Neutron Flux and Criticality in Two-Group and Eight-Group Examples

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Introduction

Neutron balance is a very important factor to consider in a nuclear reactor. The balance of neutrons from one generation to the next determines the criticality of the reactor, which in turn describes the reactor's efficiency. If the criticality is too low, the reactor will not sustain itself, but if it's too high, the reactor is a nuclear bomb. Important to criticality is the flux of neutrons through the reactor, which increases many of the factors used to determine criticality [1]. Therefore, it is important to calculate both neutron flux and criticality.

Problem 1

Description

In an infinite reactor with multiple different energy groups, how does one go about finding the criticality and neutron flux? If, for now, we assume 2 energy groups, E_1 and E_2 where E_1 is greater than E_2, the equations for balance are:

$$\Sigma_{a1}\varphi_1 + \Sigma_{1\to 2}\varphi_1 = \frac{\chi_1}{k}\left(\vartheta\Sigma_{f1}\varphi_1 + \vartheta\Sigma_{f2}\varphi_2\right) + \Sigma_{2\to 1}\varphi_2 \quad (1)$$

$$\Sigma_{a2}\varphi_2 + \Sigma_{2\to 1}\varphi_2 = \frac{\chi_2}{k}\left(\vartheta\Sigma_{f2}\varphi_2 + \vartheta\Sigma_{f1}\varphi_1\right) + \Sigma_{1\to 2}\varphi_1 \quad (2)$$

Then these equations can be converted into matrices:

$$\begin{bmatrix}\Sigma_{a1} & 0 \\ 0 & \Sigma_{a2}\end{bmatrix}\begin{bmatrix}\varphi_1 \\ \varphi_2\end{bmatrix} + \begin{bmatrix}\Sigma_{1\to 2} & 0 \\ 0 & \Sigma_{2\to 1}\end{bmatrix}\begin{bmatrix}\varphi_1 \\ \varphi_2\end{bmatrix} = \frac{1}{k}\begin{bmatrix}\chi_1\vartheta\Sigma_{f1} & \chi_1\vartheta\Sigma_{f2} \\ \chi_2\vartheta\Sigma_{f1} & \chi_2\vartheta\Sigma_{f2}\end{bmatrix}\begin{bmatrix}\varphi_1 \\ \varphi_2\end{bmatrix} + \begin{bmatrix}0 & \Sigma_{2\to 1} \\ \Sigma_{1\to 2} & 0\end{bmatrix}\begin{bmatrix}\varphi_1 \\ \varphi_2\end{bmatrix} \quad (3)$$

Which is a form where criticality k and flux φ can be more easily derived.

Derivation

If one defines the matrices as the following:

$$Absorbtion\ A = \begin{bmatrix} \Sigma_{a1} & 0 \\ 0 & \Sigma_{a2} \end{bmatrix} \tag{4}$$

$$Outscattering\ S_{out} = \begin{bmatrix} \Sigma_{1\to2} & 0 \\ 0 & \Sigma_{2\to1} \end{bmatrix} \tag{5}$$

$$Inscattering\ S_{in} = \begin{bmatrix} 0 & \Sigma_{2\to1} \\ \Sigma_{1\to2} & 0 \end{bmatrix} \tag{6}$$

$$Fission\ F = \begin{bmatrix} \chi_1\vartheta\Sigma_{f1} & \chi_1\vartheta\Sigma_{f2} \\ \chi_2\vartheta\Sigma_{f1} & \chi_2\vartheta\Sigma_{f2} \end{bmatrix} \tag{7}$$

One can then move the inscattering over to the left side of equation (3) and rearranges the equation so that k is in the numerator, the equation becomes:

$$k\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = [A + S_{out} - S_{in}]^{-1}[F]\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$$

This is the form of an eigenvector/eigenvalue problem, where k is the eigenvalue and the φ matrix is the eigenvector. Of final note is defining two matrices:

$$Migration\ Matrix\ M = [A + S_{out} - S_{in}] \tag{8}$$

$$B\ matrix\ B = [A + S_{out} - S_{in}]^{-1}[F] \tag{9}$$

The migration matrix is important for how it describes the loss of neutrons in the reactor, and the B matrix describes the number of neutrons created over the number of neutrons lost. In a sense words, it describes the effective multiplication factor k, which determines the criticality.

Problem 2

Description

This problem describes an infinite reactor with 2 energy groups, much like the problem before it. Also like the problem before it, one needs to find the criticality of the reactor and the flux of each energy group. This can be done in one of three ways: computers, power iteration, and by hand. A computer is quick, and python’s linalg package can give all the eigenvalues and eigenvectors of a matrix. Power iteration involves iterating through the equations:

$$\varphi_{i+1} = \frac{B\varphi_i}{||B\varphi_i||_2}$$

$$k_{i+1} = \frac{(B\varphi_{i+1})^T \varphi_{i+1}}{({\varphi_{i+1}}^T \varphi_{i+1})}$$

Where φ_1 and k_1 are a random vector matrix and constant respectively, however in this problem the φ_1 vector is all 1 and k_1 is 1. The more iterations run, the closer to the largest eigenvalue and eigenvector one comes. However, the power iteration only gives one eigenvalue and eigenvector, so it is not universally applicable. That is when one can calculate the eigenvalues and eigenvectors by hand. This starts with the equation:

$$\det([B] - \lambda[I]) = 0 \tag{12}$$

This gives a quadratic equation that can be solved for the eigenvalues λ. After calculating the eigenvalues, they get plugged into this equation:

$$\left[[B] - \lambda_i[I]\right] \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = 0 \tag{13}$$

From this point, one transforms the matrix created into row-echelon form, which gives:

$$\begin{bmatrix} (a - \lambda_i)\varphi_1 & b\varphi_2 \\ 0 & (c - \lambda_i)\varphi_2 \end{bmatrix} = 0 \tag{14}$$

Which can be solved for the eigenvector Φ_1 in terms of φ_2 as:

$$\Phi_1 = \varphi_2 \begin{bmatrix} d \\ e \end{bmatrix} \tag{15}$$

Where φ_2 can be any constant and d and e are related in a ratio defined by the solution.

Results

The migration and fission matrices are, respectively:

$$M = \begin{bmatrix} 0.0294 & 0 \\ -0.0202 & 0.0932 \end{bmatrix} \tag{16}$$

$$F = \begin{bmatrix} 0.0046 & 0.1139 \\ 0 & 0 \end{bmatrix} \tag{17}$$

The eigenvalues and eigenvectors as a result from hand calculation are:

$$k_1 = 0, \;\; k_2 = 0.9963 \tag{18}$$

$$\Phi_1 = \begin{bmatrix} -24.83 \\ 1 \end{bmatrix}, \qquad \Phi_2 = \begin{bmatrix} 4.61 \\ 1 \end{bmatrix} \tag{19}$$

Python's calculation of the eigenvalues and eigenvectors are:

$$k_1 = 0, \quad k_2 = 0.9961 \tag{20}$$

$$\Phi_1 = \begin{bmatrix} -0.992 \\ 0.04035 \end{bmatrix}, \qquad \Phi_2 = \begin{bmatrix} -0.977 \\ -0.212 \end{bmatrix} \tag{21}$$

Finally, the power iterated calculation of the largest eigenvalue and eigenvector after 2 iterations is:

$$k = 0.9961 \tag{22}$$

$$\Phi = \begin{bmatrix} 0.977 \\ 0.212 \end{bmatrix} \tag{23}$$

Discussion

These results show one that the reactor is close criticality if one takes the largest eigenvalue. The number of neutrons in the next generation is 99.61% of the number of neutrons in the current one. These results also show that python's linear algebra eigenfunction and eigenvalue calculation function is not extremely reliable. This is because it returns negative values for the φ_2 vector, which should not happen in the theoretical infinite reactor this problem assumes. It would be impossible for the assumed isotropic release of neutrons be give a negative flux. The hand calculation assumes that $\varphi_2 = 1$. It should be noted that the hand calculation eigenvectors are multiples of the computer calculated eigenvectors. This is because python's eig function normalizes the eigenvectors to the unit vector.

Problem 3

Description

This problem is the same situation as problem 2, except with more energy groups. Therefore, the same calculations and equations apply, with the exception of the hand calculation of the eigenvalues and eigenvectors. That said, all the matrices in the B matrix are eight by eight matrices, as is the B matrix itself. The flux vector is also an eight by one matrix instead of a two by one as it was in problem 2. Besides that though, practically all the operations are the same.

Results

The migration and fission matrices are:

$$M = \begin{bmatrix} .0888 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -.053 & .1193 & 0 & 0 & 0 & 0 & 0 & 0 \\ -.0301 & -.1159 & .0813 & 0 & 0 & 0 & 0 & 0 \\ -1e-4 & -5e-4 & -.0769 & .2152 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1.9e-3 & -.1961 & .2529 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5e-3 & -.1737 & .3437 & -2.3e-3 & 0 \\ 0 & 0 & 0 & -7e-4 & -.0246 & -.2707 & .417 & -.0275 \\ 0 & 0 & 0 & -1e-4 & -7.3e-3 & -.055 & -.3589 & .2073 \end{bmatrix} \quad (24)$$

$$F = \begin{bmatrix} .0047 & .002 & .00039 & .0023 & .0077 & .0078 & .0315 & .0751 \\ .0055 & .0030 & .00045 & .0028 & .0090 & .0091 & .0368 & .0879 \\ .0032 & .0030 & .00026 & .0016 & .0053 & .0053 & .0214 & .0511 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (25)$$

As a minor note, the χ values for energy groups above 3 are all 0, which means that their full row is also 0 due to how the fission matrix is constructed. The python calculated eigenvalues and eigenvectors are:

$$k_1 = 1.093, \quad k_2 = 2.25e-3, \quad k_3 = 6.77e-6, \quad k_4 = 2.39e-17,$$
$$k_5 = 7.72e-18+7.69e-18i,$$
$$k_6 7.72e-18-7.69e-18i, \qquad k_7 = -3.66e-15+1.33e-18i,$$
$$k_8 = -3.66e-15+1.33e-18i \tag{26-33}$$

$$\Phi_1 = \begin{bmatrix} -.26 \\ -.34 \\ -.78 \\ -.28 \\ -.22 \\ -.12 \\ -.1 \\ -.22 \end{bmatrix}, \Phi_2 = \begin{bmatrix} -.76 \\ .64 \\ .07 \\ .026 \\ .021 \\ .011 \\ 9.7e-3 \\ .02 \end{bmatrix}, \Phi_3 = \begin{bmatrix} -.98 \\ 1.5e-3 \\ .15 \\ .055 \\ .044 \\ .023 \\ .021 \\ .043 \end{bmatrix}, \Phi_4 = \begin{bmatrix} -.28 \\ -1.4e-14 \\ .63 \\ .56 \\ .40 \\ .14 \\ -.18 \\ .018 \end{bmatrix},$$

$$\Phi_5 = \begin{bmatrix} -.2-.1i \\ -3.6e-15-1.8e-15i \\ -.34-.31i \\ .39+.21i \\ -.13+.037i \\ .69 \\ .046+.15i \\ -.075-.065i \end{bmatrix}, \Phi_6 = \begin{bmatrix} -.2+.1i \\ -3.6e-15+1.8e-15i \\ -.34+.31i \\ .39-.21i \\ -.13-.037i \\ .69 \\ .046-.15i \\ -.075+.065i \end{bmatrix}, \tag{34-41}$$

Cont.

$$\Phi_7 = \begin{bmatrix} -.061+.021i \\ -1.5e-16+23-15i \\ .86 \\ .12-.42i \\ -.005+.12i \\ -.028-.124i \\ .007+.023i \\ -.004-.009i \end{bmatrix} \Phi_8 = \begin{bmatrix} -.061-.021i \\ -1.5e-16-23-15i \\ .86 \\ .12+.42i \\ -.005-.12i \\ -.028+.124i \\ .007-.023i \\ -.004+.009i \end{bmatrix}$$

The power iterated eigenvalue and eigenvector after two iterations are:

$$k = 1.093 \tag{42}$$

$$\Phi = \begin{bmatrix} .260 \\ .345 \\ .782 \\ .280 \\ .223 \\ .118 \\ .105 \\ .220 \end{bmatrix} \tag{43}$$

Discussion

Much like in problem 2, the python calculated solutions give values that cannot be obtained in a “real” infinite isotropic reactor. However, it is interesting that the formula gave imaginary values for the lower energy fluxes. This does not seem to stem from the fission matrix's rows of zeroes for energy groups less than group 3, because the group 4 eigenvector does not have any imaginary values. The imaginary numbers would therefore seem to stem from some values in the migration matrix. Similarly, it is of note that where φ_5's and φ_6's imaginary values are complex conjugates, as are φ_7's and φ_8's imaginary values. This would imply that these energy groups are coupled in such a way that they are defined by each other. On a more realistic note, the power iterated k value is above 1. This means that the reactor in this problem is supercritical and could be in danger of melting down if left unchecked.

References

[1] Shultis, J. Kenneth, Richard E. Faw. 2017. "Chapter 10: Radioactivity" *Fundamentals of Nuclear Science and Engineering Third Edition* : 321-369.