n\to\infty} (h \circ \mu_X)(M_n) = \lim_{n\to\infty} (h \circ \mathbf{P}(h))(M_n) = (h \circ \mathbf{P}(h))(M_0).$$

This proves the claim. $\square$
We have established

Proposition 3 *The algebras $\langle X, h\rangle$ for the Giry monad for Polish spaces X are exactly the convex k.u.s.c. partitions $\{G(x) \mid x \in X\}$ into closed subsets of $\mathbf{P}(X)$ such that $\delta_x \in G(x)$ for all $x \in X$ holds.*

We characterize the category $\mathfrak{Alg}$ of all algebras for the Giry monad. To this end we package the properties of partitions representing algebras into the notion of a G-partition. They will form the objects of category $\mathfrak{GPart}$.

Definition 2 *$\mathcal{G}$ is called a* G-partition for X *iff*

1. $\mathcal{G} = \{G(x) \mid x \in X\}$ *is a convex partition for* $\mathbf{P}(X)$ *into closed sets indexed by* X*,*
2. $\delta_x \in G(x)$ *holds for all* $x \in X$*,*
3. *the set-valued map* $x \mapsto G(x)$ *is k.u.s.c.*

Define the objects of category $\mathfrak{GPart}$ as pairs $\langle X, \mathcal{G}\rangle$ where X is a Polish space, and $\mathcal{G}$ is a G-partition for X. A morphism f between $\mathcal{G}$ and $\mathcal{G}'$ will map elements of $G(x)$ to $G'(f(x))$ through its associated map $\mathbf{P}(f)$. Thus an element $\tau \in G(x)$ will correspond to an element $\mathbf{P}(f)(\tau) \in G'(f(x))$.

Definition 3 *A morphism for* $\mathfrak{GPart}$ $f : \langle X, \mathcal{G}\rangle \to \langle X', \mathcal{G}'\rangle$ *is a continuous map* $f : X \to X'$ *such that* $G(x) \subseteq \mathbf{P}(f)^{-1}[G'(f(x))]$ *holds for each* $x \in X$*.*

Define the functor $F : \mathfrak{Alg} \to \mathfrak{GPart}$ by associating each algebra $\langle X, h\rangle$ its Giry partition $F(X, h)$ according to Proposition 3. Assume that $f : \langle X, h\rangle \to \langle X', h'\rangle$ is a morphism in $\mathfrak{Alg}$, and let $\mathcal{G} = \{G(x) \mid x \in X\}$ and $\mathcal{G}' = \{G'(x') \mid x' \in X'\}$ be the corresponding partitions. Then the properties of an algebra morphism yield

$$\begin{aligned}
\tau \in \mathbf{P}(f)^{-1}\left[G'(f(x))\right] &\Leftrightarrow \mathbf{P}(f)(\tau) \in G'(f(x)) \\
&\Leftrightarrow \left(h' \circ \mathbf{P}(f)\right)(\tau) = f(x) \\
&\Leftrightarrow (f \circ h)(\tau) = f(x).
\end{aligned}$$

Thus $\tau \in \mathbf{P}(f)^{-1}[G'(f(x))]$, provided $\tau \in G(x)$. Hence f is a morphism in $\mathfrak{GPart}$ between $F(X, h)$ and $F(X', h')$. Conversely, let $f : \langle X, \mathcal{G}\rangle \to \langle X', \mathcal{G}'\rangle$ be a morphism in $\mathfrak{GPart}$ with $\langle X, \mathcal{G}\rangle = F(X, h)$ and $\langle X', \mathcal{G}'\rangle = F(X, h')$. Then

$$\begin{aligned}
h(\tau) = x &\Leftrightarrow \tau \in G(x) \\
&\Rightarrow \mathbf{P}(f)(\tau) \in G'(f(x)) \\
&\Leftrightarrow h'(\mathbf{P}(f)(\tau)) = f(x),
\end{aligned}$$

thus $h' \circ \mathbf{P}(f) = f \circ h$ is inferred. Hence f constitutes a morphism in category $\mathfrak{Alg}$.
Summarizing, we have shown

Proposition 4 *The category* $\mathfrak{Alg}$ *of algebras for the Giry monad is isomorphic to the category* $\mathfrak{GPart}$ *of G-partitions.*

3.3 Smooth Relations

The characterization of algebras so far encoded the crucial properties into a partition of $\mathbf{P}(X)$, thus indirectly into an equivalence relation on that space. We can move directly to a particular class of these relations when looking at an alternative characterization of the algebras through smooth equivalence relations.

Definition 4 *An equivalence relation α on a Polish space A is called* smooth *iff there exists a Polish space B and a Borel measurable map $f : A \to B$ such that*

$$a_1 \, \alpha \, a_2 \Leftrightarrow f(a_1) = f(a_2)$$

holds.

Smooth equivalence relations are a helpful tool in the theory of Borel sets [16]. They have some interesting properties that have been capitalized upon in the theory of labelled Markov transition processes [2] and stochastic relations [4, 5, 6].
Some basic notations and constructions first: Denote for an equivalence relation α on A by A/α the factor space, i.e., the set of all equivalence classes $[a]_\alpha$, and by

$$\varepsilon_\alpha : A \to A/\alpha$$

the canonical projection. If A is a Polish space, then let $\mathcal{T}/\alpha$ be the final topology on A/α with respect to the given topology and ε_α, i.e., the largest topology on A/α which makes ε_α continuous. Clearly a map $g : A/\alpha \to B$ for a topological space B is continuous with respect to $\mathcal{T}/\alpha$ iff $g \circ \varepsilon_\alpha : A \to B$ is continuous w.r.t. the given topologies. We will need this observation in the proof of Proposition 5.
Now let $\langle X, h\rangle$ be an algebra for the Giry monad. Obviously

$$\tau_1 \, \alpha_h \, \tau_2 \Leftrightarrow h(\tau_1) = h(\tau_2)$$

defines a smooth equivalence relation on the Polish space $\mathbf{P}(X)$. Its properties are summarized in

Proposition 5 *The equivalence relation α_h is convex, each equivalence class $[\tau]_{\alpha_h}$ is closed and convex, and the factor space $\mathbf{P}(X)/\alpha_h$ is homeomorphic to X when the former is endowed with the topology $\mathcal{T}/\alpha_h$.*

Proof 1. Convexity of α_h follows from the properties of h exactly as in the proof of Proposition 1, from this, also convexity of the classes is inferred. Continuity of h implies that the classes are closed sets.
2. Define $\chi_h([\tau]_{\alpha_h}) := h(\tau)$ for $\tau \in \mathbf{P}(X)$. Then $\chi_h : X/\alpha_h \to X$ is well defined and a bijection. Let $G \subseteq X$ be an open set, then $\varepsilon_{\alpha_h}^{-1}\left[\chi_h^{-1}[G]\right] = h^{-1}[G]$. Because $\mathcal{T}/\alpha_h$ is the largest topology on $\mathbf{P}(X)/\alpha_h$ that renders ε_{α_h} continuous, and because $h^{-1}[G] \subseteq \mathbf{P}(X)$ is open by assumption, we infer that $\chi_h^{-1}[G]$ is $\mathcal{T}/\alpha_h$-open. Thus χ_h is continuous. On the other hand, if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X converging to $x_0 \in X$, then $\delta_{x_n} \rightharpoonup_w \delta_{x_0}$ in $\mathbf{P}(X)$, thus $[\delta_{x_n}]_{\alpha_h} \to [\delta_{x_0}]_{\alpha_h}$ in $\mathcal{T}/\alpha_h$ by construction. Consequently χ_h^{-1} is also continuous. $\Box$
Thus each algebra induces a G-triplet in the following sense

Definition 5 *A* G-triplet $\langle X, \alpha, \chi\rangle$ *is a Polish space X with a smooth and convex equivalence relation α on $\mathbf{P}(X)$ such that $\chi : \mathbf{P}(X)/\alpha \to X$ is a homeomorphism with $\chi([\delta_x]_\alpha) = x$ for all $x \in X$. Here $\mathbf{P}(X)/\alpha$ carries the final topology with respect to the weak topology on $\mathbf{P}(X)$ and ε_α.*

Now assume that a G-triplet $\langle X, \alpha, \chi\rangle$ is given. Define $h(\tau) := \chi([\tau]_\alpha)$ for $\tau \in \mathbf{P}(X)$. Then $\langle X, h\rangle$ is an algebra for the Giry monad: $h(\delta_x) = x$ follows from the assumption, and because $h = \chi \circ \varepsilon_\alpha$, holds, the map h is continuous. An argument very similar to that used in the proof of Proposition 1 shows that $h \circ \mu_X = h \circ \mathbf{P}(h)$ holds; this is so since α is assumed to be convex.

Definition 6 *The continuous map $f : X \to X'$ between the Polish spaces X and X' constitutes a* G-triplet morphism $f : \langle X, \alpha, \chi\rangle \to \langle X', \alpha', \chi'\rangle$ *iff these conditions hold:*

1. $\tau\,\alpha\,\tau'$ *implies* $\mathbf{P}\,(f)\,(\tau)\,\alpha'\,\mathbf{P}\,(f)\,\tau'$,
2. *the diagram*

$$\begin{tikzcd} \mathbf{P}\,(X)/\alpha \arrow[r, "\mathbf{P}\,(f)_{\alpha,\alpha'}"] \arrow[d, "\chi"'] & \mathbf{P}\,(X')/\alpha' \arrow[d, "\chi'"] \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

commutes, where

$$\mathbf{P}\,(f)_{\alpha,\alpha'}\,([\tau]_\alpha) := [\mathbf{P}\,(f)\,(\tau)]_{\alpha'}$$

G-triplets with their morphisms form a category $\mathfrak{GTrip}$.

Lemma 4 *Each algebra morphism $f : \langle X, h\rangle \to \langle X', h'\rangle$ induces a G-triplet morphism $f : \langle X, \alpha_h, \chi_h\rangle \to \langle X', \alpha_{h'}, \chi_{h'}\rangle$.*

Proof 1. It is an easy calculation to show that $\tau\,\alpha_h\,\tau'$ implies $\mathbf{P}\,(f)\,(\tau)\,\alpha_{h'}\,\mathbf{P}\,(f)\,(\tau)$. This is so because f is a morphism for the algebras.
2. Since for each $\tau \in \mathbf{P}\,(X)$ there exists $x \in X$ such that $[\tau]_{\alpha_h} = [\delta_x]_{\alpha_h}$ (in fact, $h(\tau)$ would do, because $h(\tau) = h\left(\delta_{h(\tau)}\right)$, as shown above), it is enough to demonstrate that

$$\chi'_{h'}\left(\mathbf{P}\,(f)_{\alpha_h,\alpha'_{h'}}\left([\delta_x]_{\alpha_h}\right)\right) = f(\chi_h([\delta_x]_{\alpha_h}))$$

is true for each $x \in X$. Because $\mathbf{P}\,(f)\,(\delta_x) = \delta_{f(x)}$, a little computation shows that both sides of the above equation boil down to $f(x)$. $\square$
The morphisms between G-triplets are just the morphisms between algebras (when we forget that these games play in different categories).

Proposition 6 *Let $f : \langle X, \alpha, \chi\rangle \to \langle X', \alpha', \chi'\rangle$ be a morphism between G-triplets, and let $\langle X, h\rangle$ resp. $\langle X', h'\rangle$ be the associated algebras. Then $f : \langle X, h\rangle \to \langle X', h'\rangle$ is an algebra morphism.*

Proof Given $\tau \in \mathbf{P}\,(X)$ we have to show that $(f \circ h)(\tau)$ equals $(h' \circ \mathbf{P}\,(f))(\tau)$. Since $h(\tau) = \chi([\tau]_\alpha)$, we obtain

$$\begin{aligned} (f \circ h)(\tau) &= f\,(\chi([\tau]_\alpha)) \\ &= \chi'\left(\mathbf{P}\,(f)_{\alpha,\alpha'}\,([\tau]_\alpha)\right) \\ &= \chi'\,([\mathbf{P}\,(f)\,(\tau)]_{\alpha'}) \\ &= (h' \circ \mathbf{P}\,(f))(\tau) \end{aligned}$$

$\square$
Putting all these constructions with their properties together, we obtain

Proposition 7 *The category $\mathfrak{Alg}$ of algebras for the Giry monad is isomorphic to the algebra $\mathfrak{GTrip}$ of G-triplets.*

After these general investigations we will turn to a more specific discussion by having a look at some examples including the finite and the countable case.

4 Examples

Identifying the algebras for the Manes monad on which set theoretic relations are based leads to the elegant characterization through sup-semilattices that has been mentioned above. Unfortunately, this situation does not carry over to the probabilistic case. The construction of examples for algebras is somewhat unwieldy and displays a wide spectrum. This becomes clear as we discuss a few of them.
We first show that the monad carries an instance of an algebra with it for each Polish space. Then we prove that in the finite case an algebra exists only in the case of a singleton set. Finally a geometrically oriented example is discussed by investigating the barycenter of a probability in a compact and convex subset of $\mathbb{R}^n$.

Example 1 The pair $\langle \mathbf{P}(X), \mu_X \rangle$ is always an algebra. We know from Lemma 1 that $\mu_X : \mathbf{P}(\mathbf{P}(X)) \to \mathbf{P}(X)$ is continuous. Because $\langle \mathbf{P}, \eta, \mu \rangle$ is a monad, the natural transformation $\mu : \mathbf{P}^2 \stackrel{\bullet}{\to} \mathbf{P}$ satisfies

$$\mu \circ \mathbf{P}\mu = \mu \circ \mu\mathbf{P}$$

in the category of functors with natural transformations as morphisms, see the diagram at the end of Section 2. Since $(\mathbf{P} \circ \mu)_X = \mathbf{P}(\mu_X)$ and $(\mu \circ \mathbf{P})_X = \mu_{\mathbf{P}(X)}$, this translates to

$$\mu_X \circ \mathbf{P}(\mu_X) = \mu_X \circ \mu_{\mathbf{P}(X)}.$$

Because the equation $\mu_X \circ \eta_X = id_{\mathbf{P}(X)}$ is easily established through a simple computation, the defining diagrams are commutative. —

The finite case can easily be characterized: there are no algebras for $\{1, \ldots, n\}$ unless $n = 1$. This will be shown now. As a byproduct we obtain a simple geometric description as a necessary condition for the existence of algebras.
We need a wee bit elementary topology for this.

Definition 7 *A metric space A is called* connected *iff the decomposition $A = A_1 \cup A_2$ with disjoint open sets A_1, A_2 implies $A_1 = \emptyset$ or $A_2 = \emptyset$.*

Thus a connected space cannot be decomposed into two non-trivial open sets. The connected subspaces of the real line $\mathbb{R}$ are just the open, half-open or closed finite or infinite intervals. The rational numbers $\mathbb{Q}$ are not connected. A subset $\emptyset \neq A \subseteq \mathbb{N}$ of the natural numbers which carries the discrete topology (because we assume that it is a Polish space) is connected as a subspace iff $A = \{n\}$ for some $n \in \mathbb{N}$.
The following facts about connected spaces are well known, cp. [7, Chapter 6.1] (or any other standard reference to topology).

Lemma 5 *Let A be a metric space.*

1. *If A is connected, and $f : A \to B$ is a continuous and surjective map to another metric space B, then B is connected.*
2. *If two arbitrary points in A can be joined through a connected subspace of A, then A is connected.*

This has as a consequence

Corollary 1 *If $\langle X, h\rangle$ is an algebra for the Giry monad, then X is connected.*

Proof If $\tau_1, \tau_2 \in \mathbf{P}(X)$ are arbitrary probability measures on X, then the line segment $\{c \cdot \tau_1 + (1-c) \cdot \tau_2 \mid 0 \leq c \leq 1\}$ is a connected subspace which joins τ_1 and τ_2. This is so because it is the image of the connected unit interval $[0, 1]$ under the continuous map $t \mapsto t \cdot \tau_1 + (1-t) \cdot \tau_2$. Thus $\mathbf{P}(X)$ is connected by Lemma 5. Since h is onto, its image X is connected. $\Box$
Consequently it is hopeless to search for algebras for, say, the natural numbers or a non-trivial subset of it:

Corollary 2 *A subspace $A \subseteq \mathbb{N}$ has algebras for the Giry monad iff A is a singleton set.*

Proof It is clear that a singleton set has an algebra. Conversely, if A has an algebra, then A is connected by Lemma 1, and this can only be the case when A is a singleton. $\Box$
The next example deals with the unit interval:

Example 2 The map

$$h : \mathbf{P}([0, 1]) \ni \tau \mapsto \int_0^1 t\ \tau(dt) \in [0, 1]$$

defines an algebra $\langle [0, 1], h\rangle$. In fact, $h(\tau) \in [0, 1]$ because τ is a probability measure. It is clear that $h(\delta_x) = x$ holds, and — by the very definition of the weak topology — that $\tau \mapsto h(\tau)$ is continuous. Thus by Proposition 2 it remains to show that the partition induced by h is convex. Let $h(\tau_i) = x_i = h(\tau_i')$ for $i = 1, 2.$ and $0 \leq c \leq 1$. Then

$$\begin{aligned} h(c \cdot \tau_1 + (1-c) \cdot \tau_2) &= c \cdot \int_0^1 t\ \tau_1(dt) + (1-c) \cdot \int_0^1 t\ \tau_2(dt) \\ &= c \cdot x_1 + (1-c) \cdot x_2 \\ &= h(c \cdot \tau_1' + (1-c) \cdot \tau_2'). \end{aligned}$$

Consequently, the partition induced by h is a G-partition, showing that h is indeed the morphism part of an algebra. —

The final example has a decidedly geometric touch to it. We deal with bounded and closed subsets of some Euclidean space and show that the construction of a barycenter yields an algebra. Fix $X \subseteq \mathbb{R}^n$ as a bounded, closed and convex subset of the Euclidean space $\mathbb{R}^n$ (for example, X could be a closed ball or a cube in $\mathbb{R}^n$).
Denote for two vectors $x, x' \in \mathbb{R}^n$ by $x \star x' := \sum_{i=1}^n x_i \cdot x_i'$ their inner product. Then $\lambda x.x \star x'$ constitutes a continuous linear map on $\mathbb{R}^n$ for fixed x'. In fact, each linear functional on $\mathbb{R}^n$ can be represented in this way.

Definition 8 *The vector* $x^* \in \mathbb{R}^n$ *is called a* barycenter *of the probability measure* $\tau \in \mathbf{P}(X)$ *iff*

$$x \star x^* = \int_X x \star y \; \tau(dy)$$

holds for each $x \in X$.

Because X is compact, the integrand is bounded on X, thus the integral is always finite. We collect some basic facts about barycenters and refer the reader to [8] for details.

Lemma 6 *The barycenter of* $\tau \in \mathbf{P}(X)$ *exists, it is uniquely determined, and it is an element of* X.

Proof Once we know that the barycenter exists, uniqueness follows from the well-known fact that the linear functionals on $\mathbb{R}^n$ separate points. Existence of the barycenter is established in [8, Theorem 461 E], its membership in X follows from [8, Theorem 461 H]. $\square$
These preparations help in establishing that the barycenter constitutes an algebra:

Proposition 8 *Let* $h(\tau)$ *be the barycenter of* $\tau \in \mathbf{P}(X)$. *Then* $\langle X, h\rangle$ *is an algebra for the Giry monad.*

Proof 1. $h : \mathbf{P}(X) \to X$ is well defined by Lemma 6. From the uniqueness of the barycenter it is clear that $h(\delta_x) = x$ holds for each $x \in X$.
2. Assume that $(\tau_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbf{P}(X)$ with $\tau_n \rightharpoonup_w \tau_0$. Put $x_n^* := h(\tau_n)$ as the barycenter of τ_n, then $(x_n^*)_{n\in\mathbb{N}}$ is a sequence in the compact set X, thus has a convergent subsequence (which we take w.l.g. as the sequence itself). Let x_0^* be its limit. Then we have for all $x \in X$:

$$x \star x_n^* = \int_X x \star y \; \tau_n(dy) \to \int_X x \star y \; \tau_0(dy) = x \star x_0^*$$

Hence h is continuous.
3. It remains to show that the partition induced by h is convex. This, however, follows immediately from the linearity of $y \mapsto \lambda x.x \star y$. $\square$
It should be mentioned that this example can be generalized considerably to metrizable topological vector spaces. The terminological effort is, however, somewhat heavy, and the example remains essentially the same. Thus we refrain from a more general discussion.
These examples show that a concise characterization of the algebras for the Giry monad is not easily available, in contrast to the Manes monad as the counterpart for the set theoretic case. This supports the observation again that easily stated facts for set theoretic relations are sometimes rather hard to establish for their probabilistic cousins.

5 Further Work

We characterize the algebras for the Giry monad which assigns each Polish space its space of probabilities. The morphisms in this category are continuous maps between Polish spaces. Continuity is technically crucial for the argumentation: when investigating an algebra, we start with discrete measures, for which the equation defining an algebra gives some characteristics that render their treatment feasible. Since discrete measures are dense, and since we are dealing with continuous maps only, these characteristics can be carried over to general measures through a convergence argument. This approach will not work for the case that general

Borel measurable maps serve as morphisms between Polish spaces (although these maps are fairly interesting from the point of view of applications), thus a more general characterization for these algebras is desirable.
Continuity plays also a crucial role in some of the examples that are discussed. Through the geometric argument of connectedness we could show that for the discrete case no algebras exist, except in the very trivial case of a one point space. This argument also does work only when the morphisms involved are continuous. So it is desirable to find algebras for the general case of Borel maps over finite domains (probably they do not exist usually there either: one would also like to know that). The last example hints at a connection between these algebras and barycenters for compact convex sets in topological vector spaces. It ends here where the fun begins there, viz., when looking at Choquet's theory of integral representations. There is room for further work exploring this avenue. The examples show that the world of algebras for this monad is quite colorfully polymorphous.
The most interesting question, however, addresses the expansion of the characterization given here for Borel measurable maps which are based on Polish spaces, or, going one crucial step further, on analytic ones.

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